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A reiterated homogenization problem for the p-Laplacian equation in corrugated thin domains

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Abstract

In this paper, we study the asymptotic behavior of the solutions of the p-Laplacian equation with mixed homogeneous Neumann-Dirichlet boundary conditions. It is posed in a two-dimensional rough thin domain with two different composites periodically distributed. Each composite has its own periodicity and roughness order. Here, we obtain distinct homogenized limit equations which will depend on the relationship among the roughness and thickness orders of each one.

1. Introduction

Recently, a lot of attention has been given to the study of Partial Differential Equations (PDE) under singular perturbation of domains, [1–30]. Such PDE's have represented successful ways to model important phenomena in physics, chemistry, engineering and several other sciences [7–9,11,15,28]. With focus to the applications, for instance, in lubrication and microfluidics [29] we may give special care to PDE's posed in thin domains (domains in which one direction is much larger than the remaining ones).

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Thereby, with thin domains in mind, we notice that most materials have a microstructure which is not perfectly flat. Indeed, they have a lot of irregularities on its surface and also can be composed by several materials [17]. Moreover, different material must present different types of irregularities on its surface. Hence, taking such aspect in account, we are interested here in analyzing the asymptotic behavior of the solutions of the following p-Laplacian problem

$$\begin{cases} -\Delta_p u^\varepsilon = f^\varepsilon \quad \text{in} \quad R^\varepsilon, \\ |\nabla u^\varepsilon|^{p-2}\frac{\partial u^\varepsilon}{\partial \eta^\varepsilon} = 0 \quad \text{on} \quad \partial R^\varepsilon \setminus \partial_l R^\varepsilon, \\ u^\varepsilon = 0 \quad \text{on} \quad \partial_l R^\varepsilon, \end{cases} \tag{1.1}$$

where η_ε is the unit outward normal vector to the boundary ∂R^ε, $1 < p < \infty$ with $p^{-1} + p'^{-1} = 1$, the family of forcing terms $f^\varepsilon \in L^{p'}(R^\varepsilon)$ is uniformly bounded and converges, in some sense, to $\bar{f} \in L^{p'}(0,1)$. $\Delta_p \cdot = \operatorname{div}\left(|\nabla \cdot|^{p-2}\nabla \cdot\right)$ is the p-Laplacian operator, $\partial_l R^\varepsilon$ is the lateral boundary of R^ε where R^ε is the 2-dimensional thin domain

$$R^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in (0,1),\ 0 < y < \varepsilon g_\varepsilon(x)\right\} \quad 0 < \varepsilon \ll 1 \tag{1.2}$$

with

$$g_\varepsilon(x) = \begin{cases} g_1\left(\dfrac{x}{\varepsilon^\alpha}\right) & \text{in} \quad \Gamma_1^\varepsilon \\ g_2\left(\dfrac{x}{\varepsilon^\beta}\right) & \text{in} \quad \Gamma_2^\varepsilon \end{cases}. \tag{1.3}$$

The parameters α and β are positive, g_1 and $g_2 : \mathbb{R} \to \mathbb{R}$ are respectively two positive, L_1 and L_2 periodic functions satisfying $0 < g_i^m \le g_i(x) \le g_i^M$ for all $x \in (0,1)$ with g_i^m and g_i^M set by

$$g_i^m = \min_{x\in\mathbb{R}} g_i(x) \quad \text{and} \quad g_i^M = \max_{x\in\mathbb{R}} g_i(x) \quad \text{for } i = 1, 2.$$

Also, the sets Γ_i^ε from (1.3) are defined as follows. Let us take $\gamma > 0$ such that

$$p'\gamma < \min\{\alpha, \beta\}. \tag{1.4}$$

We divide the interval (0, 1) in k^ε subintervals of length ε^γ with k^ε denoting the biggest integer less than or equal to $1/\varepsilon^\gamma$ for $0 < \varepsilon \ll 1$. Hence, we define Γ_i^ε as

$$\Gamma_1^\varepsilon = \bigcup_{i=0}^{k^\varepsilon - 1}\left[i\varepsilon^\gamma, (2i+1)\frac{\varepsilon^\gamma}{2}\right] \quad \text{and} \quad \Gamma_2^\varepsilon = \bigcup_{i=0}^{k^\varepsilon - 1}\left[(2i+1)\frac{\varepsilon^\gamma}{2}, (i+1)\varepsilon^\gamma\right].$$

See that each Γ_i^ε is the union of k^ε subintervals of length $\varepsilon^\gamma/2$. On them, we take the functions $\varepsilon g_i(\cdot/\varepsilon^{\mu_i})$ for $\mu_1 = \alpha$ and $\mu_2 = \beta$ defining the thin domain R^ε as illustrated by Fig. 1. We still have

$$R^\varepsilon = \operatorname{int}\left(\overline{\mathcal{R}_1^\varepsilon} \cup \overline{\mathcal{R}_2^\varepsilon}\right) \quad \text{where}$$

$$\mathcal{R}_i^\varepsilon = \left\{(x,y) : x \in \Gamma_i^\varepsilon,\ 0 < y < \varepsilon g_i\left(\frac{x}{\varepsilon^{\mu_i}}\right)\right\} \quad \text{for } i = 1, 2.$$

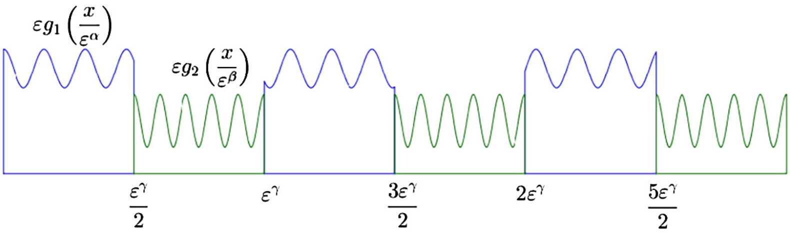

Fig. 1. A thin domain with two different scales of roughness.

Notice that here we are dealing with four scales which can be described as follows. The first one, is that one, given by the compression scale ε in the vertical direction. The second acts on the horizontal direction which is given by ε^γ and the sets Γ_i^ε. Finally, we have the third and fourth scales established by ε^α and ε^β which are responsible for the roughness on the top boundary of the thin domain R^ε. See also that we are mixing two different sorts of roughness whose profile is defined by the functions g_i.

Moreover, it is worth saying that the form of constructing the thin domain R^ε actually makes possible to precisely use the reiterated homogenization discussed for instance in [8,13]. This happens due to the fact that the subscales ε^α and ε^β are much smaller than ε^γ allowing us to divide the intervals $\left[i\varepsilon^\gamma, (2i+1)\frac{\varepsilon^\gamma}{2}\right]$ and $\left[(2i+1)\frac{\varepsilon^\gamma}{2}, (i+1)\varepsilon^\gamma\right]$ in m_α^ε and m_β^ε subintervals respectively of the type $[j\varepsilon^\alpha, (j+1)\varepsilon^\alpha]$ and $\left[j\varepsilon^\beta, (j+1)\varepsilon^\beta\right]$ with

$$\frac{m_\alpha^\varepsilon}{\varepsilon^\alpha} \sim \frac{m_\beta^\varepsilon}{\varepsilon^\beta} \sim \frac{1}{\varepsilon^\gamma}.$$

All of the scales depend on the exponents α, β and γ. These distinct scales allow us to capture different limit regimes as $\varepsilon \to 0+$. In fact, it is the core of this work. In the sequel we describe the achieved results. As we will see, the limit equation is the same for all positive values of α, β and γ. What changes is the homogenized coefficient q. Without loss of generality, we can assume from now on that

$$\alpha \le \beta.$$

Consequently, condition (1.4) becomes

$$p'\gamma < \alpha.$$

The homogenized equation is a one dimensional p-Laplacian equation with homogeneous Dirichlet boundary condition of the form

$$\begin{cases} -q\frac{\partial}{\partial x}\left(\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\right) = \bar{f} \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0. \end{cases} \tag{1.5}$$

If $\gamma < 1 = \alpha = \beta$ in (1.2), the homogenized coefficient is

$$q = \sum_{j=1}^{2} \int_{Y_j^*} \frac{1}{2L_j} \left| (1,0) + \nabla_{y_1 y_2} X^j \right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^j \right) (1,0) dy_1 dy_2 \tag{1.6}$$

where X^j is the solution of the auxiliary problem

$$\int_{Y_j^*} \left| (1,0) + \nabla_{y_1 y_2} X^j \right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^j \right) \nabla \psi_j dy_1 dy_2 = 0, \quad \forall \psi_j \in W_\#^{1,p}(Y_j^*)$$

$$\text{with} \quad \int_{Y_j^*} X^j dy_1 dy_2 = 0.$$

Y_j^* is the representative cell set by the function g_j

$$Y_j^* = \{ (y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L_j \text{ and } 0 < y_2 < g_j(y_1) \}$$

and $W_\#^{1,p}(Y_j^*)$ is the space of the functions in $W^{1,p}(Y_j^*)$ which are periodic on variable y_1. Notice that here one can recover the classic situation with just one profile and same order of roughness given respectively by $g_1 = g_2$ and $\alpha = \beta$. See for instance [1,18] for $p = 2$ and [2] for $p \in (1, \infty)$. Also, we emphasize the explicit dependence of the coefficient q given by (1.6) on both profiles set by the functions g_1 and g_2. It is possible to identify the effect of each part of the geometry of the thin domain in the limit equation. As we will see, this will happen for all the other cases which will also depend on α, β and γ.

If $\gamma < \alpha \le \beta < 1$, then

$$q = \frac{1}{2} \sum_{j=1}^{2} \left\langle 1/g_j^{p'-1} \right\rangle^{1-p}$$

in (1.5), and we generalize the weakly oscillatory case performed in [2] for $g_1 = g_2$ and $\alpha = \beta < 1$. As in the previous case (1.6), here we obtain a homogenized coefficient which combine the effect of the both profiles. More precisely, we get a homogenized coefficient which combines weak roughness and two distinct profiles.

On the other side, if we suppose $\gamma < \alpha < \beta = 1$, another kind of homogenized coefficient is established. We get a mixture of the weak and resonant homogenized coefficients first computed in [2]. We obtain

$$q = \frac{1}{2} \left\langle 1/g_1^{p'-1} \right\rangle^{1-p} + \frac{1}{2L_2} \int_{Y_2^*} \left| (1,0) + \nabla_{y_1 y_2} X^2 \right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^2 \right) (1,0) dy_1 dy_2.$$

Once again, we are able to identify all the effect of each part of the geometry of the thin domain in the homogenized equation.

Now, when one of the subscales sets a strong roughness, specifically if $\gamma < \alpha < 1 < \beta$, we get

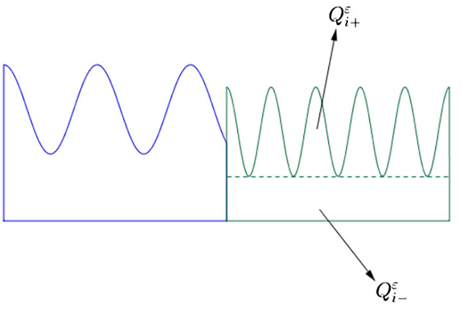

Fig. 2. Resonant roughness on the left side and strong on the right.

$$q = \frac{1}{2}\left\langle 1/g_1^{p'-1}\right\rangle^{1-p} + \frac{g_2^m}{2}$$

and a mixture of the weak and strong corrugated cases are attached. On the other way, if we assume $\gamma < \alpha = 1 < \beta$, the resonant and strong oscillatory case are combined obtaining

$$q = \frac{1}{2L_1}\int_{Y_1^*} \left|(1,0) + \nabla_{y_1y_2}X^1\right|^{p-2}\left((1,0) + \nabla_{y_1y_2}X^1\right)(1,0)dy_1dy_2 + \frac{g_2^m}{2}.$$

We point out that in the both previous mentioned cases $\gamma < \alpha \leq 1 < \beta$, we actually find that there is no diffusion in the second subscale, specifically in the region Q^{ε}_{i+} set by the Fig. 2. This mechanism (first observed in [4]) makes the diffusion to be led only by the region Q^{ε}_{i-} which corresponds to the term $\frac{g_2^m}{2}$ in the definition of q.

Finally, let us discuss the cases $1 < \alpha \leq \beta$ with $\gamma \leq 1$ and $1 < \gamma < \alpha$. See that such assumptions set strong roughness on the whole top of the boundary. We first remark that, if $g_1^m = g_2^m$, that is, if the minimum value of g_1 and g_2 are the same, then the homogenized coefficient is constant and equal to this minimum value for any $1 < \alpha \leq \beta$ and $0 < \gamma < \alpha$. We obtain

$$q = g_1^m = g_2^m. \tag{1.7}$$

On the other side, if $g_1^m \neq g_2^m$, other three different regimes are established setting three other different homogenized coefficients. As in the case (1.7), it is possible to realize that the diffusion in the high oscillatory regions P^{ε}_{i+} and Q^{ε}_{i+} set by Fig. 3 will not affect the homogenized coefficient also (although, we will remain with a microstructure given by $P^{\varepsilon}_{i-} \cup Q^{\varepsilon}_{i-}$). This three different cases are the following.

First, for $1 < \alpha \leq \beta$ and $\gamma < 1$, we have

$$q = \left\langle 1/H^{p'-1}\right\rangle^{1-p} \quad \text{where} \quad H(x) = \begin{cases} g_1^m & \text{for} \quad x \in (0, 1/2] \\ g_2^m & \text{for} \quad x \in (1/2, 1) \end{cases}. \tag{1.8}$$

Notice that here, H is a piecewise constant function which combines the minimum values of the profiles.

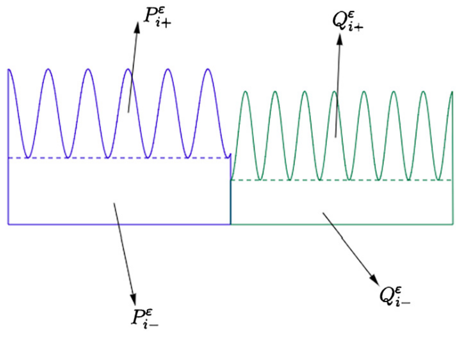

Fig. 3. Subscales set by different orders of strong roughness: $1 < \alpha \le \beta$.

Next, if we take $\gamma = 1$, still assuming $1 < \alpha \le \beta$, we get

$$q = \int_Z \left|(1,0) + \nabla_{y_1 y_2}\mathbb{X}\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2}\mathbb{X}\right)(1,0) dy_1 dy_2 \tag{1.9}$$

with $\mathbb{X}$ solving

$$\int_Z \left|(1,0) + \nabla_{y_1 y_2}\mathbb{X}\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2}\mathbb{X}\right) \nabla\psi_2 dy_1 dy_2 = 0, \quad \forall \psi_1 \in W^{1,p}_\#(Z)$$

$$\text{with} \quad \int_Z \mathbb{X} dy_1 dy_2 = 0 \quad \text{and} \quad Z = \{(y_1, y_2) : y_1 \in (0,1), 0 < y_1 < H(y_1)\}.$$

Here, $\mathbb{X}$ is an auxiliar solution, which is set in the representative cell Z defined just by the minimum values of the profiles g_i. And finally, if we set $\gamma > 1$ with $1 < \alpha \le \beta$, we obtain

$$q = \min\{g_1^m, g_2^m\}. \tag{1.10}$$

Notice that here the parameter γ somehow predominates on α and $\beta > 1$. Indeed, taking account the results obtained in [2], as $\gamma < 1$, q assume the form (1.8) which is associated to the weakly oscillatory case; if $\gamma = 1$, q assumes the form (1.9) which is now associated to the resonant case, and finally, when $\gamma > 1$, the high roughness case at the limit is established by (1.10). Also, it is worth pointing out that the rough part of the thin domain R^ε does not appear in the expression of the homogenized coefficients emphasizing the effect of α and β bigger than 1.

