FINITE $\mathcal{A}$-DETERMINACY OF GENERIC HOMOGENEOUS MAP GERMS IN $\mathbb{C}^3$

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Abstract. Denote by $H(d_1, d_2, d_3)$ the set of all homogeneous polynomial mappings $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{C}^3$, such that $\deg f_i = d_i$. We show that if $\gcd(d_i, d_j) \leq 2$ for $1 \leq i < j \leq 3$ and $\gcd(d_1, d_2, d_3) = 1$, then there is a non-empty Zariski open subset $U \subset H(d_1, d_2, d_3)$ such that for every mapping $F \in U$ the map germ $(F, 0)$ is $\mathcal{A}$-finitely determined. Moreover, in this case we compute the number of discrete singularities (0-stable singularities) of a generic mapping $(f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{C}^3$, where $\deg f_i = d_i$.

1. Introduction

Let $\Omega(d_1, \ldots, d_n)$ denote the set of all polynomial mappings $F = (f_1, \ldots, f_n) : \mathbb{C}^n \to \mathbb{C}^n$ such that $\deg f_i = d_i$. We have proved in [1] that there is an Zariski open subset $U \subset \Omega(d_1, \ldots, d_n)$ such that for every $F \in U$ the mapping F is transversal to the Thom-Boardman varieties and satisfies the normal crossings property. Moreover, by [3] all such mappings are topologically equivalent, in particular they have the same number of discrete singularities. If $U_0 \subset \Omega(d_1, \ldots, d_n)$ is the maximal Zariski open subset with these properties (i.e., for every $F \in U_0$ the mapping F has constant topological type and it is transversal to the Thom-Boardman varieties and satisfies the normal crossings property) then we say that every mapping $F \in U_0$ is a generic mapping.

Let $F \in \Omega(d_1, \ldots, d_n)$ be a generic polynomial mapping. In particular in Mathers nice dimensions (see [6]) F is a stable mapping. In [1] we have computed the number of cusps and nodes for F in dimension $n = 2$. Now we would like to compute the number of discrete singularities (0-stable singularities) in dimension $n = 3$.

Note that a generic polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^n$ can be defined at infinity only if $d_1 = d_2 = \ldots = d_n = d$. However even in this case the mapping F (if non-linear) has to be degenerate at infinity, i.e., the whole hyperplane at infinity is a component of the critical set of F. Indeed the topological degree of F is $\mu(F) = d^n$, but the mapping F restricted to the infinity has topological degree at most d^{n-1}. Hence the critical set of F is not smooth and consequently such a mapping can never be stable as a mapping from $\mathbb{P}^n$ to $\mathbb{P}^n$. In particular we cannot use here global techniques based on Thom polynomials.

However, in some cases we can apply local methods using Thom polynomials described by Ohmoto [9] (see also [7], [8], [4]). Indeed, let $F : \mathbb{C}^3 \to \mathbb{C}^3$ be a generic mapping. Since the pair $(3, 3)$ is a pair of nice dimensions, the mapping F is stable. For $F = (f_1, f_2, f_3) \in \Omega(d_1, d_2, d_3)$ we denote by $\overline{f}_i$ the homogeneous part of f_i of degree d_i and set $F_0 = (\overline{f}_1, \overline{f}_2, \overline{f}_3)$. Hence F_0 has a stable deformation $F_t(x) = (t^{d_1} f_1(x/t), t^{d_2} f_2(x/t), t^{d_3} f_3(x/t))$. Assume that $(F_0, 0)$ is a finitely $\mathcal{A}$-determined germ. Since the deformation F_t contracts all

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discrete singularities to 0 as $t \to 0$, we can compute the number of discrete singularities of F using the local formulas of Ohmoto for the mapping F_0. Hence the fundamental problem here is to describe finitely $\mathcal{A}$-determined homogeneous mappings $H : \mathbb{C}^3 \to \mathbb{C}^3$. We denote by $H(d_1, d_2, d_3)$ the set of all homogeneous polynomial mappings $F = (f_1, f_2, f_3) : \mathbb{C}^3 \to \mathbb{C}^3$, such that $\deg f_i = d_i$. Our first main result is:

Theorem 1.1. *If* $\gcd(d_i, d_j) \leq 2$ *for* $1 \leq i < j \leq 3$ *and* $\gcd(d_1, d_2, d_3) = 1$ *then there is a non-empty Zariski open subset* $U \subset H(d_1, d_2, d_3)$ *such that for every mapping* $F \in U$ *the map germ* $(F, 0)$ *is finitely* $\mathcal{A}$*-determined.*

On the other hand if $\gcd(d_i, d_j) > 2$ *for some* $i, j \in \{1, 2, 3\}$, $i \neq j$ *or* $\gcd(d_1, d_2, d_3) > 1$, *then there are no finitely* $\mathcal{A}$*-determined homogeneous map germs with degrees* d_1, d_2, d_3.

