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TOPOLOGY OF ORIENTED MATROIDS

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TOPOLOGY OF ORIENTED MATROIDS

By

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A B S T R A C T

This thesis is about the connexions between oriented matroids and sphere systems, combinatorial objects on one hand and topological objects on the other.

Oriented matroids arise as a combinatorial abstraction of linear algebra and geometry over ordered fields. An o.m. is a collection of signed vectors on a finite set E (maps $E \to \{0,+,-\}$). It is required to satisfy certain axioms which reflect obvious properties of any family of signed vectors that results from a vector space of real valued vectors upon forgetting magnitudes.

A sphere system (a.k.a. "arrangement of pseudohemispheres") on the unit sphere S^n is a finite family $\{S_e^0 | e \in E\}$ of images of S^{n-1} under homeomorphisms of S^n, together with labels S_e^+, S_e^- on the complementary domains of each S_e^0, so that the intersection of any subcollection of S_e^0's is a topological sphere that crosses any S_e^0 not containing it. The sphere system is said to be linear if each S_e^0 is the intersection of S^n with a hyperplane through the origin.

For each point $x \in S^n$ let $\sigma(x)$ be the signed vector on E defined by $\sigma(x)_e = j \iff x \in S_e^j$, $j = 0,+,-$. Let $C = \{\sigma(x) | x \in S^n\} \cup \{0\}$.

It turns out that C is an oriented matroid and that every oriented matroid arises in this way. This is the Representation Theorem of Folkman and Lawrence.

The first half of this thesis, chapters 2-6 contains a new proof of that theorem, which entails a detailed analysis of some posets associated to oriented matroids and of the topology of sphere systems. The main result is that oriented matroids are PL-spheres (PL≡ "Piecewise Linear"). Among other things that implies that for any sphere system there exists a self-homeomorphism of S^n and a radial projection onto a polytope boundary which combined send each S_e^0 onto a union of faces of that polytope.

As parts of this development we have:

a) The use of constructibility, a decomposition property, as a means for crossing the borderline between Combinatorics and Topology. A particular result is that oriented matroids are constructible.

b) A combinatorial development of Piecewise-Linear Topology, in which the emphasis is in complexes rather than spaces.

c) A proof of the improved Representation Theorem which entails the fact that many structural properties of sphere systems previously taken as definition requirements are consequences of the simple crossing properties.

d) A proof of the Upper Bound Conjecture for the oriented matroid counterparts of polytope lattices.

In the rest of the thesis those results are applied: in Chapter 7 for the construction of some extensions of oriented matroids; in Chapter 8 in the development of "surgery", a technique that highlights the distinction between general sphere systems and linear ones; in Chapter 9 as part of the way one studies affine matroids.

An affine matroid is an oriented matroid looked at from a viewpoint which makes it an encoding of the way a family of (pseudo) hyperplanes partitions euclidean space. A main result is the connexion between an extension property related to "Euclid's parallels postulate" and the monotonicity of linear functionals in $\mathbb{R}^n$ along lines. Using these concepts and surgery we are then able to produce infinitely many minor minimal nonlinear oriented matroids whose underlying matroids are all linear. These in turn give evidence that some properties of Euclidean space have no oriented matroid counterpart. Finally, special cases of conjectures by Murty and Las Vergnas are proved.

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CHAPTER 1

A GENERAL OVERVIEW†

This thesis explores the connexions between oriented matroids and sphere systems.

Oriented matroids arise as an abstraction of the combinatorics of linear algebra over ordered fields. What is special about ordered fields from this viewpoint is that inequalities are as important as equations.

1) Let E be a finite set, indexing the rows of a homogeneous linear system $A.x=0$ in $\mathbb{R}^n$. One may regard each equation $A_e.x=b_e$ as defining a hyperplane H_e, whose sides get labels $H_e^+=\{x|A_e.x>0\}$, $H_e^-=\{x|A_e.x<0\}$. We shall use the term <u>homogeneous linear system</u> for any finite collection $A=\{H_e|e\epsilon E\}$ of hyperplanes in some $\mathbb{R}^n$, with sides labelled H_e^+, H_e^-, and by an abuse of notation think of A also as a matrix. The half-spaces of the system are $\overline{H}_e^+=H_e^+\cup H_e$ and $\overline{H}_e^-=H_e^-\cup H_e$. We allow some of the H_e to be copies of $\mathbb{R}^n$, which may be thought of as degenerate hyperplanes.

For each point $x\epsilon\mathbb{R}^n$ the set $<x>=\cap\{\overline{H}_e^j|x\epsilon\overline{H}_e^j\}$ is a polyhedral cone. The set $C(A)=\{<x>|x\epsilon\mathbb{R}^n\}$ is the <u>complex</u> of A. One easily verifies that $<x>\subsetneq<y>$ if $<x>$ is a face of $<y>$ and the intersection of any two members of $C(A)$ is in $C(A)$ also and is a face of both.

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†This chapter is purely introductory; nothing in the rest of this thesis depends on it for definitions or anything else.

A main motivation for the study of oriented matroids is the desire to comprehend the inclusion order of $C(A)$, and as part of the deal, to get a better understanding of face lattices of cones (or of polytopes). One can encode that partial order of $C(A)$ very conveniently in this way:

2) A signed vector on E is a vector indexed by E, whose components may have values $0, +, -$; i.e., a map from E to $\{0,+,-\}$. For each vector $x \in R^E$, let $\sigma(x)$ be the signed vector defined by: $\sigma(x)_e = 0, +$ or $-$ according as $x_e = 0$, $x_e > 0$ or $x_e < 0$. The inclusion order of $C(A)$ can be represented by the family of signed vectors $C = \{\sigma(Aw)\ w \in R^n\}$, the oriented matroid of A: it follows that $\sigma(Ax) = \sigma(Ay)$ iff $\langle x \rangle = \langle y \rangle$, and that $\langle x \rangle$ is a face of $\langle y \rangle$ iff $\sigma(Ax)$ can be obtained from $\sigma(Ay)$ by changing some components to zero.

From the fact that $\{Ax | x \in R^n\}$ is closed under linear combinations, one may derive some simple combinatorial properties its oriented matroid must satisfy. Out of these properties comes this axiomatic definition.

3) An oriented matroid on E is a set C of signed vectors on E such that

a) $\underline{0} \in C$, where $\underline{0}$ is the signed vector all whose components equal 0.

b) $X \in C$ implies $-X \in C$, where $-X$ is defined by $(-X)_e = 0,+,-$ according as $X_e = 0,-,+$.

c) $X, Y \in C$ implies $X.Y \in C$, where $(X.Y)_e = X_e$ if $X_e \neq 0$,

$= Y_e$ otherwise.

d) For any $X, Y \in C$ and $e \in E$, if $(X_e, Y_e) = (+,-)$, then there exists a $Z \in C$ such that $Z_e = 0$, and for every $f \in E$ such that $(X_f, Y_f) \neq (+,-)$, $Z_f = (X.Y)_f$.

4) The set family $M(C) = \{E - \underline{X} | X \in C\}$, where $\underline{X} = \{e \in E | X_e \neq 0\}$ is shown in section (2.I) to be the family of closed sets of a matroid, which is said to underlie C. This observation helps proving some initial results, but after a while becomes quite secondary to our development. A reason for this is that the desire to bring out the geometry underlying the concepts here presented leads to a conceptualization and proof methods which have little to do with those of matroid theory. This contrasts and complements earlier works of Bland, Las Vergnas, Folkman and Lawrence in which oriented matroids were studied as an extension of matroid theory. In order to make this thesis readable for those with no knowledge of matroid theory, section (2.I) contains a short introduction to that subject.

5) Every oriented matroid is partially ordered by: $X \leq Y$ iff X results from Y by changing some components to zero. A <u>cell</u> of C is a set of form $[X] = \{Y \in C | Y \leq X\}$, and the <u>complex</u> of C is the set of its cells. Clearly the inclusion order of the complex is isomorphic to the order of C. It is also isomorphic

to the inclusion order of $C(A)$ if C is the oriented matroid of a homogeneous linear system A.

In particular, each cell of a linear oriented matroid, with the restricted order, is isomorphic to a polytope (cone) lattice. Hence one may study polytope lattices by studying oriented matroid cells. It was shown by Lawrence (unpublished, see [Mu]), that there exist oriented matroid cells which are not isomorphic (as posets) to any polytope lattice, so there are some limitations to this idea.

6) Chapter 2 describes some elementary properties of oriented matroid partial orders. In particular, it is shown that they have the Jordan-Dedekind (JD) chain condition, so we can associate to each vector (cell) of C an integer "dimension", such that $d(\underline{0}) = -1$ and $d(X) = d(Y) + 1$ for every pair (X,Y) such that X covers Y.

Given a set $\mathcal{D}$ of signed vectors on E, a <u>flat</u> of $\mathcal{D}$ is a set of form $\mathcal{D}/I = \{X \in \mathcal{D} | X_e = 0$ for every $e \in I\}$, where $I \subseteq E$. A <u>half-space</u> of $\mathcal{D}$ is a set of form $\overline{\mathcal{D}}_e^j = \{X \in \mathcal{D} | X_e = 0$ or $j\}$, $j = +$ or $-$. A <u>supercell</u> of $\mathcal{D}$ is an intersection of half-spaces which is not a flat.

Part of Chapter 2 also studies the flats and supercells of oriented matroids. In particular, the flats are shown to form a lattice, with JD property. Also, a characterization of oriented matroids by properties of their supercells is obtained in section (2.V).

7) Chapter 3 is about constructibility. A complex is a family K of sets called cells for which the sets $\{i_K(p)|p\in K\}$ are pairwise disjoint, where $i_K(p) = p - \cup\{q\in K|q \subsetneq p\}$. Each poset P will be identified with the complex whose cells are $[p] = \{q\in P|q \le p\}$. The space of a complex K is the union of its cells, denoted $s(K)$. A complex $L \subseteq K$ with the property that for any $p\in L$, $\{q\in K|q \subseteq p\} \subseteq L$, is said to be a subcomplex of K. In particular, each cell of a poset is a subcomplex. Two complexes K, L are order isomorphic if there exists a bijection $f: K \to L$ such that $f(p) \subseteq f(q)$ iff $p \subseteq q$. A complex K with minimum cell $\underset{\sim}{0}$ is pure dimensional if there is a function $d: K \to \mathbb{Z}$ such that $d(\underset{\sim}{0}) = -1$, $d(p) = d(q) + 1$ whenever $p \supsetneq q$ and no member of K is between them, and, $d(p)$ has the same value n for each maximal member of K. We say that the dimension of K is n, $d(K) = n$. In this case, denote $\beta(K) = \{p\in K|d(p) = n-1$ and p is contained in exactly one maximal member of $K\}$; the boundary of K is $\partial K = \{p\in K|p \subseteq q$ for some $q\in\beta(K)\}$.

A pure-dimensional complex is said to be S-constructible if its dimension is -1, or if it is the union of two B-constructible subcomplexes K_1, K_2 such that $K_1 \cap K_2 = \partial K_1 = \partial K_2$. A pure-dimensional complex of dimension $n \ge 0$ is said to be B-constructible if it has a unique maximal cell and S-constructible boundary, or if it is the union of two B-constructible subcomplexes of dimension n whose intersection is B-constructible of dimension $n-1$ and contained in the boundary of each.

(3.III.1) Theorem: Every oriented matroid is S-contructible and each of its supercells is B-constructible.

It is also part of (3.III.1) that cells of oriented matroids are "shellable", which consists of a strong form of constructibility. This extends a result of Bruguesser and Mani [BM] about polytopes. That result and a strengthening of it by Klee and Danaraj [DK1], which also holds for oriented matroid cells (3.IV.4) are the most restrictive combinatorial properties of polytope lattices ever published, yet in view of (3.IV.4) and Lawrence's result, they fail to characterize polytope lattices.

8) It has been demonstrated in [LV1], [BLV], [FL] that not all oriented matroids arise from linear systems. Nevertheless, a topological generalization of linear systems provides the exact scope of the oriented matroid axioms.

Denote: $S^n = \{x \in R^{n+1} | x_1^2 + x_2^2 + \ldots + x_{n+1}^2 = 1\}$, and identify S^{n-1} with $\{x \in S^n | x_{n+1} = 0\}$. An n-sphere is a topological space homeomorphic to S^n. A subset K of an n-sphere S is a hypersphere of S if it is the image of S^{n-1} by a homeomorphism of S^n to S; the components of $S - K$ are the sides of K.

A sphere system is a triple (S,E,H) where S is a topological sphere, E is a finite set and $H = \{S_e^j | e \in E,\ j \in \{0,+,-\}\}$ is a family of subsets of S satisfying:

a) For each $e \in E$, either $(S_e^0, S_e^+, S_e^-) = (S, \phi, \phi)$ or S_e^0 is a hypersphere of S with sides S_e^+, S_e^-.

b) For every nonempty subset A of E, the subspace $\cap\{S_e^0 | e \in A\}$ is a sphere. This is called a flat of the system.

c) For every flat F and hypersphere S_e^0 not containing it, $S_e^0 \cap F$ is a hypersphere of F with sides $S_e^+ \cap F$ and $S_e^- \cap F$.

d) Denote $\bar{S}_e^j = S_e^j \cup S_e^0$. For every collection B of such sets either $\cap B = \cap\{S_e^0 | \bar{S}_e \in B$ or $\bar{S}_e^- \in B\}$ or $\cap B$ is a ball.

One is led to considering sphere systems by the observation that for a homogeneous linear system A all positive scalar multiples of each vector of $\mathbb{R}^n$ determines the same signed vector in the corresponding oriented matroid. Thus, no information is lost by looking only at points of S^n, and substituting each hyperplane H_e of A by the hypersphere $H_e \cap S^n$ of S^n. The definition of sphere system reflects some simple properties of that kind of system.

Given a sphere system (S,E,H) (denoted simply by S if no confusion arises) we associate to each point $x \in S$ the signed vector $\sigma(x)$ given by: $\sigma(x)_e = j$ iff $x \in S_e^j$; $\sigma(S)$ denotes the set of all such signed vectors. The degeneracy of S is one plus the dimension of the sphere $\cap\{S_e | e \in E\}$. In particular the degeneracy of S is 0 iff $\underline{0} \notin \sigma(S)$. Two sphere systems S and T with same index set E are said to be equivalent if there exists a homeomorphism $h: S \to T$ such that $h(S_e^j) = T_e^j$, for every e, j.

Clearly equivalent sphere systems satisfy $\sigma(S)=\sigma(T)$.

The relationship between sphere systems and oriented matroids was established by Folkman and Lawrence [FL], in the following Representation Theorem:

(4.I.2) "For any sphere system S, $\sigma(S)\cup(\underline{0})$ is an oriented matroid, and any oriented matroid can be obtained in this way. Two sphere systems are equivalent iff they have the same oriented matroid and same degeneracy."

The second part (Chapters 4,5,6) of this thesis reproves this theorem, in a way that deepens one's understanding of both the Combinatorics and the Topology involved. These results had a strong contribution of Jack Edmonds, whose insight led to the formulation and proof of much of what is in these chapters. The contributions are mainly:

- Our proof that every oriented matroid arises from a sphere system is by an induction using constructibility; it entails the fact that oriented matroids are PL-spheres. This fact (whose meaning we detail in a few paragraphs from now) has important consequences both combinatorially and topologically.
- Theorem (6.II.1): condition (8.d) in the definition of sphere systems is redundant. This fact makes it much easier the task of recognizing whether a given collection of hyperspheres is a sphere system. Proof of the Theorem involves again constructibility.

These two aspects are given in Chapter 6. Chapter 4 is an introduction to sphere systems. It explains how the linearly motivated oriented matroid axioms still hold in the topological setting. It is also shown that sphere systems can also be considered as systems of pseudohyperplanes in projective space, by producing an involutory self-homeomorphism h of S such that $h(S_e^+) = S_e^-$, $h(S_e^-) = S_e^+$, for all $e \in E$.

10) Chapter 5 is an introduction to Piecewise-Linear (PL) Topology with emphasis on the associated combinatorics.

A PL-sphere is any complex order isomorphic to a complex K whose space is the boundary of a polytope and such that for every cell $p \in K$ there is a polytope P and a homeomorphism $f: p \to P$ such that: a) $f(p - i_K(p))$ is the boundary of P; b) there is a decomposition of p as a union $p_1 \cup p_2 \cup \ldots \cup p_k$ of polytopes such that on each of these f is an affine map. An alternative description of the requirements for K involves polyhedral complexes, which are complexes whose cells are polytopes and all their faces and such that any two cells intersect in a face of both. A PL-sphere may also be defined (5.VII.7) as any complex order isomorphic to a complex K for which there exists a polyhedral complex L satisfying: a) $s(L) = s(K)$ and it is the boundary of a polytope (and indeed one may even require that L be the boundary complex of a polytope); b) Each cell p of K is the space of a subcomplex $L(p)$ of L such that

(b_1): $\cup L(p) = \cup\{L(q) \mid q \in K,\ q \subsetneq p\}$ and (b_2): $L(p)$ is order isomorphic to a polyhedral complex whose space is a polytope.

PL-balls can be defined similarly.

(5.I.5) Theorem: S-constructible complexes are PL-spheres and B-constructible complexes are PL-balls.

(5.I.1) Theorem: Oriented matroids are PL-spheres,

Although the definitions of PL-spheres seem rather cumbersome, they express some topological simplicity as all the homeomorphisms involved result of piecing together affine maps. We review in section (5.I) some of the reasons which have led topologists to introduce these concepts, and us to delve on them.

In one point, Chapter 5 strongly diverges from the traditional focus of PL-topology ([Hu],[RS]). We are more interested in the order structure of the complexes which comprise the spaces than in the topology of the spaces. This is reflected in our slightly unorthodox terminology, and in the way we restate some classical result. It is felt that even those already acquainted with PL-topology can find some interesting novelties in this approach.

New results are Theorem (5.VII.7) which turns (5.I.5) inside out and shows that PL-balls (and PL-spheres) can be defined entirely in terms of constructible complexes, without any reference to topology or polyhedra. Another result is (5.I.8), the Upper Bound Conjecture ([Mo],[MM],[St2]) for PL-spheres in which the intersection of any two cells is a cell.

11) The last two parts apply many of the results and techniques of the first two. In view of the Representation Theorem, topological proofs of combinatorial results on oriented matroids are possible.

If X is a signed vector on E, and $I \subseteq E$, $X\backslash I$ denotes the restriction of X to $E - I$. For a collection C of signed vectors, $C\backslash I$ denotes $\{X\backslash I \mid X \in C\}$. When C is an oriented matroid, so is $C\backslash I$. If $I = \{e\}$, we omit the brackets and write $C\backslash e$, $X\backslash e$.

An <u>extension</u> of a sphere system S is any other sphere system obtained by adjunction of some hyperspheres. Similarly, an oriented matroid $\mathcal{D}$ is an <u>extension</u> of C if $C = \mathcal{D}\backslash I$ for some set I.

A theorem (7.I.3) of Las Vergnas [LV2] establishes a bijection between extensions of an oriented matroid by a single element and certain partitions of the set of minimal nonzero vectors. We have found it a most useful result. In particular, combining in different proportions Las Vergnas' and the Representation Theorem, we obtain three intriguingly different proofs of:

(7.I.7) <u>Theorem</u>: Suppose that oriented matroids C, $\mathcal{D}$ on E are such that $C \subseteq \mathcal{D}$ and $d(\mathcal{D}) = d(C) + 1$. Then there exists an extension E of $\mathcal{D}$ by a single element e such that $C = \{X\backslash e \mid X \in E,\ X_e = 0\}$.

This Theorem, inspired by its obvious converse, is a very special case of a conjecture of Las Vergnas [LV4]. It is subsequently used in a characterization of "general position".

The rest of Chapter 7 contains a discussion of the germane concepts of general position and erections. We say that $e \in E$ is in general position in C if no nonzero flat of C contained in $C_e = \{X \in C \mid X_e = 0\}$ is an intersection of members of $\{C_f \mid f \in E - e\}$. We say that $\mathcal{D}$ is an erection of E if there exists an extension C of $\mathcal{D}$ by an element e in general position such that $C/e = E$.

We have some methods for producing extensions in general position, and erections of uniform oriented matroids (those all whose elements are in general position).

12) Those concepts come together in Chapter 8, where we investigate the following operation:

Start with a sphere system (S, E, H) of degeneracy 0, and let (v, w) be a 0-sphere which is a flat of S. Choose a disc neighbourhood D of v in S such that the intersections of the boundary of D with the hyperspheres of the system which contain v comprise a sphere system. Suppose that one has another sphere system T on $A \cup \{\infty\}$, where $A = \{e \in E \mid v \in S_e^0\}$ and $\infty \notin E$, in which the system induced on T_∞^0 is equivalent to that in the boundary of D. Then one can remove D and replace it by T_∞^+ via a homeomorphism that makes each $S_e^0 \cap D$ match $T_e^0 \cap T_\infty^0$. We also do a symmetric operation in a neighbourhood of w.

The Surgery Theorem (8.I.4) states that provided ∞ is in general position in the oriented matroid of T, the construction succeeds in obtaining a new sphere system.

A companion oriented matroid Surgery Theorem (8.I.1) describes the resulting oriented matroid in terms of S and T.

One use of surgery is in constructing examples of oriented matroids which cannot arise from any linear system. That includes orientations of the Vamos matroid [BLV], and other well-known examples like the non-Pappus and non-Desargues' matroids. In Chapter 9 some infinite families of "bad" oriented matroids are presented with the use of surgery.

13) The last part of this work studies aspects of real affine geometry as reflected by oriented matroids.

An affine linear system is an indexed family $\{H_e | e \in E\}$ of affine hyperplanes in some $\mathbb{R}^n$, with sides labelled H_e^+, H_e^-. To one such system we associate a homogeneous linear system in $\mathbb{R}^{n+1}$, $A = \{\overline{H}_e | e \in E \cup \{\infty\}\}$ as follows: $\overline{H}_e$ is the linear subspace of $\mathbb{R}^{n+1}$ spanned by $H_e \times \{1\}$, while $\overline{H}_\infty$ is given by the equation $x_{n+1} = 0$. Side labels on each $\overline{H}_e$ are inherited from the H_e. From this, one can obtain a sphere system in the usual way.

Reversing the construction, we are led to define an affine system as the configuration A induced on the positive side of a hypersphere S_∞^0 of a sphere system S. Similarly, an affine matroid is a nonempty set of signed vectors of the form $A = \{X \in C | X_\infty = +\}$, where C is an oriented matroid on $E \cup \{\infty\}$. We denote those relationships as $A = (S, \infty)$, $A = (C, \infty)$. A hyperplane of A (not to be confused with matroid hyperplanes) is a set of the form

$A_e = \{x \in A | X_e = 0\}$, provided $\phi \neq A_e \neq A$. A flat of A is any intersection of hyperplanes. A flat which is a singleton is a vertex of A, and any flat which covers a vertex in the inclusion order is a line of A.

Consider an affine system $A = (S, \infty)$. Clearly S^+_∞ is homeomorphic to some $\mathbb{R}^n$, and properties of hyperspheres imply that the image of each $S^0_e \cap S^+_\infty$ is a "pseudohyperplane", that is, an image of $\mathbb{R}^{n-1}$ by a self-homeomorphism of $\mathbb{R}^n$. Indeed, each flat of A is the image of some $\mathbb{R}^m$ by a self-homeomorphism of $\mathbb{R}^n$; in particular, each line is homeomorphic to the real line.

14) The infinity of a flat of A is the intersection of the corresponding flat of S with S^0_∞. Similarly, the infinity of a flat of an affine matroid $A = (C, \infty)$ is the intersection of the corresponding flat of C with C^0_∞.

Two flats are said to be parallel if the infinity of one contains another. For affine linear systems this definition is tantamount to the geometrical meaning of parallelism.

An affine system is said to be euclidean if given any hyperplane H_e and any vertex V there exists an extension of the system with a hyperplane through V parallel to H_e. A similar definition may be formulated of euclidean (affine) matroids, so that both concepts become equivalent via the Representation Theorem.

In $\mathbb{R}^n$, given a hyperplane H and a point v not in H, there is a (unique) affine hyperplane through v, parallel to

H. That very basic property surely implies that affine linear systems are euclidean. It comes at first as a shock the realization that some affine matroids fail to be euclidean. Section (9.IV) does some damaging surgery to certain linear oriented matroids in order to produce some noneuclidean ones. On the positive side, 2-dimensional affine matroids are euclidean; this fact is equivalent to a theorem of Levi [Le] on arrangements of pseudolines.

When an affine matroid is not euclidean, it must exhibit a well-characterized configuration. We explain that in terms of affine systems, under the simplifying hypothesis that every line meets every hyperplane.

A line, being homeomorphic to the real line, has two possible orientations. Given a fixed hyperplane H_e of the system A, the e-orientation of each line of A not contained in H_e is that along which one moves from H^- to H^+. The e-orientation is well defined because each line of A not in H_e meets H_e at exactly one point, at which they cross.

(9.III.6) Theorem: An affine system is euclidean iff it has no configuration consisting of a hyperplane H_e and a sequence of vertices $V^0, V^1, \ldots, V^k$, $k \geq 3$ such that $V^k = V^0$ and for $i = 1, 2, \ldots, k$, V^{i-1} and V^i lie on a line of A, and V^{i-1} precedes V^i in the e-orientation of that line.

Some noneuclidean matroids we construct are real show-offs in their non-linear behavior. Section (9.V) examines several statements about affine systems which are well known to be true for affine linear systems but are not true in general.

15) The last section shows that every oriented matroid which can be made euclidean after an extension in general position by an element to be put at infinity has a configuration consisting of a maximal cell order isomorphic to a simplex, and a vertex V of that cell which disappears upon deletion of some element of C. That gives a partial positive answer to conjectures of Las Vergnas and Murty, and brings those conjectures together.

16) How far "sphere systems" are from "linear systems" is shown in section (9.IV), where the construction of noneuclidean matroids brings an extra information. First we remark that in the definition of homogeneous linear system and linear oriented matroid the field of real numbers can be substituted by any other ordered field or ordered division ring.Thus, one may speak of an oriented matroid as being "linear over the ordered division ring R".

It can be shown that if C is linear over R then so are any of its proper minors, i.e., oriented matroids of form $(C\setminus I)/J$, where I, J are disjoint subsets of E, $I \cup J \neq \phi$. It is also true that an oriented matroid that is linear over the rationals is linear over any ordered division ring or field.

Theorems (9.IV.12) and (9.IV.20) give an idea of how far the oriented matroid axioms are from characterizing linearity. Each of these theorems describe an infinite family of oriented matroids which proves:

"There exist infinitely many oriented matroids which are not linear over any division ring, but whose underlying matroid and all whose proper minors are linear over the rationals."

A remark about cross references inside this work. Chapters are numbered by arabic numerals and sections within each chapter by roman numerals. Within each section, all numeration is sequential, including theorems, lemmas, formulae and occasional paragraphs. A reference to within the same section is by number only, and if to other section of the same chapter, we omit the chapter number.

CHAPTER 2

BASIC THEORY

Axiomatic definitions of oriented matroids abound and we present three in this chapter. In section I, a definition based on the signed vectors of a real vector space is used to introduce the theory. The notion of supercells and an axiom system based on topological properties of a partition of space by hyperplanes are presented in section V. Last, in section VI, the definition based on minimal vectors, which is the one previous authors have preferred.

We associate two posets to an oriented matroid: The lattice of flats is studied in section I together with the unerlying matroid. The complex of the oriented matroid is the object of section IV, where several known facts about polytope lattices are proved in the context of oriented matroids.

Section II connects oriented matroids and geometry, and section III just develops machinery.

1) ORIENTED MATROIDS AND THEIR FLATS

Throughout this work E is a finite set. A signed vector on E is a map $X : E \to \{0, +, -\}$, and we write X_e for the image of $e \in E$. The support of X is $\underline{X} = \{e \in E | X_e \neq 0\}$. The opposite of X is $-X$ defined by $(-X)_e = -X_e$, where $-(0, +, -) = (0, -, +)$. An element $e \in E$ separates X and Y if $X_e = -Y_e \neq 0$. The product $X \cdot Y$ of X and Y is defined by: $(X \cdot Y)_e = X_e$ if $X_e \neq 0$, $= Y_e$ otherwise. The (signed) vector $\underline{0}$ has all its components 0 and is indexed by whatever appropriate set we want.

1) An oriented matroid on E is a collection C of signed vectors on E satisfying:

(1.a) $\underline{0} \in C$.

(1.b) If $X \in C$, then $-X \in C$.

(1.c) If $X, Y \in C$, then $X \cdot Y \in C$.

(1.d) If $X, Y \in C$ and $e \in E$ separates them, then there exists $Z \in C$ such that $Z_e = 0$ and for every $f \in E$ which does not separate X and Y, $Z_f = (X \cdot Y)_f = (Y.X)_f$. We say that Z results from the "elimination of e between X and Y".

We shall add the following basic and very useful property to the axioms list, though it is redundant.

(1.e) Given $X, Y \in C$ with $\underline{Y} \subseteq \underline{X}$ and such that at least one component separates X from Y, then there exists $Z \in C$ such that $Z \neq X$, $Z_e = X_e$ for every e not separating X from Y, and no component separates X and Z. We say that Z is a "Y-approximation of X".

2. The axioms of course are inspired on properties of vector spaces. Let R be an ordered field, let R^E be the vector space of mappings from E to R. For $x \in R^E$ let $\sigma(x)$ be the signed vector on E given by $\sigma(x)_e = 0, +, -$ according as $x_e = 0$, $x_e > 0$ or $x_e < 0$. Let V be a linear subspace of R^E and $C = \sigma(V)$. The axioms (1.a) - (1.e) all describe simple properties of this C.

The first two are obvious. For the other three, let $x, y \in V$ be such that $X = \sigma(x)$, $Y = \sigma(y)$, and let $D = \{e \in E | x_e \cdot y_e < 0\}$ $= \{e \in E | e$ separates X and $Y\}$. Then:

with $0 < \alpha < \min\{|x_e|/|y_e| \mid e \in D\}$,

$\sigma(x+\alpha y) = X \cdot Y$, thus $X \cdot Y \in C$;

with $z = |y_e|x + |x_e|y$, $Z = \sigma(z)$ is as in (1.d);

with $\alpha = \max\{|y_e|/|x_e| \mid e \in D\}$, $z = \alpha x + y \in V$

and $Z = \sigma(z)$ is as in (1.e).