In order to determine such asymptotic models we make use of the periodic unfolding method. It was first developed in the context of oscillating coefficients, next for periodically perforated domains and later for thin domains with oscillating boundary [4–6,12,13,25]. Here, we shall adapt the techniques from [5,6] for the context of reiterated homogenization. As is well known, when we make use of the unfolding operator, we somehow transform the weak formulation of a singularly perturbed domain into an equivalent formulation posed on a fixed domain. In this way, we generally say that we '*unfolded the weak formulation*'. When one '*unfolds the weak formulation*', there is an issue that usually appear due to the periodicity. This issue is actually due to the scaling properties of the domain, since we periodically distribute the basic cell Y^* to

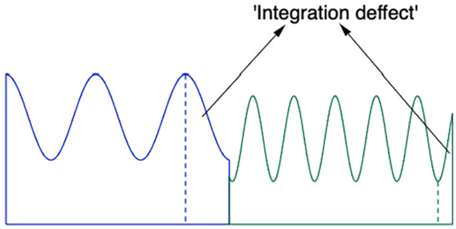

Fig. 4. Integration deffect.

construct the domain and with this periodic distribution of Y^*, we have that not always an exact number of rescaled cells Y^* fits in the domain. Thus, '*unfolding a weak formulation*' generates what we call '*integration deffect*' corresponding to the non exact Y^* cell fitting in the singularly perturbed domain. Here the '*integration deffect*' actually appears in each part of the subscales and we deal it properly in this work (see Fig. 4).

To finish the introduction, let us give a brief historical remark in related works. We start by mentioning the pioneering work [16] whose focus was the study of the dynamics of parabolic equations in standard thin domains (those ones that do not present roughness on the boundary). The first published works dealing with elliptic and parabolic equations in thin domains with rough boundaries are respectively [18] and [1]. In [18] a linear elliptic equation is studied and in [1] the dynamic of a semilinear parabolic equations is considered. In this direction, it is worth also mentioning [9] where nonlinear elasticity problems in thin domains were considered and dealt with Γ-convergence techniques.

Indeed, if one takes $p = 2$ in (1.1), which represents the Laplace differential operator, some previous works using different techniques and methods were developed. In this direction, the unfolding operator for thin domains is introduced in [5,6] where the weakly, resonant and strongly oscillatory cases are treated. The case with fast oscillatory boundary ($\alpha > 1$) was first obtained in [4] by decomposing the domain in two parts separating the oscillatory boundary. There, the authors also consider more general and complicated geometries which are not given as the graph of certain smooth functions (in this direction see also [3] where a locally periodic thin domain is considered).

On the other side, the p-Laplacian equation in thin domains was first considered in [27] (see also [26]) but without roughness. In [2], the authors were able to deal with equation (1.1) on non-smooth oscillating thin domains for any $1 < p < \infty$ and any order of oscillation combining techniques from [5,6] and [14]. Our goal here is to generalize [2] considering different profiles and mixing then with different scales. The present work, with the best of the authors knowledge, is a pioneering one regarding reiterated homogenization and rough thin domains.

We also have to mention many other works where rough thin domains are considered in a variety of problems. In this direction we mention [10,19–24] and references there in where reaction–diffusion problems, p-Laplacian equations from fluid mechanics are studied.

2. The unfolding operator

In this preliminary section we will introduce the iterated unfolding. Thereunto, let us first recall the unfolding operator introducing our functional setting. Let $g : \mathbb{R} \to \mathbb{R}$ be a function strictly positive, lower semicontinuous and L-periodic with $g \in L^\infty(\mathbb{R})$. Also, if $g_0 = \min_{x\in\mathbb{R}} g(x)$ and $g_1 = \max_{x\in\mathbb{R}} g(x)$, let us assume $0 < g_0 \le g(x) \le g_1$ for all $x \in \mathbb{R}$. We set

$$I = \{x \in \mathbb{R} : c < x < c+d\} = (c, c+d) \quad \text{and} \quad Q^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in I, 0 < y < \varepsilon g(x/\varepsilon^\kappa)\right\}$$

for some positive parameter κ. Y^* is the representative cell given by

$$Y^* = \{(y_1, y_2) \in \mathbb{R}^2 : 0 < y_1 < L \text{ and } 0 < y_2 < g(y_1)\}.$$

The average of $\varphi \in L^1_{loc}(\mathbb{R}^2)$ on a measure set $\mathcal{O} \subset \mathbb{R}^2$ is denoted by $\langle\varphi\rangle_{\mathcal{O}}$. We will also need the following functional spaces:

$$L^p_\#(Y^*) = \{\varphi \in L^p(Y^*) : \varphi(y_1, y_2) \text{ is } L\text{-periodic in } y_1\},$$
$$L^p_\#\left(I \times Y^*\right) = \{\varphi \in L^p(I \times Y^*) : \varphi(x, y_1, y_2) \text{ is } L\text{-periodic in } y_1\},$$
$$W^{1,p}_\#(Y^*) = \{\varphi \in W^{1,p}(Y^*) : \varphi|_{\partial_{left} Y^*} = \varphi|_{\partial_{right} Y^*}\} \quad \text{etc.}$$

where $\cdot_\#$ denotes the periodicity of the functions in the variable $y_1 \in (0, L)$.

By $[a]_L$ we mean the unique integer number that satisfies $a = [a]_L L + \{a\}_L$ with $\{a\}_L \in [0, L)$. Then, for each $\varepsilon > 0$ and any $x \in \mathbb{R}$, we have

$$x = \varepsilon^\kappa \left[\frac{x}{\varepsilon^\kappa}\right]_L L + \varepsilon^\kappa \left\{\frac{x}{\varepsilon^\kappa}\right\}_L \quad \text{where } \left\{\frac{x}{\varepsilon^\kappa}\right\}_L \in [0, L).$$

Also, we denote

$$I_\varepsilon = \text{ Int } \left(\bigcup_{k=0}^{N_\varepsilon^{c,d}-1} \left[c + kL\varepsilon^\kappa, c + (k+1)L\varepsilon^\kappa\right] \right),$$

where $N_\varepsilon^{c,d}$ is the largest integer such that $\varepsilon^\kappa L(N_\varepsilon^{c,d} + 1) \leq d$. Finally, we set

$$\begin{gathered} \Lambda_\varepsilon = I \backslash I_\varepsilon = [\varepsilon^\kappa L(N_\varepsilon^{c,d} + 1), d), \quad Q_0^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in I_\varepsilon, 0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\kappa}\right)\right\} \\ \text{and} \quad Q_1^\varepsilon = \left\{(x,y) \in \mathbb{R}^2 : x \in \Lambda_\varepsilon, 0 < y < \varepsilon g\left(\frac{x}{\varepsilon^\kappa}\right)\right\}. \end{gathered} \tag{2.1}$$

Definition 2.1. *Let $\varphi \in \mathcal{M}(Q^\varepsilon)$, where $\mathcal{M}(Q^\varepsilon)$ is the set of Lebesgue-measurable functions in Q^ε. The unfolding operator $\mathcal{T}_\varepsilon : \mathcal{M}(Q^\varepsilon) \to \mathcal{M}(I \times Y^*)$ is defined by*

$$\mathcal{T}_\varepsilon\varphi(x, y_1, y_2) = \begin{cases} \varphi\left(\varepsilon^\kappa \left[\frac{x}{\varepsilon^\kappa}\right]_L L + \varepsilon^\kappa y_1, \varepsilon y_2\right) \text{ a.e. } (x, y_1, y_2) \in I_\varepsilon \times Y^*, \\ 0 \text{ a.e. } (x, y_1, y_2) \in \Lambda_\varepsilon \times Y^*. \end{cases}$$

Next, we summarize some properties of $\mathcal{T}_\varepsilon$.

Proposition 2.2. *The unfolding operator satisfies the following properties:*

1. *$\mathcal{T}_\varepsilon$ is linear and satisfies $\mathcal{T}_\varepsilon(\varphi\psi) = \mathcal{T}_\varepsilon(\varphi)\mathcal{T}_\varepsilon(\psi)$;*
2. *Let $\varphi \in L^p(Y^*)$ be a Lebesgue function in Y^* extended periodically in the y_1 direction. Then, $\varphi^\varepsilon(x, y) = \varphi\left(\frac{x}{\varepsilon^\kappa}, \frac{y}{\varepsilon}\right)$ is measurable in Q^ε, $\mathcal{T}_\varepsilon(\varphi^\varepsilon)(x, y_1, y_2) = \varphi(y_1, y_2)$ and $\varphi^\varepsilon \in L^p(Q^\varepsilon)$;*

3. *For all* $\varphi^\varepsilon \in L^1(Q^\varepsilon)$*, we have*

$$\frac{1}{L}\int_{I\times Y^*} \mathcal{T}_\varepsilon(\varphi)(x, y_1, y_2)dxdY = \frac{1}{\varepsilon}\int_{Q^\varepsilon} \varphi(x, y)dxdy - \frac{1}{\varepsilon}\int_{Q_0^\varepsilon} \varphi(x, y)dxdy,$$

where $dY = dy_1dy_2$.
4. *Assume* $\varphi_\varepsilon \in L^p(Q^\varepsilon)$ *with* $\varepsilon^{-1/p}\,||\cdot||_{L^p(Q^\varepsilon)}$ *uniformly bounded satisfies the unfolding criterion for integrals (u.c.i.), that is,*

$$\frac{1}{\varepsilon}\int_{Q_1^\varepsilon} |\varphi_\varepsilon|dxdy \to 0.$$

Moreover, if ψ_ε *is given by*

$$\psi_\varepsilon(x, y) = \psi\left(\frac{x}{\varepsilon^\kappa}, \frac{y}{\varepsilon}\right)$$

for some $\psi \in L^q(Y^*)$ *with* $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$*, and* $\phi \in L^{p'}(I)$*, then* $(\varphi_\varepsilon\psi_\varepsilon)$ *and* $\varphi_\varepsilon\phi$ *satisfies (u.c.i.).*
5. *Let* $\varphi \in W^{1,p}(R^\varepsilon)$*. It holds*

$$\frac{\partial \mathcal{T}_\varepsilon\varphi}{\partial y_1} = \varepsilon^\kappa \mathcal{T}_\varepsilon\frac{\partial\varphi}{\partial x} \quad \textit{and} \quad \frac{\partial \mathcal{T}_\varepsilon\varphi}{\partial y_2} = \varepsilon\mathcal{T}_\varepsilon\frac{\partial\varphi}{\partial y} \quad \textit{a.e. in} \quad I\times Y^*.$$

6. *Let* $\varphi \in L^p(I)$*. Then,* $\mathcal{T}_\varepsilon\varphi \to \varphi$ *strongly in* $L^p(I\times Y^*)$*.*
7. *Let* (φ_ε) *be a sequence in* $L^p(I)$ *such that* $\varphi_\varepsilon \to \varphi$ *strongly (respectively weakly) in* $L^p(I)$*. Then,* $\mathcal{T}_\varepsilon\varphi_\varepsilon \to \varphi$ *strongly (respectively weakly) in* $L^p(I\times Y^*)$*.*

Proof. See [4] for the proof. □

Notice that, due to the order of the height of the thin domain the factor $1/\varepsilon$ appears in properties 3 and 4. Thus, it makes sense to consider the following rescaled Lebesgue measure in the thin domains

$$\rho_\varepsilon(\mathcal{O}) = \frac{1}{\varepsilon}|\mathcal{O}|, \forall\mathcal{O}\subset R^\varepsilon,$$

which is widely considered in works involving thin domains. As a matter of fact, from now on, we use the following rescaled norms in the thin open sets

$$|||\varphi|||_{L^p(R^\varepsilon)} = \varepsilon^{-1/p}\,||\varphi||_{L^p(R^\varepsilon)} \quad \forall\varphi \in L^p(R^\varepsilon),\ 1\le p<\infty,$$
$$|||\varphi|||_{W^{1,p}(R^\varepsilon)} = \varepsilon^{-1/p}\,||\varphi||_{W^{1,p}(R^\varepsilon)} \quad \forall\varphi \in W^{1,p}(R^\varepsilon),\ 1\le p<\infty.$$

For completeness we may denote $|||\varphi|||_{L^\infty(R^\varepsilon)} = ||\varphi||_{L^\infty(R^\varepsilon)}$.

Now we are in position to introduce the iterated unfolding. For this sake, divide 1, the size of the interval (0, 1), by ε^γ. Using the notation presented in the beginning of this section, we have $1 = \left[\frac{1}{\varepsilon^\gamma}\right] + \left\{\frac{1}{\varepsilon^\gamma}\right\}$. Let

$$R_0^\varepsilon = \left\{(x,y) : x \in H_0^\varepsilon,\ 0 < y < \varepsilon g_\varepsilon(x)\right\} \quad \text{where } H_0^\varepsilon = \bigcup_{k=0}^{\left[\frac{1}{\varepsilon^\gamma}\right]-1} \left[k\varepsilon^\gamma, (k+1)\varepsilon^\gamma\right] \quad \text{and}$$

$$R_1^\varepsilon = \left\{(x,y) : x \in H_1^\varepsilon,\ 0 < y < \varepsilon g_\varepsilon(x)\right\} \quad \text{with } H_1^\varepsilon = (0,1) \setminus H_0^\varepsilon.$$

Then, if $k^\epsilon = \left[\frac{1}{\varepsilon^\gamma}\right]$ and $\varphi \in L^1(R^\varepsilon)$, we have

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dx dy &= \frac{1}{\varepsilon}\sum_{i=0}^{k^\varepsilon-1} \int_{i\varepsilon^\gamma}^{(i+1)\varepsilon^\gamma} \int_0^{\varepsilon g_\varepsilon(x)} \varphi(x,y) dy dx \\
&= \frac{1}{\varepsilon}\sum_{i=0}^{k^\varepsilon-1} \left[\int_{i\varepsilon^\gamma}^{\frac{2i+1}{2}\varepsilon^\gamma} \int_0^{\varepsilon g_1\left(\frac{x}{\varepsilon^\alpha}\right)} \varphi(x,y) dy dx + \int_{\frac{2i+1}{2}\varepsilon^\gamma}^{(i+1)\varepsilon^\gamma} \int_0^{\varepsilon g_2\left(\frac{x}{\varepsilon^\beta}\right)} \varphi(x,y) dy dx \right] \\
&= \sum_{i=0}^{k^\varepsilon-1} \left[\frac{1}{\varepsilon}\int_{Q_i^\varepsilon} \varphi(x,y) dy dx + \frac{1}{\varepsilon}\int_{P_i^\varepsilon} \varphi(x,y) dy dx \right]
\end{aligned}$$

where

$$Q_k^\varepsilon = \left\{(x,y) : x \in I_k^\varepsilon,\ 0 < y < \varepsilon g_1\left(\frac{x}{\varepsilon^\alpha}\right)\right\} \quad \text{and}$$
$$P_k^\varepsilon = \left\{(x,y) : x \in J_k^\varepsilon,\ 0 < y < \varepsilon g_2\left(\frac{x}{\varepsilon^\beta}\right)\right\}$$

with $I_k^\varepsilon = \left[k\varepsilon^\gamma, \frac{2k+1}{2}\varepsilon^\gamma\right]$ and $J_k^\varepsilon = \left[\frac{2k+1}{2}\varepsilon^\gamma, (k+1)\varepsilon^\gamma\right]$. Next, we apply Proposition 2.2 to obtain

$$\begin{aligned}
\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dx dy = \sum_{i=0}^{k^\varepsilon-1} \Bigg[& \frac{1}{L_1}\int_{I_i^\varepsilon \times Y_1^*} \mathcal{T}_\varepsilon^1 \varphi dx dy_1 dy_2 + \frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon} \varphi dx dy \\
& + \frac{1}{L_2}\int_{J_i^\varepsilon \times Y_2^*} \mathcal{T}_\varepsilon^2 \varphi dx dy_1 dy_2 + \frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon} \varphi dy dx \Bigg]
\end{aligned}$$

where the unfolding operators $\mathcal{T}_\varepsilon^1$ and $\mathcal{T}_\varepsilon^2$ are respectively associated with the family of thin domains Q_k^ε and P_k^ε. The sets Q_{k1}^ε and P_{k1}^ε are analogous defined as in (2.1).