This is an extension of a part of a well-known two-dimensional result of Gaffney-Mond [2] to dimension three. In fact Gaffney and Mond provide a classification of finitely $\mathcal{A}$-determined quasi-homogeneous $(\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ map germs of corank 1 and determine the admissible weights and degrees for germs of corank 2. Note that our method is ill-suited for the weighted-homogeneous case. In the homogeneous case we use an action of the linear group to vastly simplify the necessary computations. In the weighted-homogeneous case the action is no longer available and the computations become prohibitively complicated.

Theorem 1.1 has the following nice application:

Theorem 1.2. *If* $\gcd(d_i, d_j) \leq 2$ *for* $1 \leq i < j \leq 3$ *and* $\gcd(d_1, d_2, d_3) = 1$ *then there is a non-empty Zariski open subset* $U_1 \subset \Omega(d_1, d_2, d_3)$ *such that for every mapping* $F \in U_1$ *we have:*

- *F is stable, in particular the discrete mono- or multi-singularities are of type A_3, A_2A_1 or A_1^3,*
- *F has precisely $\#A_3 = c_1^3 + 3c_1c_2 + 2c_3$ singularities of type A_3,*
- *F has precisely $\#A_2A_1 = (P-3)s_1\#A_2 - 3\#A_3$ singularities of type A_2A_1,*
- *F has precisely $\frac{1}{6}\left[(P^2 - 3P + 2)s_1^3 - 6\#A_2A_1 - 6\#A_3 - 3s_1\#A_1^2 - 4s_1\#A_2\right]$ singularities of type A_1^3.*

Here $s_1 = d_1 + d_2 + d_3 - 3$, $s_2 = (d_1 - 1)(d_2 - 1) + (d_1 - 1)(d_3 - 1) + (d_2 - 1)(d_3 - 1)$, $s_3 = (d_1 - 1)(d_2 - 1)(d_3 - 1)$, $P = d_1d_2d_3$, $c_1 = s_1$, $c_2 = s_2 - s_1$, $c_3 = s_3 - 2s_2 + s_1$, $\#A_2 = c_1^2 + c_2$ *and* $\#A_1^2 = (P-2)s_1^2 - 2\#A_2$.

Remark 1.3. The proof works only for finitely $\mathcal{A}$-determined map germs, i.e., for d_1, d_2, d_3 as in Theorem 1.1. However, we intend to prove in a separate paper, by using global methods rather then local, that the formula for the number of A_3 singularities holds for all degrees. However the formulae for the numbers of A_2A_1 and A_1^3 singularities depend on $\gcd(d_1, d_2, d_3)$.

2. Main result

For a polynomial mapping $F : \mathbb{C}^n \to \mathbb{C}^m$ let us denote by $C(F)$ the set of critical points of F and by $\Delta(F) = F(C(F))$ the discriminant of F.

Moreover, we call a line through the origin a *ray*. We will denote by $(\mathbb{C}^n)^{*t}$ the set $\{(p_1, \ldots, p_t) : \ p_i \in \mathbb{C}^n,\ p_i \neq 0 \text{ and } p_i \neq p_j \text{ for } i \neq j\}$. If $p \in (\mathbb{C}^n)^*$ then we denote by $\mathbb{C}p$ the unique ray passing through p. Here we prove that for certain mappings the critical set is smooth outside 0 and the discriminant has only simple normal crossings outside 0.

Lemma 2.1. *Assume that* $\gcd(d_i, d_j) \leq 2$ *for* $1 \leq i < j \leq 3$ *and* $\gcd(d_1, d_2, d_3) = 1$. *There is a non-empty Zariski open subset* $U \subset H(d_1, d_2, d_3)$ *such that for every mapping* $F = (f_1, f_2, f_3) \in U$*:*

(1) $F^{-1}(0) = \{0\}$,
(2) *if* d_1, d_2, d_3 *are pairwise co-prime then* F *restricted to any ray contained in* $C(F)$ *is injective, if they are not co-prime, i.e.,* d_i *is odd and the other two are even, then* F *restricted to any ray contained in* $C(F) \setminus V(f_i)$ *is injective and* F *restricted to any of the finite number of rays contained in* $C(F) \cap V(f_i)$ *is* $2:1$,
(3) $F_{|C(F)}$ *is injective outside a finite set of rays,*
(4) *if* $p \in \Delta(F)$ *then* $\#(F^{-1}(p) \cap C(F)) \leq 2$,
(5) *outside the origin the singularities of* F *are either folds or cusps, in particular* $C(F) \setminus \{0\}$ *is smooth,*
(6) *if* F *has a cusp at* p *then* $F^{-1}(F(p)) \cap C(F) = \{p\}$,
(7) *if* $\#(F^{-1}(p) \cap C(F)) = 2$ *then the surface* $\Delta(F)$ *has a normal crossing at* p.