An oriented matroid C is <u>linear</u> if there exists a vector space $V \subseteq \mathbb{R}^E$ ($\mathbb{R}$ = real field) such that $C = \sigma(V)$. Note that nothing distinguishing the reals from other ordered division rings is involved in the axioms; it is however more convenient to have this familiar field present in the geometrical discussion of the next section. It can also be proven that if an oriented matroid C is of the form $\sigma(V)$, where $V \leq R^E$, R an arbitrary ordered field, then C is linear (the proof is a straightforward application of a theorem of Tarski [Ta];the method is implicit in [Li]); that means, the reals provide the most general concept of "linear" as far as fields go.

3. Linear oriented matroids are in fact much more than a family of examples: they are the main motivation of the theory. From this viewpoint, any property which can be immediately proven for linear oriented matroids can be taken as an axiom. It so happens that all candidates for axioms we have found are consequences of (1.a) - (1.d). We shall see later in chapter 9 another, more complicated property, which does not follow from these axioms.

Before the theory really develops, let us satisfy ourselves that these axioms are independent. We give for each of the axioms a list of signed vectors which fail to satisfy that axiom while satisfying the others. For simplicity, signed vectors are written as sequences of signs, without brackets.

(1.a): the empty set.

(1.b): 00, ++ .

(1.c): 00, +0, -0, 0+, 0- .

(1.d): 00, +0, -0, ++, --, -+, +- .

As we have said, (1.e) is redundant. Indeed it could substitute (1.d). We show it in two propositions:

4. PROPOSITION: If a set C of signed vectors satisfies (1.d), then it satisfies (1.e).

PROOF: Let X, Y be given as in (1.e). Choose $Z \in C$ such that $Z \neq X$, $Z_e = X_e$ for every $e \in E$ which does not separate X and Y, and such that Z is separated from X by as few components as possible. If some component e separates X and Z, eliminating e between X and Z we obtain a $Z' \in C$ which contradicts the choice of Z, as it is separated from X by fewer component than Z was, and $Z'_f = X_f$ whenever $Z_f = X_f$. Thus Z is not separated from X, and is therefore a Y-approximation of X. □

5. PROPOSITION: If a set C of signed vectors satisfies (1.c) and (1.e), then it satisfies (1.d).

PROOF: Let X, Y, e be given as in (1.d). Let $X^1 = X \cdot Y \in C$. Choose $X^2 \in C$ such that $X^2_e = X^1_e$, $X^2_f = X^1_f$ for every f which does not separate X and Y, and such that X^2 and Y are separated by as few components as possible. Let $Z \in C$ be a Y-approximation of X^2, which exists as $\underline{Y} \subseteq X^2$ and $X^2_e = -Y_e = 0$. Clearly Z is separated from Y by fewer components than X^2 is; thus by the choice of X^2, $Z_e = 0$. It follows that Z results from eliminating e between X and Y. □

6. For the remainder of the section, let us fix an oriented matroid on E. It is convenient to partition E in two types: $e \in E$ is a <u>loop</u> of C if $X_e = 0$ for every $X \in C$; the remaining

elements are the <u>equators</u> of C. Thus the set of equators is $\bullet\{\underline{X} | X \in C\}$.

Let us define the following relation among signed vectors on E : $X \le Y$ if for every $e \in E$, $X_e \ne 0$ implies $Y_e = X_e$; as usual, $X < Y$ stands for ($X \le Y$ and $X \ne Y$). We can use this notation to reformulate (1.e): Z is a <u>X-approximation</u> of Y if $Z < Y$ and $Z_e = Y_e$ for every e which does not separate X and Y.

The restriction of $\le$ to C is a partial order; this poset $(C, \le)$ will be studied extensively here. One basic property it has is the Jordan-Dedekind chain condition. We express this in the more convenient form of the following definition.

7. A poset P is JD (for Jordan-Dedekind) if it has a minimum element 0, and there is a function $r : P \to \mathbb{Z}$ such that

(7.1) for every $p,q \in P$, if q covers p, $r(q) = r(p) + 1$ (q <u>covers</u> p if $p < q$ and there is no $s \in P$ such that $p < s < q$).

Such an h exists iff for each $p \in P$, all saturated chains from 0 to p have the same length and this length is $r(p) - r(0)$. Thus h is uniquely determined by its value at 0. The <u>rank</u> function of P is the only such r satisfying $r(0) = 0$; the <u>dimension</u> function of P, which shall be more used later, is defined by $d(p) = \text{rank}(p) - 1$.

The proof of "C is JD" will be given in section IV. Meanwhile, as a step towards this, we initiate the study of flats of C and C's underlying matroid.

Notation: $F + e$ instead of $F \cup \{e\}$ and $F - e$ instead of $F - \{e\}$.

8. A matroid on E is collection M of subsets of E called closed sets satisfying

(8.a) $E \in M$

(8.b) If $F, G \in M$ then $F \cap G \in M$

(8.c) If $F \in M$, $e, f \in E - F$ and there exists $G \in M$ containing $F + e$ but not f then there exists $H \in M$ containing $F + f$ but not e .

A matroid M, ordered by inclusion, is a lattice, which is JD. This is a well-known fact and we prove it at the end of this section for the benefit of the unacquainted reader.

9. The zero-set of $X \in C$ is $\{e \in E | X_e = 0\}$. Let $M(C)$ be the collection of all zero-sets of members of C , $M(C) = \{E - \underline{X} | X \in C\}$.

10. PROPOSITION: $M(C)$ is a matroid on E .

PROOF: Properties (8.a) and (8.b) follow from (1.a) and (1.c) respectively in an obvious way. To show (8.c), let F, e, f, G be given as in the statement. Let $X, Y \in C$ be such that $F = E - \underline{X}$, $G = E - \underline{Y}$.

Since $F+e \subseteq G$, X_e, X_f, Y_f are nonzero and $Y_e = 0$. Substituting if necessary Y by $-Y$, we can assume that $Y_f = -X_f$. Eliminating f between X and Y, we obtain $Z \in C$, which satisifes: $Z_f = 0$, $Z_e \neq 0$ (as e does not separate X and Y), and $\underline{Z} \subseteq (\underline{X} \cup \underline{Y}) - f = (E-F)-f = E-(F+f)$. Thus, with $H = E - \underline{Z}$, $H \in M(C)$, $F+f \in H$ and $e \notin H$. □

A matroid N is <u>orientable</u> if $N=M(C)$ for some oriented matroid C.

For an oriented matroid, it is more interesting to consider another lattice, which is intimately related to the underlying matroid.

The <u>flat</u> of C with zero-set S is the collection C/S of vectors of C whose zero-set contain S; thus $C/S = \{X \in C \mid \underline{X} \cap S = \emptyset\}$.

11. PROPOSITION: Let $L(C)$ be the set of flats of C. Then:

(11.1) C is a flat.

(11.2) If $F, G \in L(C)$, then $F \cap G \in L(C)$.

(11.3) $L(C)$ ordered by inclusion is a lattice.

(11.4) The correspondence $S \to C/S$ is a bijection between $M(C)$ and $L(C)$, and if $S, T \in M(C)$, $S \subseteq T$ iff $C/T \subseteq C/S$.

(11.5) $L(C)$ is JD, and its rank function is

$$r(C/S) = \rho(E) - \rho(S),$$

where ρ is the rank function of $M(C)$.

PROOF: (11.1) Let X be the product, in some order, of all vectors of C. Then $X \in C$, and its zero-set K is the set of loops of C.

Clearly $C = C/K$, hence $C \in L(C)$.

(11.2) It is clear from the definition that every flat is closed under products. Let X be the product, in some order, of all vectors of $F \cap G$. By the remark above, $X \in F \cap G$. We claim that $F \cap G = C/S$, where S is the zero-set of X. Indeed, as $\underline{X} = \cup\{\underline{Y} | Y \in F \cap G\}$, S is contained in the zero-set of each member of $F \cap G$, so $F \cap G \subseteq C/S$. For the reverse inclusion, let us write $F = C/T$. As $X \in F$, $T \subseteq S$. But then, $Y \in C/S \Rightarrow \underline{Y} \cap S = \phi \Rightarrow \underline{Y} \cap T = \phi \Rightarrow Y \in F$; thus $C/S \subseteq F$, and similarly $C/S \subseteq T$.

(11.3) The meet in $L(C)$ is intersection, the join is $F \vee G = \cap\{H \in L(C) \mid F \subseteq H \text{ and } G \subseteq H\}$, which is a flat of C by (11.1) and (11.2).

(11.4) Clearly the given map is onto and inclusion reversing. Now, given $F = C/S \in L(C)$, let $Z = Z(F) = \cap\{E - \underline{X} \mid X \in F\}$. Then $F \mapsto Z(F)$ is the inverse of the map $S \to C/S$. Indeed, as zero-sets are closed under intersection, Z is a zero-set. Since $S = E - \underline{X}$ for some $X \in F$, $Z \subseteq S$, but as the zero-set of each member of F contains S, so does Z, hence $Z = S$.

Finally one must check that $F \subseteq G$ implies $Z(G) \subseteq Z(F)$; but this follows immediately from the definition of Z.

(11.5) The minimum flat of C is of course $\{\underline{0}\} = C/E$, and $r(\{\underline{0}\}) = \rho(E) - \rho(E) = 0$. Thus it is enough to show that r as defined in (11.5) satisfies (7.1). Let C/S, C/T be flats such that C/S covers C/T in $L(C)$. By (11.4), T covers S in $M(C)$.

thus $\rho(T) = \rho(S) + 1$. One immediately obtains $r(C/S) = r(C/T) + 1$
□

12. Remark: It turns out later to be more geometrically appealing to handle the dimension of flats instead of their rank. Recall that dimension $\equiv$ rank $- 1$. Thus we have the formula:

$$d(C/S) = \rho(E) - \rho(S) - 1 .$$

An n-flat of C is a flat of dimension n . We shall repeat this terminological pattern with all JD posets we encounter: an "n-object" is an "object" of dimension n . □

13. Remark: Let $V \subseteq \mathbb{R}^E$ be a subspace and $C = \sigma(V)$ be its oriented matroid. If $F \subseteq C$ is a flat, it is easy to see that $W = \{x \in V | \sigma(x) \in F\}$ is a subspace of V . One can also prove that $r(F)$ is the dimension of W as a vector space. In particula $r(C) = \dim V$. □

For the time being, this is all to be said about flats. We conclude this section with a short introduction to matroids. The unprov results can be found in [We] .

Let M be a matroid on E , as defined in (8) . The closure of $A \subseteq E$ is $cl(A) = \cap\{F \in M | A \subseteq F\}$. By (8.b), $cl(A) \in M$. A set $I \subseteq E$ is independent if for every $e \in I$, $e \notin cl(I-e)$. A basi of $A \subseteq E$ is a maximal independent subset of A .

We shall prove the next two results:

14. PROPOSITION: All bases of each subset of E have the same cardinality.

The <u>rank</u> $\rho(A)$ of $A \subseteq E$ is defined as the common cardinality of its bases.

15. PROPOSITION: Consider M as a poset, ordered by inclusion. Then M is JD and a lattice, its rank function being ρ, restricted to flats.

A maximal proper flat of M is a <u>hyperplane</u>; thus a flat is a hyperplane iff its rank is $\rho(M) - 1$.

16. PROPOSITION: Every flat is an intersection of hyperplanes. □

The complement of a hyperplane is a <u>cocircuit</u> of M. Clearly no cocircuit contains another. A minimal dependent set is a <u>circuit</u> of M.

17. PROPOSITION: Let C be a collection of subsets of a set E such that no member of C includes another. The following are equivalent:

(17.1) C is the family of cocircuits of a matroid on E.

(17.2) For every $X, Y \in C$, $e \in X \cap Y$, $f \in X - Y$, there exists $Z \in C$ such that $f \in Z \subseteq X \cup Y - e$.

(17.3) For every $X, Y \in C$, $e \in X \cap Y$, there exists $Z \in C$ such that $Z \subseteq X \cup Y - e$.

(17.4) C is the family of circuits of a matroid on E. □

18. PROPOSITION: A matroid is completely determined given any one of the following families of sets:

a) Its independent sets.

b) Its bases.

c) Its hyperplanes.

d) Its cocircuits.

e) Its circuits. □

Now we proceed to the proof of (15). First we prove (14), for which the following preliminary is needed:

19. LEMMA: Let M be a matroid on E. Then:

(19.1) For every $A \subseteq E$, $A \subseteq cl(A) = cl(cl(A))$; $A = cl(A)$ iff A is closed

(19.2) For every $A \subseteq B \subseteq E$, $cl(A) \subseteq cl(B)$

(19.3) For all $A \subseteq E$, $e,f \in E - cl(A)$, if $e \notin cl(A+f)$ then $f \notin cl(A+e)$.

(19.4) If $I \subseteq E$ is independent and $e \notin cl(I)$, then $I + e$ is independent.

(19.5) If $I \subseteq A \subseteq E$ is independent then I is a base of A iff $cl(I) = cl(A)$.

PROOF: (19.1) and (19.2) are obvious. To verify (19.3), first notice that a closed set contains A iff it contains $cl(A)$, hence a closed set contains $A + f$ iff it contains $cl(A) + f$; so (19.3) is a restatement of (8.c). For (19.4), let $J = I + e$. Then $e \notin cl(J-e)$ by hypothesis for every $f \in I$, as $e \notin cl((I-f)+f)$

and $f \notin cl(I-f)$ (since I is independent) it follows from (19.3) that $f \notin cl((I-f)+e) = cl(J-f)$; thus J is independent. About (19.5), notice that since necessarily $cl(I) \subseteq cl(A)$ (by 19.2)) the condition $cl(A) = cl(I)$ is equivalent to $cl(A) \subseteq cl(I)$, which in turn is equivalent to $A \subseteq cl(I)$. Suppose $A \subseteq cl(I)$. Then $I + e$ is not independent for any $e \in A - I$, hence I is a base of A . Conversely, if $A \not\subseteq cl(I)$, then we can choose $e \in A - cl(I)$ and $I + e$ is independent, by (19.4), showing that I is not a base. □

20. PROOF of (14): Suppose the statement false and choose $A \subseteq E$ and bases B_1, B_2 of A such that $|B_1| < |B_2|$ and $|B_1 \cap B_2|$ is largest possible subject to these conditions. Choose $e \in B_2 - B_1$. Then, by definition of independent sets, $cl(B_2-e) \subsetneq cl(B_2) = cl(A)$. On the other hand, $cl(B_1) = cl(A)$, thus, as $cl(B_1) \not\subseteq cl(B_2-e)$, $B_1 \not\subseteq cl(B_2-e)$ by (19.2). Choose $f \in B_1 - cl(B_2-e)$, and let $B_3 = B_2 - e + f$. By (19.4), B_3 is independent, so it is contained in a base B of A . As $B_3 \subseteq B$ and $|B_3| = |B_2|$, $|B| \geq |B_2| > |B_1|$. On the other hand, $B_1 \cap B \supseteq B_1 \cap B_2 + f$, thus $|B_1 \cap B| > |B_1 \cap B_2|$, contradicting the choice of B_1 and B_2 . □

21. PROOF of (15): The fact that M is a lattice follows from (8.a) and (8.b): given $A,B \in M$, their meet is $A \cap B$ and their join is $\cap\{F \in M | A \cup B \subseteq F\}$.

Let now $\underline{0} = 0M$ be the minimum flat. Then $\underline{0} = cl(\phi)$ so ϕ is a base of $\underline{0}$ and $\rho(\underline{0}) = 0$. Hence to prove that M is

JD with rank function ρ , we only need to prove that ρ satisfies (7.1).

Let F , G be flats such that G covers F . Choose $e \in G - F$ and a base I of F . Then, as $cl(I) = cl(F) = F$, $e \notin cl(I)$; therefore by (19.4) $I + e$ is independent. As $e \notin F$ and $I + e \subseteq G$, $F \subsetneq cl(I+e) \subseteq G$. Since G covers F , $cl(I+e) = G$, thus by (19.5) $I + e$ is a base of G . Hence $\rho(G) = |I + e| = \rho(F) + 1$. □

II) LINEAR SYSTEMS

Linear oriented matroids are particularly useful in describing certain geometrical aspects of systems of linear inequalities and equations.

1. The hyperplane of $\mathbb{R}^n$ determined by the pair (c,b), where $\underset{\sim}{0} \neq c \in \mathbb{R}^n$, $b \in \mathbb{R}$, is $H = \{w \in \mathbb{R}^n \mid c \cdot w = b\}$ ($(c \cdot w)$ is the inner product $\sum c_i w_i$). The sides of H are $H^+ = \{w \in \mathbb{R}^n \mid c \cdot w > b\}$ and $H^- = \{w \in \mathbb{R}^n \mid c \cdot w < b\}$. Note that H can be determined by several pairs (c,b); the partition (H^+, H^-) of $\mathbb{R}^n - H$ depends only on H but the labels H^+, H^- may be reversed if we use a negative multiple of (c,b) to describe H. A signed hyperplane is a hyperplane H whose sides have been labelled H^+, H^-, and it is determined by any pair (c,b) which induces such labelling. A hyperplane is homogeneous if it contains the origin; thus it is determined by a pair $(c,0)$. For convenience, we drop this trailing 0 and say that H is determined by c.

We shall use the term "linear system" to describe an indexed family of hyperplanes. For the present theory it is important to distinguish between two types, which we call "homogeneous" and "affine".

2. A homogeneous linear system in $\mathbb{R}^n$, indexed by E, is a family $\{H_e \mid e \in E\}$ of homogeneous signed hyperplanes and copies of $\mathbb{R}^n$.

To each matrix $A \in \mathbb{R}^{E \times n}$, we associate the homogeneous system whose members are determined by the rows $(A_e | e \in E)$; thus $H_e = \{w \in \mathbb{R}^n | A_e.w = 0\}$, and $H_e = \mathbb{R}^n$ when $A_e = 0$. We avoid special notations, and call this sytem A . Clearly every homogeneous system is of this form.

For each $w \in \mathbb{R}^n$, the vector $\alpha(w) = \sigma(Aw)$ denotes the position of w relative to each member of A : $\alpha(w)_e = 0, +, -$ accordingly as $w \in H_e, H_e^+$ or H_e^- . Thus the oriented matroid $C = \sigma(V)$, where $V = \{x \in \mathbb{R}^E | x = Aw$, some $w\}$ is the column space of A , encodes all possible positions of points of $\mathbb{R}^n$ relative to A .

3. The concepts we developed for oriented matroidsin general have significant interpretation in the context of linear systems.

First of all, equators and loops: $e \in E$ is an equator of C iff H_e is a hyperplane; thus the loops of C are those $e \in E$ such that $H_e = \mathbb{R}^n$. This is actually a justification for the aesthetically unappealing presence of copies of $\mathbb{R}^n$ in the description of linear systems. Adding a demand of no loops to the defining axioms for oriented matroids would unduly complicate the theory, as will become patent in later developments. That is why we allow loops to occur in linear systems.

4. A <u>flat</u> of A is a linear subspace of $\mathbb{R}^n$ which is an intersection of members of A . The flats of A correspond naturally to the flats of C : if C/S is a flat of C , then $\{w \in \mathbb{R}^n | \sigma(Aw) \in C/S\}$

is a flat of A, the intersection $\cap(H_e | e \in S)$. Conversely, if $S \subseteq E$, $F = \cap(H_e | e \in S)$, the set $\{\sigma(Aw) | w \in F\}$ is a flat of . Indeed, this flat is C/G, where $G = \{e \in E | H_e \supseteq F\}$.

This last expression also gives the meaning of the underlying matroid $M(C)$ of : the closure of $S \subseteq E$ in $M(C)$ is $\{e \in E | F \subseteq H_e\}$, where $F = \cap(H_e | e \in S)$. A set $I \subseteq S$ is a base of S if I is minimal with the property that the hyperplanes it indexes intersect in F. Also, $I \subseteq E$ is independent iff the corresponding rows of A are linearly independent.

Let $S \subseteq E$ be a zero set of C, and $I = \{e_1, e_2, \ldots, e_k\}$ be a base of S. Define

$$I_i = \{e_1, e_2, \ldots, e_i\} \text{ and } F_i = \cap(H_e | e \in I_i) .$$

As each I_i is independent, $F_1 \supsetneq F_2 \supsetneq \ldots \supsetneq F_k$. Since F_{i+1} is the intersection of F_i with a hyperplane, $\dim F_{i+1} = \dim F_i - 1$ ($\dim F$ is the dimension of F as a linear subspace). We conclude that $\dim F_k = n - k$. Now, F_k corresponds to the flat C/S of C. The rank of C/S is $r(C/S) = \rho(E) - \rho(S) = \rho(E) - k$. Thus, $\dim F_k = r(C/S) + (n - \rho(E))$, and we see that the correspondence between flats of A and flats of C preserves rank up to an additive constant.

This constant can be directly computed from A. Denote by $N = \{w \in \mathbb{R}^n | Aw = 0\}$ the intersection of all hyperplanes of A . The corresponding flat of $\mathcal{C}$ is $\{\underline{O}\} = \mathcal{C}/E$, and has rank 0 . We conclude that $\dim N = n - \rho(E)$. Thus we may finally express, with $S \subseteq E$ and $F = \cap\{H_e | e \in S\}$:

(4.1) $\dim F = r(\mathcal{C}/S) + \dim N$.

5. An important aspect of linear systems inspired the definition of the partial order on $\mathcal{C}$:

The cells of A are the sets $\{\bar{\psi}(X) | X \in \mathcal{C}\}$, defined by: $\bar{\psi}(X) = \{w \in \mathbb{R}^n | \sigma(Aw) \leq X\}$. Each cell is a cone, a solution of a homogeneous system of linear inequalities and equations:

(5.1) $\bar{\psi}(X) = \{w \in \mathbb{R}^n | A_e w = 0 \ , \geq 0 \ , \leq 0$ according as $X_e = 0, +, -\}$.

Now, to A one can associate several systems of linear inequalities and equations, other than those specified by $\mathcal{C}$, by choosing for each $e \in E$ one of $A_e w = 0$, $A_e w \geq 0$, $A_e w \leq 0$. The complex of A , $C(A)$ is the set of all the cones obtainable as solutions to these systems.

6. PROPOSITION: The complex of A is the set of cells of A . Moreover, for $X,Y \in \mathcal{C}$, $\bar{\psi}(X) \subseteq \bar{\psi}(Y)$ iff $X \leq Y$.

PROOF: Clearly every cell of A belongs to $C(A)$. Conversely, let P be a cell of A and let $S = \{\sigma(Aw) | w \in P\}$. We can represent the defining system of P as $\bar{\psi}(X)$, extending the notation of (5.1), where X is a signed vector not necessarily in $\mathcal{C}$. Let $S = \{\sigma(Aw) | w \in P\}$, and let Y be the product of all vectors in S , in any order. Observe that $Y_e = 0$ iff $\sigma(Aw)_e = 0$ for every $w \in P$, and if $Y_e \neq 0$ then $Y_e = X_e$; thus $Y \leq X$. Then $P = \bar{\psi}(Y)$: if $w \in \bar{\psi}(Y)$, then $\sigma(Aw) \leq Y \leq X$, hence $w \in P$ and conversely, if $w \in P$, then $\sigma(Aw) \in S$, hence $\sigma(Aw) \leq Y$.

The fact that $\bar{\psi}(X) \subseteq \bar{\psi}(Y)$ iff $X \leq Y$ is easily verified and we omit the details. □

This result gives the main interpretation of the partial order on an oriented matroid: in case where C is the oriented matroid of a homogeneous linear system A, the complex C(A), partially ordered by inclusion, is isomorphic to the poset C.

Given a cell $\bar{\psi}(X) \in C(A)$, the corresponding set $\{\sigma(Aw) | w \in \bar{\psi}(X)\}$ of signed vectors is, by Proposition 6, $[X] = \{Y \in C / Y \leq X\}$. Accordingly, we call such a set [X] a <u>cell</u> of C.

The <u>faces</u> of a cone $\bar{\psi}(X)$ are the cells of A it contains. Alternatively, $K \subseteq \bar{\psi}(X)$ is a face if there exists $c \in \mathbb{R}^n$ such that $c.w \geq 0$ for every $w \in \bar{\psi}(X)$ and $K = \{w \in \bar{\psi}(X) | c.w = 0\}$. The faces of $\bar{\psi}(X)$, ordered by inclusion, form a lattice, the <u>face-lattice</u> of $\bar{\psi}(X)$. It follows from Proposition 6 that this lattice is isomorphic (as a poset) to [X]. This motivates the study of oriented matroid cells as an abstraction of face-lattices of cones.

7. An <u>affine linear system</u> in $\mathbb{R}^n$ indexed by E, is a family $\{H_e | e \in E\}$ of signed hyperplanes and copies of $\mathbb{R}^n$.

As with homogeneous systems, each affine system will be represented by a pair (A,b) where $A \in \mathbb{R}^{E \times n}$, $b \in \mathbb{R}^E$, and $H_e = \{w \in \mathbb{R}^n | A_e w = b_e\}$ (thus we require that $b_e = 0$ whenever $A_e = \underset{\sim}{0}$; th..t is for those e such that $H_e = \mathbb{R}^n$).

As in the homogeneous case, to each point $w \in \mathbb{R}^n$, we associate a signed vector $X(W)$ on E which indicates how w is positioned relative to (A,b): $X(w) = \sigma(Aw-b)$. The collection $\mathcal{D}$ of all such signal vectors is not an oriented matroid in general; unless $b = \underset{\sim}{0}$, the negativity axiom (I.1.b) will fail. It can, however, be fruitfully regarded as "part" of an oriented matroid:

Let us identify $\mathbb{R}^n$ with $\mathbb{R}^n \times \{1\} \subseteq \mathbb{R}^{n+1}$, and consider the homogeneous system comprised by the homogeneous hyperplanes spanned by each H_e, and as a matter of convenience, let us add $\mathbb{R}^n \times \{0\}$ to the system, signed so that its positive side is $\{w | w_{n+1} > 0\}$. That is, we consider the homogeneous system with matrix $A' = \begin{pmatrix} A & \vdots & -b \\ \underset{\sim}{0} & \vdots & -1 \end{pmatrix}$, where the last row of A' is indexed by a new element, ∞. This is essentially the usual process of projectivizing affine space by introducing a "hyperplane at infinity".

The oriented matroid C of A' is the one we associate with (A,b), and call it the oriented matroid of (A,b). Let

$\mathcal{A} = \{X \in \mathcal{C} \mid X_\infty = +\}$. We observe that $\mathcal{A}$ is precisely the set $\{(Y,+) \mid Y \in \mathcal{D}\}$ of vectors obtained by adding a new component + to each vector in $\mathcal{D}$. Indeed, if $Y \in \mathcal{D}$, $Y = \sigma(Aw-b)$, then $(Y,+) = \sigma(A'(w,1)) \in \mathcal{A}$; and if $(Y,+) = \sigma(A'w) \in \mathcal{A}$, then $Y = \sigma(Aw'-b) \in \mathcal{D}$, where $w' = (1/w_{n+1})(w_1, w_2, \ldots, w_n)$.

Consonantly, we define an affine matroid to be an oriented matroid with a distinguished equator and study these objects as an abstraction of affine linear systems. In particular, one can define flats and the notion of parallelism in a natural way. This is done in Chapter 7, and we do not delve on these points right now.

Still something must be noted about affine matroids; their connection with face-lattices of convex polyhedra.

A (convex) polyhedron is a subset of $\mathbb{R}^n$ which is the solution set of a system of linear inequalities and equations. A polytope is a bounded polyhdron. One may specify a polyhedron by a triple (A,b,X), where (A,b) is an affine linear system indexed by E and X is a signed vector. The polyhedron thus specified is $P = \{w \in \mathbb{R}^n \mid \sigma(Aw-b)) \leq X$. The complex of (A,b) is the collection of all polyhedra definable from (A,b) as above; the cells of (A,b) are the polyhedra (A,b,X) where $X \in \mathcal{D}$. As in Proposition 6, we have:

PROPOSITION: The complex of (A,b) is precisely the set of cells of (A,b). Moroever, for $X,Y \in \mathcal{D}$,

$$(A,b,X) \subseteq (A,b,Y) \quad \text{if} \quad X \le Y .$$

PROOF: Similar to that of (6) . □

From this result, every polyhedron is a cell of an affine linear system. The faces of (A,b,X) are those cells of (A,b) which are contained in (A,b), and also the empty set is said to be a face of (A,b,X) . Ordered by inclusion, the faces of (A,b,X) form a lattice, which is isomorphic to $\{Y \in C \mid Y \le X\}$.

Polytopes are, among polyhedra, those whose face-lattices have deserved more attention. We shall be proving results about these directly in the oriented matroid context, way before the study of affine matroids begins. This is justified by the next proposition.

10. PROPOSITION: Let P be a polytope, given as a cell (A,b,X) of an affine linear system (A,b) . Then the face-lattice of P is isomorphic to the cell $[(X,+)]$ of the oriented matroid of (A,b) . In short, every polytope-lattice is isomorphic to a cell of an oriented matroid.

PROOF: Let $\mathcal{D}, A', C$ be defined for (A,b) as in (7) . By (9), we have that the face-lattice of P is isomorphic with $\{Y \in \mathcal{D} \mid Y \le X\} \cup \{\underline{0}\}$. Thus the result will be proved if we show that $[(X,+)] = \{(Y,+) \mid Y \in \mathcal{D},\ Y \le X\} \cup \{\underline{0}\}$.

From the discussion on (7), $[(X,+)]$ contains all vectors on the right, and every vector in $[(X,+)]$ whose last component is $+$

lies in the set on the right. Thus we need only to show that: if $(Y,0) \in C$, $(Y,0) \leq (X,+)$, then $(Y,0) = \underset{\sim}{0}$.

If that was not the case, there would be a $w' \in \mathbb{R}^{n+1}$ such that $\sigma(A'w') \leq (X,+)$ and $w'_{n+1} = 0$. Let $\bar{w} = (w'_1, w'_2, \ldots, w'_n)$. Then $\bar{w} \neq 0$, and for every $w \in P$, $\lambda \geq 0$, we have that $w + \lambda\bar{w} \in P$, since $\sigma(A(w+\lambda\bar{w})-b) = \sigma(Aw-b+\lambda A\bar{w}) \leq X$, as both $\sigma(Aw-b) \leq X$ and $\sigma(A\bar{w}) \leq X$. Since this is true for arbitrary positive λ, we get a contradiction to the hypothesis that P is bounded. □

11. Remark: An affine linear system all whose members contain the origin may be thought of as a homogeneous system. However, as is patent from the preceeding discussion, we handle the system differently in either case. For instance the oriented matroid of the affine system has one element more than the oriented matroid of the homogeneous system. Thus, although formally homogeneous systems form a special class of affine systems, those two classes have distinct (albeit related) theories. This accounts for our use of the modifier "affine" in what apparently could be termed just "linear systems".

