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A canonical Ramsey theorem with list constraints in random graphs

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Abstract

The celebrated canonical Ramsey theorem of Erdős and Rado implies that for a given graph H, if n is sufficiently large then any colouring of the edges of K_n gives rise to copies of H that exhibit certain colour patterns, namely monochromatic, rainbow or lexicographic. We are interested in sparse random versions of this result and the threshold at which the random graph $\mathbf{G}(n, p)$ inherits the canonical Ramsey properties of K_n. Our main result here pins down this threshold when we focus on colourings that are constrained by some prefixed lists. This result is applied in an accompanying work of the authors on the threshold for the canonical Ramsey property (with no list constraints) in the case that H is an even cycle.

Keywords: random graph; canonical Ramsey; Ramsey theory

1. Introduction

For $r \in \mathbb{N}$ and graphs G and H, we say G has the *r-Ramsey property* (with respect to H), denoted $G \xrightarrow{r} H$, if every colouring of the edges of G with r colours results in a *monochromatic* copy of H, that is, a copy of H with all its edges in the same colour. The classical theorem of Ramsey [12], from which the term *Ramsey theory* stems, states that if n is large enough in terms of r and H, then $K_n \xrightarrow{r} H$. In a highly influential work, Erdős and Rado [5] explored what colour patterns are guaranteed when one colours a graph with no restriction on the number of colours. Clearly,

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monochromatic copies of H are no longer guaranteed as one can colour each edge of K_n with a unique colour. Graphs with such edge-colourings, where each edge receives a unique colour, are said to be *rainbow*.

If H is a forest, every colouring of K_n induces either a monochromatic or a rainbow copy of H, provided that n is sufficiently large.[1] If H contains a cycle, it is also not the case that every colouring of K_n induces either a monochromatic or rainbow copy of H. Indeed, one can associate a unique colour $c(i)$ to each vertex i in $[n] = V(K_n)$ and colour each edge $ij \in E(K_n)$ by $c(\min\{i, j\})$. Then any copy of H in K_n is neither monochromatic nor rainbow but is coloured *lexicographically*.

Definition 1.1 (Lexicographic colouring)**.** Let H be a graph, σ an ordering of $V(H)$ and $\chi : E(H) \to \mathbb{N}$ an edge colouring of H. We say that the pair (H, χ) is *lexicographic* with respect to σ if there exists an injective assignment of colours $\phi : V(H) \to \mathbb{N}$ such that for every edge $e = uv \in E(H)$ with $u <_\sigma v$, we have that $\chi(e) = \phi(u)$. If χ is clear from the context, we simply say that H is lexicographic with respect to σ.

The celebrated *canonical Ramsey theorem* of Erdős and Rado [5] implies that if n is large enough in terms of $m \in \mathbb{N}$, then any colouring of K_n results in a copy of K_m that is either monochromatic, rainbow or lexicographic. This theorem serves as a beautiful example of the popular Ramsey theory maxim that there is an inevitable order amongst chaos. Applying the canonical Ramsey theorem with $m = v(H)$ implies the existence of H with certain colour patterns. The following definition captures this behaviour.

Definition 1.2 (The canonical Ramsey property)**.** Given a graph H and an ordering σ of $V(H)$, a coloured copy of H is *canonical with respect to* σ if it is monochromatic, rainbow or lexicographic with respect to σ. A graph G has the *H-canonical Ramsey property*, denoted $G \xrightarrow{\text{can}} H$, if for every edge colouring $\chi : E(G) \to \mathbb{N}$ and every ordering σ of $V(H)$, there is a copy of H which is canonical with respect to σ.