Using a change of variable in x, we get

$$\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dx dy = \sum_{i=0}^{k^\varepsilon-1} \left[\frac{\varepsilon^\gamma}{L_1} \int_0^{\frac{1}{2}} \int_{Y_1^*} \mathcal{T}_\varepsilon^1 \varphi\left(\varepsilon^\gamma i + \varepsilon^\gamma z_1, y_1, y_2\right) dy_1 dy_2 dz_1 + \frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon} \varphi dx dy \right.$$

$$+\frac{\varepsilon^\gamma}{L_2}\int_{\frac{1}{2}}^{1}\int_{Y_2^*}\mathcal{T}_\varepsilon^2\varphi\left(\varepsilon^\gamma i+\varepsilon^\gamma z_1,y_1,y_2\right)dy_1dy_2dz_1+\frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon}\varphi(x,y)dydx\Bigg]$$

$$=\sum_{i=0}^{k^\varepsilon-1}\Bigg[\frac{1}{L_1}\int_{i\varepsilon^\gamma}^{(i+1)\varepsilon^\gamma}\int_0^{\frac{1}{2}}\int_{Y_1^*}\mathcal{T}_\varepsilon^1\varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right)dy_1dy_2dz_1dx$$

$$+\frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon}\varphi dxdy+\frac{1}{L_2}\int_{i\varepsilon^\gamma}^{(i+1)\varepsilon^\gamma}\int_{\frac{1}{2}}^{1}\int_{Y_2^*}\mathcal{T}_\varepsilon^2\varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right)dy_1dy_2dz_1dx$$

$$+\frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon}\varphi(x,y)dydx\Bigg] \tag{2.2}$$

$$=\frac{1}{L_1}\int_{H_0^\varepsilon}\int_0^{\frac{1}{2}}\int_{Y_1^*}\mathcal{T}_\varepsilon^1\varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right)dy_1dy_2dz_1dx+\sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon}\varphi dxdy$$

$$+\frac{1}{L_2}\int_{H_0^\varepsilon}\int_{\frac{1}{2}}^{1}\int_{Y_2^*}\mathcal{T}_\varepsilon^2\varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right)dy_1dy_2dz_1dx$$

$$+\sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon}\varphi(x,y)dydx.$$

Now, with this expression in mind, we introduce the following unfolding operators.

Definition 2.3. *Let*

$$\mathbb{I}_\varepsilon=\bigcup_{k=0}^{k^\varepsilon-1}I_k^\varepsilon \quad \text{and} \quad \mathbb{J}_\varepsilon=\bigcup_{k=0}^{k^\varepsilon-1}J_k^\varepsilon.$$

Take $\varphi\in L^1(\mathbb{I}_\varepsilon\times Y_1^*)$ and $\psi\in L^1(\mathbb{J}_\varepsilon\times Y_2^*)$. Then, consider the following unfolding operators $\mathbf{T}_\varepsilon^1: L^1(\mathbb{I}_\varepsilon\times Y_1^*)\to L^1((0,1)\times(0,1/2)\times Y_1^*)$ and $\mathbf{T}_\varepsilon^2: L^1(\mathbb{J}_\varepsilon\times Y_2^*)\to L^1((0,1)\times(1/2,1)\times Y_2^*)$ given by

$$\mathbf{T}_\varepsilon^1\varphi(x,z_1,y_1,y_2)=\begin{cases}\varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right) & in \quad H_0^\varepsilon\times(0,1/2)\times Y_1^*\\ 0 \quad in \quad H_1^\varepsilon\times(0,1/2)\times Y_1^*\end{cases} \quad and$$

$$\mathbf{T}_\varepsilon^2\psi(x,z_1,y_1,y_2)=\begin{cases}\psi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right]+\varepsilon^\gamma z_1,y_1,y_2\right) & in \quad H_0^\varepsilon\times(1/2,1)\times Y_2^*\\ 0 \quad in \quad H_1^\varepsilon\times(1/2,1)\times Y_2^*.\end{cases}$$

We just mention that this unfolding operators also satisfy the properties from Proposition 2.2 with obvious changes. Due to (2.2), we still have

$$\frac{1}{\varepsilon}\int_{R_0^\varepsilon}\varphi dxdy = \frac{1}{L_1}\int_{(0,1)\times(0,1/2)\times Y_1^*}\mathbf{T}_\varepsilon^1\mathcal{T}_\varepsilon^1\varphi\,(x,z_1,y_1,y_2)\,dy_1dy_2dz_1dx + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon}\varphi dxdy$$
$$+\frac{1}{L_2}\int_{(0,1)\times(1/2,1)\times Y_2^*}\mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^2\varphi\,(x,z_1,y_1,y_2)\,dy_1dy_2dz_1dx$$
$$+\sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon}\varphi(x,y)dydx.$$

Notice that, according to [13], relationship (2.3) establishes via the operators $\mathbf{T}_\varepsilon^1\mathcal{T}_\varepsilon^1$ and $\mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^2$ what is known iterated unfolding. Before proceeding with their convergence properties, it is worth noticing that the iterated unfolding (2.3) works properly in the cases where the periodic oscillations on $\mathcal{R}_1^\varepsilon$ or $\mathcal{R}_2^\varepsilon$ are resonant or weak, that is, when $\alpha \le 1$ or $\beta \le 1$. When they present strong oscillatory behavior ($\alpha > 1$ or $\beta > 1$), we will need a slightly different approach which we describe in the following.

Suppose that $\gamma < \alpha \le 1 < \beta$. We decompose P_k^ε in two parts:

$$P_{k+}^\varepsilon = \left\{(x,y)\in P_k^\varepsilon : \varepsilon g_2^m \le y \le \varepsilon g_2\left(\frac{x}{\varepsilon^\alpha}\right)\right\} \quad \text{and}$$
$$P_{k-}^\varepsilon = \left\{(x,y)\in P_k^\varepsilon : 0 \le y \le \varepsilon g_2^m\right\}.$$

Then, from item 3 at Proposition 2.2, we have

$$\frac{1}{\varepsilon}\sum_{i=0}^{k^\varepsilon-1}\int_{P_i^\varepsilon}\varphi(x,y)dydx = \frac{1}{\varepsilon}\sum_{i=0}^{k^\varepsilon-1}\left[\int_{P_{i+}^\varepsilon}\varphi(x,y)dydx + \int_{P_{i-}^\varepsilon}\varphi(x,y)dydx\right]$$
$$=\sum_{i=0}^{k^\varepsilon-1}\left[\frac{1}{L_2}\int_{J_i^\varepsilon\times Y_{2+}^*}\mathcal{T}_\varepsilon^{2+}\varphi dxdy_1dy_2 + \frac{1}{\varepsilon}\int_{P_{i+0}^\varepsilon}\varphi dxdy + \frac{1}{\varepsilon}\int_{P_{i-}^\varepsilon}\varphi\right]$$
$$=\frac{1}{L_2}\int_{(0,1)\times(1/2,1)\times Y_{2+}^*}\mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^{2+}\varphi dxdy_1dy_2 + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i+0}^\varepsilon}\varphi dxdy$$
$$+\int_{(0,1)\times Z_2^-}\Pi_2^\varepsilon\varphi dxdzdy_2,$$

where $\mathcal{T}_\varepsilon^{2+}$ is the unfolding in the thin domain P_{k+}^ε,

$$Z_2^- = (1/2,1)\times(0,g_2^m)$$

and Π_2^ε is the unfolding operator introduced below.

Definition 2.4. *The unfolding operator* $\Pi_2^\varepsilon : L^1(\mathcal{R}_{2-}^\varepsilon) \to L^1\left((0,1)\times Z_2^-\right)$ *is given by*

$$\Pi_2^\varepsilon\varphi(x,z,y_2) = \begin{cases} \varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z, \varepsilon y_2\right) & in \quad H_0^\varepsilon \times Z_2^- \\ 0 \quad in \quad H_1^\varepsilon \times Z_2^-. \end{cases}$$

Notice that Π_2^ε is basically the same as the unfolding $\mathcal{T}_\varepsilon$ introduced in (2.1). They only differ on the micro scale that they are acting.

Now, taking $\gamma \le 1 < \alpha \le \beta$ or $1 < \gamma < \alpha \le \beta$, we shall have a little different approach. Anyway, we can perform analogous arguments as above.

Definition 2.5. *Let* $\varphi \in L^1(R^\varepsilon)$, *then we set the unfolding operator* $\Pi^\varepsilon : L^1(R^\varepsilon) \to L^1((0,1)\times Z)$, *with* $Z = \operatorname{int}\left(\left([0,1/2]\times[0,g_1^m]\right)\cup\left([1/2,1]\times[0,g_2^m]\right)\right)$, *by*

$$\Pi^\varepsilon\varphi(x,z,y_2) = \begin{cases} \varphi\left(\varepsilon^\gamma\left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z, \varepsilon y_2\right) & in \quad H_0^\varepsilon \times Z \\ 0 \quad in \quad H_1^\varepsilon \times Z. \end{cases}$$

Next, we summarize the integration formulas depending on the case we study:

1. If $\gamma < \alpha \le \beta \le 1$,

$$\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dxdy = \frac{1}{L_1}\int_{(0,1)\times(0,1/2)\times Y_1^*} \mathbf{T}_\varepsilon^1\mathcal{T}_\varepsilon^1\varphi\left(x,z_1,y_1,y_2\right)dy_1dy_2dz_1dx + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon}\varphi dxdy$$
$$+\frac{1}{L_2}\int_{(0,1)\times(1/2,1)\times Y_2^*} \mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^2\varphi\left(x,z_1,y_1,y_2\right)dy_1dy_2dz_1dx + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1}^\varepsilon}\varphi(x,y)dydx. \tag{2.3}$$

2. If $\gamma < \alpha \le 1 < \beta$,

$$\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dxdy = \frac{1}{L_1}\int_{(0,1)\times(0,1/2)\times Y_1^*} \mathbf{T}_\varepsilon^1\mathcal{T}_\varepsilon^1\varphi\left(x,z_1,y_1,y_2\right)dy_1dy_2dz_1dx + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1}^\varepsilon}\varphi dxdy$$
$$+\frac{1}{L_2}\int_{(0,1)\times A_2\times Y_{2+}^*} \mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^{2+}\varphi dxdy_1dy_2 + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1+}^\varepsilon}\varphi dxdy + \int_{(0,1)\times Z_2^-}\Pi_2^\varepsilon\varphi dxdzdy_2; \tag{2.4}$$

3. If $\gamma \leq 1 < \alpha \leq \beta$

$$\begin{aligned}\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dxdy &= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y^*_{j+}} \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \varphi dxdy_1dy_2 + \int_{(0,1)\times Z} \Pi^\varepsilon \varphi dxdzdy_2 \\ &+ \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{P^\varepsilon_{i1+}} \varphi dxdy + \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1+}} \varphi dxdy.\end{aligned} \tag{2.5}$$

4. If $1 < \gamma < \alpha \leq \beta$,

$$\begin{aligned}\frac{1}{\varepsilon}\int_{R_0^\varepsilon} \varphi dxdy &= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y^*_{j+}} \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \varphi dxdy_1dy_2 + \int_{R_m} \mathcal{S}_\varepsilon \varphi dxdy \\ &+ \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{P^\varepsilon_{i1+}} \varphi dxdy + \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1+}} \varphi dxdy,\end{aligned} \tag{2.6}$$

where, in this case, $Y^*_{j+} = \{(y_1, y_2) : 0 < y_1 < L_j,\ \min\{g_1^m, g_2^m\} < y_2 < g_j(y_1)$ and $\mathcal{S}_\varepsilon : (0,1) \times \varepsilon \min\{g_1^m, g_2^m\} \to (0,1) \times (0, \min\{g_1^m, g_2^m\}) = R_m$ denotes the rescaling operator defined by

$$\mathcal{S}_\varepsilon \varphi(x, y) = \varphi(x, \varepsilon y), \quad (x, y) \in R_m.$$

Remark 2.6. *We remark that if* $|||\varphi_\varepsilon|||_{W^{1,p}(R^\varepsilon_m)}$, $R^\varepsilon_m = (0,1) \times (0, \varepsilon \min\{g_1^m, g_2^m\})$, *then there is* $\varphi \in W^{1,p}(0,1)$ *such that* $\mathcal{S}_\varepsilon \varphi_\varepsilon \rightharpoonup \varphi$ *weakly in* $W^{1,p}(R_m)$.

Now that we have defined the unfolding operators which will iterate with $\mathcal{T}^1_\varepsilon$ and $\mathcal{T}^2_\varepsilon$, we need to analyze the 'integration defect' produced by such iteration. The next result actually proves that these repeated defects do not affect the limit behavior of the transformed functions as ε goes to zero.

Proposition 2.7. *Let* $\varphi \in L^p(R^\varepsilon)$ *be such that* $\varepsilon^{-1/p}||\varphi||_{L^p(R^\varepsilon)}$ *is uniformly bounded. Also, let* $\phi \in C_0^\infty(0,1)$ *and* $\psi_i \in L^{p'}_\#(Y^*_i)$ *for* $i = 1, 2$. *Suppose* $\beta \geq \alpha > p'\gamma$. *Then,*

$$\sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1}} \varphi\phi dxdy \to 0, \qquad \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{P^\varepsilon_{i1}} \varphi\phi dydx \to 0,$$

$$\sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1}} \varphi\psi_1^\varepsilon dxdy \to 0, \quad \text{and} \quad \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon}\int_{P^\varepsilon_{i1}} \varphi\psi_2^\varepsilon dydx \to 0$$

as $\varepsilon \to 0$ *where* $\psi_1^\varepsilon(x, y) = \psi_1(x/\varepsilon^\alpha, y/\varepsilon)$, $\psi_2^\varepsilon(x, y) = \psi_2(x/\varepsilon^\beta, y/\varepsilon)$.

Proof. Notice that for each $i = 0, \dots, k^\varepsilon - 1$, we have

$$\frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1}} \varphi\phi dxdy \le \varepsilon^{-1}\|\varphi\|_{L^p(Q^\varepsilon_i)}\|\phi\|_{L^{p'}(Q^\varepsilon_i)}.$$

Also,

$$\begin{aligned}
\|\phi\|^{p'}_{L^{p'}(Q^\varepsilon_i)} &= \int_{Q^\varepsilon_{i1}} |\phi|^{p'} dxdy \\
&\le \|\phi\|^{p'}_{L^\infty(0,1)} \int_{(N^i_\varepsilon-1)L_1\varepsilon^\alpha}^{\frac{2i+1}{2}\varepsilon^\gamma} \varepsilon g_1\left(\frac{x}{\varepsilon^\alpha}\right)dx \\
&\le \varepsilon\|\phi\|^{p'}_{L^\infty(0,1)} g_1^M\left(\frac{2i+1}{2}\varepsilon^\gamma - (N^i_\varepsilon - 1)L_1\varepsilon^\alpha\right).
\end{aligned}$$

See that, due to the construction of our thin domain, we have that the order of $(N^i_\varepsilon - 1)L_1\varepsilon^\alpha$ is the same as $\frac{2i+1}{2}\varepsilon^\gamma$. Thus, $\left(\frac{2i+1}{2}\varepsilon^\gamma - (N^i_\varepsilon - 1)L_1\varepsilon^\alpha\right)$ is of order ε^α. Then,

$$\sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q^\varepsilon_{i1}} \varphi\phi dxdy \le \|\varphi\|_{L^p(R^\varepsilon)}\|\phi\|_{L^\infty(0,1)}(g_1^M)^{1/p'}\sum_{i=0}^{k^\varepsilon-1}\left[O(\varepsilon^\alpha)\right]^{1/p'} = O(\varepsilon^{\frac{1}{p}-\gamma+\frac{\alpha}{p'}}).$$

As $\alpha > p'\gamma$, $\varphi\phi$ satisfy the (u.c.i.).

Now,

$$\|\psi_1^\varepsilon\|^{p'}_{L^{p'}(Q^\varepsilon_i)} \le \int_{(N^i_\varepsilon-1)L_1\varepsilon^\alpha}^{N^i_\varepsilon L_1\varepsilon^\alpha}\int_0^{\varepsilon g_1\left(\frac{x}{\varepsilon^\alpha}\right)} \left|\psi_1\left(\frac{x}{\varepsilon^\alpha}, \frac{y}{\varepsilon}\right)\right|^{p'} dydx = \varepsilon^\alpha\int_0^{L_1}\int_0^{g_1(y_1)} |\psi_1(y_1, y_2)|^{p'} dy_2 dy_1.$$

Thus, $\varphi\psi_1^\varepsilon$ also satisfy (u.c.i.). For the integrals on P^ε_{i1}, we can perform analogous arguments to show the results. It is left to the interested reader. □

Next, we prove some useful convergence results.