12. A homogeneous linear system A is pointed if the null space of A is $\{\underset{\sim}{0}\}$. From (4.1), if A is pointed, then for every flat F of A, if C/S is the corresponding flat of the oriented matroid of A,

(12.1) $\dim F = r(C/S)$.

In particular, if A is located in $\mathbb{R}^n$, $r(C) = n$.

As far as linear oriented matroids are concerned, we may restrict ourselves to pointed systems. For, if $C = \sigma(V)$, with $V \subseteq \mathbb{R}^E$, then let A be a matrix whose columns form a basis of V. One verifies easily that the homogeneous system A is pointed and has C for its oriented matroid. Thus:

(12.2) Every linear oriented matroid is the oriented matroid of a pointed homogeneous linear system.

13. One may wish to be able to think of linear systems in a coordinate free form. We sketch some ideas on it for homogeneous systems.

Let V be a real vector space and $L = \{\pi_e | e \in E\}$ a set of linear functionals on V. We may call L a homogeneous linear system on V, comprised of the signed hyperplanes. $H_e = \{v \in V | \pi_e(v) =$ with $H_e^+ = \{v \in V | \pi_e(v) > 0\}$, and copies of V for those $e \in E$ such that $\pi_e = 0$. Let $: V \to \mathbb{R}^E$ be the linear map defined by $\pi(v) = (\pi_e(v))_{e \in E}$. Then $\pi(V)$ is a subspace of $\mathbb{R}^E$ and its oriented matroid C is the oriented matroid of L. All concepts like flats and cells of linear systems and their relationship to translate easily to this context.

If V is finite dimensional, we recuperate the original definition by choosing an isomorphism $h = \mathbb{R}^n \to V$. Then L can be considered as a homogeneous system in $\mathbb{R}^n$, whose matrix is the matrix of $h\pi$ relative to the canonical basis of $\mathbb{R}^n$ and $\mathbb{R}^E$.

III) MINORS AND CHANGE OF SIGNS

We describe some basic machinery which is constantly used in what follows. It is a collection of operations with simple interpretation in the context of linear systems, and, with the exception of change of signs, extend usual operations on matroids.

Let X be a signed vector on E and $S \subseteq E$. We denote by $X \backslash S$ the signed vector on $E - S$ defined by: $(X\backslash S)_e = X_e$ for every $e \in E - S$. Let C be an oriented matroid on E. Define:

1. $C \backslash S = \{X\backslash S \mid X \in C\}$,

2. $C/S = \{X\backslash S \mid X \in C,\ \underline{X} \cap S = \phi\}$

It is easily verified that both $C\backslash S$ and C/S are oriented matroids on $E - S$. They are said to be obtained from C by, respectively, deleting S and contracting S. Deletions and contractions can be effected one element at a time and in any order, the end result depending only on the sets of elements which were deleted or contracted.

The reader will have noticed the double meaning of C/F when F is a zero-set. It can mean a contraction or a flat. This ambiguity will be usually resolved in the context. However, for most purposes it is convenient, in case $E \subseteq E'$, to consider each signed vector on E as a signed vector on E' in which the new components are all 0. Thus an oriented matroid on E may be considered as an oriented matroid on E', where the members of $E' - E$ are all loops. In this sense, the flat C/F and the contraction C/F become the same object.

3. In the case where C is the oriented matroid of a homogeneous linear system A, deletions and contractions have the following interpretation:

   $C \setminus S$ is the oriented matroid of the homogeneous system obtained from A by deleting the rows (hyperplanes, copies of $\mathbb{R}^n$) indexed by members of S.

   C/S is the oriented matroid of the homogeneous system on the flat of A which is the intersection of the members of A indexed by S. One should use the coordinate free definition of homogeneous system given in (II.13).

4. Deletions and contractions are common operations in matroid theory. If M is a matroid on E, and $S \subseteq E$, one defines:

$$M \setminus S = \{F - S \mid F \in M\} ,$$
$$M/S = \{F - S \mid F \in M , S \subseteq F\} .$$

   It follows:

$$M(C \setminus S) = M(C) \setminus S$$

   and

$$M(C/S) = M(C)/S .$$

5. Another useful operation is that of <u>reversing signs</u>. One reverses the signs of some hyperplanes of a homogeneous system A by multiplying the corresponding rows of A by -1. The analogous oriented matroid operation is as follows:

With $S \subseteq E$, X a signed vector on E, ${}_{\bar{S}}X$ is the signed vector Z given by: $Z_e = -X_e$ if $e \in S$, X_e if $e \notin S$. The result of reversing S on C is: ${}_{\bar{S}}C = \{{}_{\bar{S}}X \mid X \in C\}$.

It is quite clear that ${}_{\bar{S}}C$ is an oriented matroid. One also notices:

(5.1) The map ${}_{\bar{S}} : C \to {}_{\bar{S}}C$ given by $X \to {}_{\bar{S}}X$ is a bijection which preserves the partial order, products, opposites, approximations, eliminations, flats, rank, etc...

Indeed, almost all aspects of oriented matroids we are interested in are preserved under reversal of signs; one could consider C and ${}_{\bar{S}}C$ as isomorphic objects.

A remark about notation: when $S = \{e\}$, we write $C/e, C\setminus e, {}_{\bar{e}}C$ and so on, omitting the brackets for clarity.

6. PROPOSITION: Let e, f be equators of C. The following are equivalent:

(6.1) $C/e \subseteq C/f$ (i.e., for every $X \in C$, $X_e = 0$ implies $X_f = 0$)

(6.2) Either for every $X \in C$, $X_e = X_f$

or for every $X \in C$, $X_e = -X_f$

(6.3) $C/e = C/f$

PROOF: The implications (6.2) => (6.3) => (6.1) are trivial. Let us assume (6.1) and show (6.2).

As f is an equator, there is an $X \in C$ such that $X_f \neq 0$. From (6.1), $X_e \neq 0$. There are two possibilities, namely $X_e = X_f$ and $X_e = -X_f$. We deal with the first only as the second is similar. Let us show that for every $Y \in C$, $Y_e = Y_f$.

If this was not true, there would be a $Y \in C$ such that $Y_e \neq Y_f$. From (6.1) we can conclude that $Y_e \neq 0$, and, substituting if necessary Y by $-Y$, we may assume that $Y_e = -X_e$. Let $Z \in C$ be obtained by eliminating e between X and Y. As $Y_f \neq Y_e$, either $Y_f = 0$ or $Y_f = -Y_e = -(-X_e) = X_e = X_f$. In either case, $Z_f = X_f \neq 0$, while $Z_e = 0$, contradicting (6.1). □

Equators e,f satisfying the conditions above are called coincident. It is clear from (6.3) that coincidence is an equivalence relation. The correspondent concept in matroid theory is that of parallel elements; we reserve this term for a later use.

An oriented matroid is simple if it has no loops and no two elements are coincident. Recall that a loop is an element $e \in E$ such that $Y_e = 0$ for every $Y \in C$. Equivalently, C is simple if it does not have loops and $\{e\}$ is a zero-set for each $e \in E$ (from (6.1)).

PROPOSITION: Let C be an oriented matroid on E, and $S \subseteq E$ contain all loops and all but one equator of each coincidence class. Then $C \backslash S$ is simple and isomorphic to C as a poset via: $X \to X \backslash S$.

PROOF: Clearly $C \backslash S$ has no loops. Also, if $e, f \in E - S$, as they do not coincide in C, there is a $X \in C$ such that $X_e = 0$, $X_f = 0$, by (6.1); then $X \backslash S \in C \backslash S$ shows that they do not coincide in $C \backslash S$. Hence $C \backslash S$ is simple.

The map $X \to X \backslash S$ is clearly onto and order preserving; (6.2) shows that it is 1-1, and that the inverse bijection is also order preserving. □

An oriented matroid obtained from C as above is a <u>simplification</u> of C. One easily checks that if C_1 and C_2 are simplifications of C, then one can obtain C_2 from C_1 by relabeling some equators and changing some signs.

IV) THE COMPLEX OF AN ORIENTED MATROID

Throughout this section, $\mathcal{C}$ is an oriented matroid on E. The complex of $\mathcal{C}$ is the collection $\{[X]/X \in \mathcal{C}\}$, where $[X] = \{Y \in \mathcal{C}/Y \leq X\}$ is a cell of $\mathcal{C}$. Note that the inclusion order on the complex of $\mathcal{C}$ is isomorphic to the order $\leq$ on $\mathcal{C}$. We occasionally abuse the language and refer to vectors of $\mathcal{C}$ as cells.

1. PROPOSITION: The intersection of two cells is a cell.

PROOF: Let $[X]$, $[Y]$ be cells of $\mathcal{C}$, and $K = [X] \cap [Y]$.

As $[X]$ and $[Y]$ are closed under products, so is K. Let Z be the product of all members of K, in some order; thus $Z \in K$. So $[Z] \subseteq K$ as $Z \leq X$ and $Z \leq Y$. Conversely, if $W \in K$ then $\underline{W} \subseteq \underline{Z}$, as clearly $\underline{Z} = \cup\{\underline{T}|T \in K\}$. For every $e \in \underline{W}$, since both $W,Z \leq X$, $W_e = X_e$ and $Z_e = X_e$; thus $W_e = Z_e$. That means that $W \leq Z$. □

2. COROLLARY: Each cell of $\mathcal{C}$ is a lattice, with the restricted partial order.

PROOF: Let $[X]$ be a cell. It is clear that the product of two vectors in $[X]$ is their least upper bound in $\mathcal{C}$; and it is in $[X]$. The meet of Y and $Z \in [X]$ is the vector W such that $[W] = [Y] \cap [Z]$. □

The results above reflects the well known fact that the faces of a polytope or cone form a lattice. All results here about cells translate directly into statements about polytope lattices. The terminology used will make it clear.

The lattice of flats of an oriented matroid has been shown to be JD by relating it to the lattice of the underlying matroid. We now extend this result to cells.

For every vector $X \in C$, $E - \underline{X}$ is the zero set of a flat F . Let us define the number $r(X)$ as $r(F)$. A more intrinsic expression is, as in (I.11.5):

3. $$r(X) = \rho(E) - \rho(E-\underline{X})$$

where ρ is the rank function of the underlying matroid $M(C)$.

4. THEOREM: C is JD and its rank function is r , as defined in (3).

PROOF: Clearly $r(\underset{\sim}{0}) = 0$. Let us show that if $X^1, X^2 \in C$, and X^2 covers X^1 , then $r(X^2) = r(X^1) + 1$. For a contradiction, suppose that there exist $X^1, X^2 \in C$ such that X^2 covers X^1 and $r(X^2) \neq r(X^1) + 1$. Let F_i be the flat with zero-set $E - \underline{X}^i$, $i = 1,2$. As $X^1 < X^2$, $E - \underline{X}^2 \subseteq E - \underline{X}^1$, so $F_1 \subseteq F_2$. Hence $r(F_1) < r(F_2)$ and as $r(F_1) = r(X^1)$, $r(F_1)+1 < r(F_2)$. As the lattice of C is JD with rank function r , there is a flat F between F_1 and F_2 . Choose a vector X such that $E - \underline{X}$ is the zero set of F ; thus $E - \underline{X}^2 \subsetneq E - \underline{X} \subsetneq E - \underline{X}^1$. As $X^1 \cdot X$ has the same zero-set as X, we may assume that $X^1 < X$. Moreover, $\underline{X} \subsetneq \underline{X}^2$. Let Y be an X-approximation of X^2 (cf. (I.1.5)); then $X^1 < Y < X^2$. This contradicts the hypothesis that Y^2 covers Y^1, and completes the proof. □

Now that we have established (4), we shall prefer to use dimension instead of rank. For $X \in C$,

5. $$d(X) = \rho(E) - \rho(E-\underline{X}) - 1 .$$

An n-<u>cell</u> is a cell of dimension n (the dimension of $[X]$ equals $d(X)$).

At this point of development, the reasons for preferring dimension over rank may be unclear. It will turn out when topology enters the picture that "dimension" has the meaning our intuition and normal topological use requires. In this section this will be most apparent in Theorem (18) : we prove that a 1-dimensional oriented matroid can be identified with a polygon, of which the 0-cells are the vertices and the 1-cells are the edges. It seems more natural to refer to this as 1-dimensional than rank 2.

A <u>tope</u> is a maximal vector of C.

6. PROPOSITION: A vector $T \in C$ is a tope iff its support is the set of equators of C. All topes have dimension equal to $d(C)$.

PROOF: If $\underline{T}$ is the set of equators, it is clear that T is maximal. Conversely, if T is maximal, let e be an equator and $X \in C$ be such that $X_e \neq 0$. Then $T \leq T \cdot X$ and by maximality, $T = T \cdot X$. Since the support of $T \cdot X$ is $\underline{T} \cup \underline{X}$, we conclude that $e \in \underline{T}$. Dimension follows as the zero-set of T is the same as that of C. □

A tope-lattice is a lattice isomorphic to a maximal cell of an oriented matroid.

. PROPOSITION: Every cell, and every interval of C is a tope-lattice.

PROOF: If $X \in C$, and S is its zero-set, then $[X]$ is isomorphic to the cell $[X \setminus S]$ of C/S (and we shall usually identify them, as suggested in the last section). Clearly $X \setminus S$ is a tope of C/S. Intervals will be handled in more detail in Lemma (14). □

Let us adapt the terminology of polytopes to C. We say that a signed vector X is a face of $Y \in C$ if $X \in C$ and $X \leq Y$; it is a proper face if it is not Y. A facet of Y is a maximal paper face of Y; from (4), $X \in C$ is a facet of Y iff $X \leq Y$ and $d(X) = d(Y) - 1$.

. PROPOSITION: Suppose C is simple, T is a tope, $e \in E$.

Let Y be the signed vector obtained from T by changing its e-component to 0 $(Y = T \setminus e)$. The following are equivalent:

(8.1) $Y \in C$.

(8.2) Y is a facet of T .

(8.3) $_{\bar{e}}T \in C$. (recall that $_{\bar{e}}T$ is the vector resultant of reversing the sign of the e-component of T) .

Moreover, every facet of T is of the form $T\backslash e$ for some equator e .

PROOF: The equivalence of (8.1) and (8.2) is clear. If $Y \in C$, as $-T \in C$ we conclude that $_{\bar{e}}T = Y \cdot (-T) \in C$; hence (8.1) implies (8.3). Conversely, if $T \in C$, elimination of e between T and $_{\bar{e}}T$ has Y as the only possible result, hence $Y \in C$.

If X is a facet of T , then $d(X) = n - 1$, where $n = d(T) = d(C)$. From the dimension formula (5), we conclude that the zero-set T of X has rank 1 in $M(C)$. Since C is simple, that set must be a singleton $\{e\}$, thus $X = T\backslash e$. □

In a simple oriented matroid, when T and e are related as above, we say that e is a facet equator of T ; (8) shows that to each facet Y of T corresponds an unique facet equator of T , the supporting equator of Y , and this correspondence is bijective. It is an accident to language the fact that the supporting equator of a facet is the only equator not in its support.

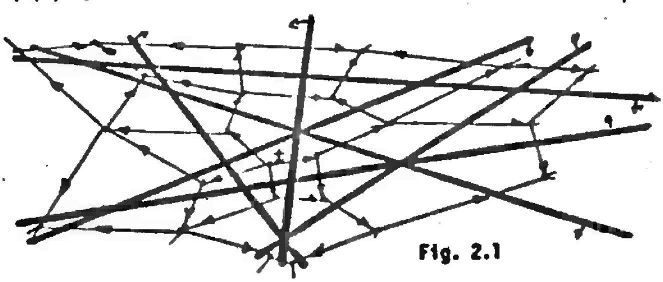
9. THEOREM: Every $(d(C) - 1)$-cell is a facet of exactly two topes.

PROOF: We may assume that C is simple. Let X be a $(d(C) - 1)$-cell. Thus X is not maximal and $X < T$ for some tope T. As $d(X) = d(T) - 1$, X is a facet of T. By (8.3), ${}_{\bar{e}}T$ is another tope of C covering X, where e is the supporting equator of Y. Clearly no other signed vector on E is $> X$, therefore T and ${}_{\bar{e}}T$ are the only topes covering X. □

10. THEOREM: Let T, T' be distinct topes of C, which we assume is simple. Then some facet equator of T separates it from T'.

PROOF: Let X be a T'-approximation of T. Since $X < T$, there is a facet Y of T such that $X \leq Y$. The zero-set of X consists only of equators separating T and T', and the supporting equator of Y is one of them. □

This result can be visually interpreted via the tope graph of C at T. This is a directed graph $TG(C,T)$ whose vertices are the topes and with an edge directed from T^1 to T^2 whenever they are separated by an unique equator e and $T^1_e = T_e$; see fig. (2.1). Note that by (9), one can identify the edges of $TG(C,T)$ with the $(d(C)-1)$-cells of C.



TG(C,±)

Fig. 2.1

11. THEOREM: With C simple, $TG(C,T)$ is acyclic, has a unique source, T, and an unique sink, $-T$.

PROOF: It is clear that $TG(C,T)$ is acyclic, T is a source and $-T$ is a sink. If $W \neq T$, then some facet equator e of W separates it from T, so ${}_{\bar{e}}W \to W$ is an edge of $TG(C,T)$, and W is not a source. Similarly, it is not a sink, unless $W = -T$. □

12. COROLLARY: It is possible to order the equators of a simple C as $e_1, e_2, \ldots, e_k$ so that all vectors $T^1, T^2, \ldots, T^k = -T$ obtained from T by successively reversing those equators are topes.

PROOF: Let $T, T^1, T^2, \ldots, T^k = -T$ be a path in $TG(C,T)$., which exists, as it follows easily from (11). The sequence $e_1, e_2, \ldots, e_k$ is obtained in the obvious way, from the edges of the path. □

The tope graph will show up again in the next chapter in connection with shellability.

The next result is a key property of length 2 intervals, which entails a characterization of 1-dimensional oriented matroids. We prove it modulo Lemma (14), whose proof follows afterwards.

13. THEOREM: Let X, Y be vectors of $\mathcal{C}$, with $X < Y$ and $d(Y) = d(X) + 2$. Then there exist precisely two facets of Y containing X.

14. LEMMA: For every cell $X \in \mathcal{C}$, the function $\phi : W \mapsto W \backslash \underline{X}$ is a poset isomorphism between $st(X,\mathcal{C}) = \{W \in \mathcal{C} \mid W \geq X\}$ (star of X) and $\mathcal{C} \backslash \underline{X}$. Moreover, $d(\mathcal{C} \backslash \underline{X}) = d(\mathcal{C}) - d(X) - 1$.

15. PROOF OF (13) (modulo Lemma (14)): As we are looking only at vectors $\leq Y$, we may contract $E - \underline{Y}$; thus assume that Y is a tope. By Lemma (14), the interval $\{W \in \mathcal{C}/X \leq W \leq Y\}$ is isomorphic as a poset to the cell $[Y \backslash \underline{X}]$ of $\mathcal{C} \backslash \underline{X}$, and $d(\mathcal{C} \backslash \underline{X}) = 1$. As $Y \backslash \underline{X}$ is a tope of $\mathcal{C} \backslash \underline{X}$, we are reduced to prove:

(15.1) Every tope of a 1-dimensional oriented matroid has exactly two facets (0-cells).

Starting the notation anew, let $\mathcal{C}$ be a 1-dimensional oriented matroid, which we can assume to be simple, and let T be a tope of $\mathcal{C}$. By reversing the signs of all negative components of T, we can assume that $T = \underline{+}$. Let W be a facet of T, with supporting equator w.

Let V be a $(-W)$-approximation of T. Then $V < T$ and $V_w = T_w = +$. As $\underline{0} < V < T$, $d(V) = 0$, hence V is a facet of T supported by the equator v. This shows already two facets of T , as $V \neq W$. Suppose for a contradiction that T has a third facet Z , supported by z . We resume the information about W,V,Z as:

$$W_v = Z_v = + \ , \ V_w = Z_w = + \ , \ W_z = V_z = + \ , \ W_w = V_v = Z_z = 0 \ .$$

Eliminating w between V and -Z , we obtain $X \in C$ with $X_v = -$, $X_w = 0$, $X_z = +$. Eliminating v between X and W we obtain $Y \in C$ such that $Y_v = 0 = Y_w$, $Y_z = +$. Then $\underline{0} < Y < Y.V < Y.T$ is a chain of length 3 in C , contradicting the hypothesis that C has dimension 1 (rank 2). □

16. PROOF of 14: Clearly ψ maps st(X) into $[Y\backslash\underline{X}]$, it is injective, and for all $W,Z \in st(X)$, $\psi(W) \leq$ (Z) iff $W \leq Z$. Let us now show that ψ is onto.

Choose $Z \in [Y\backslash\underline{X}]$. Then $Z = W\backslash\underline{X}$ for some $W \in C$. As $X.W \in C$ and $X \leq X.W$, we see that $Z = \psi(X.W)$. This completes the proof that ψ is bijective.

To check the dimension formula, recall that as $Y\backslash\underline{X}$ is a tope of $C\backslash\underline{X}$, $d(C\backslash\underline{X})+1 = r(Y\backslash\underline{X})$ (as a vector of $Y\backslash\underline{X}$) , and it is the length of a saturated chain from $\underline{0}$ to $Y\backslash X$. As ψ is an isomorphism, the inverse images of the vectors in the chain form a saturated chain from X to Y in C . The length of this chain is $d(Y)-d(X)$, which yields to desired result. □

We may combine (9) and (13) as follows: if we adjoin a maximum element to C, in the resulting poset every length 2 interval has the form of a diamond  .

We can now present a useful characterization of 1-dimensional oriented matroids, lines. It is a specialization of the more general Theorem (3.III.1); we present it here for its simplicity and because of the important role of lines in later chapters. A similar characterization appears in [LV6].

17. LEMMA: Let C be a line without loops, $V \in C$ a 0-cell with zero-set S. Then

(17.1) V is contained in exactly two 1-cells, T^1 and T^2.

(17.2) $T^2 = {}_{\bar{S}}T^1$

(17.3) If X^1 and X^2 are respectively the other 0-cells in T^1 and T^2, the equators which separate X^1 and X^2 are precisely those in S.

(17.4) Let $e \in S$, $f \in \underline{V}$. If $W \in C$, $W_e = 0$ and $W_1 = V_2$, then $W = V$.

PROOF: (17.1) is a special case of Theorem (9). (17.2) is a special case of (8), before simplification; it also follows by noticing that $T^1 = V.(-T^1) \in C$, and $V <_{\bar{S}}T^1 \neq T^1$.

To prove (17.3), notice that if e separates X^1 and X^2, it also separates T^1 and T^2, hence $e \in S$. Conversely, if $e \in S$, as $T^1 = V.X^1$, $X^1_e = T^1_e$ and similarly $X^2_e = T^2_e$; since

$T_e^1 = -T_e^2 \neq 0$, e separates T^1 and T^2 .

(17.4) Since $W_e = 0$, $W \in C/e$. As $d(C/e) = 0$, $C/e = \{V, -V, \underline{0}\}$, and since $W_f = V_f \neq 0$, $W = V$. □

If G is an even polygon with vertices numbered cyclically $v_1, v_2, \ldots, v_{2n}$, we say that, for $i = 1,2,\ldots,2n$, v_i and v_{i+n} are <u>opposite</u> (indices modulo 2n) .

Let C be an 1-dimensional oriented matroid without loops. Define the graph Skel(C) whose vertices are the 0-cells of C, X,Y being adjacent iff they are contained in a 1-cell.

18. THEOREM: Skel(C) is an even polygon in which the opposite of each V in Skel(C) is $-V$. Furthermore, if $X^1, X^2, \ldots, X^{n-1}$ are the internal vertices in one of the paths in Skel(C) between V and $-V$, and e is an equator in the zero set of V , then $X_e^1 = X_e^2 = \ldots = X_e^{n-1} \neq 0$.

PROOF: Let V be a 0-cell. By (17.1), V is contained in exactly two 1-cells. By (15.1) each of these contains exactly one 0-cell distinct from V . These two 0-cells are the only ones adjacent to V , and are distinct by (17.3). Thus each vertex in Skel(C) is adjacent to exactly two others. Hence each component of Skel(C) is a polygon.

Let T be a 1-cell containing V , and let X^1 be the other

1-cell of T. Define $X^2, X^3, \ldots$ sequencially as follows: if $X^i \neq -V$, let X^{i+1} be the 0-cell of $X^i.(-V)$ which is distinct from X^i. Clearly $X^n = -V$ for some n. By construction, $V, X^1, X^2, \ldots, X^{n-1}, -V$ are the vertices of a path in Skel(C). Since X, Y adjacent in Skel(C) implies $-X, -Y$ adjacent, $V, -X^{n-1}, \ldots, -X^2, -X^1, -V$ is also a path in Skel(C). It will follow that Skel(C) is an even polygon, and the opposite of V is $-V$ if we show that those two paths use all its vertices.

Indeed, if W is a 0-cell, $W \neq V, -V$, and e is in its zero-set, then $V_e \neq 0$. Let i be maximum such that $X_e^i = V_e$ ($X^0 \equiv V$). Since X^{i+1} is adjacent to X^i, it cannot happen that $X_e^{i+1} = -V_e$, hence $X_e^{i+1} = 0$. It follows from (17.4) that either $W = X^{i+1}$ or $W = -X^{i+1}$.

To complete the proof, let e be an equator in the zero-set of V. Then, for $i = 1, 2, \ldots, n-1$, $X_e^i \neq 0$, by (17.4) and since X^i is adjacent to X^{i+1}, $X_e^i = X_e^{i+1}$ ($i<n-1$). □

A converse of this theorem is given by the following construction. Let P be an even polygon, E a finite set, and for each $e \in E$ let P_e^0 be a pair of opposite vertices of P. It is required that each vertex of P be in at least one P_e^0. Choose for each e one of the paths between the vertices of P_e^0, and let P_e^+ be the set of its internal vertices; let P_e^- be the set of opposites of those members of P_e^+. Note that no edge has both a vertex in P_e^+ and one in P_e^- .

Define, for each vertex v of P the signed vector $\sigma(v)$ on E by: $\sigma(v)_e = j \iff V \in P_e^j$. Define also for each edge α the signed vector $\sigma(\alpha)$ by: $\sigma(\alpha)_e = j \iff \alpha$ has a vertex in P_e^j , $j = +, -$.

19. THEOREM: With the construction above,

$$= \{0\} \cup \{\sigma(v) \;/\; v \text{ a vertex of } P\} \cup \{\sigma(\alpha)/\alpha \text{ an edge of } P\}$$

is a 1-dimensional oriented matroid, and vertices v, w of P are adjacent iff $\sigma(v)$ and $\sigma(w)$ are adjacent in Skel($\mathcal{C}$).

PROOF: We sketch the verification of the oriented matroid axioms, and leave the remaining details for the reader.

If v is a vertex of P , and V' is its opposite, then $\sigma(v') = -\sigma(v)$. If α is an edge, with vertices v, w, then the opposites v', w' are incident to an edge α', and $\sigma(\alpha') = -\sigma(\alpha)$. Let x, y be vertices or edges, and suppose that $\sigma(x).\sigma(y) \neq \sigma(x)$. Then x is a vertex and there is a unique shortest path in P containing both x and y . Let α be the edge of this path incident to x , then $\sigma(\alpha) = \sigma(x).\sigma(y)$. Finally, suppose that $\sigma(x)_e = +$, $\sigma(y)_e = -$ and $\sigma(x) \neq -\sigma(y)$. Choose again the shortest path containing both x and y . Since x has (or is) a vertex in P_e^+, and y has (or is) a vertex in P_e^-, that path contains one vertex $v \in P_e^0$; $\sigma(v)$ is the result of eliminating e between

$\alpha(x)$ and $\alpha(y)$. □

From the two results above, 1-dimensional oriented matroids can be thought of an even polygons, and there is no advantage on distinguishing between C and Skel(C). This will be explored later.

V. SUPERCELLS

Consider a linear system B obtained by deleting some members of another system A. Then each cell of B is a union of cells of A. This suggests considering each cell of B as a "generalized cell" of A. The corresponding concept in oriented matroids is what we call supercells.

The main result in this section is a characterization of oriented matroids by properties of their supercells. That way one obtains a definition of oriented matroids as a set of signed vectors with certain topological properties (Theorem (9)). For that purpose, some notions, will be introduced for arbitrary sets of signed vectors and then specialized to oriented matroids.

In this section, unless otherwise mentioned, $\mathcal{D}$ denotes a set of signed vectors on the finite set E.

For each $e \in E$, $j \in \{0, +, -\}$, let $\mathcal{D}_e^j = \{X \in \mathcal{D} | X_e = j\}$, and let $\overline{\mathcal{D}}_e^j = \mathcal{D}_e^j \cup \mathcal{D}_e^0$. Thus $\overline{\mathcal{D}}_e^j = \mathcal{D}_e^0 = \overline{\mathcal{D}}_e^+ \cap \overline{\mathcal{D}}_e^-$. Sets of form $\overline{\mathcal{D}}_e^+$ and $\mathcal{D}_e^-$ are called the <u>half-spaces</u> of $\mathcal{D}$. Given $I \subseteq E$, the set $\mathcal{D}/I = \{X \in \mathcal{D} | X_e = 0$ for every $e \in I\}$ is the <u>flat</u> of $\mathcal{D}$ it determines. Note that $\mathcal{D}/I = \cap\{\mathcal{D}_e^0 | e \in I\} = \cap\{\overline{\mathcal{D}}_e^j | e \in I,\ j = +, -\}$, that is, any flat is an intersection of half-spaces. The set $\mathcal{D}$ is also a flat $\mathcal{D}/\phi = \mathcal{D}$.

A subset K of $\mathcal{D}$ which is an intersection of half-spaces, but not a flat, is called a <u>supercell</u> of $\mathcal{D}$.

EXAMPLE: Let $A = \{H_e | e \in E\}$ be a homogeneous system in $\mathbb{R}^n$, and let be its matroid. Denote by $\alpha(w)(\in C)$ the signed vector indicating the position of $w \in \mathbb{R}^n$ relative to A, as in (II.2). Then for each half-space $\overline{C}_e^j$, $\alpha^{-1}(\overline{C}_e^j)$ is a (closed) half-space in $\mathbb{R}^n$, of form $H_e^j \cup H_e$. If K is a supercell of C, $\alpha^{-1}(K)$ is a cell of the linear system one obtains by deleting from A those H_e for which C_e^0 does not contain K. □

PROPOSITION: If K is either a flat or a supercell of C, and $Y \in K$, then for every $X \in K$, $X \le Y$ implies $X \in K$.