Note that in the case that there are neither monochromatic nor rainbow copies of H, our definition requires copies of H with *all* possible lexicographic colourings. As observed by Jamison and West [7], the canonical Ramsey theorem implies that for any H, a set S of colour patterns of H is unavoidable in all colourings of large enough cliques if and only if S contains the monochromatic pattern, the rainbow pattern and at least one lexicographic pattern. Hence our notion of G having the H-canonical Ramsey property is as strong as possible; any set of colour patterns that is unavoidable in large enough cliques should be unavoidable in all colourings of G. Alternative definitions of canonical Ramsey properties are discussed in Section 1.4.

1.1. Sparse Ramsey theory and random graphs

Returning to the setting of colourings with a bounded number of colours, a prominent theme in Ramsey theory has been to explore the existence of *sparse* graphs G that are r-Ramsey with respect to H; see for example [11] and the references therein. One famous example is the work of Frankl and Rödl [6], who used a random graph to construct a K_4-free graph G such that $G \xrightarrow{2} K_3$. This prompted Łuczak, Ruciński and Voigt [9] to initiate the study of thresholds for Ramsey properties in random graphs, which has since become a prominent theme in probabilistic combinatorics. It turns out that the threshold for $\mathbf{G}(n, p)$ having the Ramsey property with respect to a graph H is governed by the following parameter of H.

Definition 1.3. Given a graph H with at least 2 edges, the *maximum 2-density* of H is defined by

$$m_2(H) := \max\left\{\tfrac{e(F)-1}{v(F)-2} : F \subseteq H,\ v(F) > 2\right\}.$$

In a seminal series of papers, Rödl and Ruciński [13, 14, 15] established the threshold for the Ramsey property when the random graph is coloured with a bounded number of colours. Here and throughout, we say that a function

[1] One can deduce this fact from an application of the Erdős-Rado Theorem (see Theorem 1.2).

$\hat{p} = \hat{p}(n)$ is the *threshold* for a monotone increasing graph property $\mathcal{P}$ if

$$\lim_{n\to\infty} \Pr(\mathbf{G}(n,p) \text{ satisfies } \mathcal{P}) = \begin{cases} 0 & \text{if } p = o(\hat{p}), \\ 1 & \text{if } p = \omega(\hat{p}). \end{cases}$$

We refer to $\hat{p}$ as *the* threshold for $\mathcal{P}$, although it is only defined up to order of magnitude.

Theorem 1.4 (Rödl–Ruciński [13, 14, 15])**.** *Let $r \geq 2$ be an integer and H be a nonempty graph which is not a star forest. Then $n^{-1/m_2(H)}$ is the threshold for the property $\mathbf{G}(n,p) \xrightarrow{r} H$.*

1.2. Canonical Ramsey properties of random graphs

The motivation for the current work is to establish the threshold for the canonical Ramsey property with respect to a given graph H. Note that for any H as in Theorem 1.4 that is not a triangle, the threshold for the property that $\mathbf{G}(n,p) \xrightarrow{\text{can}} H$ is at least $n^{-1/m_2(H)}$. Indeed, for any such H, there is an ordering σ of its vertices such that the lexicographic colouring of H with respect to σ uses at least 3 colours. Using Theorem 1.4, we have that when $p = o\left(n^{-1/m_2(H)}\right)$, asymptotically almost surely (a.a.s. from now on) there is a 2-colouring of $\mathbf{G}(n,p)$ that avoids monochromatic copies of H. Moreover, such a colouring avoids rainbow and lexicographic copies of H with respect to σ, simply because there are not enough colours available for such colour patterns. This shows that $\mathbf{G}(n,p)$ does not have the canonical Ramsey property for H for such p.

We believe that this lower bound is in fact the correct threshold for the canonical Ramsey property and that when $p = \omega\left(n^{-1/m_2(H)}\right)$, a.a.s. $\mathbf{G}(n,p) \xrightarrow{\text{can}} H$. Here, we provide evidence for this by focusing on colourings that are constrained to be compatible with a given list assignment.