Proof. We will consider the sets $X_1, \ldots, X_7 \subset (\mathbb{C}^3)^{*t} \times H(d_1, d_2, d_3)$, where $t \in \{1, 2, 3\}$, consisting of points and mappings that do not satisfy the assertions above. We will show that $\dim(X_1), \ldots, \dim(X_7) \leq \dim(H(d_1, d_2, d_3))$ and consider the projections $X_i \to H(d_1, d_2, d_3)$. The inequality between dimensions shows, that there is a non-empty Zariski open subset $U \subset H(d_1, d_2, d_3)$ over which the fibers of the projections are finite. However, since we consider homogeneous mappings if a point (in $(\mathbb{C}^3)^{*t}$) is in the fiber then the whole ray through this point must also be in the fiber, i.e., the fibers are either empty or infinite. Consequently mappings in U satisfy the desired properties.

The sets X_i will be invariant under linear transformations in the following sense: if $T \in GL(3)$ and $(p_1, \ldots, p_t, F) \in X_i$ then $(T(p_1), \ldots, T(p_t), F \circ T^{-1}) \in X_i$. Consequently, to compute $\dim(X_i)$ we will only have to compute the dimensions of selected fibers (in most cases only one) of the projection $X_i \to (\mathbb{C}^3)^t$.

We denote by $a_{i,j;k}$ the parameters in $H(d_1, d_2, d_3)$ giving the coefficients of f_k at $x^{d_k-i-j}y^iz^j$.

The proofs of all assertions follow the same pattern, thus in later assertions we will omit the details explained in the proofs of earlier ones. When relevant we will first assume that d_1, d_2, d_3 are pairwise co-prime and later consider the case when they are not. By symmetry we may assume that when d_1, d_2, d_3 are not pairwise co-prime then d_1 and d_2 are even and d_3 is odd.

(1) Consider $X_1 = \{(p, F) \in (\mathbb{C}^3)^* \times H(d_1, d_2, d_3) : F(p) = (0, 0, 0)\}$. As explained above we have to show that $\dim(X_1) \leq \dim(H(d_1, d_2, d_3))$ and this follows from the fact that $\dim(X_1 \cap \{(1, 0, 0)\} \times H(d_1, d_2, d_3)) \leq \dim(H(d_1, d_2, d_3)) - 3$. Let X_1' denote $X_1 \cap \{(1, 0, 0)\} \times H(d_1, d_2, d_3)$, we will treat it as a subset of $H(d_1, d_2, d_3)$. We obtain the equations of X_1' by substituting $(1, 0, 0)$ into the equations of X_1, we have $f_1(1, 0, 0) = \sum a_{i,j;k}x^{d_k-i-j}y^iz^j(1, 0, 0) = a_{0,0;1} = 0$ and $f_2(1, 0, 0) = a_{0,0;2} = 0$ and $f_3(1, 0, 0) = a_{0,0;3} = 0$. Thus $X_1' = V(a_{0,0;1}, a_{0,0;2}, a_{0,0;3})$ has codimension 3 in $H(d_1, d_2, d_3)$, as required.

(2) Let $F = (f_1, f_2, f_3)$ and $p \in (\mathbb{C}^3)^*$. Note that if any two of f_1, f_2, f_3 are nonzero at p, say $f_1(p), f_2(p) \neq 0$, then F restricted to $\mathbb{C}p$ is injective. Indeed, if $q = \lambda p$ and $F(p) = F(q)$, then $f_1(p) = f_1(\lambda p) = \lambda^{d_1} f_1(p)$. Thus $\lambda^{d_1} = 1$ and similarly $\lambda^{d_2} = 1$, if $\gcd(d_1, d_2) = 1$ then it follows that $\lambda = 1$. If $\gcd(d_1, d_2) = 2$ then $\lambda = 1$ or $\lambda = -1$.

Thus we have to show that for a generic F we have $C(F) \cap V(f_i, f_j) = \{0\}$. We show the proof for f_1 and f_2, the other two pairs follow by symmetry. Consider $X_2 = \{(p, F) \in (\mathbb{C}^3)^* \times H(d_1, d_2, d_3) : f_1(p) = f_2(p) = J(F)(p) = 0\}$. Where by $J(F)$ we denote the

Jacobian of F. Similarly as in the proof of (1) we define $X_2' = X_2 \cap \{(1,0,0)\} \times H(d_1,d_2,d_3)$ and treat it as a subset of $H(d_1,d_2,d_3)$. We have to show, that X_2' has codimension 3. As for X_1', the first two equations of X_2' are $a_{0,0;1} = 0$ and $a_{0,0;2} = 0$. The third equation is

$$J(F)(1,0,0) = \det \begin{bmatrix} d_1 a_{0,0;1} & a_{1,0;1} & a_{0,1;1} \\ d_2 a_{0,0;2} & a_{1,0;2} & a_{0,1;2} \\ d_3 a_{0,0;3} & a_{1,0;3} & a_{0,1;3} \end{bmatrix} = 0,$$

after substituting $a_{0,0;1} = a_{0,0;2} = 0$ it simplifies to $d_3 a_{0,0;3}(a_{1,0;1}a_{0,1;2} - a_{0,1;1}a_{1,0;2}) = 0$. The three equations are clearly independent, thus X_2' has codimension 3 in $H(d_1,d_2,d_3)$, as required.