PROOF: This is clear for any half-space, thus it is also true for any intersection of half-spaces. □

The span of a set $K \subseteq \mathcal{D}$ is defined as the intersections of all flats containing K. Thus the span of K equals $\cap\{\mathcal{D}_e^0 | K \subseteq \mathcal{D}_e^0\}$, provided K is contained in some $\mathcal{D}_e^0$.

PROPOSITION: Let C be an oriented matroid on E and let K be a supercell of C, with span F.

Then:

(3.1) X, $Y \in K$ implies $X.Y \in K$.

(3.2) All maximal vectors of K have support E-Z, where Z is the zero-set of F.

(3.3) All maximal vectors of K have dimension equal to d(F).

PROOF: As each half-space is a submonoid of C, so are intersections of half-spaces, and (3.1) follows. To verify (3.2), note that if X is maximal in K, and $Y \in K$, since $X \le X.Y \in K$, we have that $X = X.Y$, thus $\underline{Y} \subseteq \underline{X}$. Hence $K \subseteq C_e^0$ iff $e \notin \underline{X}$, that is $Z = E - \underline{X}$.

Combining (3.2) with the dimension formula (IV.5), (3.3) follows. □

We introduce a notation for an intersection of half-spaces. Let Y be a signed vector on E, and $I \subseteq E$. Denote:

$$\mathcal{D}(Y,I) = \cap\{\bar{\mathcal{D}}_e^{Y_e} \mid e \in E-Y\}$$

$$= \{X \in \mathcal{D} \mid X \backslash I \leq Y \backslash I\} \quad .$$

4. PROPOSITION: The supercells of an oriented matroid C are precisely the sets $C(Y,I)$ where $Y \in C$ and $\underline{Y}-I \neq \phi$.

PROOF: Suppose that $Y \in C$ and $\underline{Y}-I \neq \phi$. Then $Y \in C(Y,I)$, while $-Y \notin C(Y,I)$. Hence, $C(Y,I)$ cannot be a flat, so it is a supercell.

Conversely, let K be a supercell of C . Let $I = \{e\ E \mid C_e^+ \cap K \neq \phi \neq C_e^- \cap K\}$, and choose a maximal $Y \in K$. Suppose that $\bar{C}_e^j \supset K$. Then either $Y_e = 0$ and by (3.2) $K \subseteq C_e^0$, or $Y_e \neq 0$ and then $Y_e = j$, thus $e \in I$. That implies that $K = C(Y,I)$. Of course, $\underline{Y}-I \neq \phi$, otherwise one would have $K = C/(E-I)$, a flat.

We shall need a few more concepts which are motivated by topological considerations about linear systems. We now topologize each $\mathcal{D}$ by defining a set $K \subseteq \mathcal{D}$ to be open iff it satisfies: for every $X \in \mathcal{D}$, and $Y \in K$, if $X \geq Y$ then $X \in K$. Clearly K is closed if $X \leq Y \in K$ implies $X \in K$, whenever $X \in \mathcal{D}$.

Note that for any $F \in \mathcal{D}$, the inclusion mapping is an embedding, i.e., the open sets of F are precisely the intersections with F of open sets of $\mathcal{D}$. Similarly for closed sets.

5. EXAMPLE: In the notation of Example (1), the map $\alpha: R^n \to C$ is continuous, since for each closed set K, $\alpha^{-1}(K) = \alpha^{-1}(\cup\{[X] \mid X \in K\}) = \cup\{\alpha^{-1}([X]) \mid X \in K\}$ = finite union of closed polyhedra. This map is also open. □

2.V

Consider a polyhedron $K = \{x \in R^n | Ax \geq b\}$, with the rows of A indexed by E. Let $I = \{e \in E | A_e . x = b_e$ for every $x \in K\}$. Then the span (affine hull) of K is $\{x \in R^n | A_I x = b_I\}$. The relative interior of K may be defined in two different ways: (a) as the topological interior of K in its span; (b) as $\{x \in K | A_e x > b_e$ for every $e \in E-I\}$. It is easy to show that those two definitions are equivalent.

Each of (a) and (b) suggests a definition for the relative interior of a supercell. For oriented matroids those coincide, as Proposition (7) shows. However, this is not the case for an arbitrary $\mathcal{D}$ and the definition of relative interior we have found effective is the one derived from (b), and we present it below.

The <u>relative interior</u> of a supercell K of $\mathcal{D}$ is defined by:

$\mathrm{relint}(K) = (\cap\{\mathcal{D}_e^j | K \subseteq \bar{\mathcal{D}}_e^j, K \not\subseteq \mathcal{D}_e^0\}) \cap K.$

Note that, for instance if $\mathcal{D} = \{+0, ++, -0, --\}$, and $K = \{+0, ++\}$, then $\mathrm{relint}(K) = \{++\}$, while K is open in its span $\mathcal{D}$.

6. PROPOSITION: Let F be the span of K and write $K = \mathcal{D}(Y, I)$, with Y minimal. The following two expressions equal relint(K):

(6.1) $(\cap\{\mathcal{D}_e^j | K \subseteq \mathcal{D}_e^j, K \not\subseteq \mathcal{D}_e^0\}) \cap F.$

(6.2) [illegible]

PROOF: Denote by K_1 and K_2 the sets in (6.1) and (6.2). Since $K \subseteq F$, $\mathrm{relint}(K) \subseteq K_1$. Now we show that $K_1 \subseteq K_2$. Let $X \in K_1$, and $e \in E-I$. That means that $K \subseteq \mathcal{D}_e^j$, where $j = Y_e$. If $j=0$, then

$X \in \bar{\mathcal{D}}^0_e$, i.e., $X_e = 0$. If $j \neq 0$, minimality of Y implies that $K \not\subseteq \mathcal{D}^0_e$. Then $K_1 \subseteq \bar{\mathcal{D}}^j_e$, $X_e = j$. Reversing the argument one gets $K_2 \subset K_1$. Finally, we obtain: relint$(K) = K_1 \cap K = K_2 \cap K = K_2$, since $K_2 \subseteq K$. □

7. PROPOSITION: If K is a supercell of the oriented matroid $\mathcal{C}$, then relint(K) is the topological interior of K in its span. Further, if $K = \mathcal{C}(Y,I)$, with $Y \in K$, then relint$(K) = \{X \in \mathcal{C} \mid X \backslash I = \dot{Y} \backslash I\}$.

PROOF: Let F be the span of K. The first assertion means that relint (K) is the union of all open sets of F contained in K. Since each $\mathcal{D}^j_e$ is an open set, (6.1) shows that relint(K) is an open set of $\mathcal{D}$. Conversely suppose that $A \subseteq K$ is open in F and let us show that A relint(K). It is enough to show that if $K \subseteq \mathcal{D}^j_e$ and $K \not\subseteq \mathcal{D}^0_e$, then $A \subseteq \mathcal{D}^j_e$.

Suppose that this fails for a specific $\mathcal{D}^j_e$. Thus there exists an $X \in A \cap \mathcal{D}^0_e$. Since $K \not\subseteq \mathcal{D}^0_e$, e is not a loop, hence there exists a $W \in \mathcal{D}$ such that $X < W$ and $W_e = -j$. Since A is open, $W \in A$, contradicting $A \subseteq K$.

Consider now the expression $K = (Y,I)$. Since $Y \in K$, for every $e \in E-I$, we have that, $K \subseteq \bar{C}^{Y_e}_e$ and $K \not\subseteq C^0_e$ iff $Y_e \neq 0$. Thus relint$(K) = (\cap\{C^{Y_e}_e \mid e \in \underline{Y}-I\}) \cap K$. That is equivalent to the desired conclusion. □

A set $\mathcal{D}$ is (topologically) <u>connected</u> if it is not the union of two disjoint non-empty open sets. It is <u>weakly connected</u> if there is no $e \in E$ such that $\mathcal{D}_e^+ \neq \phi \neq \mathcal{D}_e^-$ and $\mathcal{D}_e^0 = \phi$.

8. PROPOSITION: A connected set $\mathcal{D}$ is weakly connected. The two types of connectivity can be characterized as follows:

(8.1) $\mathcal{D}$ is weakly connected iff the following graph is connected: vertices $= \mathcal{D}$, X and Y adjacent if they are not separated by any component.

(8.2) $\mathcal{D}$ is connected iff the following graph is connected vertices $= \mathcal{D}$, X and Y adjacent iff $X<Y$ or $Y<X$ (the "comparability graph" of $\mathcal{D}$).

PROOF: If $\mathcal{D}$ is not weakly connected, the partition $(\mathcal{D}_e^+, \mathcal{D}_e^-)$ shows that $\mathcal{D}$ is not connected. We omit the proof of (8.1), which is similar to (8.2).

To prove (8.2), consider disjoint open sets A_1, A_2 of $\mathcal{D}$. Then no member of A_1 is adjacent to a member of A_2 in the graph. Thus every topological component of $\mathcal{D}$ induces a component of the graph and vice-versa. □

Now we can finally present a characterization of oriented matroids. Note that the closure of a set $K \subseteq \mathcal{D}$ is $\{X \in \mathcal{D} | X \leq Y$ for some $Y \in K\}$.

9. THEOREM: Let $\mathcal{D}$ be a collection of signed vectors on E. Then $\mathcal{D} + \underset{\sim}{0}$ is an oriented matroid iff $\mathcal{D}$ satisfies.

(9.1) For every flat F, $\mathcal{D}_e^+ \cap F \neq \phi$ iff $\mathcal{D}_e^- \cap F \neq \phi$.

(9.2) Every supercell is the closure of its relative interior.

(9.3) The relative interior of each supercell is (weakly) connected.

PROOF: We prove necessity first. Let $\mathcal{D} + \underset{\sim}{0}$ be an oriented matroid.

One obtains (9.1) immediately from the existence of opposites. To check (9.2) and (9.3) let K be a supercell of $\mathcal{D}$, $K = \mathcal{D}(Y,I)$, where $Y \in K$. Then, if $X \in K$, $X \cdot Y \in K$ by (3.1); comparing supports, $(X.Y) \setminus I = Y \setminus I$, hence $X.Y \in \text{relint}(K)$, by (7). As $X \leq X.Y$, X is in the closure of relint(K) . Thus $K \subseteq$ closure relint(K) $\subseteq K$, where the last inclusion holds since K is closed. Finally, suppose for a contradiction that relint(K) is disconnected, so it can be partitioned into disjoint open sets A_1, A_2. Choose $X^1 \in A_1$, $X^2 \in A_2$ such that they are separated by as few components as possible. The same hypotheses are still satisfied if we substitute X^1 by $X^1.X^2$ and X^2 by $X^2.X^1$. Thus we assume that $\underline{X}^1 = \underline{X}^2$. If e separates X^1 and X^2, elimination produces a vector still in relint(K), and separated from either by fewer components. It follows that X^1 and X^2 are not separated, hence $X^1 = X^2$, contradiction. Thus (9.3) holds.

Conversely, we assume that (9.1) - (9.3) are satisfied by $\mathcal{D}$. We establish first:

(a) If $\mathcal{D}(Y,I)$ contains a vector W such that $W_e = Y_e \neq 0$ for some $e \in E-I$, then $\mathcal{D}(Y,I)$ is a supercell.
<u>Proof</u>: Let $j = Y_e$. Since $\mathcal{D}^j_e \cap \mathcal{D}(Y,I) \neq \phi$ but $\mathcal{D}^j_e \cap (Y,I) = \phi$, (9.1) implies that $\mathcal{D}(Y,I)$ is not a flat. //
Now for the oriented matroid axioms.

(b) If $X,Y \in \mathcal{D}$, then $X.Y \in \mathcal{D}$.
<u>Proof</u>: Let I be the set of components separating X and Y.

let $W = X.Y$, and consider $K = \mathcal{D}(W,I)$. Since both $X,Y \in \mathcal{D}(W,I)$, and for each $e \in E-I$, either $X \in \mathcal{D}_e^{W_e} \cap K$ or $Y \in \mathcal{D}_e^{W_e} \cap K$, $\text{relint}(K) = (\cap\{\mathcal{D}_e^{W_e} \mid e \in E-I\}) \cap K = \{Z \in K \mid Z \backslash I = W \backslash I\}$.

By the definition of closure, and (9.2), there exists $Z \in \text{relint}(K)$ such that $X \leq Z$. Thus, $X \leq Z$ and $Z \backslash I = W \backslash I$. This shows that $Z = W$, since $I \subseteq \underline{X}$, and $X \leq W$. Hence $X.Y = W \in \text{relint}(K)$. //

(c) Elimination holds in $\mathcal{D} + \underline{0}$.

Proof: Let $X, Y \in \mathcal{D}$ and let e be a component separating them.

The presence of $\underline{0}$ allows us to assume that $X \neq -Y$. Consider again $K = \mathcal{D}(W,I)$, with W, I as in part (b), and recall that $\text{relint}(K) = \{Z \in K \mid Z \backslash I = W \backslash I\}$. By (b), $W = X.Y$, and $T = Y.X$ are both in $\mathcal{D}$, and are therefore in $\text{relint}(K)$. Since W_e and T_e are opposite, we have that $\mathcal{D}_e^+ \cap \text{relint}(K)$ and $\mathcal{D}_e^- \cap \text{relint}(K)$ are both non-emtpy. By weak connectivity, there exists $Z \in \mathcal{D}_e^0 \cap \text{relint}(K)$. That Z results from eliminating e between X and Y. //

(d) $X \in \mathcal{D}$ implies $-X \in \mathcal{D}$.

Proof: We prove it by contradiction. Suppose that there exists an $X \in \mathcal{D}$ such that $-X \notin \mathcal{D}$, and choose such an X with minimal support. Clearly $X \neq \underline{0}$. Let $F = \mathcal{D}/I$, where $I = E - \underline{X}$, and choose $e \in \underline{X}$. By (9.1), there exists $Y \in F$ such that $Y_e = -X_e$. Since $Y \in F$, $\underline{Y} \subseteq \underline{X}$. Let $W \in \mathcal{D}$ be an Y-approximation of X, which exists by (c) and (I.4). Then $W < X$ and the minimality of X implies $-W \in \mathcal{D}$. As for every $f \in \underline{X}$, either $W_f = X_f$ or $Y_f = -X_f$, $(-W).Y = -X$. Since $\mathcal{D}$ has products, $-X \in \mathcal{D}$, contradicting the choice of X. This contradiction proves (d), and the theorem. □

(I) VERTICES

A vertex of C is a minimal nonzero vector, equivalently, a 0-cell. We say that a vertex V is a vertex of a cell Y if $V \leq Y$.

1. PROPOSITION: Every nonzero vector of C is the product of its vertices.

PROOF: It is enough to show that a cell is the product of some of its vertices. Suppose this is not the case, and choose a cell Y minimal with respect to not being the product of its vertices.

Let X be a vertex of Y. Let Z be a $(-X)$-approximation of Y; thus $Z < Y$ and $Z \backslash \underline{X} = Y \backslash \underline{X}$. By the minimality of Y, Z is the product of its vertices. Since every vertex of Z is also a vertex of Y, $Y = X.Z$ is the product of its vertices, contradicting the way it was chosen. □

It is a corollary of this, as C is a monoid, that $C - \underline{0}$ consists of all signed vectors which are products of vertices. Thus, the collection of vertices of C is enough data for a complete description of C. It turns out that there are very simple conditions which determine when a collection of signed vectors is the set of vertices of an oriented matroid. These conditions are quite similar to the ones describing a set of cocircuits; a fact which is not a coincidence, as the supports of vertices of C are the cocircuits of $M(C)$:

2. PROPOSITION: A vector V is a vertex iff there is no cell Y with $\phi \neq \underline{Y} \subsetneq \underline{V}$.

PROOF: If there is no such Y, of course V is minimal. Converse if there is such a Y, either Y or a Y-approximation of V shows, that V is not a vertex. □

This result shows that V is a vertex iff its zero-set is not contained in any other zero-set besides E, that is, E - $\underline{V}$ is a hyperplane of M(C). Hence V is a vertex iff $\underline{V}$ is a cocircuit of M(C).

3. PROPOSITION: Let V be the set of vertices of an oriented matroid. Then

(3.1) $\underline{0} \notin V$; $X \in V$ implies $-X \in V$; $X,Y \in V$ and $\underline{X} \subseteq \underline{Y}$ implies $X = Y$ or $X = -Y$.

(3.2) For every $X,Y \in V$, given any $e \in E$ separating X and Y and any $f \in \underline{X} \cup \underline{Y}$ not separating them, there is a $Z \in V$ such that $Z_e = 0$, $Z_f \neq 0$ and for every other $g \in E$, Z_g equals either X_g or Y_g or 0.

PROOF: (3.1): $\underline{0} \notin V$ be definition. As $\underline{X} = -\underline{X}$, from (2) it follow that $X \in V$ implies $-X \in V$. If $X,Y \in V$, $\underline{X} \leq \underline{Y}$ and X is not Y or -Y, then there is an $e \in \underline{Y}$ separating them and one which does not. Eliminating e between X and Y, we obtain a nonzero cell Z, with $\underline{Z} \subsetneq \underline{Y}$. This contradicts (2).

(3.2) Let W be a cell obtained by eliminating e between X and Y. As f does not separate X and Y, $W_f \neq 0$. From (1) it follows that there is a vertex Z of W such that $Z_f = W_f$. This Z clearly satisfies the required condition. □

It is a quite simple exercise to show that if a collection V of signed vectors satisfies (3.1) and (3.2), it is the set of vertices of an oriented matroid. This is analogous to the equivalence of (1.17.1) and (1.17.2).

More significantly, an "oriented" version of (1.17.3) is available, Theorem (4) below. That is taken as the basic axiomatic definition of oriented matroids by Bland and Las Vergnas [BLV], and Folkman and Lawrence [FL]. The reader can find there a proof that a set V satisfying the conditions below also satisfies (3.2).

4. THEOREM: Let V be a collection of signed vectors satisfying:

(4.1) $\underline{0} \notin V$; $X \in V$ implies $-X \in V$; $X,Y \in V$ and $\underline{X} \le \underline{Y}$ implies $X = Y$ or $X = -Y$.

(4.2) For every $X,Y \in V$ such that $X \neq -Y$ and any $e \in E$ separating them, there is a $Z \in V$ such that $Z_e = 0$ and for every $f \in E - e$, $Z_f = X_f$ or Y_f or 0.

Then V is the set of vertices of an oriented matroid. □

When C is the oriented matroid of an affine linear system (A,b) in $\mathbb{R}^n$ whose columns are linearly independent, the vertices of C which correspond to points of $\mathbb{R}^n$ correspond to "basic points". Those are the points which are determined as intersection of hyperplanes of (A,b).

The vertex axioms (also called "circuit axioms") arise naturally in connection with directed graphs. One can associate to a directed graph a pair of oriented matroids, reflecting the circuits and the cutsets of the graph. Those matroids are linear. Unfortunately, the results in this thesis do not seem to yield graph-theoretic results, when specialized to these matroids. We refer the reader to [BLV].

We conclude with the connection between our geometrical interpretation of linear oriented matroids and that presented in [BLV].

Let $A \in \mathbb{R}^{E \times n}$, and let C be its oriented matroid. Let $W = \{y \in \mathbb{R}^e | yA = 0\}$ and let $\mathcal{D} = \sigma(W)$. The vertices of $\mathcal{D}$ are called the <u>circuits</u> of C; C and $\mathcal{D}$ are said to be <u>orthogonal</u>. We may consider the rows of A as points in $\mathbb{R}^n$, and then the circuits of C reflect the linear dependencies of those points. In this model, each zero-set F of C indexes a maximal collection of those points lying in a subspace of $\mathbb{R}^n$, of dimension $\rho(F)$.

In general, for any oriented matroid C on E, the collection D
signed vectors X such that for every Y∈C either X.Y≠Y.X or $\underline{X}\cap\underline{Y}=\phi$
an oriented matroid. This is called the <u>orthogonal</u> or <u>dual</u> of C .
e relationship between C and its dual is studied in detail in [BLV]
d [FL]. We shall make just a passing use of duality; this tool is
sent from this thesis otherwise.

In comparing our results with those earlier articles, it is
portant to understand that in those works an oriented matroid is
scribed by the vertices of its dual matroid, called the <u>circuits</u> of C .

VII) REMARKS

The foundations of oriented matroids were independently laid by Las Vergnas [LV1], Bland [B1] and Folkman and Lawrence [FL] (this last paper contains Lawrence's development of the theory initiated by Folkman, and was published after Folkman's death). In all these works, oriented matroids are developed as an extension of matroid theory. That explains a preference for the vertex axioms, other than the grounds that these are simpler than the set we have chosen.

The crux of the difference of approaches is however the nature of the geometrical model for linear matroids. It becomes very apparent when one compares the results of section IV with the geometric theory of Las Vergnas [LV3]. To help in comparing viewpoints let us consider a polytope P in $\mathbb{R}^n$ given as the set of solutions of the system $Ax \geq (1,1,\ldots,1)$; let Q be the convex hull of the rows of A, and let C be the oriented matroid of the affine linear system (A,1).

The face-lattice of C as defined by Las Vergnas is the collection of all zero-sets of vectors in $[\underset{\sim}{+}]$, ordered by inclusion. It corresponds to sets of rows of A whose convex hull is a face of Q, and is thus isomorphic to the face-lattice of Q. On the other hand, we consider $[\underset{\sim}{+}]$ as our abstraction of polytope, modelling the lattice of P. Those two lattices are related by an order reversing bijection, and some results here appear in [LV3] in ""upside-down" form. For instance, (IV.4) is essentially the same as (1.1) of [LV3] . Some other results are harder to recognize; for instance [LV3,1.2] is essentially the same as (IV.10).

In Folkman and Lawrence [FL], the linear system model appears as a guide for the theory. So one sees there the presence of the whole set C and its partial order as an object of study. In particular, (IV.4) appears as the Lemma following Theorem 19, and the core of (IV.13) is Theorem 17 of [FL]. We shall have many more points of contact with [FL] in Chapter 4.

Recent work of Munson [Mu] contains a more detailed comparison between tope lattices and Las Vergnas lattices. In particular, she proves that the two classes of lattices are not the same. That is an interesting contrast to polytope lattices.

Munson's result is partly based on a construction of J.Lawrence (unpublished) of tope lattices which are not isomorphic to any polytope lattice.

CHAPTER 3

CONSTRUCTIBILITY OF THE MATROID COMPLEX

The main result of this chapter is Theorem (III.1) which asserts that oriented matroids are constructible. This is the stepping stone for achieving the topological representation of oriented matroids. The proof is spread over sections III-V, and the main part of it is the proof that tope-lattices are shellable.

Constructible and shellable complexes are defined and studied in section II. Section I introduces the notion of complexes. Those are little more than families of sets, and are basically an alternative to posets.

The last section refines the proof that topes are shellable with the help of the tope graph. We actually show that each tope can be proven shellable in several different ways - that is, it has several "shellings".

I

COMPLEXES

A complex is a finite collection K of sets called cells, such that the sets $\{i_K(p) \mid p \in K\}$ are pairwise disjoint, where $i_K(p) = p - \cup\{q \in K \mid q \subsetneq p\}$ is the interior of p in K. The space of K, denoted $s(K)$, is the union of all cells of K.

To every poset P, one associates the complex whose cells are the sets $[p] = \{q \in P \mid q \leq p\}$, $p \in P$. Given a set S and a function $f : S \to P$, the set of inverse images of cells of P is a complex; the interior of $f^{-1}([p])$ is $f^{-1}(p)$.

Example: Let A be a homogeneous linear system in $\mathbb{R}^n$, with oriented matroid C. Define $F : \mathbb{R}^n \to C$ by $f(w) = \sigma(Aw)$. Then the complex defined by f is the complex $C(A)$ of the linear system. □

Example: The complex of a poset P is determined by the identity function on P. □

Example: Ball-complex: it is a collection K of topological balls in $\mathbb{R}^n$, whose relative interiors are disjoint and such that the boundary of each member of K is a union of members of K (see fig. 3.1). Thus, for each $p \in K$, $i_K(p)$ equals its ball interior. Note that the definition is not through a map to a poset. Ball complexes are the main motivation for the introduction of the notion of complexes, and their role will become apparent in the next chapters.

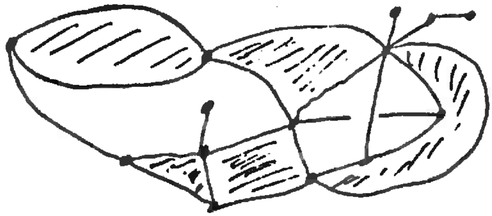


Figure 3.1

5. Example: The collection of faces of a polytope forms a complex. This is both a special type of ball-complex and a subcomplex (definition later) of the complex of any affine linear system associated with the given polytope. □

6. PROPOSITION: Let K be a finite collection of sets whose union is S. The following are equivalent.

(6.1) K is a complex.

(6.2) The intersection of any two members of K is a union of members of K (possibly empty).

(6.3) For each $x \in S$, the intersection of all members of K containing x is in K.

(6.4) For all $p \in K$, $x \in p$, $x \in i_K(p)$ iff every member of K containing x also contains p.

PROOF: (6.1) => (6.2) . Let $p,q \in K$ and $x \in p \cap q$. We must show that there exists $a \in K$ such that $x \in a \subseteq p \cap q$. Let p' be a cell, minimal so that $x \in p'$ and $p' \subseteq p$; clearly $x \in i_K(p')$. Obtain q' similarly; it also follows that $x \in i_K(q')$. So $i_K(p') \cap i_K(q') \neq \phi$, and since K is a complex, $p' = q'$. Thus $a = p'$ is as required.

(6.2) => (6.3) Clear.

(6.3) => (6.4) . Let q be the intersection of all members of K containing x . Clearly $q \subseteq p$, and $x \notin i_K(p)$ iff the inclusion is proper.

(6.4) => (6.1) . If $i_K(p) \cap i_K(q) \neq \phi$, then $p \subseteq q$ and $q \subseteq p$, by (6.4). □

The connection between complexes and posets goes both ways. A complex K , ordered by inclusion, is a poset, and we refer to it as "the poset of K" . Define the map f from $s(K)$ to the poset of K by $f(x)$ = the cell of K whose interior contains x (so $x \in f(x)$) .

7. PROPOSITION: The complex defined by f above is K , and indeed, for each $p \in K$, $f^{-1}([p]) = p$.

PROOF: Only the last statement needs verification, as it clearly implies the first. Let $p \in K$. If $x \in p$, then by (6.4) $f(x) \subseteq p$, hence $f(x) \in [p]$; thus $x \in f^{-1}([p])$. Conversely, if $x \in f^{-1}([p])$, then $f(x) \subseteq p$ and as $x \in f(x)$, $x \in p$. □

From this correspondence one can easily translate concepts from posets to complexes and vice-versa. For instance, a JD complex is one whose poset is JD . These translations will be made tacitly in what follows. Actually, most properties of complexes we study are properties of their partial order, and irrespective of the contents of cells.

An <u>order isomorphism</u> between two complexes K , L is an inclusion preserving bijection $f : K \to L$ (i.e. $p \subseteq q$ implies $f(p) \subseteq f(q)$) whose inverse is inclusion preserving. This is of course the same as an isomorphism of the posets of K and L . Without further comment, we speak of order isomorphisms between complexes and posets.

Let us introduce some terminology, to be soon used. A <u>subcomplex</u> of a complex K is a complex $L \subseteq K$, such that for every $p \in L$, $q \in K$ if $q \subseteq p$ then $q \in L$ (this is sometimes called an order ideal, for posets). Unions and intersection of subcomplexes are also subcomplexes. If $Q \subseteq K$, the subcomplex <u>induced</u> by Q is $[Q] = \{p \in K | p \subseteq q \text{ for some } q \in Q\}$, the intersection of all subcomplexes of K containing Q . In particular, if $Q = \phi$, then $[Q] = \phi$, even if K has an empty cell. Note that if $p \in K$, the subcomplex $[p]$ is just the cell corresponding to p in the poset of K .

A JD complex K is pure dimensional if all of its maximal cells have the same dimension; that is denoted $d(K)$, the dimension of K. If $d(K) = n$, we say that K is an n-complex, for short.

Example: An oriented matroid C is pure dimensional, by (2.IV.6). Every supercell and every flat of C is a pure dimensional subcomplex, by (2.V.3).

II) CONSTRUCTIBLE COMPLEXES

In the next sections we shall prove that oriented matroids are constructible. Here we introduce constructible complexes. Throughout this section, all complexes are supposed to be pure dimensional (thus JD).

For an n-dimensional complex K, let $\beta(K)$ denote the set of those (n-1)-cells which are covered by exactly one n-cell of K. The boundary of K is the subcomplex induced by $\beta(K)$, and is denoted ∂K. Thus:

$\partial K = \{p \in K \mid$ there exists $q \in K$, with $p \leq q$ and q covered by exactly one n-cell$\}$.

In particular, if K has only one maximal cell p, i.e. $K = [p]$, $\partial K = K - p$. Note that either $\partial K = \phi$ or $d(\partial K) = d(K) - 1$.

1. Let K be a pure n-dimensional complex.
   (1.a) K is S-constructible if $n = -1$ or if $n \geq 0$ and K is the union of two n-dimensional B-constructible subcomplexes, whose intersection is the boundary of each.
   (1.b) K is B-constructible if $n \geq 0$, and if either K has a unique maximal cell, with S-constructible boundary, or if K is the union of two n-dimensional B-constructible subcomplexes, whose intersection is the intersection of their boundaries and is B-constructible and (n-1)-dimensional.

We say that K is <u>constructible</u> if it is either S-constructible or B-constructible, and its <u>type</u> is S or B.

<u>Examples</u>: We characterize constructible complexes of small dimension, up to order isomorphism.

$d(K) = -1$: K has only one cell, and is S-constructible.

$d(K) = 0$: If K has only one 0-cell, its poset is represented by the Hasse diagram  . It is B-constructible, as $\partial[p] = \{\hat{0}\}$, which is S-constructible.

Other 0-dimensional complexes have diagrams:

With two maximal elements  , K is S-constructible, as $[p] \cap [q] = \{0\} = \partial[p] = \partial[q]$. It is clearly not B-constructible.

Now, if $d(K) = 0$ and K has more than two maximal cells, in any decomposition of K as the union of two 0-dimensional subcomplexes, at least one of these has at least two maximals. It follows that no such K is constructible.

$d(K) = 1$: If K has only one 1-cell, it is constructible iff ∂K is S-constructible; as $d(\partial K) = 0$, by the former case the only possibility for ∂K is $\emptyset$. Thus the only B-constructible 1-cell is  .

If K has more than one 1-cell, and is constructible, then every 1-cell subcomplex is constructible. As above, every one cell contains exactly two 0-cells. Define the graph Skel(K), whose vertices are the 0-cells and whose edges are the 1-cells, the ends of an edge being the two 0-cells it contains. It follows easily by induction that if K is B-constructible, then Skel(K) is a path whose end vertices are the only two 0-cells in $\beta(K)$. From this, if K is S-constructible, then Skel(K) is a polygon (digon also possible). The similarity with (2.IV.18) is not accidental; oriented matroids are S-constructible.

