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REPRESENTATIONS OF PRIMITIVE POSETS

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Abstract

We give a new proof of the Kleiner's theorem for the case of primitive posets.

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Key words: finite partially ordered set (poset), primitive poset, representations of pair posets

1 Introduction

An important problem in the theory of representations of finite dimensional algebras (or f.d. algebras, in short) is to obtain the full list of different kinds of algebras which are of a finite representation type (or finite type, or f.r.t., in short) The first classes of associative f.d. algebras of f.r.t which were described was the class of algebras with zero square radical and hereditary algebras over algebraically closed fields.

There are different approaches to study the representations of f.d. algebras. One of them is approach of P.Gabriel [4], which reduces the study of representations of algebras to study of representations of quivers. Other approach was first introduced by L.A.Nazarova and A.V.Roiter [17]. This approach consists in reducing this problem to solving "matrix problems", that is the reducing some classes of matrices by means of admissible transformations, and then the description of representations of corresponding finite partially ordered sets (shortly, posets). And the third approach is the Auslander-Reiten approach which is connected with technique of almost split sequences.

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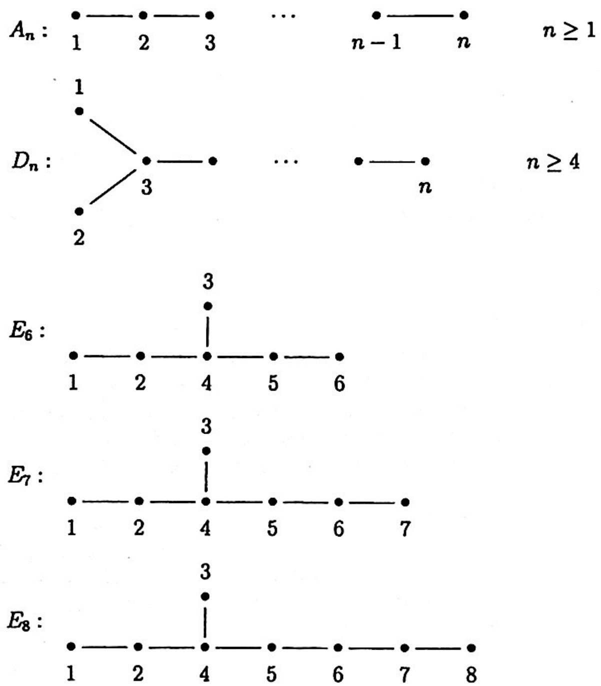
The classification of finite dimensional algebras of f.r.t. with zero square radical was first obtained independently by P.Gabriel [4] and S.A.Kruglak [14].

For any finite quiver $Q = (VQ, AQ, s, e)$ we can construct a bipartite quiver $Q^b = (VQ^b, AQ^b, s_1, e_1)$ by the following way. Let $VQ = \{1, 2, ..., s\}$, $AQ = \{\sigma_1, \sigma_2, ..., \sigma_k\}$. Then $VQ^b = \{1, 2, ..., s, b(1), b(2), ..., b(s)\}$ and $AQ^b = \{\tau_1, \tau_2, ..., \tau_k\}$, such that for any $\sigma_j \in AQ$ we have $s_1(\tau_j) = s(\sigma_j)$ and $e_1(\tau_j) = b(e(\sigma_j))$. In other words, in the quiver Q^b from the vertex i to vertex $b(j)$ go t_{ij} arrows if and only if in the quiver Q from the vertex i to vertex j go t_{ij} arrows. As above, denote by $\overline{Q}$ an undirected graph which is obtained from Q by deleting the orientation of all arrows.

Theorem 1.1. (P.Gabriel, [4]) *Let A be a finite dimensional algebra over an algebraically closed field k with zero square radical and quiver Q. Then A is of a finite type if and only if $\overline{Q^b}$ is a finite disjoint union of Dynkin diagrams of the form A_n, D_n, E_6, E_7, E_8:*

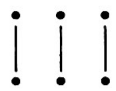


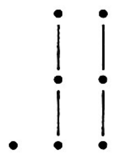
In his paper S.A.Kruglak [14] used the technique of Nazarova-Roiter of reducing to some "matrix problem". For "solving" these "matrix problems" the main role play the results obtained by L.A.Nazarova and A.V.Roiter [17] in 1972 on representations of finite posets. In this paper they gave the algorithm of "differentiation" which allows the transform from one poset to another one preserving its representation type. Using this algorithm of "differentiation" M.M.Kleiner in 1972 characterized posets of a finite type [11] and described their pairwise nonisomorphic indecomposable representations [13], see also [5].

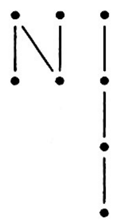
We recall that a finite poset S is called primitive if it is a cardinal sum of a linearly ordered sets $L_1, \dots, L_m$. We use the following notations: $S = L_1 \sqcup \dots \sqcup L_m$. We denote a chain C_n from n elements by (n).