If $\gcd(d_1,d_2) = 2$ then we additionally have to show that there is only a finite number of rays contained in $C(F) \cap V(f_3)$. Consider $X_{2a} = \{(p,F) \in (\mathbb{C}^3)^* \times H(d_1,d_2,d_3) : f_3(p) = J(F)(p) = 0\}$. Similarly as for X_2 we show that X_{2a} has codimension 2, hence the general fiber of the projection $X_{2a} \to H(d_1,d_2,d_3)$ has dimension 1, so it must be a finite union of rays.

(3) Consider $X_3 = \{(p_1,p_2,F) \in (\mathbb{C}^3)^{*2} \times H(d_1,d_2,d_3) : F(p_1) = F(p_2),\ J(F)(p_1) = J(F)(p_2) = 0\}$. Let X_3' be a nonempty fiber of the projection to $(\mathbb{C}^3)^{*2}$. By (2) we may assume that F is injective on rays and consider only fibers over (p_1,p_2) where p_1 and p_2 are not proportional. Since linear transformations induce isomorphisms of the fibers, we may assume that $(p_1,p_2) = ((1,0,0),(0,1,0))$. Thus the equations for X_3' are: $(a_{0,0;1}, a_{0,0;2}, a_{0,0;3}) = (a_{d_1,0;1}, a_{d_2,0;2}, a_{d_2,0;3})$ and

$$\det \begin{bmatrix} d_1 a_{0,0;1} & a_{1,0;1} & a_{0,1;1} \\ d_2 a_{0,0;2} & a_{1,0;2} & a_{0,1;2} \\ d_3 a_{0,0;3} & a_{1,0;3} & a_{0,1;3} \end{bmatrix} = \det \begin{bmatrix} a_{d_1-1,0;1} & d_1 a_{d_1,0;1} & a_{d_1-1,1;1} \\ a_{d_2-1,0;2} & d_2 a_{d_2,0;2} & a_{d_2-1,1;2} \\ a_{d_3-1,0;3} & d_3 a_{d_3,0;3} & a_{d_3-1,1;3} \end{bmatrix} = 0.$$

The first three equations define a linear subspace of codimension 3, the other two clearly do not have a common factor even after restricting to this subspace, i.e., after substituting $a_{0,0;k}$ for $a_{d_k,0;k}$ in the last equation. Thus X_3' has codimension 5 in $H(d_1,d_2,d_3)$ and consequently $\dim(X_2) \le \dim(H(d_1,d_2,d_3)) + 1$. Note that if $(p_1,p_2,F) \in X_3$ then also $(\lambda p_1, \lambda p_2, F) \in X_3$ for $\lambda \in \mathbb{C}^*$, thus the nonempty fibers of the projection $X_3 \to H(d_1,d_2,d_3)$ are infinite and so there are only finitely many of them.

(4) Here we prove that if $p \in \Delta(F)$ then at most two points from $F^{-1}(p)$ are critical points, which is the first step to prove that the discriminant has "good" self-intersections. Consider

$$X_4 = \{(p_1,p_2,p_3,F) \in (\mathbb{C}^3)^{*3} \times H(d_1,d_2,d_3) : F(p_1) = F(p_2) = F(p_3),$$
$$J(F)(p_1) = J(F)(p_2) = J(F)(p_3) = 0\}.$$

Similarly as above we consider the fibers of the projection $X_4 \to (\mathbb{C}^3)^{*3}$. However now we have to consider more than one case: if p_1,p_2,p_3 are not coplanar with the origin then we may assume that $(p_1,p_2,p_3) = ((1,0,0),(0,1,0),(0,0,1))$, if p_1,p_2,p_3 are coplanar with zero then we can only assume that $(p_1,p_2,p_3) = ((1,0,0),(0,1,0),(a,b,0))$ for some $a,b \in \mathbb{C}^*$. We denote the fiber by X_4' in the former case and by X_4^{ab} in the latter. If $\gcd(d_1,d_2) = 2$ then we must additionally consider the case when two of the points are opposite. In that case we may assume that $(p_1,p_2,p_3) = ((1,0,0),(-1,0,0),(0,1,0))$, we denote the fiber by X_4^-.

The equations for X_4' are similar to those of X_3'. First we have $(a_{0,0;1}, a_{0,0;2}, a_{0,0;3}) = (a_{d_1,0;1}, a_{d_2,0;2}, a_{d_3,0;3}) = (a_{0,d_1;1}, a_{0,d_2;2}, a_{0,d_3;3})$ which define a linear subspace of codimension 3. Then we have three equations given by determinants of a matrix. After restricting to the linear subspace the matrices have a common column: $[d_1 a_{0,0;1}, d_2 a_{0,0;2}, d_3 a_{0,0;3}]$, but otherwise contain disjoint sets of variables. Thus the equations give a transverse intersection outside $V(a_{0,0;1}, a_{0,0;2}, a_{0,0;3})$ which itself has codimension 3. So X_4' has codimension 9, as required.