To any graph $G = (V,E)$ one can associate the complex:
$\mathcal{K}(G) = \{\phi\} \cup \{\{v\} | v \in V\} \cup \{\{e,v,w\} | e \in E , v,w \text{ the ends of } e\}$.
The converse of the statement in the last paragraph holds: if G is a path, $\mathcal{K}(G)$ is B-constructible, if G is a polygon $\mathcal{K}(G)$ is S-constructible. One also notices that $\mathcal{K}$(Skel(K)) is order isomorphic to K, if K is constructible. This characterizes constructible 1-complexes up to order isomorphism as $\mathcal{K}$(path) or $\mathcal{K}$(polygon). □

3. Example: Consider the planar map in Fig. (3.2).

The letters denote the respective faces.

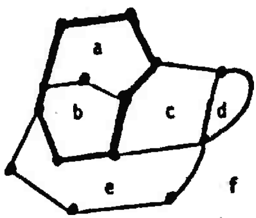


Fig. (3.2)

Consider the complex K whose cells are the faces, edges, vertices (as subsets of the plane) and the empty set. Clearly K is 2-dimensional.

By the discussion about 1-dimensional complexes, each 2-cell of K is B-constructible. Now, the subcomplex [a,b] induced by {a,b} is B-constructible, since [a,b] = [a]∪[b] and [a]∩[b] is a path in the boundary of both. Similarly, [f,d] , [c,e] are B-constructible and so is their union, [c,d,e,f] . Since the intersection [a,b]∩[c,d,e,f] is the polygon in dark lines, which is the boundary of both, it follows that K is S-constructible.

Indeed, every S-constructible 2-complex can be obtained (up to order isomorphism) from a planar map in this way, and vice-versa, provided the boundary of each face of the map is a polygon. This is a special case of the theorem to be proved in chapter 5 that every constructible complex is order isomorphic to a ball complex C, such that $s(C)$ is a topological ball or sphere. □

We now develop a few properties of constructible complexes. For convenience, if a n-complex K is the union of two n-subcomplexes K_1, K_2 such that $K_1 \cap K_2 = \partial K_1 \cap \partial K_2$, we say that K results from <u>pasting</u> K_1 and K_2.

4. PROPOSITION: Let K be an n-dimensional constructible complex. Then:

(4.1) Every cell of K is B-constructible, with the exception of the (-1)-cell.

(4.2) If K is obtained by pasting n-subcomplexes K_1 and K_2, then $\beta(K) = \beta(K_1) \Delta \beta(K_2)$, where Δ denotes symmetric difference.

(4.3) Every (n-1)-cell of K lies in at most two n-cells of K.

(4.4) If K is S-constructible then $\beta(K) = \phi$; thus every (n-1)-cell of K is a face of exactly two n-cells.

(4.5) If p, q are cells, with $p \subseteq q$, $d(q) = d(p) + 2$, then there exist exactly two cells between p and q.

PROOF: (4.1) is clear.

(4.2): Note first that as $K_1 \cap K_2$ has dimension (n-1), K_1 and K_2 have no n-cell in common. Let p be an (n-1)-cell of K and k, k_1, k_2 be respectively the number of n-cells of K, K_1, K_2 which cover p. By the argument above $k = k_1 + k_2$. Thus as $p \in \beta(K)$ iff $k = 1$, $p \in \beta(K)$ iff $(k_1, k_2) = (0,1)$, that is, iff p is in exactly one of $\beta(K_1)$, $\beta(K_2)$, which is the same as saying $p \in \beta(K_1) \Delta \beta(K_2)$.

(4.3) Proof by induction in the number of n-cells of K. If this number is 1, the result is obvious. Otherwise, K is obtained by pasting K_1 and K_2, for which the result holds by the induction hypothesis. Let p by an $(n-1)$-cell of K. If $p \in K_1 \cap K_2 = \partial K_1 \cap \partial K_2$, it follows that $p \in \beta(K_1) \cap \beta(K_2)$.

As $p \in \beta(K_i)$ $i = 1,2$, there is exactly one n-cell in K_1 and one in K_2 covering p, and this gives exactly two n-cells of K covering p, so $p \notin \beta(K)$. Otherwise p is in exactly one of K_1 and K_2, say K_1: any n-cell of K covering p must be also in K_1, since K_2 is a subcomplex, and there are at most two of these n-cells.

(4.4) We have that $K = K_1 \cup K_2$, where $\partial K_1 = \partial K_2$, K_1, K_2 n-subcomplexes. Therefore, $\beta(K_1) = \beta(K_2)$ and from (4.2), $\beta(K) = \beta(K_1) \Delta \beta(K_2) = \phi$. The remaining of the statement follows from (4.3).

(4.5) By (4.1), $\partial[q]$ is S-constructible. Since $p \in \partial[q]$ and its dimension is $d(\partial[q])-1$, the result follows from (4.4). □

As one constructs a complex by pasting, the boundary is also being "constructed" in some sense. We have not been able to prove that the boundary is constructible, though. Still, examples and the theorems associating constructible complexes with balls and spheres point towards an affirmative answer for the question:

Is the boundary of a B-constructible complex S-constructible?

One of the consequences of the topological results is a formula for the Euler characteristic of constructible complexes. That is, however, directly derivable from constructibility, as we see here.

The Euler characteristic of a JD- complex K is:

$$\chi(K) = \sum_{i=0}^{d(K)} (-1)^i . f_i(K)$$
$$= \sum \{(-1)^{d(p)} | p \in K - \emptyset\},$$

where $f_i(K)$ is the number of i-cells of K.

5. PROPOSITION: $\chi(K) = 1$ if K is B-constructible, $\chi(K) = 1 + (-1)^{d(K)}$ if K is S-constructible.

PROOF: We prove this together with $\chi(\partial K) = 1 + (-1)^{d(K)-1}$, if K is B-constructible. The proof is by induction in the dimension and then number of cells of K. True if $d(K) = -1$. For the rest, the following formulae are easily verified:

a) if K has only one maximal cell,
$\chi(K) = (-1)^{d(K)} + \chi(\partial K)$.

b) if K is obtained by pasting K_1 and K_2, with $K_1 \cap K_2$ B-constructible of dimension $d(K)-1$, 3.11

$\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)$;

$\chi(\partial K) = \chi(\partial K_1) + \chi(\partial K_2) - 2\chi(K_1 \cap K_2) + \chi(\partial(K_1 \cap K_2))$.

c) if K is obtained by pasting K_1 and K_2, $K_1 \cap K_2 = \partial K_1 = \partial K_1$,

$\chi(K) = \chi(K_1) + \chi(K_2) - \chi(K_1 \cap K_2)$.

Each formula expresses the characteristic of a complex in terms the characteristics of smaller complexes. Substitution of their val-s as assumed inductively will complete the inductive step. □

. COROLLARY: No complex is both B-constructible and S-constructible. □

Motivated by (4.3) we introduce a new definition: A complex is PM if it is pure dimensional, say, of dimension n, and every (n-1)-cell is covered by at most two n-cells. The "PM" stands to resemble "pseudo-manifolds", which are PM simplicial complexes.

/. Examples: Every JD complex with a unique maximal cell is PM. From (4.3), every constructible complex is PM. If C is an oriented matroid complex, (2.IV.9) states that C is PM, and (2.IV.13) implies that for every cell [X] of C, ∂[X] is PM. □

If we want to show that a complex is constructible, it helps knowing in advance that it is PM. This is indicated by Proposition (9). We first prove:

8. LEMMA: Suppose that K is a PM-complex of dimension n, which is the union of n-subcomplexes K_1, K_2 whose intersection is (pure) (n-1)-dimensional. Then

(8.1) $\beta(K) = \beta(K_1) \Delta \beta(K_2)$

(8.2) $K_1 \cap K_2 = \partial K_1 \cap \partial K_2$

(8.3) $\partial K = \phi$ iff $K_1 \cap K_2 = \partial K_1 = \partial K_2$.

PROOF: (8.1) The proof is the same as in (4.2).
(8.2) We only need to show that $K_1 \cap K_2 \subseteq K_1 \cap \partial K_2$, as the reverse inclusion is trivial. As both $K_1 \cap K_2$ and $\partial K_1 \cap \partial K_2$ are subcomplexes of K , it is enough to show that every maximal cell of $K_1 \cap K_2$ is in $\partial K_1 \cap \partial K_2$.

Let p be a maximal cell of $K_1 \cap K_1$. Then $d(p) = n-1$ by hypothesis. Since $d(K_1) = d(K_2) = n$, there exist n-cells $q_1 \in K_1$, $q_2 \in K_2$, both containing p . As K_1 and K_2 have no n-cell in common, $q_1 \neq q_2$. But then q_1 and q_2 are the only n-cells of K containing p , as K is PM . Thus p is covered by only one cell of K_1 , so $p \in \beta(K_1)$; similarly $p \in \beta(K_2)$. Hence $p \in \beta(K_1) \cap \beta(K_2) \subseteq \partial K_1 \cap \partial K_2$.
(8.3) From (8.1), $\partial K = \phi$ iff $\beta(K_1) = \beta(K_2)$. That is iff $\partial K_1 = \partial K_2$. The result follows from (8.2) . □

9. PROPOSITION: Suppose that a PM complex K of dimension n is the union of two B-constructible n-subcomplexes K_1, K_2 whose intersection has dimension $n-1$. Then K is B-constructible provided $K_1 \cap K_2$ is so; K is S-constructible provided $\partial K = \phi$.

PROOF: The B case follows from (8.2); the S case follows from (8.3). □

Shellable complexes are a special subclass of constructible complexes, which deserve some separate interest:

An n-complex K is S-shellable if $n = -1$ or if $n \geq 0$, and $K = K_1 \cup [p]$, where $[p]$ is a B-shellable n-cell of K, K_1 is a B-shellable n-subcomplex of K, and $K_1 \cap [p] = \partial K_1 = \partial [p]$.

An n-complex K is B-shellable ($n \geq 0$) if either $K = [p]$ and ∂K is S-shellable or if $K = K_1 \cup [p]$ where K_1, $[p]$ are B-shellable n-subcomplexes (p an n-cell of K), such that $K_1 \cap [p]$ is B-shellable of dimension n-1.

In section VII we present a less recursive alternative to this definition. We conclude with a counterpart of (9).

10. PROPOSITION: Suppose that K is n-dimensionl PM, K_1 is a B-shellable subcomplex, $[p]$ a B-shellable n-cell such that, $K = K_1 \cup [p]$ and $K_1 \cap [p]$ is (n-1)-dimensional. Then K is B-shellable provided $K_1 \cap [p]$ is so; K is S-shellable if $\partial K = \phi$.

PROOF: It follows from (8) as before. □

III) THE MATROID CONSTRUCTIBILITY THEOREM

1. THEOREM: Let C be an oriented matroid. Then

(1.1) Every nonzero cell of C is B-shellable.

(1.2) Every supercell of C is B-constructible.

(1.3) Every flat of C is S-constructible; in particular, C is S-constructible.

We shall prove this theorem in two steps. In this section, we assume (1.1), and proceed with the proof of (1.2) and (1.3). The proof of (1.1) will be presented in the next sections.

Let us first establish (2), the basic property of supercells which entail their constructibility. Recall that a supercell can be represented as $C(W,S) = \{X \in C \mid X \backslash S \leq W \backslash S\}$, where S is a set of equators and $W \in C$, $\underline{W} - S \neq \phi$. A <u>supertope</u> is a supercell which contains a tope, i.e., a supercell representable as $C(T,S)$, where T is a tope.

2. PROPOSITION: Let K be a supertope of C containing at least two topes.

Then there exist supertopes K_1, K_2 such that $K = K_1 \cup K_2$ and $K_1 \cap K_2$ is a supercell of dimension $d(C) - 1$.

PROOF: The order isomorphism in (2.III.7) from C to a simplification C' induces a bijection between supercells of C and those of C'.

Thus, it is enough to consider the case where C is simple.

Let T^1, T^2 be topes in K. We may represent K as $C(T^1,S) = C(T^2,S)$ for some nonempty set S. Note that every equator separating T^1 and T^2 is in S. By (2.IV.10), one of these equators, call it e, is a facet equator of T^1. Call X the corresponding facet of T^1; thus $X_e = 0$, $X_f = T^1_f$ for every $f \neq e$. Let $S' = S - e$, $K_1 = C(T^1,S')$, $K_2 = C(T^2,S')$.

Clearly $K_1 = \{W \in K \mid W_e = T^1_e \text{ or } 0\}$, $K_2 = \{W \in K \mid W_e = -T^1_e \text{ or } 0\}$; thus $K = K_1 \cup K_2$. Since $T^1 \in K_1$, $T^2 \in K_2$, both these supercells are supertopes. Also, $K_1 \cap K_2 = \{W \in K \mid W_e = 0\} = C(X,S')$. Thus $K_1 \cap K_2$ is a supercell, of dimension $\leq d(C)-1$, as it is contained in C/e. Indeed, as $X \in K_1 \cap K_2$ and $d(X) = d(C)-1$, it follows that $d(K_1 \cap K_2) = d(C)-1$. □

PROOF of (1.2) and (1.3) modulo (1.1).

The results are true if $d(C) = -1$ ($C = \{\underline{0}\}$). We assume inductively the results for all oriented matroids of dimension $< d(C)$. Now, every supercell which is not a supertope is a supercell of some contraction of C; also every proper flat of C is an oriented matroid of smaller dimension. Thus, the inductive hypothesis handles these supercells and flats, and we only have to worry about proving (1.2) for supertopes and (1.3) for the flat C.

(1.2) A supertope which contains only one tope is a tope cell, thus B-constructible by the assumption of (1.1). An induction in the number of topes will handle other supertopes, by a combination of Propositions (2) and (II.9).

(1.3) Let e be an equator. Let $K_1 = \mathcal{C}_e^+ = \{X \in \mathcal{C} \mid X_e = + \text{ or } X_e = 0\}$, $K_2 = \mathcal{C}_e^-$. Then K_1 and K_2 are supertopes, and $K_1 \cap K_2 = \mathcal{C}/e$ has dimension one less than $d(\mathcal{C})$. This is the hypothesis of Proposition (II.9), thus $\mathcal{C}$ is S-constructible. □

IV) ROOTING

The usefulness of simplification in proving statements about oriented matroid complexes was apparent in section (2.IV) and in Proposition (III.2). Next section we are going to discuss individual cells, and a further simplification is possible and useful.

To motivate the idea to be presented, let us consider a homogeneous linear system A in $\mathbb{R}^n$, with oriented matroid C on E. Consider the cone $P = \{w \in \mathbb{R}^n | Aw \geq 0\}$, and suppose for simplicity that there is a $w^0 \in \mathbb{R}^n$ such that $Aw^0 > 0$ (i.e., $\underset{\sim}{+} \in C$). Also suppose that C is simple, i.e., no row of A is a scalar multiple of another. An equator e is a facet equator of $\underset{\sim}{+}$ iff there exists $w \in \mathbb{R}^n$ such that $A_e w = 0$ and $A_f w > 0$ for every $f \neq e$; in this case, $\{w \in P | A_e w = 0\}$ is a facet of P. Now, consider an equator e which is not a facet equator, and let A' be the matrix obtained by removing A_e from A. The cone $P' = \{w \in \mathbb{R}^n | A'w \geq 0\}$ contains P; the fact that e does not define a facet of P implies that $P' = P$. For if $w^1 \in P' - P$, then $A'w^1 \geq 0$ and $A_e w^1 < 0$; but then $w^2 = |A_e w^1| w^0 + |A_e w^0| w^1$ satisfies: $A_e w^2 = 0$, $A'w^2 > 0$, contradicting the choice of e. That is, as far as P is concerned, e is irrelevant. Thus we obtain the well known result:

1. PROPOSITION: Let $P = \{w \in \mathbb{R}^n | Aw \geq 0\}$, and suppose that there exists a $w^0 \in \mathbb{R}^n$ such that $Aw^0 > 0$. Let B be the submatrix of A comprised of the rows of A which define facets of P. Then $P = \{w \in \mathbb{R}^n | Bw \geq 0\}$. □

This result generalizes to oriented matroids from the form "the face-lattice of P is preserved upon deletion of non-facet inequalities". It appears in [LV3] as Theorem (2.2).

2. THEOREM: Let T be a tope of a simple oriented matroid C on E and let F be the set of facet equators of T, $S = E - F$. Let $\alpha: C \to C\backslash S$ be the map $\alpha(X) = X\backslash S$. Then:

(2.1) For all $W \in C$, if $\alpha(W) \le \alpha(T)$, then $W \le T$. Equivalently, $\alpha^{-1}(\alpha([T])) = [T]$.

(2.2) The restriction of α to $[T]$ is an order isomorphism with the cell $[T\backslash S]$ of $C\backslash S$.

(2.3) All equators of $C\backslash S$ are facet equators of $T\backslash S$.

PROOF: We shall repeatedly use (2.IV.10): for T,T' topes of C, some equator separating T and T' is a facet equator of T.

(2.1) Let $W \in C$ and suppose that $\alpha(W) \le \alpha(T)$ but W is not a face of T. Then some member of S separates T from W. But then, $W.T$ is a tope of C, distinct from T, and separated from T only by members of S. By (2.IV.10), one member of S is a facet equator of T, contradicting the definition of S. This contradiction shows that W is a face of T.

(2.2) Clearly α maps $[T]$ into $[T\backslash S]$ and is order preserving. From (2.1), it is also onto. Let us show that it is 1-1:

Suppose that this is not the case, so there exist distinct

$X,Y \in [T]$ such that $X\backslash S = Y\backslash S$. Choose $e \in S$ such that $X_e \neq Y_e$; neither of X_e, Y_e can be $-T_e$, so, without loss of generality, $X_e = 0$, $Y_e = T_e$. Let $T' = X.(-Y).T$. Then $T'_e = -T_e$, so $T' \neq T$, and T' is separated from T only by $S \cap \underline{Y}$. Appealing to (2.IV.10) we obtain a contradiction. With this we have concluded that α is bijective.

It remains to be shown now that for $X,Y \in [T]$, if $\alpha(X) \leq \alpha(Y)$, then $X \leq Y$. It follows as above that if we assume that the conclusion fails for particular X,Y , one gets a contradiction by considering the tope $Y.(-X).T$. This completes the proof of (2.2).

(2.3) From (2.2), the facets of $T\backslash S$ are precisely $\{ X\backslash S \mid X$ a facet of $T\}$. Since C and $C\backslash S$ are simple, the result follows from the correspondence between facets and facet equators. □

An oriented matroid with the property that every element is a facet equator of a tope T is said to be <u>rooted at</u> T . Given any oriented matroid C and a tope T , by first simplifying and then deleting all elements which are not facet equators of the tope corresponding to T , we obtain an oriented matroid C' rooted at a tope T' , such that $[T']$ is isomorphic to $[T]$. Thus the study of tope lattices can be restricted to rooted oriented matroids. Note that these matroids are always simple.

3. PROPOSITION: If C is rooted, then for every equator e, C/e is simple.

PROOF: Since C is itself simple, C/e has no loops. Let now f, g be arbitrary equators of C/e; we shall show that they do not coincide.

By a suitable reversal of signs, we may assume that the root of C is $\underset{\sim}{+}$. Call X, Y, Z the facets of $\underset{\sim}{+}$ supported respectively by the equators f, g, e. Then $X_e = Y_e = +$, $X_g = Z_g = +$, $Y_f = Z_f = +$, and $X_f = Y_g = Z_e = 0$. Eliminating e between X and $-Y$, we obtain a W such that $W_e = 0$, $W_g = +$, $W_f = -$. As both W and Z are vectors of C/e, f and g do not coincide there. □

V) UMBRELLAS AND SHELLABILITY OF TOPES

To prove that oriented matroid cells are shellable, we need to specify subcomplexes which are going to be "shelled" in the intermediate steps of the proof. For this purpose we introduce umbrella subcomplexes. In the same vein as the proof in section III reduced to the consideration of supertopes only, here we shall be confining ourselves to topes primarily.

Let C be a simple oriented matroid, with topes T and T'. Denote by $U(T,T')$ the collection of facets of T supported by equators separating T and T'. Thus, $U(T,T) = \phi$, $U(T,-T) =$ {all facets of T}. When $T' \neq T$ or $-T$, $U(T,T')$ is called an <u>umbrella</u> of T. A subcomplex induced by an umbrella is an <u>umbrella subcomplex</u> of T.

1. THEOREM: Let T be a tope of the oriented matroid C, $d(C) \geq 0$.

   Then:

   (1.1) Every umbrella subcomplex of T is B-shellable.

   (1.2) The boundary of T is S-shellable.

   (1.3) The cell $[T]$ is B-shellable.

We need some preparation before (1) is proved. First:

2. PROPOSITION: With the notation of (IV.2), α defines a bijection between umbrellas of T and of $T\backslash S$, by $U(T,T') \mapsto U(\alpha(T),\alpha(T'))$.

PROOF: Immediate. □

When C is rooted, umbrellas and topes are in a bijective correspondence:

3. PROPOSITION: If C is rooted at T, and $U(T,T') = U(T,T'')$, then $T' = T''$.

PROOF: Let $\{F^1, F^2, \ldots, F^k\} = U(T,T')$ and e_i be the equator supporting F^i, for $i = 1,2,\ldots,k$. By the definition of $U(T,T')$, $e_1, e_2, \ldots, e_k$ are the facet equators of T separating it from T'. As C is rooted at T, these are precisely the equators separating T and T'. Since T'' is separated from T by the same equators, $T'' = T'$. □

Note the similarity between the next two results and (III.2). Theorem (2.IV.10) appears again crucially.

4. THEOREM: Suppose that T, T' are topes of C and $|U(T,T')| \geq 2$. Then, for some $F \in U(T,T')$, $U(T,T')-F$ is an umbrella of T (even if $T' = -T$).

PROOF: By (2) we can suppose that C is rooted at T.

By (2.IV.10) there is a facet equator e of T' separating T and T'; thus ${}_{\bar{e}}T'$ is also a tope. Clearly $U(T,{}_{\bar{e}}T') = U(T,T')-F$, where F is the facet of T supported by e. □

5. THEOREM: Assume that C is rooted at T. Let $U(T,T')$ be an umbrella of T and suppose that F is a facet of T not in $U(T,T')$, such that $U(T,T')+F$ is also an umbrella of T. Then the collection of faces of F which lie in the subcomplex induced by $U(T,T')$ is an umbrella subcomplex of F in C/e, where e is the supporting equator of F.

(Remark: Notice that F is a tope of C/e, which is simple by (IV.3), so we are entitled to speak about umbrellas of F in C/e.)

PROOF: Let T'' be a tope such that $U(T,T'') = U(T,T')+F$. As C is rooted at T, by Proposition (3), T'' and T' are separated only by e. Thus, e is also a facet equator of T' and $F' = T' \backslash e$ is a tope of C/e. Furthermore, as $U(T,T')$ and $U(T,T'')$ are umbrellas of T, both T and T' differ from T and $-T$, hence F' differs from F and $-F$. Let $\mathcal{W}$ be the subcomplex of C/e induced by $U(F,F')$, and let $\mathcal{U} = [U(T,T')]$. We shall prove that $\mathcal{U} \cap [F] = \mathcal{W}$.

$\mathcal{W} \subseteq \mathcal{U} \cap [F]$. It is enough to show that $U(F,F') \subseteq \mathcal{U} \cap [F]$. Let $Y \in U(F,F')$ and f be the facet equator of F supporting it. Then $F'_f = F_f$, by the definition of $U(F,F')$, and hence $T'_f = -T_f$. It follows that the facet W of T supported by f is in $U(T,T')$. As $Y < T$ and $Y_f = 0$, $Y \leq W$. Thus $Y \in \mathcal{U}$.

$\mathcal{U} \cap [F] \subseteq \mathcal{W}$; Let $Y \in \mathcal{U} \cap [F]$. Then $Y_f = 0$ for some equator $f \neq e$ separating T and T'. This f must also separate

F and F', thus $Y.F = F$. All equators separating F and $Y.F'$ are in $E-\underline{Y}$ and separate F from F'. By (2.IV.10), some facet equator g of F separates it from $Y.F'$, thus from F'. Then $F\backslash g \in U(F,F')$ and as $Y_g = 0$, $Y \leq F\backslash g$, implying $Y \in \mathcal{U}$. □

These last two results can be put together now for a proof of Theorem (1).

6. PROOF of (1): By (2.IV.13), $\partial[T]$ is PM with empty boundary; thus we may use (II.10) to prove (1.1) and (1.2). The whole proof is by induction in the dimension n of C. The case $n = 0$ is trivial thus we may assume $n > 0$, and that the theorem holds for oriented matroids of dimension $< n$. In particular, for every facet F of T, $[F]$ is B-shellable, as F is a tope of a flat of C. Note that $\partial[T]$ has dimension $n-1$.

(1.1) By (2) and (IV.2) we may assume that C is rooted at T. Let $U(T,T'')$ be an umbrella of T and $\mathcal{U} = [U(T,T'')]$. If $U(T,T''$ has just one facet of T, $\mathcal{U} = [F]$, thus is B-shellable by the induction.

We proceed with an additional induction on $k = |U(T,T'')|$. From Theorem (4), when $k \geq 2$ $U(T,T'') - F$ is an umbrella $U(T,T')$ of T for some $F \in U(T,T'')$. Let $\mathcal{U} = [U(T,T'')]$. By Theorem (5) $\mathcal{W} \cap [F]$ is an umbrella subcomplex of F in C/e, for the correct e. As $d(C/e) = n-1$, that intersection has dimension $n-2$, and is B-shellable due to its low dimension and induction. Since $|U(T,T')| = k-1$, $\mathcal{W}$ is also B-shellable, thus by (II.10) so is $U = W \cup [F]$.

(1.2) As $\partial[T] = U(T,-T)$, there is by Theorem (4) a facet F of T such that $\partial[T] - F$ is an umbrella $U(T,T')$. By (1.1), $[U(T,T')]$ is B-shellable, and as $[F]$ is shellable, (II.10) shows that $\partial[T] = [U(T,T')] \cup [F]$ is S-shellable.

(1.3) is immediate from (1.2). □

More simply restated:

7. THEOREM: Every tope lattice is B-shellable. □

8. COROLLARY: Every polytope lattice is B-shellable. □

Wee shall later need:

9. PROPOSITION: Let X be a nonzero proper face of a tope T . Then the subcomplex of $\partial[T]$ induced by the facets of T containing X is B-shellable.

PROOF: We can assume that C is simple. Now, since $X \neq \underline{0}, T$, $-X.T \neq T,-T$. It is easy to see that the complex in question is induced by $U(T,-X.T)$, whence the result follows from (1.1). □

When we talk about tope lattices a certain care must be taken about umbrellas. It is conceivable that a definition of umbrella subcomplexes in a tope lattice could be provided in an intrinsic manner, without reference to other topes of an ambient oriented matroid.

However, this cannot be done. We present an example of two rooted oriented matroids, whose roots have isomorphic lattices, with an umbrella of one which does not correspond to any umbrella of the second, in any order isomorphism.

10. EXAMPLE: Consider the cube Q in $\mathbb{R}^3$ described by the system of inequalities: $-1 \le w_1, w_2, w_3 \le 1$. Label its facets $a_1, a_2, a_3, b_1, b_2, b_3$ so that a_i is supported by $w_i = -1$, and b_i is supported by $w_i = 1$. Homogeinizing this polytope, we obtain a cone in $\mathbb{R}^4$ described by:

$$\left.\begin{array}{lllllll} a_1 & : & w_1 & & & + w_4 & \ge 0 \\ b_1 & : & -w_1 & & & + w_4 & \ge 0 \\ a_2 & : & & w_2 & & + w_4 & \ge 0 \\ b_2 & : & & -w_2 & & + w_4 & \ge 0 \\ a_3 & : & & & w_3 & + w_4 & \ge 0 \\ b_3 & : & & & -w_3 & + w_4 & \ge 0 \end{array}\right\} Aw \ge 0$$

The lattice of this cone is isomorphic to that of the cube. Consider the oriented matroid C associated to the linear system A. The vector $\underset{\sim}{+}$ is a root tope of this oriented matroid, and its lattice is isomorphic to that of the cube. Thus we have a bijection between topes of C and umbrellas of $\underset{\sim}{+}$.

Let us show that no set consisting of two opposite faces of $\underset{\sim}{+}$ plus a third face is an umbrella of $\underset{\sim}{+}$. By symmetry, it is enough to show that $\{a_1,b_1,a_2\}$ is not an umbrella. If it was an umbrella, the vector ---+++ would be a tope of C. That is, there would exist a vector w such that

$$\begin{aligned}
w_1 \quad\quad\quad &+ w_4 < 0 \\
-w_1 \quad\quad\quad &+ w_4 < 0 \\
w_2 \quad\quad &+ w_4 < 0 \\
-w_2 \quad\quad &+ w_4 > 0 \\
w_3 &+ w_4 > 0 \\
-w_3 &+ w_4 > 0 .
\end{aligned}$$

Adding together the first two inequalities and separately the last two, we immediately conclude that there is no such w.

Let us now tilt a face of Q, substituting the inequality $w_1 \geq -1$ by $w_1 + w_2 \geq -1$. Homogeneizing the new system, we obtain a cone $A'w \geq 0$, where A' results from replacing the inequality a_1 by: $w_1 + w_2 + w_4 \geq 0$.

Again the oriented matroid of A' is rooted at $\underset{\sim}{+}$ and the lattice of $\underset{\sim}{+}$ is isomorphic to that of Q in an obvious way. Note that any such lattice isomorphisms sends a pair of opposite faces to a pair of opposite faces.

Now:

$$A'(2,-4,0,1)^t = (-1,-1,-3,5,1,1)^t .$$

which shows that ---+++ is a tope of the matroid of A' . Thus $\{a_1,b_1,a_2\}$ is an umbrella of $\pm$ in this oriented matroid.

VI) SHELLINGS AND THE TOPE GRAPH

In a proof that a complex is shellable, one has to remove one maximal cell at a time, so that the resulting complex at each step is shellable. One can conveniently look at the reverse process, like adding the maximal cells one at a time. This is the more usual way in which shellability is presented.

A shelling of a pure n-dimensional JD complex is an ordering $F_1, F_2, \ldots, F_k$ of its maximal cells such that for each $1 \le i \le k$, F_i intersects the subcomplex generated by $F_1, \ldots, F_{k-1}$ in a shellable (n-1)-dimensional subcomplex, which is of type B for each $i < k$.