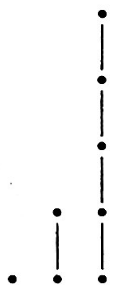
Theorem 1.2. (M.Kleiner, [11]). *A finite partially ordered set $\mathcal{P}$ is of a finite representation type if and only if $\mathcal{P}$ does not contain as a full subposet any poset from the following list:*

$(1) \sqcup (1) \sqcup (1) \sqcup (1)$:

$(2) \sqcup (2) \sqcup (2)$: 

$(1) \sqcup (3) \sqcup (3)$: 

$N \sqcup (4)$: 

$(1) \sqcup (2) \sqcup (5)$: 

The symbol N denotes the following 4-element poset $\{a_1 \prec a_2 \succ a_3 \prec a_4\}$. The posets in the list of the Kleiner theorem are called the critical subposets. Therefore all critical posets except the poset $N \sqcup (4)$ in the list of theorem 1.2 are primitive.

In 1977 A.G.Zavadskij [18] introduced a new algorithm of differentiation relative to a pair of points for computing representations of posets, and this algorithm was used for giving a new proof for classifications of posets of tame type (see for more details

[19]). In 1981 O.Kerner using this algorithm gave a new proof for classification the posets of a finite representation type [10].

Note that many authors studied the representations of the poset $(1) \sqcup (1) \sqcup (1) \sqcup (1)$ (see [2], [7], [15], [16]).

The description of the finitely generated torsionless modules over a triad [1] can be reduced to the description of the representations of the primitive poset $(1) \sqcup (1) \sqcup (1)$.

In this paper we give the criterion for primitive posets to be of a finite representation type.

Theorem 1.3. *A finite primitive partially ordered set P is of a finite representation type if and only if P does not contain as a full subposet any poset from the following list:* $(1) \sqcup (1) \sqcup (1) \sqcup (1)$; $(2) \sqcup (2) \sqcup (2)$; $(1) \sqcup (3) \sqcup (3)$; $(1) \sqcup (2) \sqcup (5)$.

Let $L_1 \sqcup L_2 \sqcup L_3$ a primitive poset of a finite type, and let S_3 be a symmetric group of degree 3. Then for any $\sigma \in S_3$ the poset $L_{\sigma(1)} \sqcup L_{\sigma(2)} \sqcup L_{\sigma(3)}$ is of a finite representation type also.

Corollary 1.1. *The following primitive posets are of a finite representation type:*

(a) any chain (n);

(b) any cardinal sum $(n) \sqcup (m)$;

(c) $(1) \sqcup (1) \sqcup (n)$;

(d) $(1) \sqcup (2) \sqcup (2)$;

(e) $(1) \sqcup (2) \sqcup (3)$;

(f) $(1) \sqcup (2) \sqcup (4)$.

Conversely, if P is a primitive poset of a finite representation type, then P is either a poset of the form (a)-(b) or a poset of the form $L_{\sigma(1)} \sqcup L_{\sigma(2)} \sqcup L_{\sigma(3)}$, where $L_1 \sqcup L_2 \sqcup L_3$ is a poset of the form (c), (d), (e), (f).

To prove this criterion we use only the trichotomy lemma which was proved by P.Gabriel and A.V.Roiter in [6], the Kleiner lemma about the representations of a pair of finite posets proved by M.Kleiner in [11] and the main construction, considered in section 6. Note that this construction in some form were introduced by L.A.Nazarova and A.V.Roiter in [17]. For convenience of the reader we also give the simple proofs of the trichotomy lemma in section 4 and the Kleiner's lemma in section 5.

2 Preliminaries

In this section we recall the main notions and definitions of the representation theory of posets. We recommend to readers [8] and [9] for the additional information.

In this section we recall the main notions and definitions of the representation theory of posets.

Let $(\mathcal{P}, \preceq)$ be a finite poset, where $\mathcal{P}$ is a finite set and $\preceq$ is a partial order relation. We denote by $x \prec y$ the strict order, i.e. the relation "$x \preceq y$ and $x \neq y$".

An element $x \in \mathcal{P}$ is called **maximal** if there is no element $y \in \mathcal{P}$ satisfying $x \prec y$. Dually, x is **minimal** if there is no element $y \in \mathcal{P}$ satisfying $y \prec x$. We say that $x \in \mathcal{P}$ is a **least element** if $x \preceq y$ for all $y \in \mathcal{P}$, and $x \in \mathcal{P}$ is a **greatest element** if $y \preceq x$ for all $y \in \mathcal{P}$.

A totally ordered poset is called a **chain**. We denote a chain of cardinality n by C_n, or (n). In this case the number n is called the **length** of the chain C_n. An **antichain** is a poset whose elements are pairwise incomparable. A **chain in a poset** $(\mathcal{P}, \preceq)$ is a sub-poset C of $\mathcal{P}$ which is totally ordered by the restriction of $\preceq$. An **antichain** A of a poset $(\mathcal{P}, \preceq)$ is its sub-poset which consists from pairwise incomparable elements of $\mathcal{P}$.