Proposition 2.8. *Let $\phi_j \in C_0^\infty\left((0,1) \times A_j\right)$, $A_1 = (0, 1/2)$ and $A_2 = (1/2, 1)$. Define $\phi^\varepsilon(x) = \phi\left(x, \left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$, $x \in (0,1)$. Then*

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \phi^\varepsilon_j(x, z, y_1, y_2) \to \phi(x, z) \quad \textit{pointwise in} \quad (0,1) \times A_j \times Y^*_j.$$

Proof. Let us perform the proof for $j = 1$. For any $(x, z, y_1, y_2) \in (0,1) \times A_1 \times Y^*_1$ we have

$$\mathbf{T}^1_\varepsilon \mathcal{T}^1_\varepsilon \phi^\varepsilon(x, z, y_1, y_2)$$

$$= \phi_1 \left(\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1, \left\{ \frac{\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1}{\varepsilon^\gamma} \right\} \right).$$

First, we claim that

$$\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1 \to x.$$

Indeed, notice that

$$\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 = \varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z - \varepsilon^\alpha \left\{ \frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right\}_{L_1} \tag{2.7}$$

and

$$\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] = x - \varepsilon^\gamma \left\{\frac{x}{\varepsilon^\gamma}\right\}.$$

Thus, since $\left\{\frac{x}{\varepsilon^\gamma}\right\} \in (0, 1)$, we have

$$\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] \to x.$$

Since $\left\{ \frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right\}_{L_1} \in [0, L_1)$ and $z \in A_1$, one has, by (2.7),

$$\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 \to x.$$

Notice that

$$\left\{ \frac{\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1}{\varepsilon^\gamma} \right\}$$

$$= \frac{\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1}{\varepsilon^\gamma} - \left[\frac{\varepsilon^\alpha \left[\frac{\varepsilon^\gamma \left[\frac{x}{\varepsilon^\gamma}\right] + \varepsilon^\gamma z}{\varepsilon^\alpha} \right]_{L_1} L_1 + \varepsilon^\alpha y_1}{\varepsilon^\gamma} \right].$$

Then, the first term of the right hand side above is written as

$$\frac{\varepsilon^{\alpha}\left[\frac{\varepsilon^{\gamma}\left[\frac{x}{\varepsilon^{\gamma}}\right]+\varepsilon^{\gamma}z}{\varepsilon^{\alpha}}\right]_{L_1} L_1+\varepsilon^{\alpha}y_1}{\varepsilon^{\gamma}}=\left[\frac{x}{\varepsilon^{\gamma}}\right]+z+\varepsilon^{\alpha-\gamma}\left\{\frac{\varepsilon^{\gamma}\left[\frac{x}{\varepsilon^{\gamma}}\right]+\varepsilon^{\gamma}z}{\varepsilon^{\alpha}}\right\}_{L_1}+\varepsilon^{\alpha-\gamma}y_1.$$

This implies for ε small enough

$$\left[\frac{\varepsilon^{\alpha}\left[\frac{\varepsilon^{\gamma}\left[\frac{x}{\varepsilon^{\gamma}}\right]+\varepsilon^{\gamma}z}{\varepsilon^{\alpha}}\right]_{L_1} L_1+\varepsilon^{\alpha}y_1}{\varepsilon^{\gamma}}\right]=\left[\frac{x}{\varepsilon^{\gamma}}\right]+z.$$

Thus,

$$\left\{\frac{\varepsilon^{\alpha}\left[\frac{\varepsilon^{\gamma}\left[\frac{x}{\varepsilon^{\gamma}}\right]+\varepsilon^{\gamma}z}{\varepsilon^{\alpha}}\right]_{L_1} L_1+\varepsilon^{\alpha}y_1}{\varepsilon^{\gamma}}\right\}=z+\varepsilon^{\alpha-\gamma}\left\{\frac{\varepsilon^{\gamma}\left[\frac{x}{\varepsilon^{\gamma}}\right]+\varepsilon^{\gamma}z}{\varepsilon^{\alpha}}\right\}_{L_1}+\varepsilon^{\alpha-\gamma}y_1\to z,$$

as $\varepsilon\to 0$, and then, we conclude the pointwise convergence. □

Remark 2.9. *Analogously to the previous Proposition, let* $\psi\in L^p_{\#}\left((0,1)\times(0,\max\{g_1^M,g_2^M\})\right)$, $g_j^M=\max_{y\in\mathbb{R}} g(y)$, *and define* $\psi_\varepsilon(x,y)=\psi\left(\frac{x}{\varepsilon^{\gamma}},\frac{y}{\varepsilon}\right)$. *Then,*

$$\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^{j+}\psi_\varepsilon\to\psi(z,y_2)\quad\textit{strongly in}\quad L^p\left((0,1)\times A_j\times Y_j^*\right),\quad j=1,2.$$

Proposition 2.10. *Let* $\psi_j\in L^p_{\#}\left(Y_j^*\right)$. *Define* $\psi_j^\varepsilon(x,y)=\psi_j\left(\frac{x}{\varepsilon^{\mu_j}},\frac{y}{\varepsilon}\right)$. *Then,*

$$\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\psi_j^\varepsilon\to\psi_j\quad\textit{strongly in}\quad L^p\left((0,1)\times A_j\times Y_j^*\right).$$

Proof. Notice that

$$\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\psi_j^\varepsilon(x,z,y_1,y_2)=\mathbf{T}_\varepsilon^j\left[\psi_j\left(y_1,y_2\right)\right]\to\psi_j(y_1,y_2)\quad\text{as }\varepsilon\to 0,$$

which proves the result. □

Now, we start by the main convergence theorems to be used throughout the paper. We start with

Lemma 2.11. *Let $\varphi \in W^{1,p}(R^\varepsilon)$ and set*

$$V_\varepsilon\varphi(x) = \frac{1}{\varepsilon g_0}\int_0^{\varepsilon g_0} \varphi(x,y)dy, \quad g_0 = \min_{i=1,2}\{g_i^m\}.$$

Then,

$$\|\varphi - V_\varepsilon\varphi\|_{L^p(R^\varepsilon)} \le c\varepsilon \left\| \frac{\partial\varphi}{\partial y} \right\|_{L^p(R^\varepsilon)}.$$

Proof. Notice that

$$\varphi(x,y) - \varphi(x,y') = \int_{y'}^{y} \frac{\partial\varphi}{\partial y}(x,s)ds.$$

Integrate the above expression in y' between 0 and εg_0 and dividing by εg_0 we obtain

$$\varphi(x,y) - V_\varepsilon\varphi(x) = \frac{1}{\varepsilon g_0}\int_0^{\varepsilon g_0}\int_{y'}^{y} \frac{\partial\varphi}{\partial y}(x,s)dsdy'.$$

Then,

$$|\varphi(x,y) - V_\varepsilon\varphi(x)|^p \le \left(\frac{1}{\varepsilon g_0}\int_0^{\varepsilon g_0}\int_0^{y} \left|\frac{\partial\varphi}{\partial y}(x,s)\right| dsdy'\right)^p \le \left(\varepsilon \max_{i=1,2} g_i^M\right)^{\frac{p}{p'}} \int_0^{\varepsilon g_\varepsilon(x)} \left|\frac{\partial\varphi}{\partial y}(x,s)\right|^p ds.$$

Finally, integrate in R^ε and get

$$\|\varphi - V_\varepsilon\varphi\|^p_{L^p(R^\varepsilon)} \le \varepsilon^p \left(\max_{i=1,2} g_i^M\right)^p \left\|\frac{\partial\varphi}{\partial y}\right\|^p_{L^p(R^\varepsilon)}. \quad \square$$

Remark 2.12. *The above Lemma, basically tells us that in order to determine the limit of a sequence $\varphi_\varepsilon \in W^{1,p}(R^\varepsilon)$, we just need to find the limit of*

$$V_\varepsilon\varphi_\varepsilon(x) = \frac{1}{\varepsilon g_0}\int_0^{\varepsilon g_0} \varphi_\varepsilon(x,y)dy.$$

Notice that this function is in $W^{1,p}(0,1)$ and therefore, we get that, there is a $\varphi \in W^{1,p}(0,1)$, such that, up to a subsequence,

$$V_\varepsilon\varphi_\varepsilon \rightharpoonup \varphi \quad \text{weakly in} \quad W^{1,p}(0,1) \quad \text{and strongly in} \quad L^p(0,1).$$

In particular, if $\varphi_\varepsilon \in W^{1,p}_{0l}(R^\varepsilon)$, we have $\varphi \in W^{1,p}_0(0,1)$ with $V_\varepsilon\varphi_\varepsilon \rightharpoonup \varphi$ weakly in $W^{1,p}_0(0,1)$.

The preceding theorem concerns to the convergence of the unfolded gradients.

Theorem 2.13. *Let $u^\varepsilon \in W^{1,p}_{0l}(R^\varepsilon)$ be a sequence with $|||u^\varepsilon|||_{W^{1,p}(R^\varepsilon)}$ uniformly bounded and let $A_1 = (0, 1/2)$ and $A_2 = (1/2, 1)$. Then, there exists $u \in W^{1,p}_0(0, 1)$ such that, up to a subsequence,*

$$V_\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}_0(0,1) \quad \text{and strongly in} \quad L^p(0,1).$$

Moreover, assume $p'\gamma < \alpha \leq \beta$.

1) If $\gamma < \alpha = \beta = 1$, then there are functions $u^1 \in L^p\left((0,1) \times A_1; W^{1,p}_\#(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1) \times A_2; W^{1,p}_\#(Y_2^*)\right)$ satisfying

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y_j^*\right), \quad j = 1, 2.$$

2) If $\gamma < \alpha < \beta = 1$, then there are functions $u^1 \in L^p\left((0,1) \times A_1; W^{1,p}_\#(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1) \times A_2; W^{1,p}_\#(Y_2^*)\right)$ satisfying $\frac{\partial u^1}{\partial y_2} = 0$ and

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y_j^*\right), \quad j = 1, 2.$$

3) If $\gamma < \alpha < 1 < \beta$, then there exists $u^1 \in L^p\left((0,1) \times A_1; W^{1,p}_\#(Y_1^*)\right)$ satisfying $\frac{\partial u^1}{\partial y_2} = 0$ and

$$\mathbf{T}^1_\varepsilon \mathcal{T}^1_\varepsilon \nabla u^\varepsilon \rightharpoonup \frac{\partial u}{\partial x} + \frac{\partial u^1}{\partial y_1} \quad \text{weakly in} \quad L^p\left((0,1) \times A_1 \times Y_1^*\right).$$

4) If $\gamma < \alpha \leq \beta < 1$, then there are functions $u^1 \in L^p\left((0,1) \times (0,1/2); W^{1,p}_\#(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1) \times (1/2,1); W^{1,p}_\#(Y_2^*)\right)$ satisfying $\frac{\partial u^1}{\partial y_2} = \frac{\partial u^2}{\partial y_2} = 0$ and

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \nabla u^\varepsilon \rightharpoonup \frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1} \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y_j^*\right), \quad j = 1, 2.$$

5) If $\gamma < \alpha = 1 < \beta$, then there exists $u^1 \in L^p\left((0,1) \times A_1; W^{1,p}_\#(Y_1^*)\right)$ satisfying

$$\mathbf{T}^1_\varepsilon \mathcal{T}^1_\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^1 \quad \text{weakly in} \quad L^p\left((0,1) \times A_1 \times Y_1^*\right)$$

Proof. Due to the previous lemma and the boundness, there is $u \in W^{1,p}(0,1)$ such that, up to a subsequence,

$$V_\varepsilon u^\varepsilon \rightharpoonup u \quad \text{weakly in} \quad W^{1,p}(0,1) \quad \text{and strongly in} \quad L^p(0,1).$$

Therefore, we just need to find the limit of the gradient. For this sake, let

$$Z_j^\varepsilon(x,z,y_1,y_2) = \frac{1}{\varepsilon^{\mu_j}}\left[\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon(x,z,y_1,y_2) - \frac{1}{|Y_j^*|}\int_{Y_j^*} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon dy_1 dy_2\right],$$

where $j=1,2$, $\mu_1=\alpha$ and $\mu_2=\beta$. See that here we are assuming that at least one of the μ_j is less or equal to one. The case both bigger than one, we shall determine the limit depending on the equation, as the reader will be able to see in the next sections.

Let us start with the case where one of the μ_j is equal to one. Due to Proposition 2.2, we have

$$\nabla_{y_1y_2} Z_j^\varepsilon = \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla u^\varepsilon.$$

Also, we obtain that $\|\nabla_{y_1y_2} Z_j^\varepsilon\|_{L^p\left((0,1)\times A_j\times Y_j^*\right)}$ is uniformly bounded. Consequently, from Poincaré-Wirtinger inequality, we have that $\|Z_j^\varepsilon\|_{L^p\left((0,1)\times A_j\times Y_j^*\right)}$ is uniformly bounded too. Now, consider the following sequence

$$Z_j^\varepsilon - y_1^c \frac{\partial u}{\partial x} \quad \text{with} \quad y_1^c = y_1 - \langle y_1\rangle_{Y_j^*}.$$

This sequence is also uniformly bounded in $L^p\left((0,1)\times A_j\times Y_j^*\right)$ and possesses average zero. Thus, passing to another subsequence of ε, there exist $u^j \in L^p\left((0,1)\times A_j\times Y_j^*\right)$ such that

$$Z_j^\varepsilon - y_1^c \frac{\partial u}{\partial x} \rightharpoonup u^j \quad \text{weakly in} \quad L^p\left((0,1)\times A_j\times Y_j^*\right).$$

Thus,

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1)\times A_j; W^{1,p}(Y_j^*)\right).$$

We remark that if $\mu_j < 1$, then one gets that $\frac{\partial u^j}{\partial y_2} = 0$, since

$$\nabla_{y_1y_2} Z_j^\varepsilon = \left(\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \frac{\partial u^\varepsilon}{\partial x}, \varepsilon^{1-\mu_j} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \frac{\partial u^\varepsilon}{\partial y}\right).$$

It remains to prove that u_j^1 is periodic in the variable y_1, however this will follow from the study of the limit of

$$Z_j^\varepsilon(x,z,y_1+L_j,y_2) - Z_j^\varepsilon(x,z,y_1,y_2)$$

and can be seen at [6, Theorems 3.1 and 4.1]. □

Remark 2.14. *The previous theorem does not contain the case $\gamma < 1 < \alpha \le \beta$ and $p'\gamma < \alpha$, since we need to use the equation to determine the limit behavior.*

Next, we prove a result concerning the unfolding operator Π_j^ε.

Proposition 2.15. *Suppose $\kappa > 1$. If $\varphi_\varepsilon \in W_{0l}^{1,p}(Q^\varepsilon)$ with $\varepsilon^{-1/p}\,||\varphi_\varepsilon||_{W_{0l}^{1,p}(Q^\varepsilon)}$ uniformly bounded, then there are $\varphi \in W_0^{1,p}(0,1)$ and $\varphi_1^j \in L^p((0,1); W_\#^{1,p}(Z_j^-))$ such that:*

$$\begin{cases} \partial_{y_1}\varphi_1^j = 0, \\ \Pi_j^\varepsilon \varphi_\varepsilon \rightharpoonup \varphi, \text{ weakly in } L^p\left((0,1); W^{1,p}(Z_J^-)\right), \\ \Pi_j^\varepsilon \nabla\varphi_\varepsilon \rightharpoonup (\partial_x\varphi, \partial_{y_2}\varphi_1^j), \text{ weakly in } L^p\left((0,1)\times Z_j^-\right). \end{cases}$$

Proof. The idea is similar to the previous Theorem by defining

$$Z_\varepsilon^j(x, y_1, y_2) = \frac{1}{\varepsilon}\left[\Pi_j^\varepsilon\varphi_\varepsilon - \int_{Z_j^-} \Pi_j^\varepsilon \varphi_\varepsilon dy_1 dy_2\right].$$

Then, one obtains that

$$Z_\varepsilon^j - y_1^c\frac{\partial u}{\partial x} \rightharpoonup u^j \quad \text{weakly in} \quad L^p((0,1); W^{1,p}(Z_j^-)).$$

Since

$$\nabla_{y_1y_2} Z_\varepsilon^j = \left(\varepsilon^{\gamma-1}\Pi^\varepsilon\frac{\partial\varphi_\varepsilon}{\partial x}, \Pi^\varepsilon\frac{\partial\varphi_\varepsilon}{\partial y}\right),$$

it follows that $\frac{\partial u^j}{\partial y_1} = 0$, i.e., $u^j = u^j(x, y_2)$. □

To finish this section, we recall an important convergence result with respect to the unfolding operator Π^ε. Their proof can be found at [6, Theorems 3.1 and 4.1].

