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BOUNDARY DELAY IN REACTION-DIFFUSION EQUATIONS: EXISTENCE AND CONVERGENCE TO EQUILIBRIUM

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Boundary Delay in Reaction-Diffusion Equations: Existence and Convergence to Equilibrium

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Abstract

We consider dissipative scalar reaction-diffusion equations that include the ones of the form $u_t - \Delta u = f(u(t))$, subjected to boundary conditions that include small delays, that is, we consider boundary conditions of the form $\frac{\partial u}{\partial n_a} = g(u(t), u(t-r))$. We show global existence and uniqueness of solutions in a convenient fractional power space, and furthermore, we show that, for r sufficiently small, all solutions are asymptotic to the set of equilibria as t tends to infinity.

1 Introduction

In recent years, there has been considerable effort devoted to the problem of stabilization and control of PDE through the application of forces on the boundary. The mathematical theory is very complete when the boundary forces are applied with no delays in time (see, for example, Lions [10]). On the other hand when the boundary forces are applied with delays in time in a nonlinear way, not much is known. See Hale [7] and references there in, for some aspects of delay on dynamics.

In this work we want to consider the equation

$$\begin{cases} u_t - \mathrm{Div}(a\nabla u) + \sum_{j=1}^{n} B_j(x)\frac{\partial u}{\partial x_j} + \mu u = f(u), & \text{in } \Omega \times \mathbb{R}^+ \\ \frac{\partial u}{\partial n_a} = g(u(t), u(t-r)), & \text{in } \partial\Omega \times \mathbb{R}^+ \\ u = \varphi, & \text{in } \Omega \times [-r, 0] \end{cases} \tag{1}$$

where Ω is an open bounded domain in $\mathbb{R}^n$, $r > 0$ is the delay, $a \in C^1(\overline{\Omega})$, $a(x) > m_0 > 0$, $x \in \Omega$, $\frac{\partial u}{\partial n_a} = (a\nabla u, \vec{n})$, $\vec{n}$ is the outward normal, λ is a positive constant and B_j is continuous in $\bar{\Omega}$, $j = 1, \cdots, n$. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ be smooth functions.

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Our aim in this work is to study one of the first questions that arises when you treat long time dynamics, stabilization and control. *Are the solutions asymptotic to the set of equilibria as t tends to infinity?*

In order to address this question we first have to check if it makes sense, that is, are *the solutions of (1) globally defined and unique?*

The method we will use to study (1) is semigroup theory, that is, to treat (1) as an evolution equation in a Banach space X, specifically, we will write (1), at least formally, in the abstract form

$$\begin{cases} \dot{u}(t) + Au(t) = H(u_t), \ \ t > 0 \\ \\ u(t) = \varphi(t), \ \ t \in [-r, 0] \end{cases} \tag{2}$$

where A will be an operator in X associated to the linear part of (1), $u_t : [-r, 0] \to X^\alpha$ denotes the function $u_t(\theta) = u(t+\theta), -r \leq \theta \leq 0$ and $H : C([-r,0], X) \to X$ will be a nonlinear term related to f and g.

Once (2) is defined, we will make use of abstract results to show existence and uniqueness of solutions of (2). Then we will apply elliptic regularity to recover the solutions of (1).

Such theory has been widely used to get well posedness of equations like (1). Most of them are based on the general results obtained by Henry [8] and applied to these problems. We will use the ideas of Amann [1, 2], combined with these general results, to get well posedness for (1). This strategy has been used successfully to study the undelayed counterpart of (1) in Oliva and Pereira [13], and in Carvalho, Oliva, Pereira and Rodriguez-Bernal [3] (without using Amann's results). The abstract results of Henry [8] have to be adapted to the case of delay equations, but this has been done by Oliveira [14].

Before we proceed, let us mention that a lot of work has been done if $g(u, v) = g(u)$ and if we introduce an interior delay that is, $f(u_t)$. Among others we can mention: Martin and Smith [12] - for existence using comparison results for equations with discrete delays; Ruan and Wu [17] - for the extension of the results of Martin and Smith [12] for equations with infinite delays; Travis and Webb [18, 19] - for existence results imposing that the nonlinearity be globally Lipschitz. In the case of boundary delay, most of the work deals with the case where the delay term is linear.

Once we have established global existence and uniqueness of (1), we can focus our attention on the main question, mentioned before. In this direction it is well-known and not hard to prove (see Hale [6], Henry [8] and Matano [15]) that for scalar reaction-diffusion equations,

$$u_t - \Delta u = f(u) \ \ (x \in \Omega \subset \mathbb{R}^n) \tag{3}$$

subject to homogeneous boundary conditions, all globally defined bounded solutions must approach the set of equilibria as t tends to infinity. This is a consequence of the fact that (3) is a gradient system (makes use of Lyapunov function) and can be easily generalized to equations with nonlinear boundary conditions.

Later on Friesecke [4] used the Lyapunov function of (3) to prove that for dissipative scalar reaction-diffusion equations, with sufficiently small delays,

$$u_t - \Delta u = f(u(t), u(t-r)) \quad (x \in \Omega \subset \mathbb{R}^n) \tag{4}$$

subject to homogeneous boundary conditions, all globally defined bounded solutions must approach the set of equilibria as t tends to infinity.

We want to extend this result when the delay appears on the boundary, that is to equation (1). We will do that following the ideas of Friesecke [4], but we do not use the Lyapunov function of (3).

As in Friesecke [4], we will assume that (1) is dissipative, without assuming any growth conditions on the undelayed part.

(H1) *Let us assume that the nonlinearity* $f : \mathbb{R} \mapsto \mathbb{R}$ *is* C^1 *and satisfy the dissipative condition*

$$\limsup_{|s| \longrightarrow \infty} \frac{f(s)}{s} \leq c_f, \tag{5}$$

where $c_f \in \mathbb{R}$.

The nonlinearity $g : \mathbb{R}^2 \mapsto \mathbb{R}$ *is assumed to be* C^2 *and to satisfy the dissipative condition*

$$\limsup_{|s| \longrightarrow \infty} \frac{g(s,t)}{s} \leq c_g(t), \tag{6}$$

for some continuous function c_g, *and all* $t \in \mathbb{R}$.

With this, we will show a decay estimate that will be enough to prove the convergence to the set of equilibria. This decay will be obtained just using the variation of constants formula and the exponential estimates of the semigroup generated by the linear part of (2).

The paper will proceed as follows: In Section 2 we will define the abstract equation equivalent to (1); In Section 3 we show existence and uniqueness of the solutions of the abstract equations, this will be done by applying the abstract results, that are stated in Subsection 3.1, to our specific equation (Subsection 3.2). After establishing global existence and uniqueness for the abstract equation, we show a smoothing property and the relation between the solution of the abstract equation and (1) in Subsection 3.2.1. Finally in Section 4 we show that all bounded solutions approach the set of equilibria as t tends to infinity, if the delay is small enough.

2 Abstract Setting

In order to define the operator in (1) we will need to extend the definition of fractional power to include negative powers. These negative fractional powers can be define using the ideas of Amann [1], [2] as done in Oliva and Pereira [13], where they consider semilinear reaction-diffusion equations with nonlinear boundary conditions, with no delays, and define the abstract equation.