Definition 1.5 (Colourings with list constraints)**.** Let $1 \leq r \in \mathbb{N}$ and $\mathcal{L} : E(K_n) \to \mathbb{N}^r$ be an assignment of lists of colours to the edges of K_n (note that we allow lists to have repeated colours). We say that a colouring $\chi : E(G) \to \mathbb{N}$ of an n-vertex graph $G \subseteq K_n$, is *compatible* with $\mathcal{L}$ if for all $e \in E(G)$, we have that $\chi(e) \in \mathcal{L}(e)$.

Our main theorem shows that for any assignment $\mathcal{L}$ of bounded lists to the edges of K_n, the threshold for the canonical Ramsey property with respect to H when considering colourings that are compatible with $\mathcal{L}$ is at most $p = n^{-1/m_2(H)}$.

Theorem 1.6. *Let H be a nonempty graph, $1 \leq r \in \mathbb{N}$ and $\mathcal{L} : E(K_n) \to \mathbb{N}^r$ be a list assignment of colours. If $p = \omega\left(n^{-1/m_2(H)}\right)$ then a.a.s. any edge colouring χ of $\mathbf{G} \sim \mathbf{G}(n,p)$ which is compatible with $\mathcal{L}$ contains a canonical copy of H with respect to σ for all orderings σ of $V(H)$.*

Note that for many assignments of lists $\mathcal{L}$ this theorem establishes the threshold for the canonical Ramsey property restricted to colourings compatible with $\mathcal{L}$. Indeed, by the same reasoning discussed above, this is the case whenever there are 2 colours that feature on all lists. We will deduce Theorem 1.6 from the more general result, namely Theorem 3.1, which deals with graphs of the form $\Gamma \cap \mathbf{G}(n,p)$ where Γ is a "locally dense graph" (see Section 2.2 for the relevant definitions).

1.3. An application for even cycles

We believe that Theorem 1.6 provides a natural stepping stone towards establishing $n^{-1/m_2(H)}$ as the threshold for the canonical Ramsey property in random graphs for all graphs H. In fact, the theorem arose naturally in a work of the authors proving a 1-statement for the canonical Ramsey property with respect to even cycles. Indeed, Theorem 1.6 (or rather its stronger version, Theorem 3.1) is a key component of the proof of the following theorem, which is given in an accompanying paper [1].

Theorem 1.7. *Let $k \geq 2$ be an integer. If $p = \omega\left(n^{-(2k-2)/(2k-1)} \log n\right)$, then a.a.s.*

$$\mathbf{G}(n, p) \xrightarrow{\text{can}} C_{2k}.$$

As we mentioned before, $n^{-1/m_2(H)}$ is a lower bound for the threshold for the canonical Ramsey property with respect to even cycles. Thus, since $m_2(C_{2k}) = (2k-1)/(2k-2)$, Theorem 1.7 establishes the threshold for the canonical Ramsey property with respect to even cycles, up to the log factor.

1.4. Alternative characterisations of the canonical Ramsey property

The canonical Ramsey theorem of Erdős and Rado [5] was stated in a different form to the one given here. Indeed, they consider *ordered* copies of K_m and K_n and prove that if n is large enough in terms of m, one can find an ordered embedding of K_m in K_n which is either monochromatic, rainbow or coloured lexicographically such that either every edge inherits its colour from its minimum vertex, or every edge inherit its colour from its maximum vertex, according to the ordering of $V(K_m)$.

For the case of $H = K_m$, this notion is strictly stronger than our notion of the canonical Ramsey property; one can guarantee a canonically coloured copy of H *and* also guarantee that the embedding of such copy respects the fixed ordering given on the vertices. One could also seek to embed ordered copies of H and define the canonical Ramsey property analogously, looking for monochromatic, rainbow, min-, or max-colourings of the ordered H. However, in general, when H is not complete, this is no longer stronger than the canonical Ramsey property defined here in Definition 1.2. Indeed, in the case that there are no monochromatic or rainbow copies of H, finding min- or max-coloured copies of all orderings of H does not necessarily guarantee copies of H with all lexicographic colour patterns. We have therefore opted to state our results here without looking for ordered embeddings so as to capture all sets of unavoidable colour patterns, as in [2, 7].