Proposition 2.16. *If $\varphi_\varepsilon \in W_{0l}^{1,p}(Q^\varepsilon)$ with $\varepsilon^{-1/p}\,||\varphi_\varepsilon||_{W_{0l}^{1,p}(Q^\varepsilon)}$ uniformly bounded, then there are $\varphi \in W_0^{1,p}(0,1)$ and $\varphi_1 \in L^p((0,1); W_\#^{1,p}(Z))$ such that:*

$$i) \quad \begin{cases} \text{If } \kappa = 1, \\ \varepsilon^{-1/p}\,||\varphi_\varepsilon - \varphi||_{L^p(Q^\varepsilon)} \to 0 \\ \Pi^\varepsilon\varphi_\varepsilon \to \varphi \text{ strongly in } L^p\left((0,1); W^{1,p}(Z_J^-)\right), \\ \Pi^\varepsilon\partial_x\varphi_\varepsilon \rightharpoonup \partial_x\varphi + \partial_{y_1}\varphi_1 \text{ weakly in } L^p\left((0,1)\times Z\right), \\ \Pi_j^\varepsilon\partial_y\varphi_\varepsilon \rightharpoonup \partial_{y_2}\varphi_1 \text{ weakly in } L^p\left((0,1)\times Z\right). \end{cases}$$

$$ii) \quad \begin{cases} \text{If } \kappa < 1,\ \partial_{y_2}\varphi_1 = 0, \\ \Pi^\varepsilon\varepsilon\varphi_\varepsilon \rightharpoonup \varphi, \text{ weakly in } L^p\left((0,1); W^{1,p}(Z)\right), \\ \Pi^\varepsilon\varepsilon\partial_x\varphi_\varepsilon \rightharpoonup \partial_x\varphi + \partial_{y_1}\varphi_1, \text{ weakly in } L^p\left((0,1)\times Z\right). \end{cases}$$

3. Main results

In this Section, we perform the proof of our main convergence results. First, we recall that throughout the whole paper $\gamma > 0$ and satisfies

$$p'\gamma < \min\{\alpha, \beta\} = \alpha \tag{3.1}$$

since we are assuming without loss of generality $\alpha \leq \beta$.

Next, let us notice that the variational formulation of the problem (1.1) is given by

$$\int_{R^\varepsilon} \left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon \nabla \varphi dxdy = \int_{R^\varepsilon} f^\varepsilon \varphi dxdy, \quad \forall \varphi \in W^{1,p}_{0l}(R^\varepsilon) \tag{VP}$$

which is equivalent to

$$\int_{R^\varepsilon} |\nabla \varphi|^{p-2} \nabla \varphi \left(\nabla \varphi - \nabla u^\varepsilon\right) dxdy \leq \int_{R^\varepsilon} f^\varepsilon \left(\varphi - u^\varepsilon\right) dxdy, \quad \forall \varphi \in W^{1,p}_{0l}(R^\varepsilon). \tag{VI}$$

We first deal with the resonant case $\alpha = \beta = 1$.

Theorem 3.1. *Let $u^\varepsilon \in W^{1,p}_{0l}(R^\varepsilon)$ be the solution of (1.1) with $\alpha = \beta = 1$. Suppose that*

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon f^\varepsilon \to f^j \quad \text{weakly in} \quad L^{p'}\left((0,1) \times A_j \times Y^*_j\right), \quad j = 1, 2.$$

Then, there are $u \in W^{1,p}_0(0,1)$, $u^1 \in L^p\left((0,1) \times A_1; W^{1,p}_\#(Y^*_1)\right)$ and $u^2 \in L^p\left((0,1) \times A_2; W^{1,p}_\#(Y^*_2)\right)$ such that

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1) \times A_j; W^{1,p}_\#(Y^*_j)\right),$$
$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y^*_j\right), \quad j = 1, 2,$$

where (u, u^1, u^2) is the solution of

$$\begin{aligned}
&\sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y^*_j} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right) \\
&\qquad \times \left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1 y_2} \psi_j\right] dxdzdy_1dy_2 \\
&\quad = \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y^*_j} f^j \rho dxdzdy_1dy_2,
\end{aligned}$$

for all $(\rho, \psi_1, \psi_2) \in W_0^{1,p}(0,1) \times L^p\left((0,1) \times A_1; W_\#^{1,p}(Y_1^*)\right) \times L^p\left((0,1) \times A_2; W_\#^{1,p}(Y_2^*)\right)$ *and* u *is the solution of the one-dimensional problem*

$$\int_0^1 q \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x} dx = \int_0^1 \bar{f} \rho dx, \forall \rho \in W_0^{1,p}(0,1),$$

where

$$q = \frac{1}{2}\sum_{j=1}^{2} \int_{Y_j^*} \left|(1,0) + \nabla_{y_1 y_2} X^j\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^j\right)(1,0) dy_1 dy_2$$

$$\bar{f}(x) = \sum_{j=1}^{2} \int_{A_j \times Y_j^*} f^j(x, z, y_1, y_2) dz dy_1 dy_2$$

and X^j, $j = 1, 2$, *is the solution of*

$$\int_{Y_j^*} \left|(1,0) + \nabla_{y_1 y_2} X^j\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^j\right) \nabla \psi_j dy_1 dy_2 = 0, \quad \forall \psi_j \in W_\#^{1,p}(Y_j^*)$$

$$\textit{with} \quad \int_{Y_j^*} X^j dy_1 dy_2 = 0, \quad j = 1, 2.$$

Proof. As usual, we start proving that the family of solutions of (1.1) is uniformly bounded. Indeed, take $\varphi = u^\varepsilon$ in (VP) and multiply it by ε^{-1}. After a Hölder's inequality and a Poincaré's inequality, is easy to get

$$|||u^\varepsilon|||_{W_{0l}^{1,p}(R^\varepsilon)}^{p-1} \le c|||f^\varepsilon|||_{L^{p'}(R^\varepsilon)},$$

where $c > 0$ is the constant independent of ε from Poincaré inequality. From Theorem 2.13, there are $u \in W_0^{1,p}(0,1)$, $u^1 \in L^p\left((0,1) \times A_1; W_\#^{1,p}(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1) \times A_2; W_\#^{1,p}(Y_2^*)\right)$ satisfying

$$\begin{aligned} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon &\to u \quad \text{strongly in} \quad L^p\left((0,1) \times A_j; W_\#^{1,p}(Y_j^*)\right), \\ \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla u^\varepsilon &\rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y_j^*\right), \end{aligned} \tag{3.2}$$

for $j = 1, 2$.

Next, we apply formula (2.3) in (VI) ignoring the terms from the 'integration deffect' and obtain

$$\begin{aligned}&\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}\left|\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\nabla\varphi\right|^{p-2}\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\nabla\varphi\left(\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\nabla\varphi-\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\nabla u^\varepsilon\right)dxdzdy_1dy_2\\&\quad\le\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j f^\varepsilon\left(\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\varphi-\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j u^\varepsilon\right)dxdzdy_1dy_2.\end{aligned}\tag{3.3}$$

Next, we define the oscillating test functions as follows: Let $\rho\in C_0^\infty(0,1)$, $\phi_j\in C_0^\infty\big((0,1)\times A_j\big)$ and $\psi_j\in C_\#^\infty(\overline{Y_j^*})$. Define $\phi_j^\varepsilon(x)=\phi_j\left(x,\left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$ and $\psi_j^\varepsilon(x,y)=\psi_j\left(\frac{x}{\varepsilon^{\mu_j}},\frac{y}{\varepsilon}\right)$, where $\mu_1=\alpha=\mu_2=\beta=1$, and

$$\varphi^\varepsilon(x,y)=\begin{cases}\rho(x)+\varepsilon^\alpha\phi_1^\varepsilon(x)\psi_1^\varepsilon(x,y) & \text{in} \quad \mathcal{R}_1^\varepsilon\\ \rho(x)+\varepsilon^\beta\phi_2^\varepsilon(x)\psi_2^\varepsilon(x,y) & \text{in} \quad \mathcal{R}_2^\varepsilon.\end{cases}$$

Using Propositions 2.8 and 2.10, we have

$$\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j\nabla\varphi^\varepsilon\to\left(\frac{\partial\rho}{\partial x},0\right)+\phi_j\nabla_{y_1y_2}\psi_j\quad\text{strongly in}\quad L^p((0,1)\times A_j\times Y_j^*),\quad j=1,2.$$

Take φ^ε as a test function in (3.3), by the above convergence and (3.2), we obtain

$$\begin{aligned}&\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}\left|\left(\frac{\partial\rho}{\partial x},0\right)+\phi_j\nabla_{y_1y_2}\psi_j\right|^{p-2}\left(\left(\frac{\partial\rho}{\partial x},0\right)+\phi_j\nabla_{y_1y_2}\psi_j\right)\\&\qquad\left[\left(\frac{\partial\rho}{\partial x},0\right)+\phi_j\nabla_{y_1y_2}\psi_j-\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1},\frac{\partial u^j}{\partial y_2}\right)\right]dxdzdy_1dy_2\\&\qquad\qquad\le\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}f^j(\rho-u)dxdzdy_1dy_2.\end{aligned}$$

By density, the variational inequality above holds for test functions $(\rho,\psi_1,\psi_2)\in W_0^{1,p}(0,1)\times\prod_{j=1}^2 L^p\left((0,1)\times A_j;W_\#^{1,p}(Y_j^*)\right)$. Thus,

$$\begin{aligned}&\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}\left|\left(\frac{\partial\rho}{\partial x},0\right)+\nabla_{y_1y_2}\psi_j\right|^{p-2}\left(\left(\frac{\partial\rho}{\partial x},0\right)+\nabla_{y_1y_2}\psi_j\right)\\&\qquad\left[\left(\frac{\partial\rho}{\partial x},0\right)+\nabla_{y_1y_2}\psi_j-\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1},\frac{\partial u^j}{\partial y_2}\right)\right]dxdzdy_1dy_2\\&\qquad\qquad\le\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*}f^j(\rho-u)dxdzdy_1dy_2.\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
&\sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j\times Y_j^*} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right) \\
&\quad \times \left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1 y_2}\psi_j\right] dxdzdy_1dy_2 \qquad (3.4)\\
&\quad = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j\times Y_j^*} f^j \rho dxdzdy_1dy_2.
\end{aligned}
$$

Recalling the auxiliary problem

$$
\int\limits_{Y_j^*} \left|(1,0) + \nabla_{y_1 y_2} X^j\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^j\right) \nabla \psi_j dy_1 dy_2 = 0, \quad \forall \psi_j \in W^{1,p}_{\#}(Y_j^*)
$$

$$
\text{with} \int\limits_{Y_j^*} X^j dy_1 dy_2 = 0, \quad j = 1, 2,
$$

taking $\rho = 0$ and $\psi_1 = 0$ in (3.4) and then $\rho = 0$ and $\psi_2 = 0$, it is not difficult to get that (see [2] for similar arguments)

$$
u^j(x, z, y_1, y_2) = \frac{\partial u}{\partial x}(x) X^j(y_1, y_2) \quad \text{in} \quad (0, 1) \times A_l \times Y_j^*, \tag{3.5}
$$

which means that u^j independs on the variable z. Using this independence in relation (3.4), we get

$$
\begin{aligned}
&\sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j\times Y_j^*} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right) \\
&\quad \times \left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1 y_2}\psi_j\right] dxdzdy_1dy_2 \\
&\quad = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j\times Y_j^*} f^j \rho dxdzdy_1dy_2,
\end{aligned}
$$

for all $(\rho, \psi_1, \psi_2) \in W_0^{1,p}(0, 1) \times \prod_{j=1}^{2} L^p\left((0, 1); W^{1,p}_{\#}(Y_j^*)\right)$.

Notice that

$$
\int\limits_{(1,0)\times A_j\times Y_j^*} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right) \left(\frac{\partial \rho}{\partial x}, 0\right) dxdzdy_1dy_2
$$

$$= \frac{1}{2}\left[\int\limits_{Y_j^*} \left|(1,0)+\nabla_{y_1y_2}u^j\right|^{p-2}\left((1,0)+\nabla_{y_1y_2}u^j\right)dy_1dy_2\right]\int\limits_0^1 \left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\left(\frac{\partial \rho}{\partial x},0\right)dx.$$

Thus, for $\psi_j = 0$, we have the following one-dimensional problem

$$\int\limits_0^1 q\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}dx = \int\limits_0^1 \bar{f}\rho dx, \forall \rho \in W_0^{1,p}(0,1),$$

where

$$q = \sum_{j=1}^{2}\frac{1}{2L_j}\int\limits_{Y_j^*}\left|(1,0)+\nabla_{y_1y_2}X^j\right|^{p-2}\left((1,0)+\nabla_{y_1y_2}X^j\right)(1,0)dy_1dy_2$$

$$\bar{f}(x) = \sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{A_j\times Y_j^*} f^j(x,z,y_1,y_2)dzdy_1dy_2. \quad \square$$

Next, let us deal with the weakly rough case.

Theorem 3.2. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the solution of (1.1) with $\alpha \le \beta < 1$. Suppose that*

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \to f^j \quad \textit{weakly in} \quad L^{p'}\left((0,1)\times A_j\times Y_j^*\right), \quad j=1,2.$$

Then, there are $u \in W_0^{1,p}(0,1)$, $u^1 \in L^p\left((0,1)\times A_1; W_\#^{1,p}(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1)\times A_2; W_\#^{1,p}(Y_2^*)\right)$ such that

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad \textit{strongly in} \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right),$$

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x},0\right)+\nabla_{y_1y_2}u^j \quad \textit{weakly in} \quad L^p\left((0,1)\times A_j\times Y_j^*\right), \quad j=1,2,$$

with $\frac{\partial u^j}{\partial y_2}=0$, where (u,u^1,u^2) is the solution of

$$\sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times(0,L_j)}\left|\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right)\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right)\left[\frac{\partial \rho}{\partial x}+\frac{\partial \psi_j}{\partial y_1}\right]g_j(y_1)dxdzdy_1$$
$$= \sum_{j=1}^{2}\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times(0,L_j)}\left(\int\limits_0^{g_j(y_1)} f^j(x,z,y_1,y_2)dy_2\right)\rho dxdzdy_1,$$

for all $(\rho, \psi_1, \psi_2) \in W_0^{1,p}(0,1) \times L^p\Big((0,1)\times A_1; W_\#^{1,p}(0,L_1)\Big) \times L^p\Big((0,1) \times A_2; W_\#^{1,p}(0,L_2)\Big)$ *and* u *is the solution of the one-dimensional problem*

$$\int_0^1 q \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x} dx = \int_0^1 \bar{f} \rho dx, \forall \rho \in W_0^{1,p}(0,1),$$

where

$$q = \sum_{j=1}^{2} \frac{1}{2L_j}\left\langle 1/g_j^{p'-1}\right\rangle^{1-p} \quad \textit{and} \quad \bar{f} = \sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*} f^j dz dy_1 f y_2.$$

Proof. The uniform bounds hold as in the previous theorem. Thanks to Theorem 2.13, there are $u \in W_0^{1,p}(0,1)$, $u^1 \in L^p\left((0,1)\times A_1; W_\#^{1,p}(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1)\times A_2; W_\#^{1,p}(Y_2^*)\right)$ satisfying $\frac{\partial u^1}{\partial y_2} = \frac{\partial u^2}{\partial y_2} = 0$ and

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right),$$
$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, 0\right) \quad \text{weakly in} \quad L^p\left((0,1)\times A_j \times Y_j^*\right), \quad j=1,2.$$

Let $\rho \in C_0^\infty(0,1)$, $\phi_j \in C_0^\infty\left((0,1)\times A_j\right)$ and $\psi_j \in C_\#^\infty(\overline{0,L_j})$. Define $\phi_j^\varepsilon(x) = \phi_j\left(x, \left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$ and $\psi_j^\varepsilon(x,y) = \psi_j\left(\frac{x}{\varepsilon^{\mu_j}}\right)$, where $\mu_1 = \alpha$ and $\mu_2 = \beta$, and

$$\varphi^\varepsilon(x,y) = \begin{cases} \rho(x) + \varepsilon^\alpha \phi_1^\varepsilon(x)\psi_1^\varepsilon(x) & \text{in} \quad \mathcal{R}_1^\varepsilon \\ \rho(x) + \varepsilon^\beta \phi_2^\varepsilon(x)\psi_2^\varepsilon(x) & \text{in} \quad \mathcal{R}_2^\varepsilon. \end{cases}$$

Using Propositions 2.8 and 2.10, we have

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \nabla \varphi^\varepsilon \to \left(\frac{\partial \rho}{\partial x} + \frac{\partial \psi_j}{\partial y_1}, 0\right) \quad \text{strongly in} \quad L^p((0,1)\times A_j \times Y_j^*), \quad j=1,2.$$

Take φ^ε as a test function in (3.3) and repeat the arguments of the previous Theorem. We obtain

$$\begin{aligned} &\sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*}\left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2}\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right)\left[\frac{\partial \rho}{\partial x} + \frac{\partial \psi_j}{\partial y_1}\right] dx dz dy_1 dy_2 \\ &\qquad = \sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j \times Y_j^*} f^j \rho dx dz dy_1 dy_2, \end{aligned} \tag{3.6}$$

for all $(\rho, \psi_1, \psi_2) \in W_0^{1,p}(0,1) \times L^p\left((0,1) \times A_1; W_\#^{1,p}(Y_1^*)\right) \times L^p\left((0,1) \times A_2; W_\#^{1,p}(Y_2^*)\right)$.