We will start by defining the spaces that will be used through out this work, namely the so called Lebesgue Spaces (we refer to Triebel [20] for all the results related to these spaces). Let $S = S(\mathbb{R}^n)$ be the set of all complex-valued rapidly decreasing infinitely differentiable functions defined on the n-dimensional real Euclidean space $\mathbb{R}^n$. As usual, $S' = S'(\mathbb{R}^n)$ denotes the space of tempered distributions, which is the dual of S. We denote by

$$(\mathcal{F}\phi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i(x,\xi)}\phi(x)dx, \ \phi \in S,$$

$(x,\xi) = \sum_{j=1}^{n} x_j\xi_j$, the Fourier transformation, and

$$(\mathcal{F}^{-1}\phi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i(x,\xi)}\phi(x)dx, \ \phi \in S,$$

the inverse Fourier transformation.

With this we can define the *Lebesgue Spaces* in $\mathbb{R}^n$, as follows

Definition 2.1 *Let $-\infty < s < \infty$ and $1 < p < \infty$. Then*

$$H_p^s(\mathbb{R}^n) = \left\{ f \in S'(\mathbb{R}^n) \big| \|f\|_{H_p^s} = \|\mathcal{F}^{-1}(1+|x|^2)^{\frac{s}{2}}\mathcal{F}f\|_{L^p} < \infty \right\}$$

Now let us define the Lebesgue Spaces in a domain $\Omega \subset \mathbb{R}^n$.

Definition 2.2 *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary (bounded or unbounded) domain. Further, let $-\infty < s < \infty$ and $1 < p < \infty$. Then $H_p^s(\Omega)$ is the restriction of $H_p^s(\mathbb{R}^n)$ to Ω,*

$$\|f\|_{H_p^s(\Omega)} = \inf_{\substack{g|_\Omega = f \\ g \in H_p^s(\mathbb{R}^n)}} \|g\|_{H_p^s(\mathbb{R}^n)}.$$

Here $g|_\Omega$ denotes the restriction of g to Ω.

Remark 2.1 *$H_p^s(\Omega)$ is a Banach Space for any s (see Triebel [20]).*

Now, let us define our spaces taking into account the boundary condition $\mathcal{B}$. To do this, we first define the admissible boundary conditions, the so called "Normal System" of boundary conditions.

Definition 2.3 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain. Further let*

$$\mathcal{B}_j f(x) = \sum_{|\alpha| \le m_j} b_{j,\alpha}(x) D^\alpha f, \ b_{j,\alpha}(x) \in C^\infty(\partial\Omega),$$

$j = 1, \cdots, k$, be differential operators on $\partial\Omega$. Then $\{\mathcal{B}_j\}_{j=1}^k$ is said to be a normal system if

$$0 \le m_1 < m_2 < \cdots < m_k$$

and if for any normal vector ν_x with respect to $\partial\Omega$ and the point $x \in \partial\Omega$ it holds that

$$\sum_{|\alpha| = m_j} b_{j,\alpha}(x)\nu_x^\alpha \neq 0, \ j = 1, \cdots, k.$$

Definition 2.4 *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^∞ domain. Further, let $\{B_j\}_{j=1}^k$ be a normal system. For $s \geq 0$ and $1 < p < \infty$, we have*

1. *If $s - \frac{1}{p} < m_1$, then*
$$H^s_{p,\{B_j\}}(\Omega) = H^s_p(\Omega)$$

2. *If for some l, $m_l < s - \frac{1}{p} < m_{l+1}$, then*
$$H^s_{p,\{B_j\}}(\Omega) = \{f | f \in H^s_p(\Omega), B_j f|_{\partial\Omega} = 0, \text{ for } j \leq l\}$$

Remark 2.2 *In our case, that is in (1), we have that $k = 1$, $m_1 = 1$, $B_1 u = Bu = \dfrac{\partial u}{\partial n_a}$. The definition takes into account the boundary condition $B_j f = 0$, as long as it makes sense, that is, as long as the m_j derivatives have trace.*

Now let us define the operator. Consider A the operator in $L^p(\Omega; \mathbb{C})$ defined by

$$D(A) = H^2_{p,\{B\}}(\Omega)$$

$$Au = -Div(a\nabla u) + \sum_{j=1}^{n} B_j(x) \frac{\partial u}{\partial x_j} + \lambda u$$

where B is the boundary operator $Bu = \dfrac{\partial u}{\partial n_a}$. Let A' (the dual operator) be the operator in $L^{p'}(\Omega; \mathbb{C})$, where $\frac{1}{p} + \frac{1}{p'} = 1$, defined by

$$D(A') = H^2_{p',\{C\}}(\Omega)$$
$$A'v = -Div(a\nabla v) - div(vB) + \lambda v,$$

where C is the boundary operator $Cv = \dfrac{\partial v}{\partial n_a} + vB \cdot \vec{n}$.

We have that (see Triebel [20], pag. 401):

- A' is an isomorphism from $H^2_{p',\{C\}}(\Omega; \mathbb{C})$ onto $L^{p'}(\Omega; \mathbb{C})$;
- Denote by A'' the dual operator of A'. Then A'' is an isomorphism from $L^p(\Omega; \mathbb{C})$ onto $(H^2_{p',\{C\}}(\Omega; \mathbb{C}))'$;
- $A'' \equiv A$, in $H^2_{p,\{B\}}(\Omega; \mathbb{C})$.

With this, let us define the operator A_{-1} in $(H^2_{p',\{C\}}(\Omega; \mathbb{C}))'$ by

$$D(A_{-1}) = L^p(\Omega; \mathbb{C})$$

$$A_{-1}u = A''u, \text{ for all } u \in L^p(\Omega; \mathbb{C}).$$

Consider the following diagram

$$\begin{array}{ccc}
L^p(\Omega;\mathbb{C}) & \xrightarrow{\quad A'' \quad} & \left(H^2_{p',\{C\}}(\Omega;\mathbb{C})\right)' \\
A \Big\downarrow & & \Big\downarrow A_{-1} \\
L^p(\Omega;\mathbb{C}) & \xrightarrow{\quad A'' \quad} & \left(H^2_{p,\{C\}}(\Omega;\mathbb{C})\right)'
\end{array}$$

If $u \in L^p(\Omega;\mathbb{C})$ then $A''^{-1}u \in H^2_{p,\{B\}}(\Omega;\mathbb{C})$. Thus $A'' \circ A \circ A''^{-1}u = A'' \circ A'' \circ A''^{-1}u = A''u = A_{-1}u$. It is also easy to check that $D(A_{-1}) = D(A'' \circ A \circ A''^{-1})$. Therefore, the diagram commutes.

Proposition 2.1 *A_{-1} is a sectorial operator, with $\rho(A) = \rho(A_{-1})$. Moreover, given $\theta \geq 0$, if we define*

$$X^\theta_{-1} = D(A^\theta_{-1})$$

then A_{-1} is also a sectorial operator in X^θ_{-1}, which we denote by $A_{\theta-1}$.

Proof: See Oliva and Pereira [13].

Let us assume the following.

(H2) *We will fix λ in such a way that A is a positive operator (as defined in Triebel [20]). This will imply that A_{-1} is also a positive operator.*

Remark 2.3 *If $B \equiv 0$, then λ can be any positive value.*

Applying Proposition 2.1 and interpolation results, we have that, if **(H2)** holds, then for $2\theta \neq 1 + \frac{1}{p}$,

$$\begin{aligned}
X^\theta_{-1} = D(A^\theta_{-1}) &= [(H^2_{p',\{C\}}(\Omega;\mathbb{C}))', (L^{p'}(\Omega;\mathbb{C}))']_\theta \\
&= [H^2_{p',\{C\}}(\Omega;\mathbb{C}), L^{p'}(\Omega;\mathbb{C})]'_\theta = (H^{2(1-\theta)}_{p',\{C\}}(\Omega;\mathbb{C}))'.
\end{aligned} \tag{7}$$

If we define $X^\theta = D(A^\theta)$, then we have the following result.