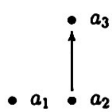
The **height** of a poset $\mathcal{P}$ is the largest cardinality of sub-posets of $\mathcal{P}$ which are chains, and the **width** of $\mathcal{P}$ is the largest cardinality of sub-posets of $\mathcal{P}$ which are antichains. In other words, the height of a poset $\mathcal{P}$ is the length of the longest chain in $\mathcal{P}$, and the width of a poset $\mathcal{P}$ is the maximal number of pairwise incomparable elements of $\mathcal{P}$. We denote the height and the width of $\mathcal{P}$ by $h(\mathcal{P})$ and $w(\mathcal{P})$ respectively.

If $\mathcal{P}$ is a finite poset, then a chain C and an antichain A of $\mathcal{P}$ have at most one element in common. Hence the least number of antichains whose union is $\mathcal{P}$ is not less then the size $h(\mathcal{P})$ of the largest chain in $\mathcal{P}$. In fact there is a partition of $\mathcal{P}$ into $h(\mathcal{P})$ antichains. There is also some kind of dual statement which is known as the Dilworth theorem.

Theorem 2.1. (Dilworth, [3]). *For a poset $\mathcal{P}$ with finite width $w(\mathcal{P})$ the minimal number of disjoint chains that together contain all elements of $\mathcal{P}$ is equal to $w(\mathcal{P})$.*

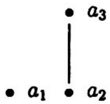
In order to visualize a poset $\mathcal{P}$ we shall use its diagram. Let x and y be distinct elements of $\mathcal{P}$. We say that y **covers** x if $x \prec y$ but there is no element z such that $x \prec y \prec z$. Recall that the **diagram** of a poset $\mathcal{P} = (p_1, \ldots, p_n)$ is the direct graph whose vertex set is $\mathcal{P}$ and the set of edges is given by the set of covering pairs (p_i, p_j) of $\mathcal{P}$, moreover, there is an edge from a vertex p_i up to a vertex p_j if and only if p_j covers p_j.

For example, the diagram below



represents a poset $(\mathcal{P}, \preceq)$ with 3 elements $\{a_1, a_2, a_3\}$ and relation $a_2 \prec a_3$.

Remark 2.1. This diagram of a poset is often called the **Hasse diagram**. Usually it is drawn in the plane in such a way that if y covers x then the point representing y is drawn higher than the point representing x. In this case we draw the Hasse diagram without arrows. For example, the Hasse diagram below



represents the same poset $(\mathcal{P}, \preceq)$ as above, i.e., $\mathcal{P} = \{a_1, a_2, a_3\}$ with relation $a_2 \prec a_3$.

Definition 2.1. *Let $(X, \preceq_1)$ and $(Y, \preceq_2)$ be any two (disjoint) posets. The* **cardinal sum** $X \sqcup Y$ *(or* **disjoint union** $X \cup Y$*) of X and Y is the set of all $x \in X$ and $y \in Y$. The relations $x \preceq_1 x_1$ and $y \preceq_2 y_1$ $(x, x_1 \in X;\ y, y_1 \in Y)$ have the same meanings and there are no another relations in $X \sqcup Y$.*

Now we give the definition of representations of a poset $\mathcal{P}$ over a division ring K which was introduced by L.A.Nazarova and A.V.Roiter in 1972 (see [17]).

Definition 2.2. *Let $(\mathcal{P}, \preceq)$ be a finite poset with n elements. A* **matrix representation** *of $\mathcal{P}$ over a division ring K is an arbitrary block matrix*

$$\mathrm{A} = \begin{array}{|c|c|c|c|}\hline \mathrm{A}_1 & \mathrm{A}_2 & \dots & \mathrm{A}_n \\ \hline \end{array}$$

partitioned horizontally into m (vertical) blocks (also called strips), where A_i is a matrix with entries in K of size $d_0 \times d_i$ for $i = 1, 2, ..., n$. Moreover, some of these matrices can be empty.

A vector $\mathrm{d} = \mathrm{d}(\mathrm{A}) = (d_0, d_1, \dots, d_{n-1}, d_n)$, where d_0 is the number of rows of a matrix A and d_i is the number of columns of a matrix A_i $(i = 1, 2, \dots, n)$, is called the **vector dimension** of A, and the number $d = d(\mathrm{A}) = d_0 + \sum\limits_{i=1}^{n} d_i$ is called the **dimension** of A.

Recall that an elementary row operation in an arbitrary matrix is one of the following:

i) interchanging of two rows;
ii) multiplying a row by a nonzero scalar;
iii) replacing a row by itself plus a scalar multiple of another row.

Analogously there can be introduced the elementary column operations.

Let A be a matrix representation of a poset $(\mathcal{P}, \preceq)$. The following elementary operations:

(a) elementary row operations on the whole matrix A;

(b) elementary column operations within each vertical strip A_i;

(c) additions of columns of a strip A_i to columns of a strip A_j if $i \prec j$ in $\mathcal{P}$

are called the **admissible elementary operations** for A. The composition of admissible elementary operations is called the **admissible operations** for A.

Notice that by a sequence of operations (b) and (c) we can add an arbitrary linear combination of columns of A_i to a column of A_j if $i \prec j$ in $\mathcal{P}$.

Let B be another matrix representation of a poset $(\mathcal{P}, \preceq)$ of the same shape as A:

$$B = \begin{array}{|c|c|c|c|} \hline B_1 & B_2 & \dots & B_n \\ \hline \end{array}$$

An **isomorphism** of matrix representations $\varphi : A \to B$ is given by a set of admissible operations (a), (b), (c) by which one can obtain the matrix B from the matrix A, and vice versa.