For $\rho = 0$ and $\psi_2 = 0$ (the case where $\psi_1 = 0$ is analogous), we have

$$\int\limits_{(0,1)\times A_1\times Y_1^*} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) \frac{\partial \psi_1}{\partial y_1} dx dz dy_1 dy_2 = 0,$$
$$\forall \psi_1 \in L^p\left((0,1) \times A_1; W_\#^{1,p}(Y_1^*)\right).$$

Since all functions do not depend on y_2, we have

$$\int\limits_{(0,1)\times A_1\times (0,L_1)} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) \frac{\partial \psi_1}{\partial y_1} g_1(y_1) dx dz dy_1 = 0,$$
$$\forall \psi_1 \in L^p\left((0,1) \times A_1; W_\#^{1,p}(0,L_1)\right).$$

Now, let $\psi \in W_\#^{1,p}(0,L_1)$, $v \in C_0^\infty(0,1)$, $\phi \in C_0^\infty(A_1)$ and take $\Phi \in C_0^\infty(A_1)$ such that $\Phi' = \phi$ $\psi_1(x,z,y_1) = v(x)\Phi(z)\psi(y_1)$. We have

$$\int\limits_{(0,1)\times A_1\times (0,L_1)} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) v(x)\Phi(z) \frac{\partial \psi}{\partial y_1}(y_1) g_1(y_1) dx dz dy_1 = 0.$$

Therefore, we have

$$\begin{aligned} 0 &= \int_0^{L_1} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) \Phi(z) g(y_1) dz \\ &= \int_0^{L_1} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) \phi'(z) g(y_1) dz, \end{aligned}$$

implying that $\left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right)$ is independent of z, that is, u^1 is independent of z. Hence,

$$\int\limits_{(0,1)\times (0,L_1)} \left|\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}\right) \frac{\partial \psi_1}{\partial y_1} g_1(y_1) dx d dy_1 = 0,$$
$$\forall \psi_1 \in L^p\left((0,1); W_\#^{1,p}(0,L_1)\right).$$

Thus, treating x as a parameter, there is $T = T(x)$ such that

$$\left(\left|\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right)\right)(x,y_1)g_1(y_1)=T(x) \quad \text{a.e. in} \quad (0,1)\times(0,L_1).$$

The above expression implies

$$\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}=\left|\frac{T}{g_1}\right|^{p'-2}\left(\frac{T}{g_1}\right)$$

integrates the above expression in y_1 between 0 and L_1 and uses that u^j is L_1-periodic in L_1, to obtain

$$L_1\frac{\partial u}{\partial x}=|T|^{p'-2}\,T\int_0^{L_1}\frac{1}{g_1^{p'-1}}dy_1,$$

that is,

$$T=\frac{1}{\left\langle 1/g_1^{p'-1}\right\rangle^{p-1}}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}.$$

Then,

$$\left(\left|\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial u^j}{\partial y_1}\right)\right)(x,y_1)g_1(y_1)=\frac{1}{\left\langle 1/g_1^{p'-1}\right\rangle^{p-1}}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}. \tag{3.7}$$

We are in position to rewrite (3.6) as follows

$$\sum_{j=1}^{2}\frac{1}{2L_j}\int_{(0,1)\times(0,L_j)}\frac{1}{\left\langle 1/g_1^{p'-1}\right\rangle^{p-1}}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\left[\frac{\partial \rho}{\partial x}+\frac{\partial \psi_j}{\partial y_1}\right]dx\,dy_1dy_2$$
$$=\sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*} f^j\rho\,dx\,dz\,dy_1dy_2.$$

In particular, the one-dimensional problem is given by

$$q\int_0^1\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}dx=\int_0^1\bar{f}\rho\,dx,$$

where

$$q=\sum_{j=1}^{2}\frac{1}{2L_j}\left\langle 1/g_j^{p'-1}\right\rangle^{1-p} \quad \text{and} \quad \bar{f}=\sum_{j=1}^{2}\frac{1}{L_j}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*} f^j dz\,dy_1 fy_2. \qquad \square$$

In next result, we just need to put together the arguments previously done, dealing $\alpha < \beta = 1$.

Theorem 3.3. *Let $u^\varepsilon \in W^{1,p}_{0l}(R^\varepsilon)$ be the solution of (1.1) with $\alpha < \beta = 1$. Suppose that*

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon f^\varepsilon \to f^j \quad \text{weakly in} \quad L^{p'}\left((0,1)\times A_j \times Y_j^*\right), \qquad j = 1, 2.$$

Then, there are $u \in W^{1,p}_0(0,1)$, $u^1 \in L^p\left((0,1)\times A_1; W^{1,p}_\#(Y_1^*)\right)$ and $u^2 \in L^p\left((0,1)\times A_2; W^{1,p}_\#(Y_2^*)\right)$ such that

$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1)\times A_j; W^{1,p}_\#(Y_j^*)\right),$$
$$\mathbf{T}^j_\varepsilon \mathcal{T}^j_\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} u^j \quad \text{weakly in} \quad L^p\left((0,1)\times A_j \times Y_j^*\right), \qquad j = 1, 2,$$

with $\frac{\partial u^1}{\partial y_2} = 0$, where (u, u^1, u^2) is the solution of

$$\begin{aligned}
&\sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j \times Y_j^*} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial u^j}{\partial y_1}, \frac{\partial u^j}{\partial y_2}\right) \\
&\qquad \times \left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1 y_2} \psi_j\right] dx dz dy_1 dy_2 \\
&\quad = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j \times Y_j^*} f^j \rho dx dz dy_1 dy_2,
\end{aligned}$$

for all $(\rho, \psi_1, \psi_2) \in W^{1,p}_0(0,1) \times L^p\left((0,1); W^{1,p}_\#(0, L_1)\right) \times L^p\left((0,1); W^{1,p}_\#(Y_2^*)\right)$ and u is the solution of the one-dimensional problem

$$\int_0^1 q \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x} \frac{\partial \rho}{\partial x} dx = \int_0^1 \bar{f} \rho dx, \forall \rho \in W^{1,p}_0(0,1), \tag{3.8}$$

where,

$$\bar{f} = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j \times Y_j^*} f^j dz dy_1 f y_2,$$

$$q = \frac{1}{2L_1} \left\langle 1/g_1^{p'-1} \right\rangle^{1-p} + \frac{1}{2L_2} \int\limits_{Y_2^*} \left|(1,0) + \nabla_{y_1 y_2} X^2\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} X^2\right)(1,0) dy_1 dy_2$$

with X^2 solving

$$\int_{Y_2^*} \left|(1,0)+\nabla_{y_1y_2}X^2\right|^{p-2}\left((1,0)+\nabla_{y_1y_2}X^2\right)\nabla\psi_2 dy_1dy_2 = 0, \quad \forall \psi_2 \in W_\#^{1,p}(Y_2^*)$$

$$\text{with} \int_{Y_2^*} X^2 dy_1 dy_2 = 0.$$

Proof. As in the previous cases, we define the oscillating test functions as follows: Let $\rho \in C_0^\infty(0,1)$, $\phi_j \in C_0^\infty\left((0,1)\times A_j\right)$, $\psi_2 \in C_\#^\infty(\overline{Y_2^*})$ and $\psi_1 \in C_\#^\infty(\overline{0,L_1})$. Define $\phi_j^\varepsilon(x) = \phi_j\left(x, \left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$, $\psi_2^\varepsilon(x,y) = \psi_2\left(\frac{x}{\varepsilon^{\mu_2}}, \frac{y}{\varepsilon}\right)$ and $\psi_1^\varepsilon(x,y) = \psi_1\left(\frac{x}{\varepsilon^{\mu_1}}\right)$, where $\mu_1 = \alpha < 1$ and $\mu_2 = \beta = 1$,

$$\varphi^\varepsilon(x,y) = \begin{cases} \rho(x) + \varepsilon^\alpha \phi_1^\varepsilon(x)\psi_1^\varepsilon(x,y) & \text{in} \quad \mathcal{R}_1^\varepsilon \\ \rho(x) + \varepsilon^\beta \phi_2^\varepsilon(x)\psi_2^\varepsilon(x) & \text{in} \quad \mathcal{R}_2^\varepsilon. \end{cases}$$

From Propositions 2.8 and 2.10, we have

$$\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^1 \nabla\varphi^\varepsilon \to \left(\frac{\partial \rho}{\partial x} + \frac{\partial \psi_1}{\partial y_1}, 0\right) \quad \text{strongly in} \quad L^p((0,1)\times A_1 \times Y_1^*),$$
$$\mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^2 \nabla\varphi^\varepsilon \to \left(\frac{\partial \rho}{\partial x}, 0\right) + \phi_2 \nabla_{y_1y_2}\psi_2 \quad \text{strongly in} \quad L^p((0,1)\times A_2 \times Y_2^*).$$

Hence, we can proceed as in the Theorems 3.1 and 3.2, passing to the limit in (3.3) and obtaining

$$\frac{1}{L_1}\int_{(0,1)\times A_1\times Y_1^*} \left|\frac{\partial u}{\partial x} + \frac{\partial u^1}{\partial y_1}\right|^{p-2}\left(\frac{\partial u}{\partial x} + \frac{\partial u^1}{\partial y_1}\right)\left[\frac{\partial \rho}{\partial x} + \frac{\partial \psi_1}{\partial y_1}\right] dx dz dy_1 dy_2$$

$$\frac{1}{L_2}\int_{(0,1)\times A_2\times Y_2^*} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial u^2}{\partial y_1}, \frac{\partial u^2}{\partial y_2}\right)\right|^{p-2}\left(\frac{\partial u}{\partial x} + \frac{\partial u^2}{\partial y_1}, \frac{\partial u^2}{\partial y_2}\right)\left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1y_2}\psi_2\right] dx dz dy_1 dy_2$$

$$= \sum_{j=1}^{2} \frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*} f^j \rho \, dx dz dy_1 dy_2.$$

Finally, using the associated auxiliar function, we can arque as in the proof of the mentioned theorem to obtain (3.8). □

Now, we consider the results which deal with strong oscillations at least in one of the profiles.

Theorem 3.4. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the solution of (1.1) with $\alpha \le 1 < \beta$. Suppose that*

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \to f^j \quad \textit{weakly in} \quad L^{p'}\left((0,1)\times A_j \times Y_j^*\right), \quad j = 1, 2.$$

Then, there are $u \in W_0^{1,p}(0,1)$ and $u^1 \in L^p\left((0,1)\times A_1; W_\#^{1,p}(Y_1^*)\right)$ such that

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad strongly\ in \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right), \quad j=1,2,$$
$$\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^1 \nabla u^\varepsilon \to \left(\frac{\partial u}{\partial x},0\right)+\nabla_{y_1y_2}u^1 \quad strongly\ in \quad L^p\left((0,1)\times A_1\times Y_1^*\right),$$
$$\mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^{2+}\left(\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon\right) \rightharpoonup 0 \quad weakly\ in \quad L^{p'}\left((0,1)\times A_2\times Y_{2+}^*\right),$$
$$\Pi_2^\varepsilon \nabla u^\varepsilon \to \left(\frac{\partial u}{\partial x},0\right) \quad strongly\ in \quad L^p\left((0,1)\times Z_2^-\right).$$

with $\frac{\partial u^1}{\partial y_2}=0$ *if* $\alpha<1$*, where* (u,u^1,u_2^-) *is the solution of*

$$\frac{1}{L_1}\int\limits_{(0,1)\times A_1\times Y_j^*}\left|\left(\frac{\partial u}{\partial x}+\frac{\partial u^1}{\partial y_1},\frac{\partial u^1}{\partial y_2}\right)\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial u^1}{\partial y_1},\frac{\partial u^1}{\partial y_2}\right)\left[\left(\frac{\partial \rho}{\partial x},0\right)+\nabla_{y_1y_2}\psi_1\right]dxdzdy_1dy_2$$
$$+\frac{g_2^m}{2}\int_0^1\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}dx=\sum_{j=1}^2\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*} f^j\rho dxdzdy_1dy_2,$$

for all $(\rho,\psi_1)\in W_0^{1,p}(0,1)\times L^p\left((0,1);W_\#^{1,p}(0,L_1)\right)$ *and* u *is the solution of the one-dimensional problem*

$$\int_0^1 q\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}dx=\int_0^1\bar{f}\rho dx,\forall\rho\in W_0^{1,p}(0,1),$$

where,

$$\bar{f}=\sum_{j=1}^2\frac{1}{L_j}\int\limits_{(0,1)\times A_j\times Y_j^*} f^j dzdy_1fy_2.$$

For $\alpha<1$*,*

$$q=\frac{1}{2L_1}\left\langle 1/g_1^{p'-1}\right\rangle^{1-p}+\frac{g_2^m}{2}$$

For $\alpha=1$*,*

$$q=\frac{1}{2L_1}\int\limits_{Y_1^*}\left|(1,0)+\nabla_{y_1y_2}X^1\right|^{p-2}\left((1,0)+\nabla_{y_1y_2}X^1\right)(1,0)dy_1dy_2+\frac{g_2^m}{2}$$

with X^1 *solving*

$$\int_{Y_1^*} \left|(1,0)+\nabla_{y_1y_2}X^1\right|^{p-2}\left((1,0)+\nabla_{y_1y_2}X^1\right)\nabla\psi_2 dy_1dy_2=0, \quad \forall\psi_1\in W_\#^{1,p}(Y_1^*)$$

$$\textit{with} \quad \int_{Y_1^*} X^1 dy_1dy_2=0.$$

Proof. The uniform bounds follow as usual. Applying Theorems 2.13 and 2.15, there are $u\in W_0^{1,p}(0,1)$, $u^1\in L^p\left((0,1)\times A_1; W_\#^{1,p}(Y_1^*)\right)$, $u^2\in L^{p'}\left((0,1)\times A_2\times Y_{2+}^*\right)$ and $u_2^-\in L^p\left((0,1); W_\#^{1,p}(Z_2^-)\right)$ with $\frac{\partial u_2^-}{\partial y_1}=0$ such that

$$\begin{aligned}
&\mathbf{T}_\varepsilon^j\mathcal{T}_\varepsilon^j u^\varepsilon\to u \quad \text{strongly in} \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right), \quad j=1,2,\\
&\mathbf{T}_\varepsilon^1\mathcal{T}_\varepsilon^1\nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x},0\right)+\nabla_{y_1y_2}u^1 \quad \text{weakly in} \quad L^p\left((0,1)\times A_1\times Y_1^*\right),\\
&\mathbf{T}_\varepsilon^2\mathcal{T}_\varepsilon^{2+}\left(\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon\right)\rightharpoonup u^2 \quad \text{weakly in} \quad L^{p'}\left((0,1)\times A_2\times Y_{2+}^*\right),\\
&\Pi_2^\varepsilon\nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x},0\right)+\nabla_{y_1y_2}u_2^- \quad \text{weakly in} \quad L^p\left((0,1)\times Z_2^-\right),\\
&\left|\Pi_2^\varepsilon\nabla u^\varepsilon\right|^{p-2}\Pi_2^\varepsilon\nabla u^\varepsilon \rightharpoonup U^- \quad \text{weakly in} \quad L^{p'}\left((0,1)\times Z_2^-\right)
\end{aligned}$$

where the unfolding $\mathcal{T}_\varepsilon^{2+}$ is the restriction of $\mathcal{T}_\varepsilon^2$ to $Y_{2+}^*=\left\{(y_1,y_2): y_2\in(0,L_2),\ g_2^m<y<g_2(y_1)\right\}$.