1.5. Related work

Until very recently, to our knowledge, there have been no results on canonical Ramsey properties in random graphs. However, simultaneously and independently to our work here and in [1], Kamčev and Schacht [8] obtained a remarkable result, completely resolving the problem for the case when $H = K_m$. Indeed, they show that for $4 \leq m \in \mathbb{N}$ and $p = \omega(n^{-2/(m+1)})$, a.a.s. $\mathbf{G}(n, p) \xrightarrow{\text{can}} K_m$. As $m_2(K_m) = (m+1)/2$, this establishes the threshold for the canonical Ramsey property with respect to complete graphs K_m. In contrast to our proofs, which appeal to the method of hypergraph containers, their proof relies on the transference principle of Conlon and Gowers [4]. Their methods also allow them to obtain partial results in the case where H is *strictly balanced*, that is when $m_2(H)$ is achieved only by H itself. For such a graph H and for $p = \omega(n^{-1/m_2(H)})$, they can prove the existence of monochromatic, rainbow or *some* lexicographic copies of H in any colouring of $\mathbf{G} \sim \mathbf{G}(n, p)$.

2. Preliminaries

In this section, we collect the necessary theory and tools needed in our proof. We will introduce the method of containers in Section 2.1 and the theory of locally dense graphs in Section 2.2. Before all of this though, we collect the relevant notation.

For simplicity, given a graph H we use $v(H)$ and $e(H)$, respectively, for $|V(H)|$ and $|E(H)|$. For a k-uniform hypergraph $\mathcal{H}$ and for $U \subseteq V(\mathcal{H})$, we let $\mathcal{H}[U]$ denote the hypergraph induced by $\mathcal{H}$ on U. Also, for any vertex subset $T \subseteq V(\mathcal{H})$, let $d_{\mathcal{H}}(T)$ denote the number of edges of $\mathcal{H}$ containing T. For $0 \leq j \leq k$, let $\Delta_j(\mathcal{H}) := \max\{d_{\mathcal{H}}(T) : T \subseteq V(\mathcal{H}), |T| = j\}$ denote the maximum degree of a vertex set of size j in $\mathcal{H}$.

The binomial random graph $\mathbf{G}(n, p)$ refers to the probability distribution of graphs on vertex set $[n]$ obtained by taking every possible edge independently with probability $p = p(n)$. We say an event happens asymptotically almost surely (a.a.s. for short) in $\mathbf{G} \sim \mathbf{G}(n, p)$ if the probability that it happens tends to 1 as n tends to infinity. We will also use standard asymptotic notation throughout, with asymptotics always being taken as the number of vertices n tends to infinity. Finally, we use the notation $a = b \pm c$ to denote a number a between $b - c$ and $b + c$ and we omit floors and ceilings throughout, so as not to clutter the arguments.

2.1. The method of containers

We will appeal to the method of hypergraph containers, developed by Balogh, Morris and Samotij [3], and independently, by Saxton and Thomason [16]. The key idea underlying this method is that if a uniform hypergraph has an edge set that is evenly distributed, then one can group the independent sets of the hypergraph into a well-behaved collection of *containers*. In more detail, these containers are vertex subsets that are almost independent (in that they induce few edges of the hypergraph), every independent set of the hypergraph lies in some container and, crucially, we have a bound on the number of containers. As there are far fewer containers than independent sets in the hypergraph, reasoning about containers rather than independent sets leads to more efficient arguments and this technique has proven to be extremely powerful. Indeed, the setting of independent sets in hypergraphs can be used to encode a wide range of problems in combinatorics and the method of hypergraph containers has been successfully exploited in a multitude of different settings. Particularly relevant to our work here are the applications of the method in sparse Ramsey theory, as done by Nenadov and Steger [10], who gave an alternative proof to the finite colour sparse Ramsey theorem using containers.