This definition we can also give in a matrix form. Introduce a block matrix Φ partitioned horizontally and vertically into n blocks:

$$\Phi = \left[\begin{array}{c|c|c|c} \Phi_{11} & \Phi_{12} & \dots & \Phi_{1n} \\ \hline \Phi_{21} & \Phi_{22} & \dots & \Phi_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \Phi_{n1} & \Phi_{n2} & \dots & \Phi_{nn} \end{array}\right],$$

where Φ_{ij} is a matrix of size $d_i \times d_j$ for $i, j = 1, ..., n$; moreover, Φ_{ii} is an inverse matrix for $i = 1, ..., n$ and $\Phi_{ij} \neq 0$ if $i \prec j$ in $\mathcal{P}$, otherwise Φ_{ij} is a zero matrix.

Then an isomorphism φ can be given by a pair of matrices (Φ_0, Φ) such that $\Phi_0 A = B\Phi$, or equivalently

$$\Phi_0 A_i = B_i \Phi_i + \sum_{i \prec j} B_i \Phi_{ij},$$

where Φ_0 is an inverse matrix of size $d_0 \times d_0$.

A matrix representation A is **equivalent** (or **isomorphic**) to a matrix representation B of a poset $(\mathcal{P}, \preceq)$ if there is an isomorphism $\phi : A \to B$.

The **direct sum** of two matrix representations A and B is a matrix representation $A \oplus B$ which is equal to:

$$A \oplus B = \begin{array}{|cc|cc|c|cc|} \hline A_1 & O & A_2 & O & \ddots & A_n & O \\ O & B_1 & O & B_2 & \ddots & O & B_n \\ \hline \end{array}$$

A matrix representation A is called **decomposable** if it is equivalent to a direct sum of two matrix representations A_1 and A_2 with $d(A_1) \neq 0$ and $d(A_2) \neq 0$. Otherwise it is called **indecomposable**.

We shall denote by $n(\mathcal{P}, K)$ the cardinal number of pairwise non-isomorphic indecomposable representations of the poset $\mathcal{P}$ over a division ring K.

Definition 2.3. *A partially ordered set $\mathcal{P}$ is said to be of* **finite representation type** *(or* **finite type,** *in short) over a division ring K if it has finitely many pairwise non-isomorphic indecomposable representations, i.e., $n(\mathcal{P}, K) < \infty$. And $\mathcal{P}$ is said to be of* **infinite representation type** *(or* **infinite type,** *in short) over a division ring K if it has infinitely many pairwise non-isomorphic indecomposable representations, i.e., $n(\mathcal{P}, K) = \infty$.*

3 Main canonical forms of matrix problems

Recall the following main definitions and facts from the course of linear algebra.

Definition 3.1. *A matrix is an* **echelon matrix** *(or is in echelon form) if:*

i) all zero rows are at the bottom;

ii) each leading (i.e. left-most) nonzero entry is to the right of the leading nonzero entry in the row above.

Example 3.1. The matrix $A \in M_{34}(K)$ with entries in a division ring K of the form

$$A = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ 0 & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ 0 & 0 & 0 & \alpha_{34} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in echelon form.

An upper-triangular matrix is a special case of an echelon matrix, and an echelon matrix is necessarily upper triangular.

Definition 3.2. *A matrix is said to be in* **row canonical form** *(or in* **reduced row echelon form***) if it is in echelon form and also each leading nonzero entry is 1 and is the only nonzero entry in its column.*

Example 3.2. The matrix $A \in M_{45}(K)$ with entries in a division ring K of

$$A = \begin{bmatrix} 1 & \alpha_{12} & 0 & 0 & \alpha_{15} \\ 0 & 0 & 1 & 0 & \alpha_{25} \\ 0 & 0 & 0 & 1 & \alpha_{35} \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row canonical form.

Row canonical form is simpler than echelon form and it is unique in the sense that it is independent of the used algorithm. And, moreover, it gives the unique form of ordered basis for the row space.

Definition 3.3. *A matrix* $\mathbf{A}$ *is said to be in* normal form *if:*

$$\mathbf{A} = \left[\begin{array}{c|c} \mathbf{E} & \mathbf{O} \\ \hline \mathbf{O} & \mathbf{O} \end{array}\right], \tag{3.1}$$

where $\mathbf{E}$ *is an identity matrix.*

From the course of linear algebra it is well-known the following three theorems. The first theorem states which that any matrix over a field k by means of elementary row operations can be reduced to an echelon form. This row reduction algorithm is often called **Gaussian elimination**. The second theorem said that any echelon matrix over a field k by means of elementary row operations can be reduced to a row canonical form. This row reduction algorithm is called **Gauss-Jordan elimination**. And the third theorem states that any matrix $\mathbf{A}$ over a field k by means of elementary row and column operations can be reduced to the normal form. It is easy to prove that these three theorems also holds for matrices over a division ring K.