For X_4^{ab} we obtain the equations

$$\begin{aligned}(a_{0,0;1}, a_{0,0;2}, a_{0,0;3}) &= (a_{d_1,0;1}, a_{d_2,0;2}, a_{d_3,0;3}) \\ &= \left(\sum a_{i,0;1}a^{d_1-i}b^i, \sum a_{i,0;2}a^{d_2-i}b^i, \sum a_{i,0;3}a^{d_3-i}b^i\right)\end{aligned}$$

which again define a linear subspace, though not as nicely as above. Furthermore we have the two equations with determinants from the definition of X_3' and a third one that is derived from $J(F)(a,b,0) = 0$. One can show that the last equation is independent from the previous ones, but in fact we do not need it. Note that the set of triples in $(\mathbb{C}^3)^{*3}$ coplanar with the origin has dimension 8, so it suffices to show that X_4^{ab} has codimension 8 in $H(d_1,d_2,d_3)$. This way we obtain a peculiar geometric fact: for a generic $F \in H(d_1,d_2,d_3)$ and $p \in \Delta(F)$ if $p_1, p_2 \in F^{-1}(p) \cap C(F)$ then none of the points in $F^{-1}(p)$ distinct from p_1, p_2 lie in the plane spanned by p_1, p_2 and the origin.

For X_4^- we have $p_2 = -p_1$ so the equation $F(p_1) = F(p_2)$ reduces to $f_3(p_1) = 0$. Furthermore the equations $J(F)(p_1) = 0$ and $J(F)(p_2) = 0$ are equivalent. Thus X_4^- is given only by 6 independent equations: $F(p_1) = F(p_3)$, $f_3(p_1) = 0$, $J(F)(p_1) = J(F)(p_3) = 0$ (note that the equation $F(p_1) = F(p_3)$ gives in fact three independent equations). However the set of points in $(\mathbb{C}^3)^{*3}$ satisfying $p_2 = -p_1$ has also dimension 6.

(5) We consider two sets:

$$\begin{aligned}X_5 &= \{(p,F) \in (\mathbb{C}^3)^* \times H(d_1,d_2,d_3) : J(F)(p) = J_{1,i}(F)(p) = J_{2,i}(F)(p) = 0\}, \\ X_{5a} &= \{(p,F) \in (\mathbb{C}^3)^* \times H(d_1,d_2,d_3) : \nabla J(F)(p) = (0,0,0)\},\end{aligned}$$

where $1 \le i \le 3$ and $J_{1,i}(F)$ is the determinant of the matrix that we obtain from the Jacobian matrix by replacing the row $\nabla f_i = [\frac{\partial f_i}{\partial x}, \frac{\partial f_i}{\partial y}, \frac{\partial f_i}{\partial z}]$ with the row $\nabla J(F)$ and similarly for $J_{2,i}(F)$ by replacing the row ∇f_i with the row $\nabla J_{1,i}(F)$. Note that X_5 describes the set of pairs (p,F) such that the singularity of F at p is worse than a cusp, e.g., is an A_n singularity with $n \ge 3$ or a singularity of corank greater than 1. However, X_5 fails to include pairs (p,F) with singular $C(F)_p$, e.g., with F_p equivalent to $(x^3,y,z)_0$ or $(x^3+y^2x,y,z)_0$. This is why we also need the set X_{5a} which describes the pairs (p,F) such that $C(F)$ is singular at p, in particular it includes also non-reduced components of $C(F)$. Thus the only singularities that are not contained in $X_5 \cup X_{5a}$ are folds and cusps. So it suffices to prove that X_5 and X_{5a} have codimension at least 3 and this can be done by considering the fibers X_5' and X_{5a}' over $p_1 = (1,0,0)$.

By taking the Laplace expansion of $J(F)(p_1)$ with respect to the second column we obtain $-a_{1,0;1}m_{1;1} + a_{1,0;2}m_{2;1} - a_{1,0;3}m_{3;1}$, where $m_{i;1}$ are the suitable minors, e.g., $m_{1;1} = d_2a_{0,0;2}a_{0,1;3} - d_3a_{0,0;3}a_{0,1;2}$. The formula for $J_{1,1}(F)(p_1)$ is too long to conveniently write down, however it is easy to see that it is the sum of $2a_{2,0;1}m_{1;1}^2$ and a polynomial that does not contain $a_{2,0;1}$. Indeed, the term $a_{2,0;1}$ can only come from $\frac{\partial^2 f_1}{\partial y^2}$ which can be only found in $\frac{\partial J(F)}{\partial y}$ by taking the derivative of $\frac{\partial f_1}{\partial y}$. Similarly, $6a_{3,0;1}m_{1;1}^3$is a summand of $J_{2,1}(F)(p_1)$. Consequently the determinant of the matrix