Theorem 2.1 *If $0 \leq \theta \leq 1$ and* **(H2)** *holds, then*

$$X^{\theta+1}_{-1} = X^\theta = H^{2\theta}_{p,\{B\}}$$

Proof: See Oliva and Pereira [13].

Notation 2.1 *Having this result in mind we will define, for all $0 \leq s \leq 1$,*

$$X^{-s} = X^{1-s}_{-1}$$

Notation 2.2 *For a given $\gamma \in \mathbb{R}$ we will define $\mathbf{C}_\gamma = \mathbf{C}([-r,0], X^\gamma)$ the Banach space of all continuous functions $\varphi: [-r,0] \to X^\gamma$ with the sup-norm.*

We observe that we can always consider the Lebesgue spaces as real Banach spaces, even though our functions are taking complex values. With this in mind, if **(H2)** holds then it follows from Proposition 2.1 and the results of Henry [8] that $A_{-\beta}$ generates an analytic semigroup in $X^{-\beta}$ for $0 < \beta < 1$ which satisfies, for $-\beta < \alpha < 1 - \beta$

$$\begin{aligned} \left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} &\leq M e^{-\epsilon t} \|u_0\|_{X^\alpha}, \quad t \geq 0 \\ \left\| e^{-A_{-\beta}t} u_0 \right\|_{X^\alpha} &\leq M e^{-\epsilon t} t^{-(\alpha+\beta)} \|u_0\|_{X^{-\beta}}, \quad t > 0. \end{aligned} \tag{8}$$

for some $\epsilon > 0, M > 0$.

We want to choose α, β and p in such a way that

1. $X^\alpha \subset \bar{C}(\Omega)$;
2. $X^{1-\beta} = H_p^{2(1-\beta)}(\Omega)$, in other words $X^{1-\beta}$ does not incorporate the boundary condition;
3. $\alpha + \beta < 1$.

So we will take p, α and β satisfying

$$\frac{n}{2p} < \alpha < 1 - \beta < 1 - \frac{1}{2p'} = \frac{1}{2} + \frac{1}{2p}. \tag{9}$$

Notice that

- 1. follows from Theorem 2.1 and embedding theorems (see Triebel [20]);
- 2. follows from Definition 2.4 and Theorem 2.1.

Remark 2.4 *It is easy to check that (9) can be realized if p is big enough (for instance $p = n$ is enough).*

Using Definition 2.4, Theorem 2.1 and duality theorems for interpolation spaces (see Triebel [20]), one can also check the following result.

Corollary 2.1 *If α, β and p satisfy (9) and* **(H2)** *holds, then*

$$X^\alpha = H_p^{2\alpha}(\Omega) \text{ and } X^{-\beta} = \left(H_{p'}^{2\beta}(\Omega)\right)'.$$

Since we are going to use the linear operator A with homogeneous boundary conditions to define the abstract problem, we need to include the nonlinear boundary conditions in the equation. Furthermore since we are working with complex valued functions, we will need to complexify f and g. This is done as follows.

Notation 2.3 *We denote by*

$$f_{\mathbb{C}}: \mathbb{C} \to \mathbb{C}, \ \ g_{\mathbb{C}}: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$$

the complexifications of f and g, respectively, that is,

$$f_{\mathbb{C}}(\zeta) := f(\zeta_R) + if(\zeta_I), \ \ \zeta = \zeta_R + i\zeta_I \in \mathbb{R} + i\mathbb{R} = \mathbb{C},$$

$$g_{\mathbb{C}}(\zeta^1, \zeta^2) := g(\zeta^1_R, \zeta^2_R) + ig(\zeta^1_I, \zeta^2_I), \ \ \zeta^j = \zeta^j_R + i\zeta^j_I \in \mathbb{R} + i\mathbb{R} = \mathbb{C}, \ \ \text{for } j = 1, 2.$$

Now, let us consider the map $g_\gamma: X^\alpha \times X^\alpha \to X^{-\beta}$ defined by

$$\langle g_\gamma(u,v), \phi\rangle := \int_{\partial\Omega} \gamma(g_{\mathbb{C}}(u,v))\gamma(\phi), \ \ \text{for all } \phi \in H^{2\beta}_{p'}(\Omega),$$

where γ denotes the trace operator.

Similarly, we define $f_\Omega: X^\alpha \to X^{-\beta}$ by

$$\langle f_\Omega(u), \phi\rangle := \int_\Omega f_{\mathbb{C}}(u)\phi, \ \text{for all } \phi \in H^{2\beta}_{p'}(\Omega).$$

We will also denote by $H: \mathbf{C}_\alpha \to X^{-\beta}$ the function defined by,

$$\varphi \in \mathbf{C}_\alpha \mapsto f_\Omega(\varphi(0)) + g_\gamma(\varphi(0), \varphi(-r)). \tag{10}$$

It is easy to show that f_Ω and g_γ are well defined.

Thus the abstract equation (2) will take the form

$$\begin{cases} \dot{u}(t) + A_{-\beta}u(t) = H(u_t), \ \ t > 0 \\ u(t) = \varphi(t), \ \ t \in [-r, 0] \end{cases} \tag{11}$$

3 Existence and Uniqueness of Solutions

We want to establish the existence and uniqueness of the solutions of (11). In order to to this, we are going to recall the abstract existence results that are going to be applied to (1).

3.1 Abstract Results

All the abstract results can be found in Oliveira [14], the adaptation to X^α is straight forward.

In this paragraph, X^γ (for any $\gamma \in \mathbb{R}$) are the Banach spaces defined in the previous section, $r \geq 0$ a real number and $\mathbf{C}_\alpha = \mathbf{C}([-r,0],X^\alpha)$ the Banach space of the continuous functions $\varphi : [-r,0] \to X^\alpha$ with the sup-norm. $\{e^{-A_{-\beta}t} : t \geq 0\}$ always means the analytic semigroup generated by the closed linear operator $A_{-\beta}$ in $X^{-\beta}$, satisfying $\|e^{-A_{-\beta}t}\|_{L(X^{-\beta})} \leq M$, for all $t \geq 0$ and some constant $M \geq 1$.

Let us consider the abstract equation

$$\dot{u}(t) + A_{-\beta}u(t) = F(t,u_t), \;\; t > 0 \tag{12}$$

where $F : \mathbb{R} \times \mathbf{C}_\alpha \to X^{-\beta}$ be continuous.

Definition 3.1 *By a solution of (12) with initial condition $u_0 = \varphi \in \mathbf{C}_\alpha$ we mean a continuous function $u : [-r,T) \to X^\alpha$, with $T > r$, such that*
(i) $u(t) = \varphi(t)$, for $-r \leq t \leq 0$,
(ii) for $0 < t \leq r$, u is a solution of the integral equation

$$u(t) = e^{-A_{-\beta}t}\varphi(0) + \int_0^t e^{-A_{-\beta}(t-s)}F(s,u_s)ds,$$

(iii) for $r < t < T$, the function u is $\mathbf{C}^1$, has $u(t) \in D(A_{-\beta})$ and $\dot{u}(t) - A_{-\beta}u(t) = F(t,u_t)$, for all $t \in (r,T)$.