Definition 3.4. *Let* $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ *be a finite poset. An indecomposable representation* V *of* $\mathcal{P}$ *with dimension vector* $\mathrm{d} = (d_0, d_1, \ldots, d_n)$ *is called* elementary *if it has one of the following form:*

1) V *is a zero representation with matrix:*

$$\left[\begin{array}{c|c|c|c|c|c} 0 & 0 & 0 & \ldots & 0 & 0 \end{array}\right], \tag{3.2}$$

and the dimension vector $\mathrm{d} = (1, 0, 0, \ldots, 0)$*;*

2) V *is a representation with matrix*

$$\left[\begin{array}{c|c|c|c|c|c|c} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{array}\right], \tag{3.3}$$

and the dimension vector $\mathrm{d} = (d_0, d_1, \ldots, d_n)$ *such that* $d_0 = d_i = 1$ *and* $d_j = 0$ *for all* $j \notin \{0, i\}$*;*

3) V *is a representation with matrix*

$$\left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c} 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \end{array}\right], \tag{3.4}$$

and the dimension vector $\mathrm{d} = (d_0, d_1, \ldots, d_n)$ *such that* $d_0 = d_i = d_j = 1$ *and* $d_s = 0$ *for all* $s \notin \{0, i, j\}$.

Lemma 3.1. *Let C_n be a chain. Then the matrix representation* $\mathbf{R}$ *of* C_n *over a division ring* K *can be reduced to the following canonical form:*

$$\mathbf{R} = \left[\begin{array}{c|c||c|c||c|c||c||c|c}
\mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \dots & \mathrm{O} & \mathrm{O} \\ \hline
\mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \dots & \mathrm{O} & \mathrm{O} \\ \hline
\mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} & \dots & \mathrm{O} & \mathrm{O} \\ \hline
\dots & \dots & \dots & \dots & \dots & \dots & \ddots & \dots & \dots \\ \hline
\mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \dots & \mathrm{E} & \mathrm{O} \\ \hline
\mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \dots & \mathrm{O} & \mathrm{O}
\end{array}\right]. \tag{3.5}$$

Therefore $n(C_n, K) = n + 1$ and all indecomposable representations are elementary of forms (3.2) or (3.3).

Proof. Let $C_n = \{p_1 \preceq p_2 \preceq \dots \preceq p_n\}$ be a chain, and let $\mathbf{R}$ be a matrix representation of C_n. Then the matrix $\mathbf{R}$ is partitioned into n vertical strips:

$$\mathbf{R} = [\ \overbrace{\mathbf{A}_1}^{p_1} \ \| \ \overbrace{\mathbf{A}_2}^{p_2} \ \| \cdots \| \ \overbrace{\mathbf{A}_n}^{p_n}\ ],$$

where a matrix $\mathbf{A}_i$ corresponds to an element p_i of the poset C_n. And so the following operations are admissible with matrix $\mathbf{R}$:

(a) elementary row operations on the whole matrix $\mathbf{R}$;
(b) elementary column operations within each vertical strip $\mathbf{A}_i$;
(c) additions of columns of a strip $\mathbf{A}_i$ to columns of a strip $\mathbf{A}_j$ if $i < j$.

We shall prove the lemma by induction on the number of n. If $n = 1$ then any matrix by row and column operations can be reduced to normal form (3.1). Assume that statement is true for all $k < n$.

Using the elementary operations (a) and (b) we first reduce the matrix $\mathbf{A}_1$ to the normal form (3.1). Next adding multiples of the columns of the first strip to the other strips of $\mathbf{A}$ we can kill all the entries in the first upper horizontal strip. Therefore we obtain the following equivalent form:

$$\mathbf{R} \simeq \left[\begin{array}{cc||c||c||c} \mathrm{E} & \mathrm{O} & \mathrm{O} & \dots & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & \mathbf{A}_{21} & \dots & \mathbf{A}_{n1} \end{array}\right] = \left[\begin{array}{cc||c} \mathrm{E} & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & \mathbf{R}_1 \end{array}\right]$$

for some matrices $\mathbf{A}_{21}, \dots, \mathbf{A}_{n1}$.

Since the matrix

$$\mathbf{R}_1 = [\ \overbrace{\mathbf{A}_{21}}^{p_2} \ \| \ \overbrace{\mathbf{A}_{31}}^{p_3} \ \| \cdots \| \ \overbrace{\mathbf{A}_{n1}}^{p_n}\ ],$$

corresponds to a matrix representation of the chain $C_{n-1} = \{p_2 \preceq p_3 \preceq \dots \preceq p_n\}$, by induction hypothesis, it can be reduced to form (3.2). And so the matrix $\mathbf{R}$ can be reduced to this form as well. $\square$

Lemma 3.2. *Let $\mathcal{P}$ be a poset of the form*

$$\begin{array}{cc} 1 & 2 \\ \bullet & \bullet \end{array}$$

then the matrix representation $\mathbf{R}$ *of $\mathcal{P}$ over a division ring K can be reduced to the following form:*

$$\mathbf{R} = \left[\begin{array}{c|c|c|c||c|c|c} \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} \end{array}\right]. \tag{3.6}$$

Therefore $n(\mathcal{P}) = 4$ and all indecomposable representations are elementary.