$$\begin{bmatrix} \frac{\partial J(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J(F)(p_1)}{\partial a_{2,0;1}} & \frac{\partial J(F)(p_1)}{\partial a_{3,0;1}} \\ \frac{\partial J_{1,1}(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_{1,1}(F)(p_1)}{\partial a_{2,0;1}} & \frac{\partial J_{1,1}(F)(p_1)}{\partial a_{3,0;1}} \\ \frac{\partial J_{2,1}(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_{2,1}(F)(p_1)}{\partial a_{2,0;1}} & \frac{\partial J_{2,1}(F)(p_1)}{\partial a_{3,0;1}} \end{bmatrix} = \begin{bmatrix} -m_{1;1} & 0 & 0 \\ \frac{\partial J_{1,1}(F)(p_1)}{\partial a_{1,0;1}} & 2m_{1;1}^2 & 0 \\ \frac{\partial J_{2,1}(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_{2,1}(F)(p_1)}{\partial a_{2,0;1}} & -6m_{1;1}^3 \end{bmatrix}$$

is equal $12m_{1;1}^6$, which proves that $X_5' \setminus V(m_{1;1})$ has codimension 3. We make identical computations for $i \in \{2,3\}$ and computation with $J_{1,1}$, $J_{2,1}$, $a_{1,0;1}$, $a_{2,0;1}$, and $a_{3,0;1}$ replaced with $J_{1,i}$, $J_{2,i}$, $a_{1,0;i}$, $a_{2,0;i}$, and $a_{3,0;i}$, respectively, to obtain that $X_5' \setminus V(m_{1;1}, m_{2;1}, m_{3;1})$ has codimension 3. The set $V(m_{1;1}, m_{2;1}, m_{3;1})$ has codimension 2, it is given by the condition that the first and the third columns of $J(F)(p_1)$ are proportional, however, we can expand $J(F)(p_1)$ with respect to the third column and obtain $a_{0,1;1}m_{1;2} + a_{0,1;2}m_{2;2} - a_{0,1;3}m_{3;2}$. Proceeding as above we obtain that $X_5' \setminus V(m_{i;2})_{1\le i\le 3}$ has codimension 3, since $V(m_{i;1}, m_{i;2})_{1\le i\le 3}$ has also codimension 3 we conclude that X_5' has codimension 3.

Let $J_x(F), J_y(F), J_z(F)$ denote the partial derivatives of $J(F)$ with respect to x, y, z, respectively. We have $(\deg J(F))J(F) = xJ_x(F) + yJ_y(F) + zJ_z(F)$, so $(d_1 + d_2 + d_3 - 3)J(F)(p_1) = J_x(F)(p_1)$. In particular we may replace $J_x(F)(p_1)$ with $J(F)(p_1)$ in the definition of X_{5a}'. Observe that

$$\begin{bmatrix} \frac{\partial J(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J(F)(p_1)}{\partial a_{2,0;1}} \\ \frac{\partial J_z(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_z(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J_z(F)(p_1)}{\partial a_{2,0;1}} \\ \frac{\partial J_y(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_y(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J_y(F)(p_1)}{\partial a_{2,0;1}} \end{bmatrix} = \begin{bmatrix} -m_{1;1} & 0 & 0 \\ \frac{\partial J_z(F)(p_1)}{\partial a_{1,0;1}} & -m_{1;1} & 0 \\ \frac{\partial J_y(F)(p_1)}{\partial a_{1,0;1}} & \frac{\partial J_y(F)(p_1)}{\partial a_{1,1;1}} & -m_{1;1} \end{bmatrix}$$

and

$$\begin{bmatrix} \frac{\partial J(F)(p_1)}{\partial a_{0,1;1}} & \frac{\partial J(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J(F)(p_1)}{\partial a_{0,2;1}} \\ \frac{\partial J_y(F)(p_1)}{\partial a_{0,1;1}} & \frac{\partial J_y(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J_y(F)(p_1)}{\partial a_{0,2;1}} \\ \frac{\partial J_z(F)(p_1)}{\partial a_{0,1;1}} & \frac{\partial J_z(F)(p_1)}{\partial a_{1,1;1}} & \frac{\partial J_z(F)(p_1)}{\partial a_{0,2;1}} \end{bmatrix} = \begin{bmatrix} m_{1;2} & 0 & 0 \\ \frac{\partial J_y(F)(p_1)}{\partial a_{0,1;1}} & m_{1;2} & 0 \\ \frac{\partial J_z(F)(p_1)}{\partial a_{0,1;1}} & \frac{\partial J_z(F)(p_1)}{\partial a_{1,1;1}} & m_{1;2} \end{bmatrix}.$$

Similarly as above we obtain that $X_{5a}' \setminus V(m_{i;1}, m_{i;2})_{1\le i\le 3}$ has codimension 3. Thus $X_5' \cup X_{5a}'$ has codimension 3, which concludes the proof of (5).