We shall point the main differences here. First, if $\alpha<1$, let $\rho\in C_0^\infty(0,1)$, $\phi_1\in C_0^\infty((0,1)\times A_1)$ and $\psi_1\in C_\#^\infty(\overline{0,L_1})$. Define $\phi_1^\varepsilon(x)=\phi_2\left(x,\left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$ and $\psi_1^\varepsilon(x,y)=\psi_1\left(\frac{x}{\varepsilon^\alpha}\right)$.

$$\varphi^\varepsilon(x,y)=\begin{cases}\rho(x)+\varepsilon^\alpha\phi_1^\varepsilon(x)\psi_1^\varepsilon(x) & \text{in} \quad \mathcal{R}_1^\varepsilon\\ 0 \quad \text{in} \quad \mathcal{R}_2^\varepsilon. \end{cases}$$

If $\alpha=1$, we define the oscillating test functions as follows. Let $\rho\in C_0^\infty(0,1)$, $\phi_1\in C_0^\infty((0,1)\times A_1)$ and $\psi_j\in C_\#^\infty(\overline{Y_1^*})$. Define $\phi_1^\varepsilon(x)=\phi_1\left(x,\left\{\frac{x}{\varepsilon^\gamma}\right\}\right)$ and $\psi_1^\varepsilon(x,y)=\psi_1\left(\frac{x}{\varepsilon},\frac{y}{\varepsilon}\right)$. Next, set

$$\varphi^\varepsilon(x,y)=\begin{cases}\rho(x)+\varepsilon^\alpha\phi_1^\varepsilon(x)\psi_1^\varepsilon(x,y) & \text{in} \quad \mathcal{R}_1^\varepsilon\\ 0 \quad \text{in} \quad \mathcal{R}_2^\varepsilon. \end{cases}$$

We repeat the arguments of the previous theorems determining the expression of u^1: (3.7) for $\alpha<1$ and (3.5) for $\alpha=1$.

To identify u^2, we shall use (VP). Let $\phi\in C_0^\infty\left((0,1)\times A_2\times(g_2^m,g_2^M)\right)$, $\psi\in C_0^\infty(0,L_2)$ and choose $\Psi\in C_0^\infty(0,L_2)$ such that $\Psi'=\psi$. Define the test functions

$$\varphi_\varepsilon(x,y)=\begin{cases}0 \quad \text{in} \quad \mathcal{R}_1^\varepsilon\\ \varepsilon^\beta\widetilde{\phi}\left(x,\left\{\frac{x}{\varepsilon^\gamma}\right\},\frac{y}{\varepsilon}\right)\Psi\left(\left\{\frac{x}{\varepsilon^\beta}\right\}_{L_2}\right) & \text{in} \quad \mathcal{R}_2^\varepsilon\end{cases}$$

It is not difficult to see that

$$\begin{aligned} \mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^2 \varphi_\varepsilon \to 0 \quad \text{strongly in} \quad L^p\left((0,1)\times A_2 \times Y_{2+}^*\right), \\ \mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^2 \frac{\partial \varphi_\varepsilon}{\partial y} \to 0 \quad \text{strongly in} \quad L^p\left((0,1)\times A_2 \times Y_{2+}^*\right), \\ \mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^2 \frac{\partial \varphi_\varepsilon}{\partial x} \to \phi\Psi' \quad \text{strongly in} \quad L^p\left((0,1)\times A_2 \times Y_{2+}^*\right). \end{aligned}$$

Taking φ_ε as a test function in (VP), using the integration formula (2.4) in (VP) and passing to the limit, one has

$$\int_{(0,1)\times A_2\times Y_{2+}^*} u^2 \phi \Psi' dx dz dy_1 dy_2 = 0.$$

Thus, one concludes that

$$\widetilde{u^2} = 0 \quad \text{in} \quad (0,1)\times A_2 \times (0,L_2)\times (g_2^m, g_2^M).$$

Now, we need to identify U^-. For this, let

$$\begin{aligned} \mathcal{I}_\varepsilon := \int_{(0,1)\times(1/2,1)\times(0,g_2^m)} & \left[\left|\Pi_2^\varepsilon \nabla u^\varepsilon\right|^{p-2} \Pi_2^\varepsilon \nabla u^\varepsilon - \left|\nabla u + \nabla_{y_1 y_2} u_2^-\right|^{p-2} \left(\nabla u + \nabla_{y_1 y_2} u_2^-\right)\right] \\ & \times (\Pi_2^\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1 y_2} u_2^-). \end{aligned}$$

and

$$\begin{aligned} \mathcal{J}_\varepsilon := \frac{1}{L_1} \int_{(0,1)\times A_1 \times Y_1^*} & \left[\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^1 \left(\left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon\right) - \left|\nabla u + \nabla_{y_1 y_2} u^1\right|^{p-2} \left(\nabla u + \nabla_{y_1 y_2} u^1\right)\right] \\ & \times \left(\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1 y_2} u^1\right). \end{aligned}$$

Recall that $\mathcal{I}_\varepsilon \geq 0$ and $\mathcal{J}_\varepsilon \geq 0$. Moreover,

$$\begin{aligned} 0 \leq & \mathcal{I}_\varepsilon + \mathcal{J}_\varepsilon \\ & + \int_{(0,1)\times A_2\times Y_{2+}^*} |\mathcal{T}_\varepsilon^2 \nabla u^\varepsilon|^p - \int_{(0,1)\times A_2\times Y_{2+}^*} \left|\Pi_2^\varepsilon \nabla u^\varepsilon\right|^{p-2} \Pi_2^\varepsilon \nabla u^\varepsilon \nabla u \\ & + \frac{1}{L_2} \int_{(0,1)\times A_2\times Y_{2+}^*} \left|\Pi_2^\varepsilon \nabla u^\varepsilon\right|^{p-2} \Pi_2^\varepsilon \nabla u^\varepsilon \nabla u \\ = & \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j (u^\varepsilon - u) \end{aligned}$$

$$- \int\limits_{(0,1)\times(1/2,1)\times(0,g_2^m)} \left|\nabla u + \nabla_{y_1y_2} u_2^- \right|^{p-2} \left(\nabla u + \nabla_{y_1y_2} u_2^- \right) (\Pi_2^\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1y_2} u_2^-)$$

$$- \frac{1}{L_1} \int\limits_{(0,1)\times A_1\times Y_1^*} \left|\nabla u + \nabla_{y_1y_2} u^1 \right|^{p-2} \left(\nabla u + \nabla_{y_1y_2} u^1 \right) \left(\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^1 \nabla u^\varepsilon - \nabla u - \nabla_{y_1y_2} u^1 \right)$$

$$\to 0,$$

as $\varepsilon \to 0$. Due to Tartar's inequality, we get

$$\|\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1y_2} u^1\|_{L^p((0,1)\times(0,1/2)\times Y_1^*)} \to 0,$$
$$\|\Pi_2^\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1y_2} u_2^-\|_{L^p((0,1)\times(1/2,1)\times(0,g_2^m))} \to 0,$$

which implies that

$$U^- = \left|\nabla u + \nabla_{y_1y_2} u_2^- \right|^{p-2} \left(\nabla u + \nabla_{y_1y_2} u_2^- \right).$$

It still remains to identify u_2^-. For this, let $\phi \in C_0^\infty(0,1)$ and $\Psi \in C_0^\infty(0, g_2^m)$. Define

$$\varrho^\varepsilon(x, y) = \varepsilon \phi(x)\, \Psi\left(\frac{y}{\varepsilon}\right).$$

Take ϱ^ε as a test function in (VP), use the integration formula (2.4) in it. Passing to the limit, as $\varepsilon \to 0$, and using the convergence properties above, leads us to

$$\int\limits_{(0,1)\times Z_2^-} \left|\nabla u(x) + \nabla_{y_1y_2} u_2^-(x, y_2)\right|^{p-2} \left(\nabla u(x) + \nabla_{y_1y_2} u_2^-(x, y_2)\right)$$
$$\times \left(0, \phi(x) \frac{\partial \Psi}{\partial y_2}(y_2)\right) dx dy_1 dy_2 = 0.$$

Thus, we conclude that

$$\left|\nabla u(x) + \nabla_{y_1y_2} u_2^-(x, y_2)\right|^{p-2} \frac{\partial u_2^-}{\partial y_2}(x, y_2)$$

independs on y_2, that is, $u_2^-(x, y_2) = u_2^-(x)$ and it will not affect the limit problem.

We are in position to identify the limit problem. Using the integration formula (2.4) in (VP) and taking test functions φ depending only on x, we obtain, as $\varepsilon \to 0$,

$$\frac{1}{L_1} \int\limits_{(0,1)\times A_1\times Y_1^*} \left|\left(\frac{\partial u}{\partial x}, 0\right) + \frac{\partial u}{\partial x} \nabla_{y_1y_2} X^1 \right|^{p-2} \left(\left(\frac{\partial u}{\partial x}, 0\right) + \frac{\partial u}{\partial x} \nabla_{y_1y_2} X^1\right) \left(\frac{\partial \varphi}{\partial x}, 0\right) dx dz dy_1 dy_2$$

$$+ \int\limits_{(0,1)\times(1/2,1)\times(0,g_2^m)} |\nabla u|^{p-2} \nabla u \left(\frac{\partial \varphi}{\partial x}, 0\right) dx dz dy_2$$

$$= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} f^j \varphi dx dz dy_1 dy_2,$$

which is rewritten as

$$q \int_0^1 \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} dx = \int_0^1 \bar{f} \varphi dx, \quad \forall \varphi \in W_0^{1,p}(0,1),$$

where

$$q = \frac{1}{2L_1} \int_{Y_1^*} \left|1 + \nabla_{y_1 y_2} X^1\right|^{p-2} \left(1 + \nabla_{y_1 y_2} X^1\right)(1,0) dy_1 dy_2 + \frac{g_2^m}{2}$$

$$\text{and} \quad \bar{f} = \sum_{j=1}^{2} \frac{1}{L_j} \int_{A_j \times Y_j^*} f_j dz dy_1 dy_2. \quad \square$$

Now, let us consider both, α and β bigger than one. In this case, as mentioned in the introduction, we still need to add some conditions on the values of γ. We get other three homogenized equations. First, we set the two first regimes taking $\gamma \le 1 < \alpha < \beta$ with γ also satisfying (3.1).

Theorem 3.5. *Let $u^\varepsilon \in W_{0l}^{1,p}(R^\varepsilon)$ be the solution of (1.1) and $\gamma \le 1 < \alpha < \beta$. Suppose that*

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \to f^j \quad \text{weakly in} \quad L^{p'}\left((0,1) \times A_j \times Y_j^*\right), \quad j = 1, 2.$$

Then, there are $u \in W_0^{1,p}(0,1)$ and $v \in L^p\left((0,1); W_\#^{1,p}(Z)\right)$ such that

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1) \times A_j; W_\#^{1,p}(Y_j^*)\right),$$

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+} \nabla u^\varepsilon \rightharpoonup 0 \quad \text{weakly in} \quad L^p\left((0,1) \times A_j \times Y_j^*\right), \quad j = 1, 2,$$

$$\Pi^\varepsilon \nabla u^\varepsilon \to \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} v \quad \text{strongly in} \quad L^p\left((0,1) \times Z\right),$$

where (u, v) is the solution of

$$\int_{(0,1)\times Z} \left|\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y_1}, \frac{\partial v}{\partial y_2}\right)\right|^{p-2} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y_1}, \frac{\partial v}{\partial y_2}\right) \left[\left(\frac{\partial \rho}{\partial x}, 0\right) + \nabla_{y_1 y_2} \psi\right] dx dz dy_1 dy_2$$

$$= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} f^j \rho dx dz dy_1 dy_2,$$

for all $(\rho, \psi) \in W_0^{1,p}(0,1) \times L^p\left((0,1); W_\#^{1,p}(Z)\right)$ *and* u *is the solution of the one-dimensional problem*

$$\int_0^1 q \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x} dx = \int_0^1 \bar{f} \rho dx, \forall \rho \in W_0^{1,p}(0,1),$$

where,

$$\bar{f} = \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} f^j dz dy_1 f y_2.$$

For $\gamma < 1$*,*

$$q = \left\langle 1/H^{p'-1}\right\rangle^{1-p}, \quad H(x) = \begin{cases} g_1^m & \textit{for} \quad x \in (0,1/2) \\ g_2^m & \textit{for} \quad x \in (1/2,1) \end{cases}.$$

If $\gamma = 1$*,*

$$q = \int_Z \left|(1,0) + \nabla_{y_1 y_2}\mathbb{X}\right|^{p-2} (1,0) + \nabla_{y_1 y_2}\mathbb{X}(1,0) dy_1 dy_2.$$

with $\mathbb{X}$ *solving*

$$\int_Z \left|(1,0) + \nabla_{y_1 y_2}\mathbb{X}\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2}\mathbb{X}\right) \nabla \psi_2 dy_1 dy_2 = 0, \quad \forall \psi_1 \in W_\#^{1,p}(Z)$$

$$\textit{with} \quad \int_Z \mathbb{X} dy_1 dy_2 = 0.$$

Proof. The uniform bound holds in this case as well. We shall use the integration formula (2.5) in (VP):

$$\begin{aligned}\sum_{j=1}^{2} \frac{1}{L_j} &\int_{(0,1)\times A_j \times Y_{j+}^*} \left|\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+} \nabla u^\varepsilon\right|^{p-2} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+} \nabla u^\varepsilon \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+} \nabla \varphi dx dy_1 dy_2 \\ &+ \int_{(0,1)\times Z} \left|\Pi^\varepsilon \nabla u^\varepsilon\right|^{p-2} \Pi^\varepsilon \nabla u^\varepsilon \Pi^\varepsilon \varphi dx dz dy_2 \\ &+ \sum_{i=0}^{k^\varepsilon - 1} \frac{1}{\varepsilon} \int_{P_{i1+}^\varepsilon} \left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon \varphi dx dy + \sum_{i=0}^{k^\varepsilon - 1} \frac{1}{\varepsilon} \int_{Q_{i1+}^\varepsilon} \left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon \varphi dx dy \qquad (3.9)\end{aligned}$$

$$= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \varphi dx dy_1 dy_2$$
$$+ \sum_{i=0}^{k^\varepsilon - 1} \frac{1}{\varepsilon} \int_{P_{i1+}^\varepsilon} f^\varepsilon \varphi dx dy + \sum_{i=0}^{k^\varepsilon - 1} \frac{1}{\varepsilon} \int_{Q_{i1+}^\varepsilon} f^\varepsilon \varphi dx dy.$$

Applying Theorem 2.15, there are $u \in W_0^{1,p}(0,1)$, $u^1 \in L^{p'}\left((0,1)\times A_1 \times Y_{1+}^*\right)$, $u^2 \in L^{p'}\left((0,1)\times A_2 \times Y_{2+}^*\right)$ and $v \in L^p\left((0,1); W_\#^{1,p}(Z)\right)$ such that

$$\begin{aligned}
&\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right), \quad j = 1, 2, \\
&\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^{1+}\left(\left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon\right) \rightharpoonup u^1 \quad \text{weakly in} \quad L^p\left((0,1)\times A_1 \times Y_{1+}^*\right), \\
&\mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^{2+}\left(\left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon\right) \rightharpoonup u^2 \quad \text{weakly in} \quad L^{p'}\left((0,1)\times A_2 \times Y_{2+}^*\right), \\
&\Pi^\varepsilon \nabla u^\varepsilon \rightharpoonup \left(\frac{\partial u}{\partial x}, 0\right) + \nabla_{y_1 y_2} v \quad \text{weakly in} \quad L^p\left((0,1)\times Z\right), \\
&\Pi^\varepsilon\left(\left|\nabla u^\varepsilon\right|^{p-2} \nabla u^\varepsilon\right) \rightharpoonup U^- \quad \text{weakly in} \quad L^{p'}\left((0,1)\times Z\right).
\end{aligned} \tag{3.10}$$