One may redefine a complex to be shellable in terms of shellings: A PM complex of dimension n is shellable if the boundary of each maximal cell is S-shellable and it admits a shelling. The type of K is S or B accordingly as $\partial K = \phi$ or not.

The definition above is easily seen to be equivalent to that in section II.

This section studies a special class of shellings the boundary of a tope supports.

1. THEOREM: Let T be a tope in an oriented matroid C and suppose the facets of T are given an ordering $F^1, F^2, \ldots, F^k$. If for $1 \le i < k$, $\{F^1, \ldots, F^k\}$ is an umbrella of T, then the given ordering is a shelling of ∂T.

PROOF: This follows from the way (V.1) was proved. □

An umbrella shelling of a tope T is a shelling of its boundary through umbrellas, in the sense of the Theorem above. These shellings can be encoded nicely by the tope graph of C, with edges coloured by equators.

Recall that the tope graph of C at T, TG(C,T) has for vertices the topes of C, and the edges are as follows: there is an edge directed from T^1 to T^2 if there is a unique equator e separating them and $T^1_e = T_e = -T^2_e$. We colour this edge with e.

2. PROPOSITION: Suppose that C is rooted at T. Let $e_1, e_2, \ldots, e_k$ be an ordering of the equators of C, and denote by F^i the facet of T supported by e_i, $i = 1,2,\ldots,k$. Then $F^1, F^2, \ldots, F^k$ is an umbrella shelling of T iff there is a path from T to -T in TG(C,T) in which the edges are coloured in order $e_1, e_2, \ldots, e_k$.

PROOF: It follows from the obvious fact that $U(T,T') = U(T,T'') \cup F$ iff there is an edge in TG(C,T) from T' to T", coloured e, where e is the equator supporting F. □

There are several such paths in TG(C,T). Denote for each tope Y of C by $\gamma(Y)$ the set of equators separating it from T.

. LEMMA: Let $T^1, T^2, \ldots, T^k$ be a sequence of topes of a simple oriented matroid C, such that $\gamma(T^1) \subset \gamma(T^2) \subset \ldots \subset \gamma(T^k)$. Then there is a path from T to $-T$ in $TG(C,T)$ that goes through $T^1, T^2, \ldots, T^k$ in this order.

PROOF: The statement and proof becomes clearer if we first do a change of signs, so that T becomes $\underline{+}$. Maintaining the labels of topes after the change of signs, we have now that $\gamma(T') = \{f \in E | T'_f = -\}$.

We can stick T to the beginning of the given sequence and $-T$ to its end, if they are not there and still have a sequence with the same property, so we assume without loss that $T^1 = T$, $T^k = -T$.

If for some i, $|\gamma(T^i)| < |\gamma(T^{i+1})|-1$, by (2.IV.10) there is an $f \in \gamma(T^{i+1}) - \gamma(T^i)$ such that $W = {}_{\bar{f}}T^{i+1}$ is a tope. Let us insert W in the sequence between T^i and T^{i+1}. As $\gamma(T^i) \subsetneq \gamma(W) \subsetneq \gamma(T^{i+1})$, the new sequence satisfies the same hypothesis. Applying this argument enough times we finally reach a sequence $T = W^1, W^2, \ldots, W^n = -T$, of which the original one is a subsequence, and such that for each $i < n$, $|\gamma(W^{i+1})| = |\gamma(W^i)|+1$. This sequence clearly determines a path in $TG(C,T)$. □

4. THEOREM: Let L be a tope lattice of dimension n. Let $\underline{0} = K^0 < K^1 < \ldots < K^{n-1} < K^n = L$ be a chain of faces of L and $F^0, F^1, \ldots, F^{n-1}$ be a sequence of facets of L such that

$K^{n-1-i} \leq F^i$ but K^{n-i} is not a face of F^i $(0 \leq i \leq n-1)$. For $2 \leq i \leq n-1$, let F_i denote the set of facets Y of L such that $K^{n-1-i} \leq Y$, $K^{n-i} \nleq Y$ and $Y \neq F^i$. Then L has a shelling following the scheme:

$$F^0 , F^1 , F_2 , F^2 , \ldots , F_{n-2} , F^{n-2} , F_{n-1} , F^{n-1} .$$

Furthermore, for every tope T of an oriented matroid such th $[T]$ is isomorphic to L, such a shelling can be obtained as an umbrella shelling of T.

PROOF: Let C be an oriented matroid with a tope isomorphic to the given lattice. We may assume that C is rooted at this tope, as umbrella shellings need only be considered in rooted matroids. We call T this tope and identify the faces of the given lattice with the faces of T to which they correspond by the isomorphism $L \to [T]$.

Let $T^i = K^{n-1-i}.(-T)$ $(0 \leq i \leq n-1)$, and let $W^i = K^{n-1-i}.(-F^i).T$ for $2 \leq i \leq n-1$. Note that $W^i = \overline{e_i} T^i$ where e_i is the equator supporting F^i.

The sequence of topes

$$T, T', T^1, W^2, T^2, W^3, T^3, \ldots, W^{n-2}, T^{n-2}, W^{n-1}, T^{n-1} = -T$$

satisfies

$$\gamma(T) \subsetneq \gamma(T^0) \subsetneq \gamma(T^1) \subsetneq \gamma(W^2) \subsetneq \ldots \subsetneq \gamma(W^{n-1}) \subsetneq \gamma(T^{n-1}) .$$

By Lemma (3), there is a directed path from T to $-T$ in $TG(C,T)$ going through these topes in this order; this path determines an umbrella shelling of T by (2) . It is routine to check that

$$U(T,T^0) = \{F^0\},\ U(T,T^1) = \{F^0,F^1\} \text{ and, for } 2 \le i \le n-1 ,$$
$$U(T,W^i) = U(T,T^{i-1}) \setminus F_i \quad ; U(T,T^i) = U(T,W^i) + F^i ,$$

which is what is required. □

VII) REMARKS

The presentation of constructibility as done here, for complexes, seems to be new, although similar definitions appear in Hochster [Ho] and Danaraj and Klee [DK1] . In these articles the definition is restricted to complexes in which the cells are polytopes and the intersection of any two cells is a face of both; the possibility of an entirely combinatorial conceptualization is not considered.

In most occurrences of "constructibility" in the literature (e.g., Stanley [St1]) the definition is given for simplicial complexes only, and in a more general form than (I.10) which does not differentiate types B and S and does not imply PM-ness. That definition properly includes the classes of complexes we have called constructible.

Shellable complexes are by far more explored than the more general constructible ones (see [DK2]) . Again here simplicial complexes get most of the attention. In [BM] and [DK1], complexes whose cells are polytopes are considered for shellability.

Bruguesser and Mani [BM] proved that polytopes are shellable. Their motivation was to provide an elementary computation of the Euler characteristic, according to the proof of (II.5). The idea behind this goes back to the last century, in the work of Schlafli. Our "umbrella proof" is essentially a combinatorialization of the ideas of [BM] .

3.VII

Theorem (IV.4) is due to Danaraj and Klee [DK1] for polytopes. We feel that there is a real simplification here in the use of the combinatorial method; the proof in [DK1] is a considerable tour-de-force of geometry.

An important point about this theorem refers to the still unsolved problem of characterizing polytope lattices. The shellings asserted by the theorem form possibly the strongest set of conditions known to be satisfied by polytope lattices. By proving them for tope lattices, we ensure that they do not suffice for a characterization of polytope lattices, as not all tope lattices are polytope lattices.

Another proof that the Euler characteristic of a tope lattice is 1 appears in[CLM].

CHAPTER 4

SPHERE SYSTEMS I

I) INTRODUCTION

Linear systems were presented as the basic geometrical motivation for the study of oriented matroids. However, as we shall see in Section IV, there are oriented matroids which are not linear. The real scope of the oriented matroid axioms is a topological generalization of linear systems, which we call sphere systems (also called "arrangements of pseudo-hemispheres" by Folkman and Lawrence [FL]).

The oriented matroid C of a linear system A in $\mathbb{R}^n$ associates to each point of the space a signed vector encoding its position relative to the hyperplanes of the system. As all positive scalar multiples of a vector are encoded by the same signed vector, one may as well look only at points in the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n \mid x_1^2 + \cdots + x_n^2 = 1\}$; with the only possible exception of $\underset{\sim}{0}$, all signed vectors of C are obtainable from points in S^{n-1}. Each hyperplane H_e in A intersects S^{n-1} in an isometric copy of S^{n-2} which separates S^{n-1} in two sides, S_e^+ and S_e^- , the intersections of S^{n-1} with the sides of H_e . Thus, as far as their oriented matroids go, linear systems in R^n are equivalent to a family of isometric copies of S^{n-2} in S^{n-1} together with a labelling of the sides of each "equator" as + or - .

Some basic properties these families of spheres have suggested the definition of sphere systems. First we need some preliminary concepts and notation.

A topological n-sphere (n-ball) is a topological space homeomorphic to S^n (respectively, $B^n = \{x \in \mathbb{R}^n | x_1^2 + \cdots + x_n^2 \leq 1\}$). Let $S_0^n = \{x \in S^n | x_{n+1} = 0\}$, $S_+^n = \{x \in S^n | x_{n+1} > 0\}$ and $S_-^n = \{x \in S^n | x_{n+1} < 0\}$. A hypersphere of an n-sphere S is a subset of S which is the image of S_0^n under a homeomorphism $f: S^n \to S$. The sets $f(S_+^n)$ and $f(S_-^n)$ are the sides of $f(S_0^n)$.

1) A sphere system is a triple (S, E, H), where S is a sphere, E is a finite index set, and $H = \{S_e^j | e \in E, j \in \{0,+,-\}\}$ is a family of subsets of S satisfying:

(1.a) For each $e \in E$, either $(S_e^0, S_e^+, S_e^-) = (S, \phi, \phi)$ or S_e^0 is a hypersphere of S, with sides S_e^+ and S_e^-.

(1.b) For every $\phi \neq A \subseteq E$, the subspace $\cap\{S_e^0 | e \in A\}$ is a sphere (possibly empty). This is called a flat of the system.

(1.c) For every flat F, and every $e \in E$, either $F \subseteq S_e^0$ or $S_e^0 \cap F$ is a hypersphere of F, whose sides are $S_e^+ \cap F$ and $S_e^- \cap F$.

(1.d) Denote $\bar{S}_e^j = S_e^j \cup S_e^0$. For every collection B of such sets either $\cap B = \cap\{S_e^0 | \bar{S}_e^j \in B \text{ for some } j\}$ or $\cap B$ is a ball.

One checks easily that the construction presented prior to the definition produces a ("linear") sphere system out of a linear system. Other kinds of sphere systems will be constructed later.

4.1

There is an intimate connection between sphere systems and oriented matroids which was first discussed by Folkman and Lawrence [FL]. This connection is the theme of this chapter.

Consider a sphere system (S, E,H), and associate to each point $x \in S$ the signed vector $\sigma(x)$ on E, defined by: $\sigma(x)_e = j$ iff $x \in S_e^j$. The degeneracy of S is $1 + \dim \sigma^{-1}(\underset{\sim}{0})$, where dim K denotes the dimension of the sphere K. Here we introduce the practice of referring to a sphere system by its supporting sphere S when that leads to no confusion. In particular, the set of all signed vectors obtained from S will be denoted $\sigma(S)$.

Two sphere systems S, T with the same index set E† will be considered equivalent if there exists a homeomorphism $h: S \to T$ such that $h(S_e^j) = T_e^j$, for every e, j. It is clear that if S and T are equivalent, then $\sigma(S) = \sigma(T)$; also, the topological invariance of dimension implies that S and T have equal dimension and same degeneracy.

If the degeneracy of S is 0, then $\underset{\sim}{0} \in \sigma(S)$. We shall often look at $\sigma(S) \div \underset{\sim}{0}$; note that still $\sigma^{-1}(\underset{\sim}{0}) = \phi = \cap\{S_e^o | e \in E\}$.

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† One can extend the notion of equivalence so as to allow different index sets, and relabellings. This would make the sets $\sigma(S)$ and $\sigma(T)$ "isomorphic" instead of identical.

4.I

2) REPRESENTATION THEOREM: The map $S \mapsto \sigma(S)+\underset{\sim}{0}$ is a bijection between equivalence classes of sphere systems of fixed degeneracy $k \geq 0$ and oriented matroids on the same index set.

This is the connexion established in [FL]. We shall present a new proof of this result which strenghtens it in several directions.

It is natural to regard Theorem (2) as making three distinct assertions about σ ; those are separated in Theorems (3),(5),(7) which together comprise a proof of Theorem (2). The simplest of the three is:

3) THEOREM: Suppose that two sphere systems have the same set of signed vectors and the same degeneracy. Then they are equivalent.

This result is proved by analyzing the partition a sphere system induces in its supporting sphere. The complex of the sphere system S is

$$C(S)= \{\sigma^{-1}([X])\mid X \in \sigma(S)+\underset{\sim}{0}\} .$$

4. PROPOSITION: The complex of a sphere system is a ball complex.

This is proved in Section III, where we slightly modify the definition of ball complex to accomodate degeneracy. An elementary property of ball complexes (III.1) is used to produce the homeomorphism asserted by (3).

The part of the proof of the Representation Theorem which actually develops the structure of oriented matroids and sphere systems is:

5. THEOREM: Given an oriented matroid C and an integer $k \geq 0$, there exists a sphere system with degeneracy k whose set of signed vectors is C .

This is proved by constructing a ball complex order isomorphic to C , and whose space is a sphere; then the several hyperspheres are identified as spaces of subcomplexes. The main difference between this and the proof in [FL] is that we also prove:

6. THEOREM: Every oriented matroid is a PL-sphere.

Roughly speaking, this says that in any ball complex order isomorphic to an oriented matroid, the cells are embedded in the whole sphere in a nice way. The concept of PL-sphere is more thoroughly discussed in the next chapter, where we prove (6). An essential part of that proof is the fact that oriented matroids are constructible. The proof of (5) is given in Chapter 6.

REMARK: All the worry about degeneracy in (2), (3) and (5) may seem like complicating for the sake of generality. For instance, Folkman and Lawrence never consider degenerate systems, so why should we?

The real reason for considering degeneracy lies in what happens when one deletes elements from a sphere system:

Select a subset $A \subseteq E$ and consider the system $S\backslash A=(S,E-A,H')$, where H' results from the removal from H of all S_e^j with $e \in A$. That is a sphere system, and $\sigma(S\backslash A)= \sigma(S)\backslash A$. Now, the degeneracy of $S\backslash A$ can be bigger than that of S; one can easily show that if the dimension of $\sigma(S)$ goes down by k upon deleting A, then the degeneracy of S goes up by k. □

A last part of the proof of (2) is showing that σ actually maps sphere systems to oriented matroids:

7. THEOREM: For any sphere system S, $\sigma(S)+\underset{\sim}{0}$ is an oriented matroid.

Proof of this is given in Section II. The basic idea is that whenever K is a supercell of $\sigma(S)$, $\sigma^{-1}(K)$ is a ball where interior is $\sigma^{-1}(\mathrm{relint}(K))$. As σ is continuous and onto, the supercell axioms (2.IV.9) follow from basic properties of balls.

Related to the proof of (7) is a question about the way sphere systems are defined, and what properties should one check to satisfy ourselves that a given triple (S,E,H) is indeed a sphere system. For instance, Folkman and Lawrence have shown that it suffices to verify (1.a,b,c) together with the conclusion of Proposition (4). They also assume the existence of an involutive homeomorphism of S, similar to the antipodality in linear sphere systems, but this is easily shown to be a redundant requirement (Theorem IV.1).

The statement that C(S) is a ball complex is essentially a weak form of (1.d), namely, it only specifies the result of intersecting a few sets of S_e^j . Thus the task of verifying that a given object is a sphere system is somewhat simplified using Folkman and Lawrence's theorem. On the other hand, one easily verifies that

(1.c) is a consequence of (1,a,b,d), therefore redundant.

Actually, a much deeper simplification is possible: the assertion (1.d) for a given system follows from (1.a), (1.b), (1.c). That is, hyperspheres which intersect nicely imply that the corresponding sides also intersect nicely. A direct proof of this is amis in Chapter (6) we shall see a proof which is a bit of a mess, involving another proof of Theorem (7) together with the full topological significance of Theorem (6).

The last two sections discuss systems of (topological) hyperplanes in projective space. Section IV shows how to identify antipodal points in a sphere system, in order to get projective systems. Two dimensional projective systems ("arrangements of pseudolines") are used to give examples of nonlinear oriented matroids. Section V discusses Levi's Theorem, which is a basic tool in the theory of arrangements of pseudolines.

II) GETTING THE ORIENTED MATROID

The axioms by which oriented matroids were defined have a faint resemblance to those properties defining sphere systems. For instance, the negativity axiom is related to the "crossing property" (I.1.c), while products and elimination reflect aspects of (I.1.d). A more clear connection appears with the use of the supercell axioms.

For this purpose and future use, it is convenient to introduce the following notion: a signed system is a triple (S,E,H) where S is a topological space, E is a finite set, and $H = \{S_e^j \mid e \in E,\ j \in \{+,-,0\}\}$ is a family of subsets of S, such that for each $e \in E$, (S_e^0, S_e^+, S_e^-) is a partition of S, S_e^0 is closed and S_e^+, S_e^- are open. A flat of the system is either S or an intersection of S_e^0's and a supercell is an intersection of $\bar{S}_e^j$ which is not a flat, where $\bar{S}_e^j = S_e^j \cup S_e^0$. The relative interior of a supercell K is defined as:

$$\mathrm{relint}(K) = \cap\{S_e^j \mid K \subseteq \bar{S}_e^j,\ K \not\subseteq S_e^0\} \cap K \ .$$

1. An abstract sphere system (one must be careful with abbreviations!) is a signed system (S,E,H) such that:

(1.a) For every flat F and $e \in E$, $S_e^+ \cap F \neq \phi$ iff $S_e^- \cap F \neq \phi$.

(1.b) Every supercell is the closure in S of its relative interior.

(1.c) The relative interior of each supercell is connected.

Every set $\mathcal{D}$ of signed vectors can be considered a system in a natural way, with the topology and sets $\mathcal{D}_e^j$ as defined in Section (2.V).

2. PROPOSITION: A set $\mathcal{D}$ of signed vectors is an abstract sphere system iff $\mathcal{D}+\underset{\sim}{0}$ is an oriented matroid.

PROOF: This is just a restatement of Theorem (2.V.9). □

3. PROPOSITION: Every sphere system is an abstract sphere system.

PROOF: Evidently, a sphere system S is a signed system. It satisfies (1.a), since $S_e^+ \cap F \neq \phi$ implies that $S_e^0 \cap F$ is a hypersphere of F and $S_e^- \cap F$ is one of its sides, therefore nonempty. The supercell properties follow from:

4. LEMMA: Let K be a supercell of S, let F be the intersection of all flats of S which contain K, and denote n = dim F. Then there exists a homeomorphism $h:B^n \to K$ such that $h(B^n - S^{n-1}) = \text{relint}(K)$.

Assume (4) for a while. Then relint(K) is connected and dense in K since $B^n - S^{n-1}$ has these properties with respect to B^n. This completes the proof of (3).

PROOF of (4): Let $\bar{F}_1, \bar{F}_2, \ldots, \bar{F}_t$ be the balls $\bar{S}_e^j \cap F$ which contain K. If $\bar{F}_i = \bar{S}_e^j \cap F$, denote $F_i^0 = S_e^0 \cap F$, $F_i = S_e^j \cap F$. Define $\bar{K}_i = \bar{F}_1 \cap \ldots \cap \bar{F}_i$, $K_i = F_1 \cap \ldots \cap F_i$; thus $K = \bar{K}_t$, $\text{relint}(K) = K_t$. No $\bar{K}_i$ is contained in any F_i^0 (otherwise, $K \subseteq K_i \subseteq F_i^0 \subsetneq F$, contradicting the definition of F), hence by (I.1.d), each $\bar{K}_i$ is a ball. Note that each K_i is an open set of F (in the relative topology), and each $\bar{K}_i$ is closed.

We show inductively that there exists a homeomorphism $h_i:B^n \to \bar{K}_i$, such that $h_i(B^n - S^{n-1}) = K_i$. The case i=1 follows from the fact that F_1^0 is a hypersphere of F, and F_1 is one of its sides.

Suppose now that the stage $i-1$ has been succesful. Since $\overline{K}_i = \overline{K}_{i-1} \cap \overline{F}_i \nsubseteq F_i^0$, $\overline{K}_{i-1} \cap F_i \neq \phi$. The map h_{i-1} implies that $\overline{K}_{i-1}$ is the closure of K_{i-1}; hence, $K_{i-1} \cap F_i \neq \phi$. Thus $\overline{K}_i$ contains the nonempty open set K_{i-1} of F. The Invariance of Domain Theorem (see [Du] , [Gr1]) implies that (since $\overline{K}_i$ is a ball) there is a homeomorphism $h_i : B^n \to \overline{K}_i$ so that $h_i(B^n - S^{n-1})$ is the union of all open sets of F contained in $\overline{K}_i$. Thus $K_i \subseteq h_i(B^n - S^{n-1})$. On the other hand, $h_i(B^n - S^{n-1}) \subseteq F_j$, for every $j \leq i$, since it is contained in $\overline{F}_j$ and is open. It follows that $h_i(B^n - S^{n-1}) \subseteq \cap F_{j \leq i} = K_i$. □

The last two propositions are tied together by:

5. PROPOSITION: If S is an abstract sphere system, so is $\sigma(S)$.

PROOF: Observe that σ is onto, and if $K \subseteq \sigma(S)$ is either a flat or a supercell, so is $\sigma^{-1}(K)$, and $\text{relint}(K) = \sigma(\text{relint}(\sigma^{-1}(K)))$. Moreover, since every closed set of $\sigma(S)$ is a union of supercells, σ is continuous.

Condition (1.a) follows for $\sigma(S)$ directly from the definition of σ and (1.a) for S, while (1.c) follows from the same for S and continuity of σ .

Since supercells are closed, any supercell contains the closure of its relative interior. Suppose that (1.b) fails for K. Thus, there exists an open set $A \subseteq \sigma(S)$, disjoint from relint(K) , such that $A \cap K \neq \phi$. Since σ onto, $\sigma^{-1}(A) \cap \sigma^{-1}(K) \neq \phi$, and as $\sigma^{-1}(A)$ is open in S and (1.b) holds for $\sigma^{-1}(K)$, $\sigma^{-1}(A) \cap \text{relint}(\sigma^{-1}(K)) \neq \phi$. Applying σ we obtain a contradiction with the choice of A which shows that (1.c) also holds in $\sigma(S)$. □

Now we reached the goal of this section:

6. PROOF of Theorem (1.7): Proposition (2) + Proposition (3) + Proposition (5). □

Folkman and Lawrence's proof of (1.7) describes the oriented matroid of S in terms of its circuits. In fact, there is a simple proof that the minimal nonzero vectors in σ(S) are the vertices of an oriented matroid. One just has to verify the conditions of Theorem (2.VI.4) as follows:

(a) If $V \in \sigma(S)$ is minimal, let $F = \cap\{S_e^0 \mid e \in E - \underline{V}\}$. It follows from (1.1.c) and minimality that $\cap\{S_e^0 \mid e \in E\}$ is a hypersphere of F. This implies (2.VI.4.1).

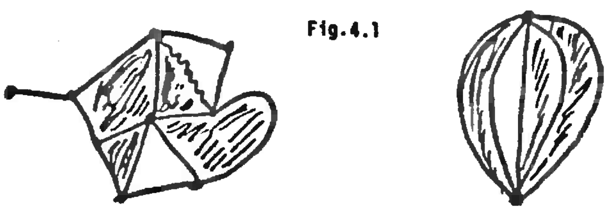
(b) (2.VI.4.2) follow easily from (1.1.d) and the fact that balls are connected . by the same method used to prove that an abstract sphere system of signed vectors satisfies the elimination property (2.V.9).

III) BALL COMPLEXES

The definition we give here differs a little from the one in (3.1) in that we allow degenerate ball complexes. By the boundary, ∂B, of a n-ball B we mean the image of S^{n-1} under a homeomorphism $B^n \to B$; the interior of B is $B-\partial B$. Intrinsic, equivalent definitions are: $\text{int}\,B = \{x \in B \mid x$ has a neighbourhood homeomorphic to $\mathbb{R}^n\}$, $\partial B = \{x \in B \mid x$ has a neighbourhood homeomorphic to $\mathbb{R}^n_+\}$.

A ball complex is a complex K with minimum $\underline{0}$, such that all members of $K-\underline{0}$ are balls in some euclidean space, and the interior of $p \in K-\underline{0}$ equals $i_K(p) = K - \cup\{q \in K \mid q \subsetneq p\}$. In particular, if p covers 0, we conclude that $\underline{0}$ is the boundary of p, hence a sphere. The degeneracy of K is $1+\dim \underline{0}$ (Fig.4.1) .

Fig.4.1



degeneracy 0 degeneracy 1

The space of K is a subspace of some $\mathbb{R}^n$, and has thus a topology, also determined by: $A \subseteq s(K)$ is closed iff its intersection with each cell is closed in the cell. Thus a map defined on $s(K)$ is continuous iff its restriction to each cell is so.

1. THEOREM: Let K, L be ball complexes of equal degeneracy and $f:K\to L$ an order isomorphism. Then there exists a homeomorphism $h:s(K)\to s(L)$ such that $h(p)=f(p)$ for every $p\in K$. Further, given any subcomplex K' of K and homeomorphism $h'=s(K')\to s(f(K'))$ compatible with f, one can construct h extending h'.

PROOF: If K' is not specified, let $K'=\{\underset{\sim}{0}_K\}$, and $h':\underset{\sim}{0}_K\to\underset{\sim}{0}_L$ be any homeomorphism. If $K'\neq K$, choose a minimal cell $p\in K-K'$. Then every $q\subsetneq p$ is in K', and we have

$$\begin{aligned} h'(\partial p) &= \cup\{h(q) \mid q\subsetneq p\} \\ &= \cup\{f(q) \mid q\subsetneq p\} \\ &= \cup\{t\in L \mid t\subsetneq f(p)\} \\ &= \partial q \quad . \end{aligned}$$

By the Lemma below, there is a homeomorphism from p to f(p) extending $h'|\,\partial p$. This map can be used to extend h' over p, thus over $K'+p$. Inductively, we extend over the whole K. □

2. LEMMA: Any homeomorphism between the boundaries of two balls can be extended to a homeomorphism of the balls.

PROOF: Suppose first that we are given a homeomorphism $h: S^{n-1}\to S^{n-1}$. We can extend it to $h: B^n\to B^n$ by defining: $h(0)=0$, $h(x)=|x|\cdot h(x/|x|)$, where $x=(x_1^2+\cdots+x_n^2)^{1/2}$. More generally, given homeomorphisms $f_1:B^n\to B_1$, $f_2:B^n\to B_2$, $h:\partial B_1\to\partial B_2$, define $h'=f_2^{-1}hf_1: S^{n-1}\to S^{n-1}$, extend h' over B^n as above and obtain the extended h as $h:f_2h'f_1^{-1}$. □

Now we return to sphere systems:

3. PROOF of (I.4): Let S be a sphere system and $X\in\sigma(S)$. Then $\sigma^{-1}([X])$ is a supercell, therefore a ball, unless $X=\underset{\sim}{0}$, in which case $\sigma^{-1}([X])$ is a sphere. When $X\neq\underset{\sim}{0}$, Lemma (II.4) says that int $\sigma^{-1}([X])$ as a ball is simply $\sigma^{-1}(X)=\sigma^{-1}([X])-\cup\{\sigma^{-1}([Y])|\ Y<X\}$, which is the interior of X in C(S). Thus C(S) is a ball complex. □

4. PROOF of (I.3): Let $(S,E,\mathcal{H})$, $(T,E,\mathcal{L})$ be sphere systems, and denote by σ_S, σ_T the corresponding maps to signed vectors. Suppose that $\sigma_S(S)=\sigma_T(T)=C$, and S and T have equal degeneracy. Thus $\sigma_S^{-1}(\underset{\sim}{0})$ and $\sigma_T^{-1}(\underset{\sim}{0})$ are homeomorphic spheres (possibly empty), and one has a homeomorphism h between them. Consider $f:C(S)\to C(T)$ defined by $f(\sigma_S^{-1}([X]))=\sigma_T^{-1}([X])$. This is clearly an order isomorphism, and in view of (I.4), combines with h to produce a homeomorphism $h:S\to T$ as in Theorem (1). It follows that for each $X\in C$, $h(\sigma_S^{-1}(X))=\sigma_T^{-1}(X)$, hence $h(S_e^j)=h(\cup\{\sigma_S^{-1}(X)|\ X_e=j\})=T_e^j$. □

IV) PROJECTIVE SPACE

A homogeneous linear system in $\mathbb{R}^n$ can be considered as a collection of hyperplanes in real projective space $\mathbb{P}^{n-1}$. There the hyperplanes do not have sides, but they still dissect $\mathbb{P}^{n-1}$ into regions. In some situations, the system on $\mathbb{P}^{n-1}$ seems to be a more comfortable geometrical object than a linear system or a homogeneous linear system. We describe a similar gimmick for sphere systems.

We consider $\mathbb{P}^n$ as the identification space $S^n/(x\sim -x)$. The identification of S^k with $S^n \cap (\mathbb{R}^{k+1} \times \underline{0})$ for $k<n$, induces a canonical inclusion $\mathbb{P}^k \subseteq \mathbb{P}^n$. A pseudo k-flat of $\mathbb{P}^n$ is the image of $\mathbb{P}^k$ by a homeomorphism of $\mathbb{P}^n$.

1. THEOREM: Let S be a sphere system. Then there exists a homeomorphism $h:S \to S$ such that for every $x \in S$, $h(x) \neq x = h(h(x))$, and such that $h(S_e^+) = S_e^-$ for every e. The identification space $S/(x \sim h(x))$ is homeomorphic to a projective space and the image of each flat of S is a pseudo-flat.

PROOF: Let $F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = S$ be a sequence of flats of S, such that $F_0 = \cap \{S_e^0 \mid e \in E\}$ and for each i, $\dim F_i = i+k$, where $k = \dim F_0$. Choose a homeomorphism $f_0: F_0 \to S^k$ (if $k=-1$, this step is vacuous), and define $h_0: F_0 \to F_0$ by $h_0(x) = f_0^{-1}(-f_0(x))$.

Suppose inductively that we have defined homeomorphisms $h_i: F_i \to F_i$, $f_i: F_i \to S^{i+k}$ such that:

(1.1) $h_i(x) \neq x = h_i(h_i(x))$,

(1.2) $f_i h_i(x) = -f_i(x)$,

(1.3) for every $X \in \sigma(F_i)$, $h_i(\sigma^{-1}(X)) = \sigma^{-1}(-X)$.