Remark 3.1 *1. As we will show bellow (cf. Theorem 3.3) if, besides continuity in $[-r,0]$, we suppose φ is locally Hölder continuous on $(-r,0]$, then a continuous function $u : [-r,T) \to X^\alpha$, with $T > 0$, satisfying (i) on $[-r,0]$ and (ii) on $[0,T)$ is a $\mathbf{C}^1$ function on $(0,T)$. In this case, our definition of solution coincides with the usual one in the evolution equations theory.*

2. The assumption $T > r$ is not too restrictive for the problem we study because, as we will see later, the hypotheses in f and g will imply that solutions are defined on arbitrarily large interval of times.

Theorem 3.1 *Suppose $F : \mathbb{R} \times \mathbf{C}_\alpha \to X^{-\beta}$ is continuous and locally Lipschitzian in the second argument. Given $(s,\varphi) \in \mathbb{R} \times \mathbf{C}_\alpha$, there exist a real number $\rho = \rho(s,\varphi) > 0$ and a unique continuous function $u : [s-r, s+\rho] \to X^\alpha$ such that $u_s = \varphi$ and*

$$u(t) = e^{-A_{-\beta}(t-s)}\varphi(0) + \int_s^t e^{-A_{-\beta}(t-\sigma)}F(\sigma,u_\sigma)d\sigma, \tag{13}$$

for all $s \leq t \leq s+\rho$.

The proof is a rather simple application of the Contraction Mapping Theorem, which the reader can supply. It is easy to see that, if $u, v : [s-r, s+\tau] \to X^\alpha$(any $\tau > 0$) are continuous solutions of (13) such that $u_s = v_s = \varphi$, then $u = v$ on $[s-r, s+\beta]$. This result allows us to consider the *maximal* solution $u(s,\varphi)$ of (13) through (s,φ) : for each (s,φ), we define $\rho^*(s,\varphi) = sup\{\rho > s : (13)$ has a continuous solution on $[s-r,\alpha]\}$

and $u(s,\varphi) : [s-r, \rho^*(s,\varphi)) \to X^\alpha$ by $u(s,\varphi)(t) = \varphi(t-s)$, if $s-r \le t \le s$ and, if $s < t < \rho^*(s,\varphi)$, then $u(s,\varphi)(t)$= the value at t of a solution of (13) satisfying $u_s = \varphi$, defined on $[s-r,\rho]$, with $t < \rho$. By the previous result, u is a well-defined continuous function on $[s-r,\rho^*(s,\varphi))$ and is a solution of (13) satisfying $u_s = \varphi$. Any other solution v of (13) satisfying the same initial condition is a restriction of $u(s,\varphi)$. Of course, the interval of existence of a maximal solution of (13) must be open to the right and the case $\rho^*(s,\varphi) = \infty$ is not excluded.

Lemma 3.1 *Suppose the solution $u = u(s,\varphi)$ of (13) with $u_s = \varphi$ is defined on $[s-r,\rho)$, for some $\rho > s$, and let T be a real number such that $s < T < \rho$. Then, there is a number $\delta > 0$ such that any solution $v = v(s,\psi)$ of (13), with $v_s = \psi$ and $\|\varphi - \psi\| < \delta$, is defined at least on $[s-r,T]$. Moreover, for a fixed t, $s \le t \le T$, the map $\varphi \mapsto u_t(s,\varphi)$ is continuous.*

Proof: See Oliveira [14].

Lemma 3.2 *Let $(s,\varphi) \in \mathbb{R} \times \mathbf{C}_\alpha$ and $u : [s-r,\rho^*) \to X^\alpha$ be the maximal solution of (13) satisfying $u_s = \varphi$. If $\rho^* < \infty$, then $\limsup_{t\to\rho^*-} |F(t,u_t)|/(1+\|u_t\|) = \infty$.*

Proof: See Oliveira [14].

Corollary 3.1 *In addition to the assumptions of Theorem 3.1, suppose F satisfies the following hypothesis: $F(B)$ is a bounded set in $X^{-\beta}$, for all bounded set B contained in $\mathbb{R} \times \mathbf{C}_\alpha$. Let $u(s,\varphi)$ and $\rho^*(s,\varphi)$ be as above and assume $\rho^*(s,\varphi) < \infty$. Then, $\limsup_{t\to\rho^*(s,\varphi)-} \|u_t(s,\varphi)\| = \infty$.*

Proof: See Oliveira [14].

Theorem 3.2 *Suppose $F : \mathbb{R} \times \mathbf{C}_\alpha \to X^{-\beta}$ is $\mathbf{C}^1$. Let $(s,\varphi) \in \mathbb{R} \times \mathbf{C}_\alpha$, $u(s,\varphi) : [s-r,\rho^*(s,\varphi)) \to X^\alpha$ be the solution of (13) through (s,φ) and $s < T < \rho^*(s,\varphi)$. Then, there exists a neighborhood U of φ such that, for all $\psi \in U$, the solution $u(s,\psi)$ of (13) with $u_s(s,\psi) = \psi$ is defined at least on $[s-r,T]$ and, for each $s \le t \le T$, the map $\psi \in U \mapsto u(s,\psi)(t) \in X^\alpha$ is $\mathbf{C}^1$ and its derivative $(\frac{\partial u}{\partial \psi}(s,\varphi)\cdot\xi)(t) \equiv v(t)$ at (s,φ) applied to ξ is the solution of*

$$v(t) = e^{-A_{-\beta}(t-s)}\xi(0) + \int_s^t e^{-A_{-\beta}(t-\sigma)}\frac{\partial F}{\partial\varphi}(\sigma, u_\sigma(s,\varphi))\cdot v_\sigma d\sigma \tag{14}$$

on $(s,T]$ and $v(t) = \xi(t-s)$ on $[s-r,s]$.

Proof: See Oliveira [14].

3.1.1 The Autonomous Case

Suppose the equation (13) is autonomous, that is, $F(t,\varphi) = H(\varphi)$ does not depend on t. If $u : [-r, \rho^*(\varphi)) \to X^\alpha$ is the maximal solution of (13) such that $u_0(\varphi) = \varphi$ and $s \in [0, \rho^*(\varphi))$, then the function $v(t) = u(\varphi)(t+s)$, defined on $[-s-r, \rho^*(\varphi)-s)$ is the solution of (13) satisfying $v_0 = u_s(\varphi)$, so $v(t) = u(u_s(\varphi))(t)$, for all $t \in [-r, \rho^*(u_s(\varphi)))$. This implies $\rho^*(\varphi) - s \leq \rho^*(u_s(\varphi))$ and $u(\varphi)(t+s) = u(u_s(\varphi))(t)$ for all $t, s \geq 0$ such that $s - r \leq t + s < \rho^*(\varphi)$. Therefore, if $-r \leq \theta \leq 0$, then $u_{t+s}(\varphi)(\theta) = u(\varphi)(t+s+\theta) = u(u_s(\varphi))(t+\theta) = u_t(u_s(\varphi))(\theta)$, and so, $u_{t+s}(\varphi) = u_t(u_s(\varphi))$ for all $t, s \geq 0$ such that $t + s < \rho^*(\varphi)$. From these considerations and the previous results we conclude that, if (13) is autonomous and the solutions $u(\varphi)$ are defined on $[-r, \infty)$ for all $\varphi \in C_\alpha$, then, the map $U(t) : C_\alpha \to C_\alpha$ given by $U(t)\varphi = u_t(\varphi)$ defines a (non-linear) strongly continuous semigroup $\{U(t) : t \geq 0\}$ on C_α.