(6) Consider $X_6 = \{(p_1, p_2, F) \in (\mathbb{C}^3)^{*2} \times H(d_1, d_2, d_3) : F(p_1) = F(p_2),\ J(F)(p_1) = J_{1,i}(F)(p_1) = J(F)(p_2) = 0\}$. We have to prove that X_6 has codimension 6. The argument is a mix of the arguments in (3) and (5). As above we focus on the fiber over $(p_1, p_2) = ((1,0,0),(0,1,0))$. The equations obtained from $F(p_1) = F(p_2)$ define a linear subspace of codimension 3. From (5) we obtain that $J(F)(p_1) = J_{1,i}(F)(p_1) = 0$ give a space of codimension 2. And the equation obtained from $J(F)(p_2)$ is independent from the previous ones outside $V(a_{0,0;1}, a_{0,0;2}, a_{0,0;3})$.

If $\gcd(d_1, d_2) = 2$ then we must additionally consider the case $p_2 = -p_1$. In this case the equation $F(p_1) = F(p_2)$ reduces to $f_3(p_1) = 0$ and the equations $J(F)(p_1) = 0$ and $J(F)(p_2) = 0$ are equivalent. Thus the fiber of X_6 over (p_1, p_2) has codimension 3, however the space of points in $(\mathbb{C}^3)^* \times (\mathbb{C}^3)^*$ satisfying $p_2 = -p_1$ has also codimension 3, so the sum of fibers of this type has codimension 6.

(7) Consider $X_7 = \{(p_1, p_2, F) \in X_2 : F(p_1) = F(p_2),\ J(F)(p_1) = J(F)(p_2) = 0,\ dF(p_1)(\mathbb{C}^3) = dF(p_2)(\mathbb{C}^3)\}$. Note that since $\Delta(F)$ is a hypersurface either the two branches at $F(p_1)$ intersect transversally or they have equal tangent spaces, which is the condition that we added in the definition of X_7. As in (3) we look at the fiber over $(p_1, p_2) = ((1,0,0),(0,1,0))$ and obtain the equations $(a_{0,0;1}, a_{0,0;2}, a_{0,0;3}) = (a_{d_1,0;1}, a_{d_2,0;2}, a_{d_2,0;3})$ and $\operatorname{rank} A \le 2$, where

$$A = \begin{bmatrix} d_1 a_{0,0;1} & a_{1,0;1} & a_{0,1;1} & a_{d_1-1,0;1} & d_1 a_{d_1,0;1} & a_{d_1-1,1;1} \\ d_2 a_{0,0;2} & a_{1,0;2} & a_{0,1;2} & a_{d_2-1,0;2} & d_2 a_{d_2,0;2} & a_{d_2-1,1;2} \\ d_3 a_{0,0;3} & a_{1,0;3} & a_{0,1;3} & a_{d_3-1,0;3} & d_3 a_{d_3,0;3} & a_{d_3-1,1;3} \end{bmatrix}.$$

After substituting $(a_{0,0;1}, a_{0,0;2}, a_{0,0;3}) = (a_{d_1,0;1}, a_{d_2,0;2}, a_{d_2,0;3})$ into A the first and the fifth columns become equal, so we may cross the fifth one out without altering the rank. We obtain a 3×5 matrix A' with variables as entries, the condition $\operatorname{rank} A \le 2$ defines a subset of codimension 3 (on the Zariski open set where a 2×2 minor is nonzero the set is given as the zero set of the three 3×3 minors containing that 2×2 minor). Together with the first three equations we obtain a set of codimension 6.

If $\gcd(d_1, d_2) = 2$ then we additionally consider the case $(p_1, p_2) = ((1,0,0),(-1,0,0))$. We obtain the equations $a_{0,0;3} = 0$ and $\operatorname{rank} B \leq 2$, where

$$B = \begin{bmatrix} d_1 a_{0,0;1} & a_{1,0;1} & a_{0,1;1} & -d_1 a_{0,0;1} & -a_{1,0;1} & -a_{0,1;1} \\ d_2 a_{0,0;2} & a_{1,0;2} & a_{0,1;2} & -d_2 a_{0,0;2} & -a_{1,0;2} & -a_{0,1;2} \\ d_3 a_{0,0;3} & a_{1,0;3} & a_{0,1;3} & d_3 a_{0,0;3} & a_{1,0;3} & a_{0,1;3} \end{bmatrix}.$$

After substituting $a_{0,0;3} = 0$ and adding columns $1, 2, 3$ to columns $4, 5, 6$, respectively we obtain

$$B' = \begin{bmatrix} d_1 a_{0,0;1} & a_{1,0;1} & a_{0,1;1} & 0 & 0 & 0 \\ d_2 a_{0,0;2} & a_{1,0;2} & a_{0,1;2} & 0 & 0 & 0 \\ 0 & a_{1,0;3} & a_{0,1;3} & 0 & 2a_{1,0;3} & 2a_{0,1;3} \end{bmatrix}.$$

The condition $\operatorname{rank} B' \leq 2$ means that either the first two rows are proportional or $a_{1,0;3} = a_{0,1;3} = 0$. Both conditions define a subset of codimension 2, together with $a_{0,0;3} = 0$ we obtain codimension 3. This is sufficient since the space of pairs $(p_1, p_2) \in (\mathbb{C}^3)^{*2}$ such that $p_2 = -p_1$ has dimension 3. □