We define now h_{i+1}, f_{i+1}:

Let $\mathcal{D} = \sigma(F_{i+1})$. There exists an $e \in E$ such that $F_i = S_e^0 \cap F_{i+1}$. Therefore, $K_+ = \{\sigma^{-1}([X]) \mid X \in \mathcal{D}_e^+\}$ is a ball complex with space $S_e^+ \cap F_{i+1}$, while $S_e^- \cap F_{i+1}$ is the space of a similar complex K_-. Certainly $g: K_+ \to K_-$ given by $\sigma^{-1}([X]) \to \sigma^{-1}([-X])$ is an order isomorphism, and h_i is a homeomorphism between the spaces of $K' = \{p \in K_+ \mid p \subseteq F_i\}$ and $g(K'_+) = \{p \in K_- \mid p \subseteq F_i\}$, such that $h_i(p) = g(p)$ for every $p \in K$. We can thus extend h_i to $t: s(K_+) \to s(K_-)$, as in (III.1), so that $t(p) = g(p)$ for every $p \in K$. Define h_{i+1} by: $h_{i+1}(x) = t(x)$ if $x \in S_e^+ \cap F_{i+1}$, $= t^{-1}(x)$ otherwise. Thus h_{i+1} is now a function $F_{i+1} \to F_{i+1}$, and it is easy to see that it is a homeomorphism and satisfies (1.1) and (1.3).

To define f_{i+1}, note first that $S_e^+ \cap F_{i+1}$ and S^{i+k+1} are balls with boundaries F_i and S^{i+k}. Thus f_i can be extended by Lemma (III.2) to a homeomorphism $w: S_e^+ \cap F_{i+1} \to S_+^{i+k+1}$. Define h_{i+1} by: $f_{i+1}(x) = w(x)$ if $x \in S_e^+ \cap F_{i+1}$, $= -wh_{i+1}(x)$ if $x \in S_e^- \cap F_{i+1}$. It is well defined since if $x \in S_e^0 \cap F_{i+1}$, $w(x) = f_i(x) = -f_i h_i(x) = -wh_{i+1}(x)$, and also continuous.

Hence (1.2) is satisfied. The homeomorphism h of the theorem is h_n and its properties follow directly from (1.1), (1.2), (1.3), with the exception of the assertion about flats:

From the construction, the image of each F_i under the identification $x \sim h(x)$ is a pseudo flat. Let F be a flat of dimension i+k. One could have started with a sequence of F_j's containing F, and the same f_o obtaining a new homeomorphism h' such that under the identification by h', the image of F is a pseudo flat. Let $e_1, e_2, \ldots, e_n$ be a base of $\sigma(S)$ and define $g: S \to S$ by:

$$g(x) = \begin{cases} x, & \text{if for the smallest } i \text{ for which } \sigma(x)_i \neq 0, \\ & \quad \sigma(x)_i = + \\ h'h(x) & \text{otherwise.} \end{cases}$$

By observing the action of g on cells of C(S) one sees that it is a homeomorphism, and from the definition, $gh = h'g$. Thus g induces a homeomorphism of the two identification spaces, hence $F/(x \sim h(x))$ is the image of the pseudo flat $F/(x \sim h'(x))$. □

Let us define a <u>projective system</u> to be an indexed family $\{\mathbb{P}_e \mid e \in E\}$ of pseudo-hyperplanes in $\mathbb{P}^n$ such that each subcollection intersects in a pseudo-flat. The preceeding theorem tells us that every sphere system can be turned into a projective system. Conversely, given a projective system in $\mathbb{P}^n$ and if $f: S^n \to \mathbb{P}^n$ is the double cover, the system lifts to a collection of hyperspheres in S^n. After labelling as +, - the sides of the hyperspheres, one can verify conditions (1.1.a,b,c) of the definition of sphere system. As we shall prove in Chapter 6, that shows that a sphere system was obtained. So, one may consider a projective system as a sphere system without +,- labelling, to which one could make correspond all oriented matroids obtained from $\sigma(S)$ by a change of signs.

A special case of projective system which has deserved a lot of attention recently is the 2-dimensional case, also called arrangement of pseudo-lines ([G3], [Co3], [Co4], [GP1]).

It follows from Theorem (1.3) that the oriented matroid of such an arrangement is linear iff the arrangement is "stretchable", i.e., can be transformed into an arrangement of lines by a homeomorphism of $\mathbb{P}^n$. In Figs.2 and 3 we show two nonstretchable arrangements. The non-stretchability of the one in Fig.4.2 is justified in [G3, p.42], and based on Pappus' Theorem. The one in Fig.4.3 is nonstretchable ; if it was, the (then) straight lines a,...,g would be part of a Desargues' configuration, thus implying that the lines AA', BB', CC' are concurrent. But those would be i, j, k, stretched, which are not concurrent.

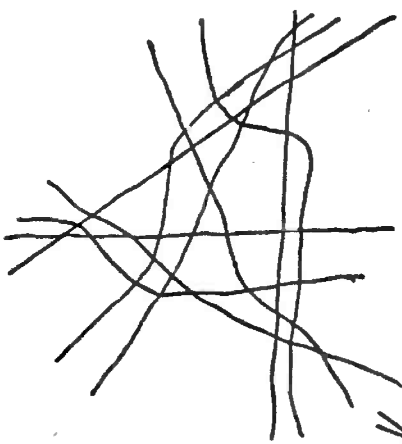


Fig.4.2

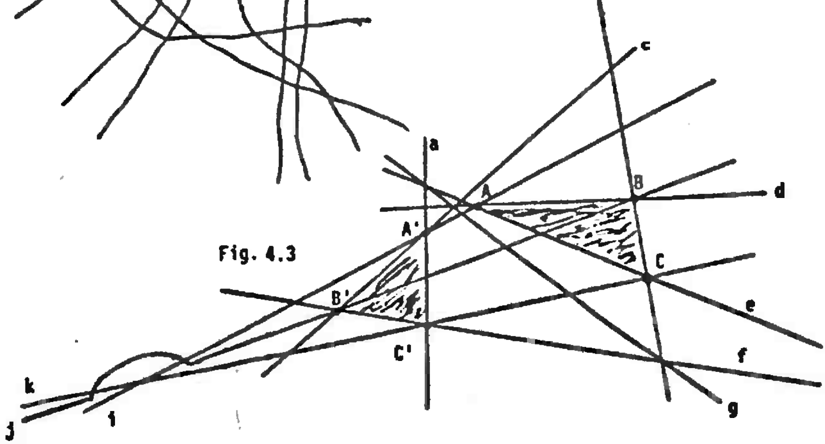


Fig. 4.3

V) LEVI'S THEOREM

In most treatments of arrangements of pseudo lines, as those quoted in last section, the following Theorem of Levi [Le] (see also [Gr2, pg.47]) is often used as the main tool:

1. THEOREM: Given an arrangement of pseudolines A in $\mathbb{P}^2$ and points p,q in $\mathbb{P}^2$, not both in one pseudoline of A, there exists a pseudoline in P^2 which contains p and q and can be adjoined to A, resulting in a new arrangement.

Given the relationship between arrangements of pseudolines and 2-dimensional sphere systems, this result is equivalent to:

2. THEOREM: Given a 2-dimensional pointed sphere system S and points p, q in S, not both in one of the hyperspheres of the system, it is possible to adjoin to S a new hypersphere S_e (with sides S_e^+, S_e^-), containing p and q, so that the new system is still a sphere system.

3. Remark : For linear sphere systems this theorem expresses an obvious property: every two points are contained in a plane through the origin. □

We shall present a proof of (2) using the Representation Theorem.

4. PROOF of (2): We use the term circle instead of "hypersphere" in this context.

Denote $C=\sigma(S)+\underline{0}$, $V=\sigma(p)$.

$W=\sigma(q)$. We shall do the case $V\neq\pm W$. The cases where p and q lie in the same cell or opposite cells of C(S) can be deduced from that easily. It helps if one views C(S) as a map in the sphere S. The geometrical dual of this map is nothing more than the tope graph of C (unoriented).

$$\text{Let } T^1 = V.W,\ T^2 = V.(-W),\ \ T^3 = W.V,\ \ T^4 = (-W).V\ .$$

Since p and q are not in any of the circles, those four vectors are topes of C . They may not be all distinct (Fig.(4.4)).

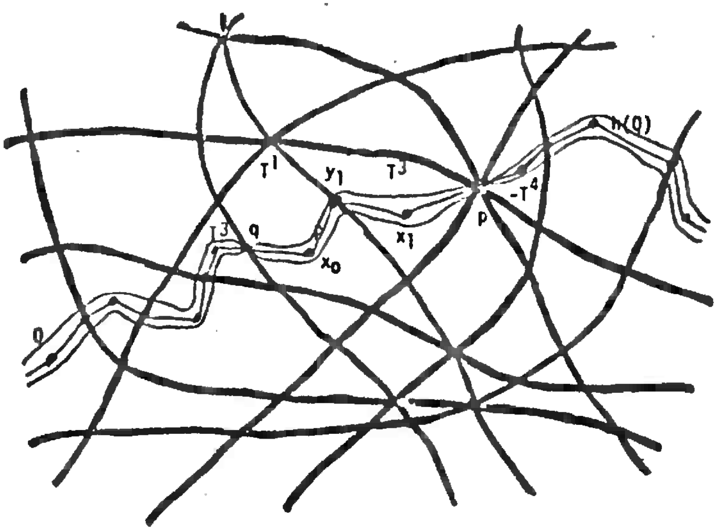


Fig. 4.4

In $TG(C, T^1)$, choose a path P from T^1 to T^2. Consider the sequence $X^0, Y^1, X^1, Y^2, \ldots, Y^n, X^n$, where $P = X^0, X^1, \ldots, X^n$, and for $i = 1, \ldots, n$, Y^i is the edge common to X^{i-1} and X^i. Choose points $x_0, y_1, \ldots, x_n$ in S such that $\sigma(x_i) = X^i$, $\sigma(y_i) = Y^i$, with the care of choosing $x_0 = p$ if $X^0 = V$, and $x_n = q$ if $X^n = W$. Join each pair of successive x,y points by an arc which, except for its y-end, is contained in the interior of the cell containing x. There is no difficulty in making successive arcs intersect only at the extremities. If $p \neq x_0$, adjoin an arc px_0 contained in $\sigma^{-1}(T^1)$ to the beginning of the sequence, and similarly adjoin an arc $x_n q$ at its end if $x_n \neq q$. Concatenating all those arcs, we get a path P in S, joining p to q, such that

a) The circles intersected by the interior of P are only those separating p and q, and P intersects each in a single point (y_i).

Let h be an involution of S as in (IV.1). In the argument above substitute q by p, p by h(q), T^1 by T^4, T^2 by T^3. There results a path Q in S from h(q) to p, satisfying the analogue of (a). Clearly $Q \cap P = \{p\}$, hence the concatenation of Q and P is a path R from h(q) to q, which crosses once each circle not containing q.

It follows that $S_e = R \cup h(R)$ is a circle containing p and q. Clearly S_e intersects each circle of S in a pair $\{x, h(x)\}$ of points, and these two circles cross each other. That is enough to guarantee that S_e and its sides can be adjoined to the system S. ∎

This proof clearly extends to the following generalization of (2).

3. THEOREM: Let S be a n-dimensional sphere system, and let p, q be points in S. Also, suppose that an involution h of S is given, as in (IV.1). Then there exists an embedding $f:S^1 \to S$ such that $f(S^1)$ contains p and q, is fixed by h and crosses each S_e not containing p and q. Thus the intersections of the S_e^j with $f(S^1)$ define a 1-dimensional sphere system on $f(S^1)$. □

This is a poor-man generalization of (2). What one really wishes is something like:

4. "If S is a pointed n-dimensional sphere system, then given any n points in S, it is possible to extend S with a new hypersphere that goes thru those points".

Unfortunately, this is false if n≥3. Counterexamples to (4) are presented in section (9.IV).

Chapter 5

P. L.

1) INTRODUCTION

A main result of this chapter, with direct bearing in the proof of the Representation Theorem (4.I.2), is:

1. THEOREM: Every oriented matroid is a PL-sphere.

The definition and study of the class of complexes called "PL-spheres", and the companion "PL-balls" is the content of this chapter.

An idea underlying the notion of "PL-sphere" is that one would like to study the ball complexes whose spaces are spheres, or balls, combinatorially. This is not too manageable, as some results one would expect to hold may fail to be true. For instance, suppose that K is a ball complex and $s(K) = S^n$, and let x be a vertex of K. There is a neighbourhood N of x is S^n which is a topological ball and meets only the cells of K which contain x. The intersection of these cells with the boundary of N determines a complex with space ∂N, and order isomorphic to $st(x,K) = \{p \in K | x \leq p\}$. One would expect that N could be chosen so that the complex in N is a ball complex, thus implying that st(x,K) has the partial order of a "spherical" ball complex. There are examples which show that this is not the case [Ed], [Can] .

The class of piecewise-linear ball complexes and corresponding notions of PL-balls and spheres avoids these pathologies. Those are ball complexes with an additional structure. We describe them after a few preliminary definitions.

A map $f: P \to \mathbb{R}^m$, where $P \subseteq \mathbb{R}^n$, is affine if for every $p_1, p_2, \ldots, p_k \in P$, $f(\lambda_1 p_1 + \ldots + \lambda_k p_k) = \lambda_1 f(p_1) + \ldots + \lambda_k f(p_k)$, whenever $\lambda_1 + \lambda_2 + \ldots + \lambda_k = 1$.

In this chapter, a polyhedron is a subset of some euclidean space which is a finite union of polytopes†. A Piecewise-Linear (PL) map between polyhedra P,Q is a 1-1 map $f : P \to Q$ such that there exists a decomposition $P = P_1 \cup P_2 \cup \ldots \cup P_t$, where the P_i's are polytopes and $f|P_i$ is affine, for each $i = 1, 2, \ldots, t$. A bijective PL-map is called a PL-homeomorphism; it is a homeomorphism in the ordinary sense, and its inverse is also a PL-map.

Compositions of PL-maps are PL-maps (II.1). Thus, PL-homeomorphisms induce an equivalence relation among polydedra. This can be alternatively described by (II.7): two polyhedra are PL-homeomorphic if they can be triangulated by isomorphic simplicial complexes.

Henceforth, we shall use PLH to abbreviate "PL-homeomorphism"

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† - The use of the term "polyhedron" here is inconsistent with the definition of unbounded convex polyhedron in chapter 2. Those definitions can be conciliated ([Hu], pg. 81) at the expense of some complication which is unnecessary here.

and "PL-homeomorphic".

2. PROPOSITION: Any two n-polytopes are PLH, and so are their boundaries.

PROOF: Given in (II.9) □

A <u>polyhedral n-ball</u> (<u>polyhedral n-sphere</u>) is a polyhedron PLH to an n-polytope (the boundary of an (n+1)-polytope). By the proposition, all polyhedral n-balls are PL-homeomorphic. The image of the boundary of a polytope by a PLH to a polyhedral ball P is the <u>boundary</u> ∂P of P; it is, of course, the boundary of P as a topological ball. We shall prove in section II that ∂P is a well defined subset of P, and relate it to triangulations of P.

<u>Remark</u>: It is common in topology to refer to polyhedral balls and spheres as PL balls and PL spheres. We reserve these names for the classes of complexes defined below.

A <u>PL ball-complex</u> is a complex K whose cells are polyhedral balls or ϕ and such that for each $p \in K$, its boundary is the union of the cells of K it properly contains. Thus $\partial p = s(\partial_K[p])$.

A <u>PL n-sphere</u> is a complex order isomorphic to a PL ball-complex whose space is the boundary of an (n+1)-polytope. Similarly, a <u>PL n-ball</u> is a complex order isomorphic to a PL ball-complex whose space is an n-polytope. Polyhedral balls and spheres can substitute the polytopes in these definitions, without any conceptual change.

Main examples: any polytope lattice is a PL-ball, and its boundary is a PL-sphere.

3. PROPOSITION: A PL n-ball is PM of dimension n, and its boundary is a PL(n-1)-sphere.

PROOF: Given in (III.6) □

4. THEOREM: Let K be a complex and suppose that K_1,K_2 are subcomplexes of K, each a PL n-ball, and such that $K_1 \cup K_2 = K$. Then,

(4.1) If $K_1 \cap K_2 = \partial K_1 = \partial K_2$, then K is a PL n-sphere.

(4.2) If $K_1 \cap K_2 = \partial K_1 \cap \partial K_2$ and is a PL (n-1)-ball, then K is a PL n-ball.

PROOF: Given in Section III. □

Similar results hold with polyhedral balls and spheres.

The presence of these two statements as parts of a single theorem is so that the analogy between them be explicit. We point out that (4.2) is actually one of the fundamental characteristics distinguishing PL-balls among other balls complexes. The analogue of (4.2), with the "PL" removed is not true in general, as can be seen from an example of Harley [Ha]. On the other hand, the proof of (4.1), presented in section III, works as well for general ball complexes whose spaces are topological balls.

5. THEOREM: Every S-constructible complex is a PL-sphere and every B-constructible complex is a PL-ball.

PROOF: The usual induction for constructible complexes applies. Theorem (4) takes care of complexes which are obtained by pasting. The case where K has only one maximum, p, and ∂K is S-constructible is handled as follows: by induction, ∂K is order isomorphic to a PL ball-complex whose space is the boundary of a polytope P. We can add P as a cell to the ball complex and extend the order isomorphism from ∂K by $p \mapsto P$. □

Theorem (1) follows directly from Theorem (5) and the constructibility of oriented matroids. By the same reason, one concludes that the boundary of a tope is a PL-sphere. Theorem (IV.8) says that the polar of a PL-sphere is a PL-sphere. Thus we can state:

6. THEOREM: Tope boundaries and their polars are PL-spheres. □

Notice that for linear oriented matroids, this theorem says just the obvious fact that polytope boundaries are PL-spheres.

The <u>star</u> of a cell p of a complex K is $st(p,K) = \{q \in K \mid p \leq q\}$; the <u>shell</u> of p is $sh(p,K) = [st(p,K)] - st(p,K)$.

We say that K' is obtained from K by pulling p at $\hat{p}$, where $\hat{p}$ is a point not in s(K-st(p,K)), if K' is order isomorphic to the complex (K-st(p,K))∪{q+$\hat{p}$|q⊂sh(p,K)}.

The operation of pulling a face is useful in the solution of some problems about polytopes. One reason is that if K is a polytope boundary, then K^1 is also isomorphic to the boundary of a polytope. Indeed, if K is the boundary complex of a polytope P, by choosing a point $\hat{p}$ separated from P by the supporting hyperplanes of facets which contain p, and no others, the convex hull of P∪{$\hat{p}$} is a polytope whose boundary is isomorphic to K^1 (see [G2, 5.2.1]). In fact this construction, and its application in the proof of the Upper Bound Conjecture for polytopes is what led us to consider "pulling".

7. THEOREM: Pulling a cell in a PL-sphere semilattice results in a PL-sphere semilattice.

PROOF: (V.6). □

A semilattice is a complex in which the intersection of any two cells is a cell. In section IV we discuss the necessity of this additional hypothesis, with the proof of the theorem.

The Upper Bound Conjecture was a (now proven) statement about the number of faces of polytopes with given number of vertices and dimension. Combining Theorem (7) with a result of Stanley [St2], we prove:

8. THEOREM: PL-spheres, and in particular, tope boundaries and their polars, satisfy the Upper Bound Conjecture. □

"Pulling" has had a honorable place in the beginnings of PL-Topology. Alexander [A1] has shown that the spaces of two simplicial complexes are PL-homeomorphic iff one can go from one complex to the other by a finite sequence of complexes such that each consecutive pair differs by a pulling.

That gives a totally combinatorial characterization of PL-balls and PL-spheres. In section VII we present another totally combinatorial characterization of PL-balls and PL-spheres in terms of constructibility and the notion of subdivisions.

In the same chapter we lay the groundwork for an approach to PL-balls and PL-spheres that further emphasizes complexes, subdivisions and order isomorphisms, as opposed to PL-maps. That section reflects ideas still being developed. It was written in order to be conceptually independent of the intervening sections (except for some concepts introduced at the beginning of Section II). Perhaps it would be a good idea for the reader to jump now to Section VII for a first appreciation of the ideas there, and then read the remaining sections with the possibility of the alternative approach in mind.

BASIC POLYHEDRAL TOPOLOGY

PROPOSITION: Compositions of PL-maps are PL-maps.

PROOF: Let $f:P\to Q$ and $g:Q\to R$ be PL-maps. Express

$P = \cup\{P_i \mid i = 1,\ldots,k\}$, $Q = \cup\{Q_j \mid j = 1,\ldots,\ell\}$, where the P_i, Q_j are polytopes, f is affine on each P_i, and g is affine on each Q_j. Then $\{f^{-1}(f(P_i)\cap Q_j)\}$ gives a decomposition of P into polytopes and gf is affine on each piece. □

A polyhedral complex is a complex K whose cells are polytopes whose interiors in K coincide with their polytope interiors and such that the intersection of any two members of K is a face of both. In particular, every face of a member of K is in K. If each member of K is a simplex, K is called a simplicial complex (a polytope P with vertices $v_0, v_1, \ldots, v_n$ is an n-simplex if every point x of P has a unique expression $x = \lambda_0 v_0 + \lambda_1 v_1 + \ldots + \lambda_n v_n$, with $0 \le \lambda_i$, $\lambda_0 + \lambda_1 + \ldots + \lambda_n = 1$). A triangulation of a polyhedron P is a simplicial complex with space P.

One can associate to each order isomorphism $f: K \to L$ of simplicial complexes a PLH also denoted $f: s(K) \to (L)$, which is as given on vertices, and is affine on each simplex of K. Note that this PLH is uniquely defined from f. This is an example of simplicial map. More generally, a PL-map $f : P \to Q$ is simplicial relative to triangulations K of P and L of Q if f maps each simplex of K affinely onto a simplex of L. In other terms, f is determined by its value on vertices, by affine extension.

We shall see now that every PL-map is simplicial relative to suitable triangulations. This combinatorial characterization of PL-maps explains for the most part their advantage over more general continuous embeddings.

Let K be a polyhedral complex and let us choose a point $\hat{p}$ in the interior of each non-empty cell p. The complex $K^* = \{conv(\hat{p}_1,\dots,\hat{p}_k) \mid \{p_1,\dots,p_k\}$ is a totally ordered set of cells of $K-\underline{\emptyset}\}$ is a (first) <u>derived</u> of K. It is a simplicial complex, $s(K^*) = s(K)$, and every cell of K is the space of a subcomplex of K^*. In fact, if $\emptyset \neq p \in K$, $p = s([p]^*)$, and we also have the recursion $[p]^* = (\partial[p])^* \cup \{conv(\sigma + \hat{p}) \mid \sigma \in (\partial[p])^*\}$. An r-th derived of K is defined inductively as the derived of an (r-1)-th derived of K (fig. 1).

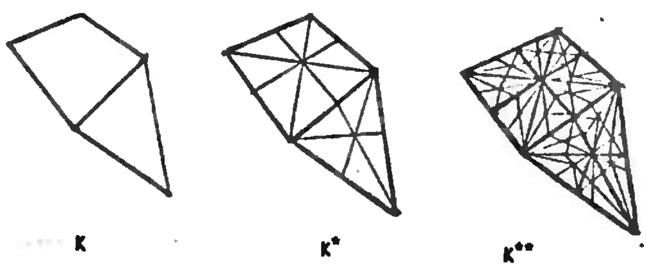

Fig. 5.1

The use of deriveds in most of the following development is not of absolute necessity, but we feel that this extra tool simplifies much of the work ahead.

For example, as polyhedral complexes are ball complexes, Theorem (4.III.1) applies. But we have the following strengthening:

PROPOSITION: Let $f: K \to L$ be an order isomorphism of polyhedral complexes. Then there exists a PL-homeomorphism $h: s(K) \to s(L)$ such that for every $p \in K$, $h(p) = f(p)$.

PROOF: As f extends naturally to an isomorphism $f^* : K^* \to L^*$; one obtains h as the simplicial map associated to f^*. □

LEMMA: Let $A_1, A_2, \ldots, A_n$ be polytopes contained in the space of a polyhedral complex K. Then K has an r-th derived containing triangulations of $A_1, A_2, \ldots, A_n$.

PROOF: Let $P_0, P_1, P_2, \ldots, P_k$ be $A_1, A_2, \ldots, A_n$ and their non-empty faces in order of non-decreasing dimension. Suppose inductively that some (r-1)-th derived $K^{(r-1)}$ contains triangulations of $P_0, P_1, \ldots, P_{r-1}$. This is clearly true for r=1, since P_0 is a point, and it is a vertex in some K^*.

Now, for each $\sigma \in K^{(r-1)}$, choose $\hat{\sigma} \in int(\sigma)$, and in $P_r \cap int(\sigma)$ if this is non-empty. Define $K^{(n)}$ as the derived of $K^{(r-1)}$ with this choice of θ's. We show now that each nonempty set $P_r \cap \sigma, \sigma \in K^{(r-1)}$ is the space of a subcomplex of $K^{(n)}$. That implies that P_r is triangulated.

This is proved by induction in the dimension of σ. If $d(\sigma) = 0$, there is nothing to prove. Suppose that $P_r \cap \tau$ is triangulated in $K^{(r)}$, for each proper face of a given $\sigma \in K^{(r)}$. Each face of the polytope $P_r \cap \sigma$ is an intersection of faces of P_r and of σ. Since each proper face P_s of P_r is triangulated in $K^{(r-1)}$, as $s>r$, $P_s \cap \sigma$ is just a face of σ. It follows that each proper face of $P_r \cap \sigma$ is the intersection with P_r of a face of σ. The induction hypothesis implies that each of these is triangulated in $K^{(r)}$, thus the boundary of $P_r \cap \sigma$ is the space of a subcomplex of L of $K^{(r)}$. Then $L \cup \{conv(\tau + \theta) | \tau \in L\}$ is a subcomplex of $K^{(r)}$ and its space is $P_r \cap \sigma$. □

The following consequences are of fundamental importance.

4. COROLLARY: Every polyhedron is the space of a simplicial complex.

PROOF: Let $P = P_1 \cup P_2 \cup \ldots \cup P_k$ be a polyhedron, where the P_i are polytopes. Choose a large simplex T such that $P \subseteq T$. By Lemma (3) some r-th derived K of the face complex of T contains triangulations of each P_i. Thus, P is the space of $\{\sigma \in K | \sigma \subseteq P\}$. □

A complex L is a <u>subdivision</u> of K if $s(L) = s(K)$ and every cell of K is the space of a subcomplex of L.

. COROLLARY: Given any finite collection of polyhedral complexes contained in a polyhedron P, there exists a triangulation of P, containing a subdivision of each of them as a subcomplex. □

. COROLLARY: Any PL-map is simplicial relative to some triangulations of the domain and codomain.

PROOF: Let $f : P \to Q$ be a PL-map. Using Lemma (3), one can triangulate P by K so that f is affine on each simplex. Triangulate now Q by L so that the image by f of each member of K is triangulated. Let $K_1 = \{f^{-1}(\tau) | \tau \in L, \tau \subseteq f(P)\}$. Then K_1 is a triangulation of P and f is simplicial relative to K_1 and L.

□

. COROLLARY: Two polyhedra are PL-homeomorphic iff they have isomorphic triangulations.

PROOF: The "if" part is obvious. For the "only if", consider a pair of polyhedra and a given PLH. The map is simplicial relative to suitable triangulations, by (6), and since it is bijective, it induces an order isomorphism of these triangulations. □

. COROLLARY: Two polyhedral complexes have PL-homeomorphic spaces iff they have isomorphic simplicial subdivisions. One of these can be chosen to be an iterated derived.

PROOF: Again, just the "only if" part requires proof. One just repeats the proof of (7), and of (6), with the following modification on (6): if P and Q are the spaces of the given complexes, when constructing K and L we impose that they are iterated deriveds of the original complexes. □

We can prove now a strong version of (I.2):

9. THEOREM: Let P,Q be n-polytopes, and $p\in\partial P$, $q\in\partial Q$ be given points. Then there exists a PL-homeomorphism $h : P \to Q$ such that $h(p) = q$ and $h(\partial P) = \partial Q$.

PROOF: Using a suitable linear transformation, one can displace and shrink P, so that we can assume that $P\subseteq \text{int}Q$ and that for some point $x\in \text{int}P$, the segment xq intersects the boundary of P in the point p. Consider now all the polyhedra $P_i\cap \text{conv}(Q_j+x)$, where Q_j is a face of Q. Triangulate the boundary of P by K so that each of these polyhedra are triangulated. Radial projection from x will map each simplex in this triangulation onto a simplex contained in ∂Q, thus giving a triangulation L of ∂Q, and an isomorphism $f : K \to L$ such that $f(\{p\}) = \{q\}$. Define now $K_1 = K\cup\{\text{conv}(\sigma+x)|\sigma\in K\}$ and L_1 analogously; then K_1 triangulates P, L_1 triangulates Q, and f extends to an isomorphism $f : K_1 \to L_1$ such that $f(x) = x$. The corresponding simplicial map is the required PLH. □

It is clear that the result above is still valid for any polyhedral n-balls P, Q.

We now relate the concepts of boundary of a complex and ball boundary:

10. THEOREM: Let K be a polyhedral complex whose space is an n-polytope. Then K is PM, $d(K) = n$ and $s(\partial K) = \partial s(K)$.

PROOF: Let $p \in K$ have dimension $<n$. Then there exists a point x in the interior of the polytope s(K) which is not in the affine flat spanned by p. Choose a point $y \in \text{int}(p)$. The interiors of cells of K partition the line segment xy into open segments, and y is an extremity of one of these, say, of $xy \cap \text{int}(q)$. Thus $y \in \partial q$, and since $y \in \text{int}(p)$, $p \subsetneq q$. This shows that if $d(p)<n$, p is not maximal, hence K is pure, $d(K) = n$.

Let $p \in K$ be an (n-1)-cell and let H be its affine hull. There is no loss of generality assuming that $s(K) \subseteq \mathbb{R}^n$, hence H is a hyperplane. Since any two n-polytopes containing p and contained in the same side of H have an interior point in common, there is at most one n-cell of K containing p on either side of H. Hence K is PM.

If $p \subseteq \partial s(K)$, then H supports a facet of s(K) and s(K) is contained in one side of H. Thus p is contained in exactly one n-cell, i.e., $p \in s(K)$. On the other hand if $p \not\subseteq \partial s(K)$, H splits int(s(K)) in two parts, and the argument of the first paragraph shows that on each side of H there is an n-cell containing p.