As we have done in the previous Theorem, we have that

$$\widetilde{u^j} = 0 \quad \text{in} \quad (0,1)\times A_j \times (g_j^m, g_j^M), \quad j = 1, 2.$$

Analogously as in the previous Theorem, one can see that

$$\mathcal{I}_\varepsilon := \int_{(0,1)\times Z} \left[\left|\Pi^\varepsilon \nabla u^\varepsilon\right|^{p-2} \Pi^\varepsilon \nabla u^\varepsilon - \left|\nabla u + \nabla_{y_1 y_2} v\right|^{p-2}\left(\nabla u + \nabla_{y_1 y_2} v\right)\right]$$
$$\times \left(\Pi^\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1 y_2} v\right) \to 0,$$

which implies, with Tartar's inequality, that

$$\|\Pi^\varepsilon \nabla u^\varepsilon - \nabla u - \nabla_{y_1 y_2} v\|_{L^p((0,1)\times Z)} \to 0,$$

and therefore,

$$U^- = \left|\nabla u + \nabla_{y_1 y_2} v\right|^{p-2}\left(\nabla u + \nabla_{y_1 y_2} v\right) \quad \text{a.e. in} \quad (0,1)\times Z.$$

Now, suppose $\gamma = 1$. Let $\phi \in C_0^\infty(0,1)$ and $\psi \in C_\#^\infty\left((0,1)\times(0, \max\{g_1^M, g_2^M\})\right)$ and define the function

$$\varphi^\varepsilon(x,y) = \varepsilon \phi(x) \psi\left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right) \quad \text{for} \quad (x,y) \in R^\varepsilon.$$

Taking φ^ε as a test function in (3.9), using convergences (3.10) and due to the fact that $u^j = 0$ $(0,1) \times A_j \times (g_j^m, g_j^M)$, $j = 1, 2$, one has

$$\int\limits_{(0,1)\times Z} \left|\nabla u + \nabla_{y_1 y_2} v\right|^{p-2} \left(\nabla u + \nabla_{y_1 y_2} v\right) \phi \nabla_{y_1 y_2} \psi \, dx dz dy_2 = 0.$$

Then, we identify v with the solution $\mathbb{X}$ of

$$\int\limits_{Z} \left|(1,0) + \nabla_{y_1 y_2} \mathbb{X}\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} \mathbb{X}\right) \nabla \psi_2 dy_1 dy_2 = 0, \quad \forall \psi_1 \in W_\#^{1,p}(Z)$$

$$\text{with} \quad \int\limits_{Z} \mathbb{X} dy_1 dy_2 = 0,$$

that is,

$$v(x, y_1, y_2) = \frac{\partial u}{\partial x}(x) \mathbb{X}(y_1, y_2), \quad (x, y_1, y_2) \in (0,1) \times Z.$$

Thus, one passes to the limit in (3.9) for test functions φ depending only on x and obtain

$$\begin{aligned} &\int\limits_{(0,1)\times Z} \left|\left(\frac{\partial u}{\partial x}, 0\right) + \frac{\partial u}{\partial x} \nabla_{y_1 y_2} \mathbb{X}\right|^{p-2} \left(\left(\frac{\partial u}{\partial x}, 0\right) + \frac{\partial u}{\partial x} \nabla_{y_1 y_2} \mathbb{X}\right) \nabla \varphi \, dx dz dy_2 \\ &\qquad = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{(0,1)\times A_j \times Y_j^*} f^j \varphi \, dx dy_1 dy_2. \end{aligned} \tag{3.11}$$

Rewriting the above equation, we have

$$\int_0^1 q \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x} \frac{\partial \varphi}{\partial x} dx = \int_0^1 \bar{f} \varphi dx, \forall \varphi \in W_0^{1,p}(0,1), \tag{3.12}$$

where

$$q = \int\limits_{Z} \left|(1,0) + \nabla_{y_1 y_2} \mathbb{X}\right|^{p-2} \left((1,0) + \nabla_{y_1 y_2} \mathbb{X}\right) (1,0) dy_1 dy_2, \quad \bar{f} = \sum_{j=1}^{2} \frac{1}{L_j} \int\limits_{A_j \times Y_j^*} f^j dz dy_1 dy_2.$$

If $\gamma < 1$, consider the test function

$$\varphi^\varepsilon(x, y) = \varepsilon^\gamma \phi(x) \psi\left(\frac{x}{\varepsilon^\gamma}\right) \quad \text{for} \quad (x, y) \in R^\varepsilon,$$

where $\phi \in C_0^\infty(0,1)$ and $\psi \in C_\#^\infty(0,1)$. With an analogous process as we have done previously, we obtain

$$\int_{(0,1)\times Z} \left|\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\right)\phi\frac{\partial \psi}{\partial z}H(z)dxdz=0, \quad H(x)=\begin{cases} g_1^m & \text{for} \quad x\in(0,1/2) \\ g_2^m & \text{for} \quad x\in(1/2,1)\end{cases}.$$

Then, using the arguments we used in Theorem 3.2, we obtain

$$\left(\left|\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\right|^{p-2}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial z}\right)\right)(x,z)H(z)=\frac{1}{\langle 1/H^{p'-1}\rangle^{p-1}}\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}.$$

Substituting the above equality in (3.11), we obtain (3.12) with

$$q=\left\langle 1/H^{p'-1}\right\rangle^{1-p}. \quad \square$$

Finally, let us finish considering the case $\gamma>1$ also satisfying (3.1).

Theorem 3.6. *Let $u^\varepsilon\in W^{1,p}_{0l}(R^\varepsilon)$ be the solution of (1.1) and $1<\gamma<\alpha<\beta$. Suppose that*

$$\mathbf{T}^j_\varepsilon\mathcal{T}^j_\varepsilon f^\varepsilon\to f^j \quad \textit{weakly in} \quad L^{p'}\left((0,1)\times A_j\times Y^*_j\right), \quad j=1,2.$$

Then, there is $u\in W^{1,p}_0(0,1)$ such that

$$\mathbf{T}^j_\varepsilon\mathcal{T}^j_\varepsilon u^\varepsilon\to u \quad \textit{strongly in} \quad L^p\left((0,1)\times A_j;W^{1,p}_\#(Y^*_j)\right),$$
$$\mathbf{T}^j_\varepsilon\mathcal{T}^{j+}_\varepsilon\nabla u^\varepsilon\rightharpoonup 0 \quad \textit{weakly in} \quad L^p\left((0,1)\times A_j\times Y^*_{j+}\right), \quad j=1,2,$$
$$\mathcal{S}_\varepsilon u^\varepsilon\rightharpoonup u \quad \textit{weakly in} \quad W^{1,p}(R_m),$$

where u is the solution of the one-dimensional problem

$$\int_0^1 q\left|\frac{\partial u}{\partial x}\right|^{p-2}\frac{\partial u}{\partial x}\frac{\partial \rho}{\partial x}dx=\int_0^1\bar f\rho dx, \forall\rho\in W^{1,p}_0(0,1),$$

where,

$$\bar f=\sum_{j=1}^2\frac{1}{L_j}\int_{(0,1)\times A_j\times Y^*_j} f^j dzdy_1fy_2$$

and

$$q=\min\{g_1^m,g_2^m\}.$$

Proof. Applying Theorem 2.15, there are $u\in W^{1,p}_0(0,1)$, $u^1\in L^{p'}\left((0,1)\times A_1\times Y^*_{1+}\right)$, $u^2\in L^{p'}\left((0,1)\times A_2\times Y^*_{2+}\right)$ and $v\in L^p\left((0,1);W^{1,p}_\#(Z)\right)$ such that

$$
\begin{gathered}
\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j u^\varepsilon \to u \quad \text{strongly in} \quad L^p\left((0,1)\times A_j; W_\#^{1,p}(Y_j^*)\right), \quad j=1,2, \\
\mathbf{T}_\varepsilon^1 \mathcal{T}_\varepsilon^{1+}\left(\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon\right) \rightharpoonup u^1 \quad \text{weakly in} \quad L^p\left((0,1)\times A_1 \times Y_{1+}^*\right), \\
\mathbf{T}_\varepsilon^2 \mathcal{T}_\varepsilon^{2+}\left(\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon\right) \rightharpoonup u^2 \quad \text{weakly in} \quad L^{p'}\left((0,1)\times A_2 \times Y_{2+}^*\right), \\
\mathcal{S}_\varepsilon u^\varepsilon \rightharpoonup u^- \quad \text{weakly in} \quad W^{1,p}(R_m), \\
\mathcal{S}_\varepsilon\left(\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon\right) \rightharpoonup U^- \quad \text{weakly in} \quad L^{p'}(R_m).
\end{gathered}
$$

Let $\phi_j \in C_0^\infty\left((0,1)\times A_j \times (\min\{g_1^m, g_2^m\}, \max\{g_1^M, g_2^M\})\right)$, $\psi_j \in C_0^\infty(0,1)$ and choose $\Psi_j \in C_0^\infty(0,1)$ such that $\Psi_j' = \psi_j$. Define the test functions

$$
\varphi_\varepsilon^j(x,y) = \varepsilon^\gamma \widetilde{\phi}_j\left(x, \left\{\frac{x}{\varepsilon^{\mu_j}}\right\}, \frac{y}{\varepsilon}\right)\Psi\left(\left\{\frac{x}{\varepsilon^\gamma}\right\}\right) \quad \text{in} \quad R^\varepsilon.
$$

From Proposition 2.8,

$$
\begin{gathered}
\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+}\varphi_\varepsilon^j \to 0 \quad \text{strongly in} \quad L^p\left((0,1)\times A_j \times Y_{j+}^*\right), \\
\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+}\frac{\partial \varphi_\varepsilon^j}{\partial x} \to \widetilde{\phi}_j(x,y_2)\Psi_j'(z) \quad \text{strongly in} \quad L^p\left((0,1)\times A_j \times Y_{j+}^*\right).
\end{gathered}
$$

Now we apply integration formula (2.6) in (VP), and have

$$
\begin{aligned}
&\sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_{j+}^*}\left|\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+}\nabla u^\varepsilon\right|^{p-2}\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+}\nabla u^\varepsilon \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+}\nabla\varphi\, dx dy_1 dy_2 \\
&\qquad + \int_{R_m}\left|\mathcal{S}_\varepsilon \nabla u^\varepsilon\right|^{p-2}\mathcal{S}_\varepsilon \nabla u^\varepsilon \mathcal{S}_\varepsilon \varphi\, dx dz dy_2 \\
&+ \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1+}^\varepsilon}\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon \varphi\, dx dy + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1+}^\varepsilon}\left|\nabla u^\varepsilon\right|^{p-2}\nabla u^\varepsilon \varphi\, dx dy \\
&\qquad = \sum_{j=1}^{2}\frac{1}{L_j}\int_{(0,1)\times A_j\times Y_j^*}\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j f^\varepsilon \mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^j \varphi\, dx dy_1 dy_2 \\
&\qquad + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{P_{i1+}^\varepsilon} f^\varepsilon \varphi\, dx dy + \sum_{i=0}^{k^\varepsilon-1}\frac{1}{\varepsilon}\int_{Q_{i1+}^\varepsilon} f^\varepsilon \varphi\, dx dy,
\end{aligned}
\tag{3.13}
$$

for all $\varphi \in W^{1,p}(R^\varepsilon)$. In particular, taking φ_ε^j as a test function in the above formulation and passing to the limit leads us to

$$\int_{(0,1)\times A_j\times Y^*_{j+}} u^j \phi_j \Psi'_j dxdzdy_1dy_2 = 0.$$

Thus, as we have done in the previous Theorem,

$$\widetilde{u^j} = 0 \quad \text{in} \quad (0,1)\times A_j \times (g_j^m, g_j^M), \qquad j = 1,2.$$

Next, observe that

$$\|\mathcal{S}_\varepsilon u^\varepsilon - \mathcal{S}_\varepsilon u\|_{L^p(R_m)} \leq |||u^\varepsilon - u|||_{R^\varepsilon} \to 0,$$

which means that

$$u^- = u \quad \text{a.e. in} \quad (0,1).$$

It remains to determine the limit problem. For this, we shall repeat the arguments we performed in the previous Theorem. Indeed, we prove that

$$\mathcal{I}_\varepsilon := \int_{R_m} \left[\left| \mathcal{S}_\varepsilon \nabla u^\varepsilon \right|^{p-2} \mathcal{S}_\varepsilon \nabla u^\varepsilon - |\nabla u|^{p-2} \nabla u \right] (\mathcal{S}_\varepsilon \nabla u^\varepsilon - \nabla u) \to 0.$$

We have

$$\begin{aligned}
\mathcal{I}_\varepsilon \leq \mathcal{I}_\varepsilon &+ \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j\times Y^*_{j+}} |\mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u^\varepsilon|^p \\
&\pm \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j\times Y^*_{j+}} \left| \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u^\varepsilon \right|^{p-2} \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u^\varepsilon \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u \\
&= \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j\times Y^*_{j}} \mathbf{T}^j_\varepsilon \mathcal{T}^{j}_\varepsilon f^\varepsilon \mathbf{T}^j_\varepsilon \mathcal{T}^{j}_\varepsilon (u^\varepsilon - u) dxdy_1dy_2 \\
&+ \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon} \int_{P^\varepsilon_{i1+}} f^\varepsilon (u^\varepsilon - u) dxdy + \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon} \int_{Q^\varepsilon_{i1+}} f^\varepsilon (u^\varepsilon - u) dxdy \\
&- \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon} \int_{P^\varepsilon_{i1+}} \left| \nabla u^\varepsilon \right|^{p-2} \nabla u^\varepsilon \nabla (u^\varepsilon - u) dxdy + \sum_{i=0}^{k^\varepsilon-1} \frac{1}{\varepsilon} \int_{Q^\varepsilon_{i1+}} \left| \nabla u^\varepsilon \right|^{p-2} \nabla u^\varepsilon \nabla ((u^\varepsilon - u)) dxdy \\
&+ \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j\times Y^*_{j+}} \left| \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u^\varepsilon \right|^{p-2} \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u^\varepsilon \mathbf{T}^j_\varepsilon \mathcal{T}^{j+}_\varepsilon \nabla u
\end{aligned}$$

$$-\int_{R_m} \left|\nabla u^-\right|^{p-2} \nabla u^- (\mathcal{S}_\varepsilon \nabla u^\varepsilon - \nabla u) \to 0,$$

as $\varepsilon \to 0$. Again, using Tartar's inequality leads us to

$$\left\| \mathcal{S}_\varepsilon \nabla u^\varepsilon - \nabla u \right\|_{L^p(R_m)} \to 0,$$

implying that

$$U^- = |\nabla u|^{p-2} \nabla u \quad \text{a.e. in} \quad R_m.$$

Hence, taking $\varphi = \varphi(x)$ in $W_0^{1,p}(0,1)$ in (3.13), we obtain

$$\int_{R_m} \left|\frac{\partial u}{\partial x}\right|^{p-2} \frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} = \sum_{j=1}^{2} \frac{1}{L_j} \int_{(0,1)\times A_j \times Y_j^*} f\varphi dx dz dy_1 dy_2,$$

obtaining the result. □

Remark 3.7. *Notice that we actually proved that*

$$\mathbf{T}_\varepsilon^j \mathcal{T}_\varepsilon^{j+} \nabla u^\varepsilon \to 0 \quad \textit{strongly in} \quad L^p\left((0,1) \times A_j \times Y_{j+}^*\right), \quad j = 1, 2,$$
$$\mathcal{S}_\varepsilon u^\varepsilon \to u \quad \textit{strongly in} \quad W^{1,p}(R_m).$$

Data availability

No data was used for the research described in the article.

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