Remark 2.2. Note that Lemma 2.1 (2) fails if $\gcd(d_i, d_j) > 2$ for some $i, j \in \{1, 2, 3\}$, $i \neq j$ or if $\gcd(d_1, d_2, d_3) > 1$. Indeed, suppose $\gcd(d_1, d_2) = d > 2$, for general $F \in H(d_1, d_2, d_3)$ the set $C(F) \cap V(f_3)$ consists of a finite and nonzero number of rays. If p is an element of such a ray then for $\varepsilon^d = 1$ we have $F(\varepsilon p) = F(p)$ and the mapping is actually $d : 1$ on that ray. If $\gcd(d_1, d_2, d_3) > 1$ then F is not generically one to one on $C(F)$.

We have the following geometric criterion for finite determinacy of homogeneous map germs (see [10]):

Theorem 2.3. *Let $F : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be a holomorphic map germ. Then F is finitely $\mathcal{A}$-determined if and only if there is a finite representative $F : U \subset \mathbb{C}^3 \to V \subset \mathbb{C}^3$ such that*

(1) $F^{-1}(0) = \{0\}$,
(2) *the restriction $F_{|U \setminus \{0\}} : U \setminus \{0\} \to V \setminus \{0\}$ is stable.*

Using Theorem 2.3, Lemma 2.1 and Remark 2.2 we can prove Theorem 1.1:

Proof of Theorem 1.1. By Lemma 2.1 any $F \in U$ is locally stable, it is also proper, since F is homogeneous and $F^{-1}(0) = 0$. Thus by [5] $F : \mathbb{C}^3 \setminus \{0\} \to \mathbb{C}^3 \setminus \{0\}$ is stable. By Theorem 2.3 $(F, 0)$ is finitely $\mathcal{A}$-determined.

The last statement follows from Remark 2.2. □

3. Counting singularities

Mappings from $\mathbb{C}^3$ to $\mathbb{C}^3$ have three types of stable discrete mono- or multi-singularities:

- A_3 – the swallowtail: $(x, y, z) \mapsto (x, y, z^4 + y^2 z + xz)$
- $A_2 A_1$ – intersection of cusp edge and fold surface:

$$\begin{cases} (x_1, y_1, z_1) \mapsto (x_1, y_1, z_1^3 + y_1 z_1) \\ (x_2, y_2, z_2) \mapsto (x_2^2, y_2, z_2) \end{cases}$$

- A_1^3 – triple self-intersection of fold surface:

$$\begin{cases}(x_1,y_1,z_1)\mapsto(x_1,y_1,z_1^2)\\(x_2,y_2,z_2)\mapsto(x_2,y_2^2,z_2)\\(x_3,y_3,z_3)\mapsto(x_3^2,y_3,z_3)\end{cases}$$

Let us denote $s_1=d_1+d_2+d_3-3$, $s_2=(d_1-1)(d_2-1)+(d_1-1)(d_3-1)+(d_2-1)(d_3-1)$, $s_3=(d_1-1)(d_2-1)(d_3-1)$ and $P=d_1d_2d_3$. Furthermore let $c_1=s_1$, $c_2=s_2-s_1$ and $c_3=s_3-2s_2+s_1$. Finally let $\#A_2=c_1^2+c_2$ and $\#A_1^2=(P-2)s_1^2-2\#A_2$. The definitions of c_1,c_2,c_3 and $\#A_2$ and $\#(A_1)^2$ have a deeper meaning, the former are related to certain quotient Chern classes, the latter to Thom polynomials. We refer the reader to a paper by Ohmoto [9] for the details.

We can now prove Theorem 1.2:

Proof of Theorem 1.2. For $F=(f_1,f_2,f_3)\in\Omega_3(d_1,d_2,d_3)$ we denote by $\overline{f}_i$ the homogeneous part of f_i of degree d_i and set $F_0=(\overline{f}_1,\overline{f}_2,\overline{f}_3)$. By [1, Theorem 2.7] there is a Zariski open set $V\subset\Omega_3(d_1,d_2,d_3)$ such that every $F\in V$ is transversal to the Thom-Boardman strata. This determines the types of singularities that F may have: A_1, A_2 and A_1^2 are the non-discrete types and A_3, A_2A_1 and A_1^3 are the discrete types. In particular F is locally stable. We let $U_1=\{F\in V:\ F_0\in U\}$, where U is the Zariski open set from Lemma 2.1. If $F\in U_1$ then F_0 is proper, so F is also proper. Since F is locally stable and proper, it is also stable. Let $F_t(x,y,z)=(t^{d_1}f_1,t^{d_2}f_2,t^{d_3}f_3)(t^{-1}x,t^{-1}y,t^{-1}z)$, then F_t is a stable deformation of F_0. Obviously for all $t\neq 0$ the mappings F_t have the same number of singularities, furthermore all the singularities tend to zero when t tends to zero. Thus by [9, Example 5.9] F has the numbers of singularities as written above. □

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