Thus, $\beta(K) = \{p \in K \mid d(p) = n-1,\ p \subseteq \partial s(K)\}$. A straightforward argument shows that the boundary complex of the polytope $s(K)$ is subdivided by a subcomplex of K. As that boundary complex is a union of (n-1)-polytopes, it follows from the first paragraph that the corresponding subcomplex of K is a union of (n-1)-cells. Thus it is generated by $\beta(K)$, and must equal ∂K. It follows that $s(\partial K) = \partial s(K)$. □

11. THEOREM: Any PL-homeomorphism of polytopes preserves boundary. Thus the boundary of a polyhedral ball is well defined.

PROOF: Let $f : P_1 \to P_2$ be a PLH, where P_1 and P_2 are polytopes.

Choose triangulations K_1 of P_1, K_2 of P_2 relative to which f is simplicial. Clearly $\sigma \in \partial K_1$ if $f(\sigma) \in \partial K_2$, thus, $s(\partial K_2) = f(s(\partial K_2$
By Theorem (10), $s(\partial K_i) = P_i$, $i = 1.2$, so $f(\partial P_1) = \partial P_2$. For a general polyhedral ball P, if $f_1 : P_1 \to P$, $f_2 : P_2 \to P$ are PLH, where P_1, P_2 are polytopes, we have that $f_2^{-1} : P_1 \to P_2$ is a PLH, thus:

$$f_2(\partial P_2) = f_2(\partial f_2^{-1} f_1(P_1))$$
$$= f_2 f_2^{-1} f_1(\partial P_1), \text{ by the first part,}$$
$$= f_1(\partial P_1).$$

□

III) PL-BALLS AND PL-SPHERES

Recall that a PL ball-complex may be defined as a pointed ball complex whose cells are polyhedral balls. That includes polyhedral complexes. In particular, Theorem (4.III.1) applies to those complexes. Actually, a stronger result holds, as expected.

1. THEOREM: Let K, L be PL ball-complexes and $f:K\to L$ be an order isomorphism. Then there exists a PL-homeomorphism $h : s(K) \to s(L)$ such that $h(p) = f(p)$ for every $p\in K$.

PROOF: The same inductive proof of (4.III.1) applies. Thus, one assumes inductively that the PLH h has been defined on cells of K of dimension $<n$, and extends h to the n-cells by the following Lemma, which is a counterpart of (4.III.5). □

2. LEMMA: Any PL-homeomorphism between the boundaries of two polyhedral balls can be extended to a PLH of the balls.

PROOF: It is enough to consider the case where one has a PLH $f : \partial I^n \to \partial I^n$. Define $f(\underline{0}) = \underline{0}$; extend f linearly over each segment $\underline{0}x$, with $x\in\partial I^n$. Then f is a homeomorphism $I^n \to I^n$. It is PLH, since f is affine on each polytope $conv(P\cup\{\underline{0}\})$ where $P\subseteq\partial I^n$ and $f|P$ is affine. □

A PL-sphere is a complex order isomorphic to a PL ball-complex whose space is a polyhedral sphere. PL-balls are defined similarly. It follows from Theorem (1) that a polyhedral complex is PL-ball (sphere) iff its space is a polyhedral ball (sphere).

3. PROPOSITION: Every closed cell (subcomplex) of a PL ball-complex is a PL-ball and its boundary subcomplex is a PL-sphere.

PROOF: Direct from the definitions. □

4. LEMMA: Every PL ball-complex K can be subdivided by a simplicial complex L so that for each $p\in K$, the subcomplex $L(p) = \{\sigma\in L|\sigma\subseteq p\}$ is order isomorphic to a subdivided polytope.

PROOF: Each cell of K, being a polyhedral ball, has a triangulation t_p isomorphic to a subdivided polytope. By (II.5), all these triangulations can be subdivided via a triangulation L of s(K). Thus, for each $p\in K$, L(p) is a subcomplex of L, and $s(L(p))=p$, which shows that L subdivides K. Further, L(p) is a simplicial subdivision of t_p; the PLH from p to a polytope P which is simplicial relative to t_p induces a subdivision of P isomorphic to L(p). □

5. THEOREM: Let K be a PL ball-complex. For each $p\in K$, define $d(p)=n$ if p is a polyhedral n-ball. Then K is JD with dimension function d. Further, let a simplicial complex L subdivide K as in Lemma (4). If L is PM, so is K and ∂L subdivides ∂K.

PROOF: Subdivide K by L as in Lemma (4). Let $p \in K$ have dimension $n \geq 1$. By (II.10), $\partial L(p)$ triangulates ∂p. Any $\sigma \in B(L(p))$ lies in some cell q of K which lies in the boundary of p. Since $n-1 = d(\sigma) \leq d(q) < d(p) = n$, $d(q) = n-1$, and we see that the maximal cells of K properly contained in p have dimension n-1. This shows that d is the dimension function of K.

We now suppose that L is PM. Call n the dimension of L. Every n-simplex is contained in a cell of K_1 which must thus have dimension n. Hence the n-cells of K contain all maximal simplices of L, so their union is s(K), and it follows that K is the union of its closed n-cells, i.e., it is pure dimensional.

Let p be an (n-1)-cell of K and choose $\sigma \in L(p)$ of dimension n-1. Every n-cell containing p must contain an n-simplex containing σ; conversely, if $q \in K$, $d(q) = n$ and $\sigma \in L(q)$, then $p \subseteq q$ and $\sigma \in L(q)$. Thus, $\sigma \in B(L)$ if there is only one q such that $\sigma \in L(q)$; equivalently, if $p \in B(K)$. Thus $\partial L = \cup\{[\sigma] \mid \sigma \in B(L)\} = \cup\{L(p) \mid p \in B(K)\}$, and this is what had to be proved. □

The next result encompasses Proposition (I.3):

6. THEOREM: A PL n-ball is PM, has dimension n, and its boundary is a PL (n-1)-sphere. A PL n-sphere is PM, has dimension n, and no boundary.

PROOF: Both assertions follow from a combination of Theorems (5) and (II.10). □

We come now to the point of proving the important Theorem (I.4). It is convenient to fix the following notation:

7. Let e_1, $e_2,\ldots,e_n$ be the cannonical basis of $\mathbb{R}^n$.

Denote = P^n = conv($\pm e_1$, $\pm e_2,\ldots,\pm e_n$),

$P^n_+ = P^n \cap \mathbb{R}^n_+ = \text{conv}(P^{n-1}+e_n)$,

$P^n_- = P^n \cap \mathbb{R}^n_- = -P^n_+$,

$Q^{n-1}_+ = \partial P^n_+ - \text{int } P^{n-1}$,

$Q^{n-1}_- = -Q^n_+$.

Note that Q^{n-1}_+ and Q^{n-1}_- are polyhedral (n-1)-balls: vertical projection onto P^{n-1} is in either case a PLH.

Recall that, in the statement of (I.4), K is a complex, K_1, K_2 are subcomplexes of K, each a PL n-ball, and $K_1 \cup K_2$ = K.

8. PROOF of (I.4.1): Suppose that $K_1 \cap K_2 = \partial K_1 = \partial K_2$. For i=1,2, let W_i be a PL ball-complex and $f_i:K_i \to W_i$ be an order isomorphism. Let $g_1:s(W_1)\to Q^n_+$ be a PLH, which exists since both are polyhedral balls. Now, restricted to ∂W_2, $f_1 f_2^{-1} : \partial W_2 \to \partial W_1$ is an order isomorphism, to which corresponds, by Theorem (1) a PLH $h:s(\partial W_2)\to s(\partial W_1)$. Thus $g_1 h$ is a PLH from $s(\partial W_2)$ to $\partial Q^b_+ = \partial P^n = \partial Q^n_-$. It can be extended by Lemma (2) to a PLH $g_2:s(W_2)\to Q^n_-$. The set $W = \{g_1(p) | p \in W_1\} \cup \{g_2(p) | p \in W_2\}$ is then a polyhedral complex with space ∂P^{n+1}, and there is an order isomorphism $f:K \to W$ given by $f(p) = g_1 f_1(p)$ or $g_2 f_2(p)$, whichever is defined (if both are, they agree). Thus K is a

PL n-sphere. □

The idea of the proof of (I.4.2) is essentially the same as above, namely, given PL n-balls K_1, K_2, we try to realize one as a ball complex with space P^n_+, the other on P^n_-, so that $K_1 \cap K_2$ is realized on $P^n_+ \cap P^n_-$. To do this, we shall need "Newman's Theorem":

9. THEOREM: If K is a PL n-sphere and a subcomplex L of K is a PL n-ball, then the subcomplex [K-L] is a PL n-ball. Similarly, if a polyhedral n-ball is removed from a polyhedral n-sphere, the closure of what remains is a polyhedral n-ball.

10. PROOF of (I.4.2.): As before, let $f_i : K_i \to W_i$ be order isomorphisms, where W_1, W_2 are ball complexes and let $h : s(f_2(K_1 \cap K_2) \to s(f_1(K_1 \cap K_2))$ be a PLH compatible as in Theorem (1) with the order isomorphism $f_1 f_2^{-1} : f_2(K_1 \cap K_2) \to f_1(K_1 \cap K_2)$. Let $g_1 : s(f_1 K_1 \cap K_2)) \to P^{n-1}$ be a PLH, which exists since by hypothesis $K_1 \cap K_2$ is a PL (n-1)-ball. Consider now $L_1 = [\partial K_1 - K_1 \cap K_2]$. By Newman's Theorem, L_1 is a PL (n-1)-ball, and clearly $\partial L_1 = \partial(K_1 \cap K_2)$, since ∂K_1 is PM without boundary by Theorem (6). Now, g_1 gives a PLH from $s(f(\partial L_1))$ to ∂Q_+^{n-1}. By Lemma (2) we can extend g_1 to a PLH of $s(f(L_1))$ to Q_+^{n-1}. So, right now, g_1 is a PLH from $s(f(\partial K_1)) = \partial s(W_1)$ to ∂P^n_+, and another application of Lemma (2) extends g_1 to a PLH $s(W_1) \to P^n_+$. By defining $g_2 : s(f_2(K_1 \cap K_2)) \to P^{n-1}$ by $g_2 = g_1 h$, we can repeat the previous argument and extend g to a PLH from $s(W_1)$ to P^n_-. Note that for each $p \in K_1 \cap K_2$,

$g_2f_2(p) = g_1hf_2(p) = g_1\, f_1(p)$, by the construction of h. Thus $\{g_1f_1(p)|p\in K_1\}\cup\{g_2f_2(p)|p\in K_2\}$ is a PL ball-complex with space P^n and order isomorphic to K. This shows that K is a PL n-ball.

□

Known proofs of Newman's Theorem require more specialized machinery which would be of little use in what follows. Proofs of the polyhedral statement can be found in Cohen [Co], Hudson [Hu, 1.2.6]. Proofs of a very different nature were given by Newman [NE1] and Alexander [A1]. The "PL" part of the theorem follows easily from the polyhedral part.

This theorem captures one of the main differences between PL-spheres and ball complexes whose space is a topological sphere. Indeed, combining results of Curtis and Zeeman [CZ] and of Edwards [Ed] and Cannon [Can], it is possible to produce a simplicial complex whose space is a 5-sphere and such that the removal of a 5-simplex leaves something other than a ball.

We shall have occasion later for using the following result. We state it in both PL (polyhedral) forms:

11. THEOREM: Let K be a PL (polyhedral) n-ball, and let L be a subcomplex (subpolyhedron) of K which is also a PL (polyhedral) n-ball, and such that $L\cap\partial K$ is a PL (polyhedral) (n-1)-ball. Then, [K-L] (the closure of K-L) is a PL (polyhedral) n-ball.

PROOF: See [Hu, 2.14] for the (disguised) polyhedral statement. The PL statement follows easily from that. □

IV) JOINS, DERIVEDS AND STARS

A set $P \subseteq \mathbb{R}^n$ and a point $v \in \mathbb{R}^n - P$ are joinable if the open segments $\overline{xy}$, $y \in P$ are pairwise disjoint. In this case, the join $x.P$ is defined by $x.P = \{\lambda x + (1-\lambda)y \mid 0 \le \lambda \le 1, y \in P\}$; and by convention, $x.\phi = \{x\}$. If P is a polytope, $x.P = \mathrm{conv}(P+x)$, thus a polytope. Hence, if P is a polyhedron, so is v.P. Also, if K is a polyhedral complex, then $v.K = K \cup \{v.p \mid p \in K\}$ is a polyhedral complex whose space is $v.s(K)$. Note that $\phi \in K$, thus $\{v\} = v.\phi \in v.K$.

The abstract join of a complex K with minimum ϕ and a point $v \notin s(K)$ is the complex $v.K = K \cup \{p+v \mid p \in K\}$. We say that a complex L is a join of K and v, if $K+\{v\}$ is a subcomplex of L and there is an order isomorphism $L \to v.K$ which is the identity on $K+\{v\}$. The context will make it clear what type of join is being handled.

If $f:P \to Q$ is a PL-map, and the joins v.P and w.Q are defined, then $\overline{f}$ given by $\overline{f}(\lambda v + (1-\lambda)y) = \lambda w + (1-\lambda)f(y)$ $(y \in P)$ is a PL-map from v.P to v.Q. If f is a PLH, so is $\overline{f}$.

In particular, if P is a polyhedral n-ball, v.P is a polyhedral (n+1)-ball. That is true, as if T is an n-simplex and the point v is joinable to T, v.T is an (n+1)-simplex. Also, since $\partial v.T = T \cup v.\partial T$, it follows from invariance of boundary that $\partial v.P = P \cup v.\partial P$. Similarly, for complexes we have:

1. PROPOSITION: Let K be a complex. The property of being JD, pure

dimensional or PM holds for v.K iff it holds for K. In the latter case, $\partial v.K = K$ if $\partial K = \emptyset$, $\partial v.K = K \cup v.\partial K$ if $\partial K \neq \emptyset$.

PROOF: Straightforward. □

2. PROPOSITION: If K is a PL n-ball or a PL n-sphere, than v.K is a PL (n+1)-ball. If K is a PL ball-complex, and v is joinable to s(K), then $v.K = K \cup \{v.P \mid p \in K\}$ is a PL ball-complex.

PROOF: If K is a PL ball-complex, then v.K is a collection of polyhedral balls, and the formulae for $\partial(v.p)$ and $\partial(v.[p])$ show that K is a PL ball-complex. For the first part, we may assume that K is a PL ball-complex and that s(K) is either an n-polytope or the boundary of an (n+1)-polytope P. Choose a point v joinable to s(K), and interior to P in the second case. Then v.K is a PL ball-complex and its space is a (n+1)-polytope, due to the choice of v. □

As an application of this result, we define the derived of a complex, and use it in the study of PL ball-complexes.

Given a complex K, a <u>derived</u> K^* of K is a simplicial complex with vertices $\{\hat{p} \mid p \in K-\underline{0}\}$ in 1-1 correspondence with non-zero cells of K, and such that $\{\hat{p}_1, \hat{p}_2, \dots, \hat{p}_k\}$ are the vertices of a member of K^* if $p_1 < p_2 < \dots < p_k$. Note that any two deriveds of K are isomorphic, thus PLH.

When a derived K^* of K is fixed, one has naturally the derived of each subcomplex of K contained as a subcomplex of K^*; we refer to these implicitly in what follows.

One easily verifies that when K_1, K_2 are subcomplexes of K,

$(K_1 \cup K_2)^* = K_1^* \cup K_2^*$,

$(K_1 \cap K_2)^* = K_1^* \cap K_2^*$,

$[p] = \hat{p}.(\partial[p])^*$

$K = \cup\{[p]^* | p \text{ maximal in } K\}$.

As we pointed out in section II, a polyhedral complex K has a derived in which $\hat{p} \in int(p)$ and $s([p]^*) = p$ for each $p \in K-Q$. For PL ball-complexes, something similar exists:

3. THEOREM: When K is a PL ball-complex there is a PL-homeomorphism $h: s(K^*) \to s(K)$ such that for each $p \in K$, $h(\hat{p}) \in int(p)$ and $h(s([p]^*)) = p$.

PROOF: Define $h(\hat{p}) = p$ for each vertex p of K. Suppose that h has been defined on $s(K_{n-1}^*)$, where $K_{n-1} = \{p \in K | d(p) \le n-1\}$. The special properties of h imply that for any subcomplex L of K_{n-1}, $h(s(L^*)) = s(L)$. In particular, if $p \in K$, $d(p) = n$, then $h(s(\partial[p])^*) = \partial p$ and $(\partial[p])^*$ is a PL-sphere (simplicial). Hence, $[p]^* = \hat{p}.Q[p])^*$ is a PL-ball, and one can extend h from its boundary to a PLH from $s([p]^*)$ to p. Clearly $h(\hat{p}) \in int(p)$. Doing this simultaneously for all n-cells of K, one extends h to a PLH from

$s(K_n^*)$ to $s(K_n)$. The result follows by induction on n. □

Indeed, this theorem still holds if one removes all "PL" from its statement; the proof is essentially the same. Thus, the following corollary also holds without any mention to PL-ness.

4. COROLLARY: The group of order automorphisms of a PL ball-complex can be faithfully represented by a group of PL self-homeomorphisms of its space, each homeomorphism compatible with the corresponding automorphism.

PROOF: Let K be the complex, and $h = s(K^*) \to s(K)$ be given by Theorem (3). For each automorphism g of K, let $\hat{g}$ be the permutation of the vertices of K* defined by $\hat{g}(\hat{p}) = \widehat{g(p)}$. Since g preserves chains in K, $\hat{g}$ induces an order automorphism of K^*, to which corresponds a PLH, still denoted $\hat{g}$, by affine extension. The representation of Aut(K) is given by $g \mapsto h\hat{g}h^{-1}$. Verification is straightforward. □

We introduce now a few more objects which are usefull to look at in a complex. Let K be a complex with minimum $\underline{0}$ and $p \in K$. There are defined (fig.5.2) the <u>star</u> of p in K, $st(p,K) = \{q \in K \mid p \leq q\}$, the <u>shell</u> of p in K, $sh(p,K) = [st(p,K)] - st(p,K)$, the <u>link</u> of p in K, $lk(p,K) = \{q \in K \mid p \wedge q = \underline{0}$ and $p \vee q$ some cell of K$\}$.

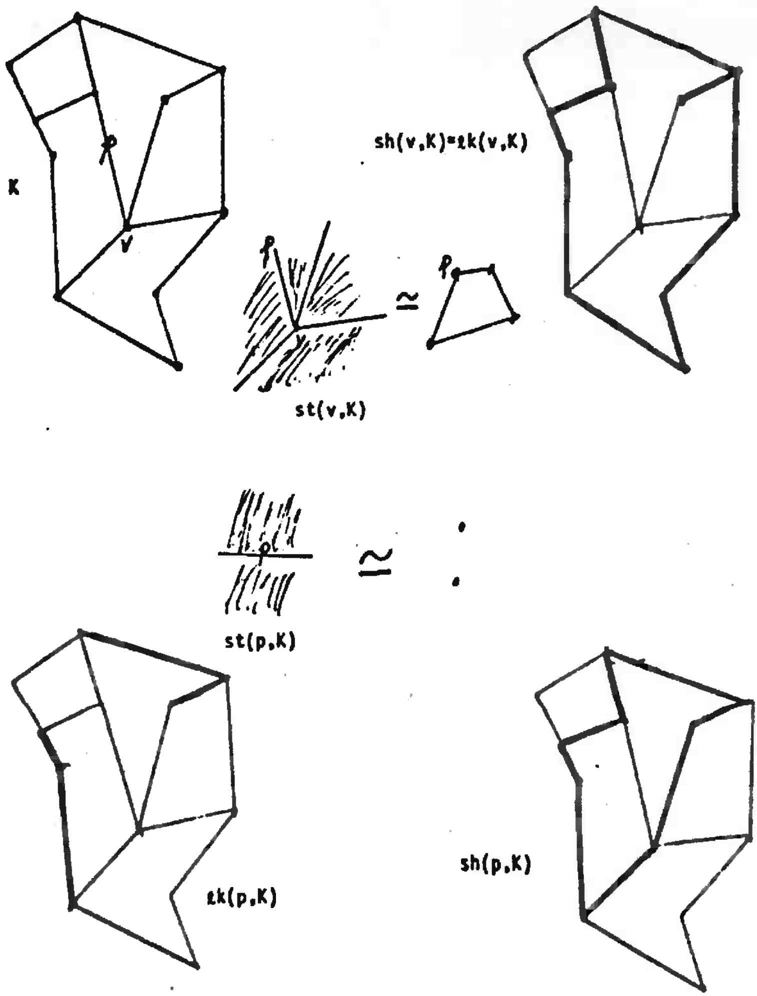

Fig. 5.2

Thus, sh(p,K) and lk(p,K) are subcomplexes of K while the star is not. However, st(p,K) is a complex on its own right and a most useful one.

We register without proof some elementary properties of these objects.

5. PROPOSITION: Let v be a vertex of a complex K. Then sh(v,K) = lk(v,K); further, if K is simplicial, then sh(v,K) is order isomorphic to st(v,K), and [st(v,K)] = v.sh(v,K). If $p \subseteq q$ then st(q,K) = st(q,st(p,K)). □

The notion of shell does not appear in the literature. What one encounters are mainly simplicial complexes and links are favoured in that context. We have little use for links, and shall work mainly with shells and stars.

6. THEOREM: The star of a k-cell in a PL n-sphere is a PL (n-k-1)-sphere. The star of a k-cell of the boundary of a PL n-ball is a PL (n-k-1)-ball.

PROOF: The proof is written for spheres only. A proof for balls requires only slight modifications.

It is enough to consider the case k=0 only, the star of a vertex. For if p_k is a k-cell of a PL n-sphere K, let $p_0 \subsetneq p_1 \subsetneq \dots \subsetneq p_k$ be a chain in K, where $k(p_i) = i$. Let K_i = st(p_i,K); then K_i = st(p_i,K_{i-1}), by part of (5) above, and p_i is a vertex of K_{i-1}. Thus, by knowing that the theorem is true for stars of vertices, one obtains the general case by iteration. So, for now we consider only the case k = 0.

The proof is by induction on n, and obvious for $n = 0$. Suppose now that $n>0$, let K be a PL n-sphere and v a vertex of K. Choose a derived K^*, and define, for each $p \in st(v,K)$,

$$t_p = sh(\hat{v},[p]^*) = \hat{p}.sh(\hat{v},\partial[p]^*).$$

For each p, t_p is a subcomplex of K^*; in particular, $t_v = \{\hat{v}\}$. Now, since each $\partial[p]$ is a PL k-sphere ($k<n$), by (III.3) so is $\partial[p]^*$, by Theorem (3); and the induction hypothesis implies that $st(v,\partial[p]^*)$ is a PL-sphere. Since $st(v,\partial[p]^*)$ and $sh(v,\partial[p]^*)$ are order isomorphic, $sh(v,\partial[p]^*)$ is a PL-sphere, and t_p is a PL-ball, by Proposition (2). One easily verifies that $T = \{s(t_p) | p \in st(v,K)\}$ is a PL ball-complex and $p \mapsto s(t_p)$ gives an order isomorphism from $st(v,K)$ to T.

Thus the result will be completed if we show that s(T) is a polyhedral (n-1)-sphere. Now, $s(T) = s(\cup\{sh(\hat{v},[p]^*) | p \in st(v,K)\})$
$= s(sh(\hat{v},K^*))$.

Since $sh(\hat{v},K^*)$ is order isomorphic to $st(\hat{v},K^*)$, what we are required to prove amounts to a special case of the theorem; we state that explicitly, in polyhedral form.

7. LEMMA: Let v be a vertex of the simplicial complex K. If K is a PL n-sphere, s(sh(v,K)) is a polyhedral (n-1)-sphere. If K is a PL n-ball and $v \in \partial K$ then s(sh(v,K)) is a polyhedral n-ball.

PROOF: We shall be only the sphere case. Let T be an (n+1)-simplex and its face complex. Since s(K) is a polyhedral n-sphere, there exists a PLH $h: s(K) \to \partial T$.

Theorem (II.9) provides a PLH $g : T \to \partial T$ such that $gh(v)$ is a vertex of T. Denote $f = gh$, and choose subdivisions K^1 of K T^1 of ∂T such that f is simplicial relative to these.

Radial projection from v induces a simplicial subdivision of sh(v,K) isomorphic to $sh(v,K^1)$. This gives a PLH $s(sh(v,K)) \to s(sh(V,K^1))$. Similarly one obtains a PLH $s(sh(f(v),T^1) \to s(sh(f(v),\partial T))$. Simpliciality of f implies that the restriction $f : s(sh(v,K^1)) \to s(sh(f(v),T^1))$ is a PLH. Combining these three PLH, one obtains a PLH $s(sh(v,K)) \to s(sh(f(v),\partial T))$. Since the shell of a vertex in the boundary of an (n+1)-simplex is the boundary of an n-simplex, the result is proved. □

Theorem (6) gives a property of PL-spheres which to a certain extent is what characterizes these among (PL) ball-complexes whose space is a topological sphere. Sullivan [Su] has proved the "Hauptvermutung" for spheres: if a PL ball-complex has for space a topological sphere and the star of each vertex is a PL-sphere, then the complex is a PL-sphere, provided the dimension is not 4. The 4-dimensional case has not been settled yet, as far as I know.

Recall that the polar of a complex K is the poset whose members are the non-zero cells of K, plus a new $\underline{0}$ element, with the order: $p<q$ iff $q<p$. As the polar of a polytope boundary is order isomorphic to a polytope boundary, one is motivated to study the polar of a

tope boundary as an abstraction of polytopes within oriented matroids. There exists a tope whose polar is not isomorphic to any tope, so the two abstractions give rise to distinct classes of complexes [Mu].

However, the polar of a tope is a PL-sphere, by virtue of:

8. THEOREM: The polar of a PL-sphere is a PL-sphere.

PROOF: The idea is that a complex and its polar have the same derived. Let K be a PL-sphere, and fix a derived K^*. If $p \in K$, $st(p,K)^*$ is a subcomplex of K^*, even though $st(p,K)$ is not a subcomplex of K. For each $p \in K$, let $W_p = \hat{p}.st(p,K)^*$, which is also a subcomplex of K^*. By Theorem (6), $st(p,K)$ is a PL-sphere, hence so is $st(p,K)^*$. It follows that each W_p is a PL-ball. It is routine now to verify that $W = \{s(W_p) \mid p \in K\}$ is a ball complex, the map $p \mapsto s(W_p)$ is an order isomorphism from the polar of p to W, and that $s(W) = s(K^*)$. Since $s(K^*)$ is a polyhedral sphere, by Theorem (3), that is all that is needed to prove the theorem. □

The previously mentioned complexes due to Edwards [Ed] and Cannon [Can] are simplicial complexes whose space is a topological sphere but such that the shells of some vertices are not simply connected. The polar of one such complex is not order isomorphic to any ball complex.

From now to the end of this section, K is a PL ball-complex and v is a vertex of K. A <u>star-neighbourhood of v in K</u> is a polyhedron $D \subset s(K)$ satisfying:

(a) For every $p \in K$, $D \cap int(p) \neq \phi$ iff $v \leq p$,

(b) For every $p \in st(v,K)-\{v\}$, $D \cap p$ and $D \cap \partial p$ are polyhedral balls, $d(p) = d(D \cap p) = d(D \cap \partial p)+1$.

9. PROPOSITION: Let $h:s(K^*) \to s(K)$ be a PLH as in Theorem (3). Then $D = h(s(\hat{v},sh(\hat{v},K^*)))$ is a star neighbourhood of v in K.

PROOF: From the properties of h listed in (3) we deduce (a). Property (b) follows from the facts:

$D \cap p = h(s(\hat{v} \cdot sh(\hat{v},[p]^*)))$,

$D \cap \partial p = h(s(\hat{v} \cdot sh(\hat{v},\partial [p]^*)))$.

Since $\partial [p]^*$ is a PL (k-1)-sphere, where $k = d(p)$, it follows from Theorem (6) that $sh(\hat{v},\partial [p]^*)$ is a PL (k-2)-sphere. Thus, $\hat{v} \cdot sh(\hat{v},\partial [p]^*)$ is PL (k-1)-ball, and $\hat{v} \cdot sh(\hat{v},[p]^*) = \hat{v} \cdot (\hat{p}.sh(\hat{v},\partial [p]^*))$ is a PL k-ball, by Proposition (2). □

For each $p \in st(v,K)-\{v\}$, let t_p denote the closure $\partial(D \cap p)-(D \cap \partial p)$; it is a polyhedral ball, by (III.11), and its interior is $t_p \cap int(p)$. Let $K_D = \{t_p \mid p \in st(v,K)\}$.

10. PROPOSITION: K_D is a PL ball-complex, and $p \to t_p$ is an order isomorphism $st(v,K) \to K_D$. Also $T(K,D) = K_D \cup \{D \cap p \mid p \in st(v,K)\}$ is a PL ball complex with space D, and order isomorphic to $v.K_D$.

PROOF: The order isomorphism $v.K_D \to T(K,D)$ is given by $p \cdot v \to D \cap p$, $p \to t_p$, for each $p \in st(v,K)$. Verification of the assertions is routine. □

In particular, it follows from (10) that any two star neighbourhoods D, D' of v in K are PLH. In fact, one can prove that there exists a self PLH of K that fixes all cells of K setwise, and throws D onto D'. Hence, all star neighbourhoods are as described in Proposition (9).

11. PROPOSITION: If K is a PL n-sphere, K_D is a PL (n-1)-sphere. If K is a PL n-ball and $v \in \partial K$, then K_D is a PL (n-1)-ball.

PROOF: Just combine (10) and (7.). □

V) SEMILATTICES AND PULLING

In spite of the strong relationship between the concepts of star, shell and link, we have no analogue to Theorem (IV.6) for shells or links with the same scope. For simplicial complexes, stars and links are the same up to order isomorphism, so Theorem (IV.6) gives information about links in simplicial PL-spheres. I have no idea about how links appear in general PL-spheres; in fact, even links in PL-spheres which are polyhedral complexes are elusive.

In contrast, shells behave much better, as long as one imposes a little restriction on the class of complexes being considered. The restriction, presented below, is not too strong, and allows for a nice development of "pulling".

A complex is called a semilattice if the intersection of any two cells is a cell. In this case, $[p \cap q] = [p] \cap [q]$.

A complex K is said to be locally semilattice at a cell p if for every $q \in sh(p,K)$, $\cap\{r \in K | p \cup q \subseteq r\}$ is a cell of K. We denote that cell as $p \vee q$; if K is a lattice under inclusion, $p \vee q$ is just the lattice join of p and q.

It is clear that a semilattice is locally semilattice at each cell. The converse is also true, but will not be needed.