Now, we will describe the relationship between $\{U(t) : t \geq 0\}$ and $\{e^{-A_{-\beta}t} : t \geq 0\}$. Let $\{T(t) : t \geq 0\}$ be the strongly continuous semigroup defined on C_α by the operator $A_{-\beta}$, that is,

$$(T(t)\varphi)(\theta) = \begin{cases} e^{-A_{-\beta}(t+\theta)}\varphi(0), & \text{if } \ t+\theta > 0 \\ \varphi(t+\theta) & , \text{if } -r \leq t+\theta \leq 0. \end{cases}$$

If $u(\varphi)$ is the mild solution of $\dot{u}(t) + A_{-\beta}u(t) = H(u_t)$ such that $u_0 = \varphi$, then

$$u(t) = \begin{cases} e^{-A_{-\beta}t}\varphi(0) + \int_0^t e^{-A_{-\beta}(t-s)}H(u_s)ds & , \text{if } \ t > 0 \\ \varphi(t) & , \text{if } -r \leq t \leq 0 \end{cases}$$

If $t \geq 0$ and $-r \leq \theta \leq 0$, we have

$$u_t(\varphi)(\theta) = \begin{cases} (T(t)\varphi)(\theta) + \int_0^{t+\theta} e^{-A_{-\beta}(t+\theta-s)}H(u_s(\varphi))ds & , \text{if } \ t+\theta > 0 \\ (T(t)\varphi)(\theta) & , \text{if } -r \leq t+\theta \leq 0 \end{cases}$$

Letting $X_0 : [-r, 0] \to L(X^{-\beta})$ be defined by $X_0(\theta) = 0$, if $-r \leq \theta < 0$ and $X_0(0) = I$, the above integral can be written as

$$\int_0^{t+\theta} e^{-A_{-\beta}(t+\theta-s)}H(u_s(\varphi))ds = \int_0^t [T(t-s)X_0](\theta)H(u_s(\varphi))ds,$$

which justifies the equality

$$u_t(\varphi) = T(t)(\varphi) + \int_0^t [T(t-s)X_0]H(u_s(\varphi))ds$$

for $t \geq 0$. Here, we define

$$[T(t)X_0](\theta) = \begin{cases} e^{-A_{-\beta}(t+\theta)} & , \text{if } \ t+\theta > 0 \\ 0 & , \text{if } \ t+\theta \leq 0 \end{cases},$$

which is (formally) the former definition.

3.1.2 Differentiability with respect to t

In this section, we will obtain sufficient conditions for a solution of (13) to be a solution of (12). We will assume that $F : \mathbb{R} \times C([-r,0], X^\alpha) \to X^{-\beta}$ is locally Hölder continuous in t and locally Lipschitzian in φ. The next result is basic and the reader can find the corresponding proof in Henry [8].

Lemma 3.3 *Suppose $\{e^{At} : t \geq 0\}$ is an analytic semigroup in a Banach space X and let $f : (0,T) \to X$ be locally Hölder continuous with $\int_0^\rho \|f(s)\|ds < \infty$ for some $\rho > 0$. For $0 \leq t < T$, define $F(t) = \int_0^t e^{A(t-s)} f(s)ds$. Then, F is continuous on $[0,T)$, continuously differentiable on $(0,T)$, with $F(t) \in D(A)$ for $0 < t < T$, $\dot{F}(t) = AF(t) + f(t)$ on $(0,T)$ and $F(t) \to 0$ in X as $t \to 0+$.*

In the next results the function $u : [-r,T] \to X^\alpha$ will be a solution of (13) on $[0,T]$ with initial condition $u_0 = \varphi$.

Theorem 3.3 *Suppose $\varphi : [-r,0] \to X^\alpha$ is continuous and locally Hölder continuous on $(-r,0]$. Then $t \mapsto u(t) : (-r,T] \to X^\alpha$ and $t \mapsto F(t,u_t) : (0,T] \to X^{-\beta}$ are locally Hölder continuous and therefore, $t \mapsto u(t)$ is C^1 on $0 < t < T$, moreover $\frac{du}{dt} \in X^\gamma$, for all $\frac{n}{2p} < \gamma < \frac{1}{2} + \frac{1}{2p}$.*

Proof. See Oliveira [14] and Henry [8].

Theorem 3.4 *Suppose $\varphi : [-r,0] \to X^\alpha$ is continuous and u is defined on $[-r,T]$, for some $T > r$. Then, u is locally Hölder continuous on $(0,r]$, and therefore, u is C^1 on $r < t \leq T$.*

Proof. See Oliveira [14].

3.2 Applying the Abstract Results

We want to apply the previous abstract results to establish existence of solutions of (11). The first thing to do is to establish the following lemma in order to get local existence for the abstract equation.

Lemma 3.4 *If f, g are Lipschitz then H is Lipschitz continuous in bounded sets of C_α.*

Proof:

Let $u, v \in V \subset \mathbf{C}_\alpha$, where V is bounded. Then we have that,

$$\begin{aligned} &\|g_\gamma(u(0), u(-r)) - g_\gamma(v(0), v(-r))\|_{X^{-\beta}} \\ = & \sup_{\left\{ \substack{\phi \in H^{2\beta}_{p'}(\Omega) \\ \|\phi\|_{H^{2\beta}_{p'}(\Omega)} = 1} \right\}} |\langle g_\gamma(u(0), u(-r)) - g_\gamma(v(0), v(-r)), \phi \rangle| . \end{aligned}$$

Since $X^\alpha \subset \bar{C}(\Omega)$ and g is C^2, we have that,

$$\begin{aligned} &|\langle g_\gamma(u(0), u(-r)) - g_\gamma(v(0), v(-r)), \phi\rangle| \\ &\leq \int_{\partial\Omega} |\gamma(g_{\mathbb{C}}(u(0), u(-r)) - g_{\mathbb{C}}(v(0), v(-r)))\gamma(\phi)| \\ &\leq \|\gamma(g_{\mathbb{C}}(u(0), u(-r)) - g_{\mathbb{C}}(v(0), v(-r))\|_{L^p(\partial\Omega)} \|\gamma(\phi)\|_{L^{p'}(\partial\Omega)} \\ &\leq K(\|(u(0) - v(0))\|_{\bar{C}(\Omega)} + \|(u(-r) - v(-r))\|_{\bar{C}(\Omega)}) \|\gamma(\phi)\|_{L^{p'}(\partial\Omega)} \\ &\leq K' \|u - v\|_{\mathbf{C}_\alpha} \|\phi\|_{H^{2\beta}_{p'}(\Omega)} \end{aligned}$$

Similarly we prove that f_Ω is Lipschitz.

■

Thus, we can apply the abstract results establish in the previous sections to get the following result.