Proof. Consider the corresponding matrix representation $\mathbf{R}$ of $\mathcal{P}$:

$$\mathbf{R} = [\ \mathbf{X} \parallel \mathbf{Y}\ ],$$

where a matrix $\mathbf{X}$ corresponds to the element 1 and a matrix $\mathbf{Y}$ corresponds to the element 2. By the elementary operations with matrix $\mathbf{X}$ we can reduce it to the normal form (3.1), and so

$$\mathbf{R} \simeq \left[\begin{array}{c|c||c} \mathrm{E} & \mathrm{O} & \mathbf{Y}_1 \\ \hline \mathrm{O} & \mathrm{O} & \mathbf{Y}_2 \end{array}\right] = \mathbf{R}_1,$$

and from this form it follows that we can add the rows of a matrix $\mathbf{Y}_2$ to the rows of a matrix $\mathbf{Y}_1$. So we can reduce the matrix $\mathbf{Y}_2$ to the normal form and then by means of the identical matrix $\mathbf{E}$ we can kill all elements which lie above in the matrix $\mathbf{Y}_1$. Therefore we obtain the required form (3.6) □

Lemma 3.3. *Let C_n be a chain. Then all indecomposable representations of a poset $\mathcal{P} = (1) \sqcup C_n$ are elementary.*

Proof. Let $\mathcal{P} = (1) \sqcup C_n$. Consider the corresponding matrix representation $\mathbf{R}$ of $\mathcal{P}$:

$$\mathbf{R} = [\ \mathbf{X} \parallel \mathbf{Y}\ ]$$

where a matrix $\mathbf{X}$ corresponds to a subposet (1) and a matrix $\mathbf{Y}$ corresponds to a chain C_n. By elementary operations we can reduce the matrix $\mathbf{X}$ to the normal form, and so

$$\mathbf{R} \simeq \left[\begin{array}{c|c||c} \mathrm{E} & \mathrm{O} & \mathbf{Y}_1 \\ \hline \mathrm{O} & \mathrm{O} & \mathbf{Y}_2 \end{array}\right] = \mathbf{R}_1.$$

Considering the matrix $\mathbf{Y}_2$ as a representation of the chain C_n, we can reduce it, by lemma 3.1, by means of elementary row operations of the matrix $\mathbf{R}_1$ and elementary column operations of the matrix $\mathbf{Y} = \left[\begin{array}{c} \mathbf{Y}_1 \\ \hline \mathbf{Y}_2 \end{array}\right]$ to the canonical form (3.5). By means of all identical matrices $\mathbf{E}$ in $\mathbf{Y}_2$ we can kill all elements which lie above in the matrix $\mathbf{Y}_1$, and so we obtain the following matrix representation:

$$\mathrm{R} \simeq \left[\begin{array}{c||c|c|c|c|c} \mathrm{X}_1 & \mathrm{O} & \mathrm{Y}_{11} & \ldots & \mathrm{O} & \mathrm{Y}_{1n} \\ \hline \mathrm{O} & \mathrm{E} & \mathrm{O} & \ldots & \mathrm{O} & \mathrm{O} \\ \hline \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \ldots & \mathrm{E} & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \ldots & \mathrm{O} & \mathrm{O} \end{array}\right] = \left[\begin{array}{c|c} \mathrm{U}_1 & \mathrm{O} \\ \hline \mathrm{O} & \mathrm{U}_2 \end{array}\right]$$

which is a direct sum of two matrix representations

$$\mathrm{U}_1 = [\ \mathrm{E} \,\|\, \mathrm{Y}_{11} \,|\, \ldots \,|\, \mathrm{Y}_{1n}\ ]$$

and

$$\mathrm{U}_2 = \left[\begin{array}{c|c|c|c} \mathrm{E} & \mathrm{O} & \ldots & \mathrm{O} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{O} & \mathrm{O} & \ldots & \mathrm{E} \\ \hline \mathrm{O} & \mathrm{O} & \ldots & \mathrm{O} \end{array}\right].$$

The representation U_2 is a direct sum of elementary representations of forms (3.2) and (3.3). Since the representation

$$[\ \mathrm{Y}_{11} \,|\, \ldots \,|\, \mathrm{Y}_{1n}\ ]$$

can be considered as a representation of the chain C_n, it is equivalent to a direct sum of elementary representations of forms (3.2) and (3.3), by lemma 3.1. Therefore an elementary representation of U_1 is one of forms (3.2), (3.3) or (3.4), as required. $\square$

4 Trichotomy lemma

Let $(\mathcal{P}, \preceq)$ be a finite poset with n elements. In this section by a representation V of a poset $\mathcal{P}$ we understand its matrix representation with dimension vector $\mathrm{d} = (d_0, d_1, \ldots, d_n)$.