Below, we state the container lemma in the form given in [3, Theorem 2.2]. Before doing so, we need to establish some terminology and definitions.

Definition 2.1 (($\varepsilon, \mathcal{H}$)-abundant set families)**.** For hypergraph $\mathcal{H} = (V, E)$ and $0 < \varepsilon \leq 1$, we say a family $\mathcal{F} \subseteq 2^V$ of subsets of V is $(\mathcal{H}, \varepsilon)$*-abundant* if the following properties hold:

1. $\mathcal{F}$ is *increasing*: for all $A, B \subseteq V$ with $A \in \mathcal{F}$ and $A \subseteq B$, we have that $B \in \mathcal{F}$;
2. $\mathcal{F}$ contains only large vertex sets: for all $A \in \mathcal{F}$, we have that $|A| \geq \varepsilon v(\mathcal{H})$;
3. $\mathcal{H}$ is $(\mathcal{F}, \varepsilon)$*-dense*: for all $A \in \mathcal{F}$, we have that $e(\mathcal{H}[A]) \geq \varepsilon e(\mathcal{H})$.

For a hypergraph $\mathcal{H} = (V, E)$, we also define $\mathcal{I}(\mathcal{H}) \subseteq 2^V$ to be the collection of independent vertex sets in $\mathcal{H}$. We now state the container theorem [3, Theorem 2.2] and we refer to the discussion in [3] for motivation and context.

Theorem 2.2 (Hypergraph Container Theorem)**.** *For every $k \in \mathbb{N}$ and $\varepsilon, D_0 > 0$, there exists $D > 0$ such that the following holds. Suppose $\mathcal{H}$ is a k-uniform hypergraph, $\mathcal{F} \subseteq 2^{V(\mathcal{H})}$ is an $(\varepsilon, \mathcal{H})$-abundant set family and $q \in (0, 1)$ is such that for each $j \in [k]$, we have $\Delta_j(\mathcal{H}) \leq D_0 q^{j-1} e(\mathcal{H})/v(\mathcal{H})$. Then there exists a family $\mathcal{S} \subseteq 2^{V(\mathcal{H})}$ of 'fingerprints' and two functions $f : \mathcal{S} \to 2^{V(\mathcal{H})} \setminus \mathcal{F}$ and $g : \mathcal{I}(\mathcal{H}) \to \mathcal{S}$ such that:*

(a) (Relatively 'small' family) $|S| \leq Dqv(\mathcal{H})$ for all $S \in \mathcal{S}$;

(b) (Containment property) for each $I \in \mathcal{I}(\mathcal{H})$, we have that $g(I) \subseteq I$ and $I \setminus g(I) \subseteq f(g(I))$.

In applications of Theorem 2.2, the set $\mathcal{C} := \{f(S) \cup S : S \in \mathcal{S}\}$ is usually referred to as the set of *containers* for the hypergraph $\mathcal{H}$. Note that property (a) can be used to bound the size of $\mathcal{C}$ whilst property (b) shows that for every independent set $I \in \mathcal{I}(\mathcal{H})$ there is some $C \in \mathcal{C}$ such that $I \subseteq C$.

2.2. Locally dense graphs

We will establish our main theorem in the context of random sparsifications of locally dense graphs, a more general setting than $\mathbf{G}(n, p)$ which will be useful for applications. Here we collect some key properties of locally dense graphs.

Definition 2.3 ((ϱ, d)-dense graphs)**.** For $\varrho, d \in (0, 1]$, we say a graph Γ is (ϱ, d)*-dense* if the following holds: for every $S \subseteq V(\Gamma)$ with $|S| \geq \varrho n$, the induced graph $\Gamma[S]$ has at least $d\binom{|S|}{2}$ edges.