Theorem 3.5 *Suppose that* **(H1)** *and* **(H2)** *hold, and that α, β and p satisfy (9). Then, given $\varphi \in \mathbf{C}_\alpha$, there exists a unique continuous solution $u : [-r, \infty) \longrightarrow X^\alpha$ of the abstract equation (11), which is C^1 for $t > r$.*

Proof: Applying Theorem 3.1, we have that there exist $\rho^* > 0$ such that there exist an unique solution u of (11) (with the nonlinearity being H defined by (10)), whose maximal interval of existence is $[-r, \rho^*]$. Therefore, we are left to prove that $\rho^* = \infty$.

But, from (5) and (6) we get that there exist $\xi \in \mathbb{R}$, such that

$$\frac{f(s)}{s+1} \leq c_0 \text{ and } \frac{g(s, s')}{s+1} \leq c_1(s'),$$

for all s with $|s| \geq \xi$, and $s' \in \mathbb{R}$, where c_1 is a continuous function.

Let us begin by assuming that $0 < \rho^* \leq r$. Thus we get that H is such that, for all t with $0 \leq t < \rho^*$,

$$\begin{aligned} \|H(u_t)\|_{-\beta} &= \|f_\Omega(u_t(0)) + g_\gamma(u_t(0), u_t(-r))\|_{-\beta} \\ &= \|f_\Omega(u_t(0)) + g_\gamma(u_t(0), \varphi(t - r))\|_{-\beta} \\ &\leq C(\varphi)(\|u_t\|_{C_\alpha} + 1) \end{aligned}$$

Thus, applying Corollary 3.2 we get a contradiction and $\delta^* > r$, moreover from the continuity of u, we get that $\|u_t\|_{C_\alpha} \leq K(\varphi, r)$, for all $t \in [0, r]$.

Therefore, if we assume that $jr < \rho^* \leq (j+1)r$, for some integer $j \geq 1$, following the same estimates above, we get again a contradiction and $\rho^* = \infty$.

■

With this, we get that the solution is defined for all times (and in particular is defined for $t > r$), and we know from Theorem 3.4 that after time $t = r$ the solution is $\mathbf{C}^1$, so without loss of generality, from now on we will assume that φ is Hölder continuous and that the solution is $\mathbf{C}^1$ for $t > 0$.

Now we need to establish a relation between the weak solution we have and solution to the original PDE.

3.2.1 Regularity Result

In this section, we want to show that the local solution is in $C^{2+\varepsilon}(\bar{\Omega})$, for some $\varepsilon > 0$, for any $t > 0$ and is, in fact, a classical solution.

Theorem 3.6 *Suppose that* **(H1)** *and* **(H2)** *hold, and that* α, β *and* p *satisfy (9). Let* $\varphi \in C_\alpha$ *be Hölder continuous and let* u *be the solution of (11). Then, there exists* $\varepsilon > 0$ *such that* $u(t,\cdot) \in C^{2+\varepsilon}(\bar{\Omega})$, *for all* $t > 0$. *Moreover, Re*$(u(t,\cdot))$ *is a classical solution of (1), for any* $t > 0$.

Proof: Applying the previous results, we know that, for $t > 0$ $u, \dfrac{du}{dt} \in C_\gamma$, for all $\dfrac{n}{2p} < \gamma < \dfrac{1}{2} + \dfrac{1}{2p}$. Therefore, using the characterization of X^γ, we have that $u, \dfrac{du}{dt} \in \mathbf{C}([-r,0]; H_p^s(\Omega))$, for all $\dfrac{n}{p} < s < 1 + \dfrac{1}{p}$. Thus using the embedding results, we obtain that, for each $t > 0$, $\dfrac{du}{dt}(t) \in \bar{C}^\delta(\Omega)$ for all $\delta < \frac{1}{p}$. Furthermore $u_t \in C([-r,0], H_p^1(\Omega)) = C([-r,0], W_p^1(\Omega))$ and thus, using the regularity of f and g; $f_{\mathbb{C}}(u(t)) - \dfrac{du}{dt}(t) \in L^p(\Omega)$, $g_{\mathbb{C}}(u(t), u(t-r)) \in W_p^1(\Omega)$, for all $t > 0$ So, by the trace theorem (see Triebel [20]), $\gamma(g_{\mathbb{C}}(u(t), u(t-r))) \in W^{1-\frac{1}{p},p}(\partial\Omega)$.

If we fix u, $t > 0$ and consider the elliptic problem,

$$\begin{cases} -\mathrm{Div}(a\nabla v) + \sum_{j=1}^{n} B_j(x)\dfrac{\partial v}{\partial x_j} + \lambda v = f_{\mathbb{C}}(u(t)) - \dfrac{du}{dt}(t), & \text{in} \quad \Omega, \\ \\ \dfrac{\partial v}{\partial n_a} = g_{\mathbb{C}}(u(t), u(t-r)), & \text{on} \quad \partial\Omega. \end{cases} \tag{15}$$

we can apply elliptic regularity results (see Lions and Magenes [11]), to conclude that $v \in W^{2,p}(\Omega) = H_p^2(\Omega)$.

Now, we want to show that $v = u(t)$. From the Green's Formula, if follows that for all $w \in H^2_p$ and $\phi \in H^2_{p',\mathcal{C}}(\Omega)$

$$\begin{aligned}\int_\Omega (Aw)(x)\phi(x)dx \; - \; \int w(x)(A'\phi)(x)dx &= -\int_{\partial\Omega} \frac{\partial w}{\partial n}(y)(\gamma(\phi))(y)dy \\ &+ \int_{\partial\Omega} (\gamma(w))(y)\left[\frac{\partial v}{\partial n_a}(y) + B(y)\cdot\vec{n}(y)\right] dy \\ &= \int_{\partial\Omega} -(\mathcal{B}w)(y)(\gamma(\phi))(y) + (\gamma(w))(y)(\mathcal{C}\phi)(y)dy \\ &= \int_{\partial\Omega} -(\mathcal{B}w)(y)(\gamma(\phi))(y)dy.\end{aligned} \tag{16}$$

Applying (16) to v and having in mind that (from (15))

$$(Av)(x) = f_{\mathbb{C}}(u(t))(x) - \frac{du}{dt}(t),$$
$$(\mathcal{B}v)(y) = (\gamma(g_{\mathbb{C}}(u(t), u(t-r))))(y),$$

for all $x \in \Omega$ and $y \in \partial\Omega$ it follows that v satisfies

$$\begin{aligned}\int_\Omega \left[f_{\mathbb{C}}(u(t))(x) - \frac{du}{dt}(t)\right]\phi(x)dx \; - \; \int v(x)(A'\phi)(x)dx \\ = \int_{\partial\Omega} -(\gamma(g_{\mathbb{C}}(u(t), u(t-r))))(y)(\gamma(\phi))(y)dy.\end{aligned} \tag{17}$$

Therefore, since $H^2_{p',\mathcal{C}}(\Omega)$ is dense in $H^{2\beta}_{p'}(\Omega)$, it follows that v satisfies, in $X^{-\beta}$, the equation

$$A_{-\beta}v = -\frac{du}{dt} + f_\Omega(u) + g_\gamma(u).$$

But u is the unique solution of (12), so $u(t) = v \in H^2_p(\Omega)$.

Applying the embedding results once more we get that $u(t) \in \bar{C}^{1+\epsilon}(\Omega)$ (and thus $u(t) \in C^{1+\epsilon}(\partial\Omega)$). Now applying regularity and existence theorems for (15) (see Ladyženskaja and Ural'ceva [9], pag.128) we conclude that $u(t) \in C^{2+\epsilon}(\bar{\Omega})$.