Definition 4.1. *Let V be a representation of a poset $\mathcal{P}$ with dimension $\mathrm{d} = (d_0, d_1, \ldots, d_n)$. The subset of $\mathcal{P}$ formed of those $p_i \in \mathcal{P}$ for which $d_i > 0$ is called the* support *of V.*

Definition 4.2. *We say that the posets X and Z form an* ordinal sum *and denote it by $X < Z$ if $x \prec z$ for all $x \in X$ and for all $z \in Z$.*

Definition 4.3. ([6]). *A* **trichotomy of a finite poset** $\mathcal{P}$ *is a triple* X, Y, Z *formed by disjoint subposets with union* $\mathcal{P}$ *such that the following conditions are satisfied:*

a) $X \neq \emptyset$, *and* $Z \neq \emptyset$;

b) $X < Z$, *i.e.,* $x \prec z$ *for all* $x \in X$ *and for all* $z \in Z$;

c) the width $w(Y) = 1$.

In this case we write $\mathcal{P} = \{X < Z\} \sqcup Y$.

Lemma 4.1. (Trichotomy Lemma, [6]). *Let* (X, Y, Z) *be a trichotomy of a poset* $\mathcal{P}$, *i.e.* $\mathcal{P} = \{X < Z\} \sqcup Y$, *where* $X, Z \neq \varnothing$, *and* $w(Y) \leq 1$. *If* R *is a representation of* $\mathcal{P}$, *then* $R \cong R_1 \oplus R_2$, *where support of* R_1 *is contained in* $X \sqcup Y$, *and support of* R_2 *is contained in* $Y \sqcup Z$.

Proof. Let $\mathcal{P}$ be a cardinal sum of posets X, Y, Z such that $X < Z$ and the width of Y is ≤ 1. If $Y = \emptyset$, the statement is obvious. Suppose that $Y = \{a_1 \prec a_2 \prec \ldots \prec a_s\}$ is a chain. Consider the corresponding matrix representation $\mathbf{R}$ of $\mathcal{P}$

$$\mathbf{R} = [\,\overbrace{\mathbf{X}}^{X} \,\|\, \overbrace{\mathbf{Y}}^{Y} \,\|\, \overbrace{\mathbf{Z}}^{Z}\,],$$

where the columns of $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$ are assigns to posets X, Y and Z respectively.

By the elementary row operations of the matrix $\mathbf{R}$ we can reduce the matrix $\mathbf{R}$ to the following form

$$\mathbf{R} \simeq \left[\begin{array}{c||c||c} \mathbf{X}_1 & \mathbf{Y}_1 & \mathbf{Z}_1 \\ \hline \mathbf{O} & \mathbf{Y}_2 & \mathbf{Z}_2 \end{array}\right] = \mathbf{R}_1$$

where $\left[\begin{array}{c} \mathbf{X}_1 \\ \hline \mathbf{O} \end{array}\right]$ is the row canonical form of the matrix $\mathbf{X}$ and the matrix $\mathbf{X}_1$ has no zero rows.

Considering the matrix $\mathbf{Y}_2$ as a representation of a chain Y, by lemma 3.1 we can reduce it by means of the elementary row operations of the matrix $\mathbf{R}_1$ and the elementary column operations of the matrix $\mathbf{Y} = \left[\begin{array}{c} \mathbf{Y}_1 \\ \hline \mathbf{Y}_2 \end{array}\right]$ to the canonical form (3.5).

By means of all identical matrices $\mathbf{E}$ in $\mathbf{Y}_2$ we can kill all the elements which lie above in the matrix $\mathbf{Y}_1$ and so we obtain the following matrix representation:

$$\mathbf{R} \simeq \left[\begin{array}{c||c|c|c|c|c||c} \mathbf{X}_1 & \mathbf{O} & \mathbf{Y}_{11} & \ldots & \mathbf{O} & \mathbf{Y}_{1s} & \mathbf{Z}_{11} \\ \hline \mathbf{O} & \mathbf{E} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{Z}_{21} \\ \hline \ldots & \ldots & \ldots & \ddots & \ldots & \ldots & \ldots \\ \hline \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{E} & \mathbf{O} & \mathbf{Z}_{2s} \\ \hline \mathbf{O} & \mathbf{O} & \mathbf{O} & \ldots & \mathbf{O} & \mathbf{O} & \mathbf{Z}_{2s+1} \end{array}\right].$$

Taking into account that the matrix $\mathbf{X}_1$ is in row canonical form without zero rows, and $X < Z$, we can kill the matrix $\mathbf{Z}_{11}$ and so we obtain the following representation:

$$
R \simeq \left[\begin{array}{c||c|c|c|c|c||c}
X_1 & O & Y_{11} & \dots & O & Y_{1s} & O \\ \hline
O & E & O & \dots & O & O & Z_{21} \\ \hline
\dots & \dots & \dots & \ddots & \dots & \dots & \dots \\ \hline
O & O & O & \dots & E & O & Z_{2s} \\ \hline
O & O & O & \dots & O & O & Z_{2s+1}
\end{array}\right] = \left[\begin{array}{c|c} U_1 & O \\ \hline O & U_2 \end{array}\right],
$$

which is a direct sum of required representations

$$
U_1 = \left[\ \overbrace{X_1}^{X}\ \Big\|\ \overbrace{Y_{11} \mid \dots \mid Y_{1s}}^{Y}\ \right]
$$

and

$$
U_2 = \begin{array}{c}
\begin{array}{cc} \overbrace{\hphantom{E \quad O \quad \dots \quad O}}^{Y} & \overbrace{\hphantom{Z_{2s+1}}}^{Z} \end{array} \\
\left[\begin{array}{c|c|c|c||c}
E & O & \dots & O & Z_{21} \\ \hline
\dots & \dots & \ddots & \dots & \dots \\ \hline
O & O & \dots & E & Z_{2s} \\ \hline
O & O & \dots & O & Z_{2s+1}
\end{array}\right]
\end{array}.
$$