Remark 2.4. *In [15] the authors observed that in order to establish that a graph Γ is (ϱ, d)-dense it suffices to establish the defining property for subsets $S \subseteq V(\Gamma)$ of cardinality exactly ϱn.*

Graphs with the (ϱ, d)-denseness property for $\varrho = o(1)$ are often called *locally dense* graphs. This property can be viewed as a weak quasirandomness property. We close this section with the following result [15, Lemma 2], which can be proven by induction.

Proposition 2.5. *For every $m \geq 2$ and $d > 0$ there exist $\varrho, c_0 > 0$ such that if Γ is an n-vertex (ϱ, d)-dense graph and n is sufficiently large, then Γ contains at least $c_0 n^m$ copies of K_m.*

3. Outline of the proof

Our proof follows the scheme of Nenadov and Steger [10] and appeals to the method of hypergraph containers (see Section 2.1). We start by giving an rough outline of the proof.

Suppose we want to find a monochromatic H in any r-colouring of $\mathbf{G} \sim \mathbf{G}(n, p)$. Our idea is to create an auxiliary hypergraph $\mathcal{H}$ whose vertex set consists of r copies of $E(K_n)$ (one copy for each colour) and whose edge set encodes the monochromatic copies of H in K_n. The key observation is the following: any colouring of $\mathbf{G}$ that avoids monochromatic copies of H can be identified with an independent set in $\mathcal{H}$. Using containers, one can efficiently group together these independent sets and identify a small set of *containers* $\mathcal{C}$ such that each independent set of $\mathcal{H}$ belongs to one such container. The proof then proceeds by showing that, for each *fixed* container $C \in \mathcal{C}$, it is very unlikely[2] that the graph $\mathbf{G}$ lies within C. By this, we mean that it is unlikely that there is a colouring of $\mathbf{G}$ that, when mapped in the obvious way to a vertex subset of $\mathcal{H}$, corresponds to a subset of C. The proof follows by performing a union bound over the choices for a container $C \in \mathcal{C}$.

This containers approach relies crucially on the fact that the hypergraph $\mathcal{H}$ that encodes the monochromatic copies of H has $v(\mathcal{H}) = O(n^2)$ and so satisfies the the degree constraints of Theorem 2.2 with $q = \Theta(n^{-1/m_2(H)})$. Having an unbounded number of copies of $E(K_n)$ in $V(\mathcal{H})$ corresponding to an unbounded number of colours available leads to an adjustment on the degree constraints and the parameter q in Theorem 2.2. This renders the upper bound on the number of containers useless and the proof falls apart. In our proof we avoid such a problem by creating a hypergraph whose vertex set is composed of r copies of $E(K_n)$ with each copy of an edge corresponding to a choice of colour of that edge according to the list of available colours. The edge set of the hypergraph then encodes the canonical copies of H. The fact that each list is bounded allows us to apply Theorem 2.2 to our hypergraph with the correct parameters. As previously discussed, we can in fact prove the following more applicable result, which is a strengthening of Theorem 1.6 and applies to random sparsifications of locally dense graphs.

Theorem 3.1 (Sparse canonical Ramsey theorem with list constraints)**.** *Let H be a graph, $1 \leq r \in \mathbb{N}$ and $d > 0$. Then there exists $\varrho, c > 0$ such that the following holds. Let σ be an ordering of $V(H)$, let Γ be an n-vertex (ϱ, d)-dense graph and let $\mathcal{L} : E(\Gamma) \to \mathbb{N}^r$ be a list assignment. If $p = \omega(n^{-1/m_2(H)})$, then with probability at least $1 - \exp(-cpn^2)$, any edge colouring χ of $\Gamma \cap \mathbf{G}(n, p)$ that is compatible with $\mathcal{L}$ contains a canonical copy of H with respect to σ.*

Theorem 1.6 follows from Theorem 3.1 applied to $\Gamma = K_n$, which is $(\varrho, 1)$-dense for all $\varrho > 0$, and a union bound over the $v(H)!$ orderings σ of $V(H)$.

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