Moreover, since $u(t) = v$ satisfies (15) and from the way we complexified f and g, we get that $Re(u)$ is a classical solution of (1).

■

Remark 3.2 *Now that we have existence for (11) and since all functions and coefficients in the equation are real, we can take the real part of the solution, and we still have a solution. Thus from now on we will suppose that X^α is the real part of functions in $H^{2\alpha}_p(\Omega)$.*

4 Convergence to Equilibrium

In this section we will show the main result of this work, namely the convergence to the set of equilibria, which in our case will take the form of the following Theorem.

Theorem 4.1 *Suppose that* **(H1)** *and* **(H2)** *hold, and that* α, β *and* p *satisfy (9). Given* $K > 0$ *there exists* r_0 *such that for* $r < r_0$ *all trajectories* u *of (1) with* $\limsup_{t\to\infty}\|u(t)\|_{X^\alpha} < K$ *satisfy*

$$dist_{X^\alpha}(u(t), E) \to 0$$

as t *tends to infinity.*

To prove this theorem we will need some auxiliary lemmas. Our first goal is to obtain appropriate decay estimates of u_t. To this end, we could not consider the Lyapunov function for the undelayed counterpart of (1), as it is done in Friesecke [4], on the other hand we will estimate directly the decay. More precisely, we get that,

Lemma 4.1 *Suppose that* **(H1)** *and* **(H2)** *hold, and that* α, β *and* p *satisfy (9). Given* $K > 0$, *there exists* r_0 *such that for* $r < r_0$, *all trajectories* u *of (1) with* $\limsup_{t\to\infty}\|u(t)\|_\alpha < K$ *satisfy*

$$\left(\int_{T-r}^{T} \|\frac{du}{dt}(t)\|_\alpha^p dt\right)^{1/p} \longrightarrow 0, \tag{18}$$

as $T \to \infty$.

Proof: Let $T_0 > 6r$, and consider $T_1 > T_0$ and $0 < h < r$, let us estimate

$$\left(\int_{T_0}^{T_1} \|u(t+h) - u(t)\|_\alpha^p \, dt\right)^{1/p}. \tag{19}$$

For this, consider $T_0 < t < T_1$, and $H(s) = H(u(s), u(s-r))$. Using the variation of constants formula, we get

$$u(t+h) = e^{-A_{-\beta}(h+r)}u(t-r) + \int_{t-r}^{t+h} e^{-A_{-\beta}(t+h-s)}H(s)ds,$$

$$u(t) = e^{-A_{-\beta}(h+r)}u(t-r-h) + \int_{t-r-h}^{t} e^{-A_{-\beta}(t-s)}H(s)ds,$$

$$= e^{-A_{-\beta}(h+r)}u(t-r-h) + \int_{t-r}^{t+h} e^{-A_{-\beta}(t+h-\tau)}H(\tau-h)d\tau.$$

Substituting this in (19) and using the exponential estimates (8) we get,

$$\left(\int_{T_0}^{T_1} \|u(t+h) - u(t)\|_\alpha^p dt\right)^{1/p} \le K_0 e^{-\epsilon(r+h)} \left(\int_{T_0}^{T_1} \|u(t-r) - u(t-r-h)\|_\alpha^p dt\right)^{1/p}$$

$$+K_0 \left(\int_{T_0}^{T_1} \left[\int_{t-r}^{t+h} (t+h-\tau)^{-(\alpha+\beta)} e^{-\epsilon(t+h-\tau)} \|H(\tau) - H(\tau-h)\|_{-\beta} d\tau\right]^p dt\right)^{1/p}, \tag{20}$$

where K_0 is independent of T_0, T_1 and h.

Let us estimate the terms in (20) separately. In order to do this, if $a, b \in \mathbb{R}$, with $a < b$, we will denote by $\chi_{[a,b]}$ the characteristic function on $\mathbb{R}$ of the interval $[a, b]$, that is a function such that

$$\chi_{[a,b]}(t) = \begin{cases} 0, & \text{if } t \notin [a,b] \\ 1, & \text{if } t \in [a,b] \end{cases}.$$

With this we get,

$$\left(\int_{T_0}^{T_1} \|u(t-r) - u(t-r-h)\|_\alpha^p dt\right)^{1/p} = \left(\int_{T_0-r-h}^{T_1-r-h} \|u(s+h) - u(s)\|_\alpha^p ds\right)^{1/p}$$

$$\leq \left(\int_{T_0-2r}^{T_1} \|u(s+h) - u(s)\|_\alpha^p dt\right)^{1/p} = | \, \|u(s+h) - u(s)\|_\alpha \, |_{L^p(T_0-2r,T_1)}$$

$$= | \, \chi_{[T_0-2r,T_0]}(s)\|u(s+h) - u(s)\|_\alpha + \chi_{[T_0,T_1]}(s)\|u(s+h) - u(s)\|_\alpha \, |_{L^p(T_0-2r,T_1)}$$

$$\leq |\chi_{[T_0-2r,T_0]}(s)\|u(s+h) - u(s)\|_\alpha|_{L^p(T_0-2r,T_1)} + |\chi_{[T_0,T_1]}(s)\|u(s+h) - u(s)\|_\alpha|_{L^p(T_0-2r,T_1)}$$

$$= \left(\int_{T_0}^{T_1} \|u(s+h) - u(s)\|_\alpha^p dt\right)^{1/p} + \left(\int_{T_0-2r}^{T_0} \|u(s+h) - u(s)\|_\alpha^p ds\right)^{1/p} \tag{21}$$

But using the exponential estimates for the derivative in a bounded interval we get that there exists K_1, independent of T_1 and h, such that

$$\left(\int_{T_0-2r}^{T_0} \|u(s+h) - u(s)\|_\alpha^p ds\right)^{1/p} \leq K_1 h \tag{22}$$

Thus combining (21) and (22) we get that

$$\left(\int_{T_0}^{T_1} \|u(t-r) - u(t-r-h)\|_\alpha^p dt\right)^{1/p} \leq K_1 h$$
$$+ \left(\int_{T_0}^{T_1} \|u(s+h) - u(s)\|_\alpha^p dt\right)^{1/p}. \tag{23}$$

Now let us estimate the other term,

$$\left(\int_{T_0}^{T_1} \left[\int_{t-r}^{t+h} (t+h-\tau)^{-(\alpha+\beta)} e^{-\epsilon(t+h-\tau)} \|H(\tau) - H(\tau-h)\|_{-\beta} d\tau\right]^p dt\right)^{1/p}$$
$$= \left(\int_{T_0}^{T_1} \left[\int_0^{r+h} s^{-(\alpha+\beta)} e^{-\epsilon s} \|H(t+h-s) - H(t-s)\|_{-\beta} ds\right]^p dt\right)^{1/p} \tag{24}$$

Now applying Hölder's inequality in the ds integral we get

$$\left(\int_{T_0}^{T_1}\left[\int_{t-r}^{t+h}(t+h-\tau)^{-(\alpha+\beta)}e^{-\epsilon(t+h-\tau)}\|H(\tau)-H(\tau-h)\|_{-\beta}d\tau\right]^p dt\right)^{1/p}$$

$$\leq\left(\int_{T_0}^{T_1}\left(\int_0^{r+h}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right)^{\frac{p}{p'}}\times\int_0^{r+h}\|H(t+h-s)-H(t-s)\|_{-\beta}^p dsdt\right)^{1/p}$$

$$\leq\left[\int_0^{2r}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right]^{\frac{p}{p'}}\left(\int_0^{r+h}\int_{T_0}^{T_1}\|H(t+h-s)-H(t-s)\|_{-\beta}^p dtds\right)^{1/p}$$

$$\leq K_2\left[\int_0^{2r}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right]^{\frac{p}{p'}}\left(\int_0^{r+h}\int_{T_0}^{T_1}\|u(t+h-s-r)-u(t-s-r)\|_\alpha^p dt\right.$$

$$\left.+\int_{T_0}^{T_1}\|u(t+h-s)-u(t-s)\|_\alpha^p dtds\right)^{1/p}$$

$$\leq K_3\left[\int_0^{2r}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right]^{\frac{p}{p'}}\left(\int_0^{r+h}\int_{T_0-3r}^{T_1}\|u(\tau+h)-u(\tau)\|_\alpha^p dtds\right)^{1/p}$$

$$\leq K_4 r\left[\int_0^{2r}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right]^{\frac{p}{p'}}\left(\int_{T_0-3r}^{T_1}\|u(\tau+h)-u(\tau)\|_\alpha^p dt\right)^{1/p}, \tag{25}$$

where K_2, K_3 and K_4 are independent of T_1 and h.

As in (21) and (22) we can use the characteristic functions and the fact that the derivative is uniformly bounded in a finite interval, which is far away from $t=0$, and the bound depends only on the Lipschitz constant of the nonlinearity, the bound of the solution and the extremes of the interval, thus there exists K_5, independent of T_1 and h, in such a way that

$$\begin{aligned}&\left(\int_{T_0}^{T_1}\left[\int_{t-r}^{t+h}(t+h-\tau)^{-(\alpha+\beta)}e^{-\epsilon(t+h-\tau)}\|H(\tau)-H(\tau-h)\|_{-\beta}d\tau\right]^p dt\right)^{1/p}\leq\\&K_4 r\left[\int_0^{2r}s^{-p'(\alpha+\beta)}e^{-p'\epsilon s}ds\right]^{\frac{p}{p'}}\left(\left(\int_{T_0}^{T_1}\|u(\tau+h)-u(\tau)\|_\alpha^p dt\right)^{1/p}+K_5 h\right).\end{aligned} \tag{26}$$

Now, combining (23) with (26) and substituting in (20) we get that

$$\left(\int_{T_0}^{T_1}\|u(t+h)-u(t)\|_\alpha^2 dt\right)^{1/2} \le K_0 e^{-\epsilon r} K_1 h$$

$$+K_0 e^{-\epsilon r}\left(\int_{T_0}^{T_1}\|u(s+h)-u(s)\|_\alpha^2 dt\right)^{1/2} \tag{27}$$

$$+K_0 K_4 r\left[\int_0^{2r} s^{-p'(\alpha+\beta)} e^{-p'\epsilon s} ds\right]^{\frac{p}{p'}}\left(\left(\int_{T_0}^{T_1}\|u(\tau+h)-u(\tau)\|_\alpha^p dt\right)^{1/p} + K_5 h\right).$$

Thus we get that

$$\left(1-K_0 e^{-\epsilon r}-K_0 K_4 r\left[\int_0^{2r} s^{-p'(\alpha+\beta)} e^{-p'\epsilon s} ds\right]^{\frac{p}{p'}}\right)\left(\int_{T_0}^{T_1}\|u(t+h)-u(t)\|_\alpha^p dt\right)^{1/p}$$

$$\le\left(e^{-\epsilon r}+r\left[\int_0^{2r} s^{-p'(\alpha+\beta)} e^{-p'\epsilon s} ds\right]^{\frac{p}{p'}}\right) K_6 h, \tag{28}$$

where K_6 doesn't depend on T_1 and h.

Now observe that

$$\lim_{r\to 0}\frac{1-K_0 e^{-\epsilon r}}{K_0 K_4 r}=\frac{1}{K_0 K_4 \epsilon}>0,$$

and

$$\lim_{r\to 0}\left[\int_0^{2r} s^{-p'(\alpha+\beta)} e^{-p'\epsilon s} ds\right]=0.$$

Thus there exists $r_0>0$, independent of T_1 and h, such that for all $r<r_0$, we have that

$$\left(1-K_0 e^{-\epsilon r}-K_0 K_4 r\left[\int_0^{2r} s^{-p'(\alpha+\beta)} e^{-p'\epsilon s} ds\right]^{\frac{p}{p'}}\right)>0.$$

So taking $r<r_0$ and taking the limit as $T_1\to\infty$ and $h\to 0$ in (28) we get that

$$\left(\int_{T_0}^{\infty}\|\frac{du}{dt}(t)\|_\alpha^p dt\right)^{1/p}<\infty,$$

thus proving the lemma.

■

Having establish this decay, we just need to prove that the w-limit set is nonempty. To do this we need the following lemma, that follows immediately from the embedding results.

Lemma 4.2 (Orbit Precompactness) *Suppose that* **(H1)** *and* **(H2)** *hold, and that α, β and p satisfy (9). Let $r>0$ be arbitrary. If $\limsup_{t\to\infty}\|u(t)\|_\alpha<\infty$, then the orbit $\{\Phi(t)\varphi\}_{t\geq 0}$ is precompact in C_α.*

Now we can follow the same idea of Friesecke [4] to prove convergence to equilibrium.

Proof of Theorem 4.1

Let u be a solution of (1) (or more precisely, of (11)), and assume that φ is Hölder continuous, $\limsup_{t\to\infty}\|u(t)\|_{X^\alpha} < K$ and $r < r_0$ (given by Lemma 4.1). According to Lemma 4.2, the associated orbit $\{\Phi(t)\varphi\}_{t\geq 0}$ is precompact and thus possesses a nonempty ω−limit set ($\omega(\varphi)$). Therefore, we need to show only that the set $\omega(\varphi)$ consists of equilibria. We do not have a Lyapunov function so that the La Salle-Hale invariance principle in its standard form cannot be applied, but the estimate in Lemma 4.1 will do just as well. Take $v_0 \in \omega(\varphi)$ and pick $t_i \to \infty$ such that $\Phi(t_i)\varphi \to v_0$ in C_α. Since $(\frac{d}{ds})[\Phi(t_i)(\varphi)](s) \to 0$ in $L^p((-r,0),X^\alpha)$ by Lemma 4.1, v_0 lies in $W^1_p((-r,0),X^\alpha)$, and $\Phi(t_i)u_0 \to v_0$ in $W^1_p((-r,0),X^\alpha)$, and therefore

$$\frac{d}{ds}v_0(s) \equiv 0, \quad s \in (-r,0).$$

Thus, since $\omega(\varphi)$ is positively invariant we get that v_0 should be an equilibrium, and this finishes the proof.

■

5 Final Remarks

1. Let us mentioned that the results obtained here, could also be done if the nonlinearity also depends on the delay (with the same delay r).

2. As shown by Friesecke [5], the assumption that the solution is bounded, cannot be ignored. since he constructs an example where solutions blow up (in infinite time). We can drop this assumption if we assume in **(H1)** that c_g is constant (as it is done in Friesecke [4]).

3. Friesecke [4] get, in the case of interior delay, r_o (in Theorem 4.1) independent of the domain Ω. In our proof, the dependence on Ω appears on the exponential estimate (8), used through out the proof.

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