□

Proposition 4.1. ([6]). *If a subposet Y of $\mathcal{P}$ consisting of all elements of $\mathcal{P}$ incomparable with element $p \in \mathcal{P}$ is a chain, then each indecomposable representation of $\mathcal{P}$ present at p is elementary.*

Proof. Let Y be a set of all elements of $\mathcal{P}$ incomparable with $p \in \mathcal{P}$. Denote by $X = \{q \in \mathcal{P} \ : \ q < p\}$, $U = \{q \in \mathcal{P} \ : \ p < q\}$. Suppose Y is a chain. Then $\mathcal{P} = X \sqcup Y \sqcup \{p\} \sqcup U$ and $X < U$. Therefore we can apply lemma 4.1 to sets X, Y, Z, where $Z = \{p\} \cup U$. From this lemma it follows that the support of p is contained in $\mathcal{P}_1 = Y \sqcup Z = Y \sqcup \{p\} \sqcup U$. Since $\mathcal{P}_1 = \{\{p\} < U\} \sqcup Y$, we can apply this lemma again for $X_1 = \{p\}$, $Y_1 = Y$ and $Z_1 = U$. Then we obtain that the support of p is contained in $\mathcal{P}_2 = \{p\} \sqcup Y$. Since Y is a chain, from lemma 3.3 it follows that all indecomposable representations of $\mathcal{P}_2$ are elementary, that required. □

Corollary 4.1. ([6]). *Any indecomposable representation of a poset $\mathcal{P}$ of width ≤ 2 is elementary. Therefore, all posets of width ≤ 2 are of a finite representation type.*

5 Kleiner's lemma

The representation of a pair of finite posets over a field were introduced by M.Kleiner in [11]. The pair of posets of a finite type and of tame type were described by M.Kleiner in [11] and [12] respectively.

Let $\mathcal{P} = (\{p_1, p_2, \dots, p_m\}, \preceq_1)$ and $\mathcal{Q} = (\{q_1, q_2, \dots, q_n\}, \preceq_2)$ be finite posets. A **representation of the pair of posets** $(\mathcal{P}, \mathcal{Q})$ over a field k is a matrix A which is partitioned into mn blocks A_{ij}:

$$\mathrm{A} = \left[\begin{array}{c|c|c|c} \mathrm{A}_{11} & \mathrm{A}_{12} & \dots & \mathrm{A}_{1n} \\ \hline \mathrm{A}_{21} & \mathrm{A}_{22} & \dots & \mathrm{A}_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{A}_{m1} & \mathrm{A}_{m2} & \dots & \mathrm{A}_{mn} \end{array}\right].$$

If the dimensions a block A_{ij} is equal to $u_i \times v_j$, then the integer vector $\mathrm{d} = (u_1, u_2, \dots, u_m; v_1, v_2, \dots, v_n)$ is called the **dimension of the partitioned matrix A.**

Let

$$\mathrm{B} = \left[\begin{array}{c|c|c|c} \mathrm{B}_{11} & \mathrm{B}_{12} & \dots & \mathrm{B}_{1n} \\ \hline \mathrm{B}_{21} & \mathrm{B}_{22} & \dots & \mathrm{B}_{2n} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{B}_{m1} & \mathrm{B}_{m2} & \dots & \mathrm{B}_{mn} \end{array}\right],$$

be another matrix representation of a pair of posets $(\mathcal{P}, \mathcal{Q})$.

Definition 5.1. *A representation* A *is* **isomorphic** *to a representation* B *of a pair of poset* $(\mathcal{P}, \mathcal{Q})$ *if* A *can be reduced to* B *by the following operations:*

(a) elementary row operations within each horizontal strip $\mathrm{A}'_i = \{\mathrm{A}_{i1}, \mathrm{A}_{i2}, \dots, \mathrm{A}_{in}\}$ *for* $i = 1, 2, \dots, m$;

(b) elementary column operations within each vertical strip $\mathrm{A}''_j = \{\mathrm{A}_{1j}, \mathrm{A}_{2j}, \dots, \mathrm{A}_{mj}\}$ *for* $i = 1, 2, \dots, n$;

(c) additions of columns of a vertical strip $\{\mathrm{A}_{1j}, \mathrm{A}_{2j}, \dots, \mathrm{A}_{mj}\}$ *to columns of a strip* $\{\mathrm{A}_{1k}, \mathrm{A}_{2k}, \dots, \mathrm{A}_{mk}\}$ *if* $j \prec_1 k$ *in* $\mathcal{P}$;

(d) additions of rows of a horizontal strip $\{\mathrm{A}_{i1}, \mathrm{A}_{i2}, \dots, \mathrm{A}_{in}\}$ *to rows of a horizontal strip* $\{\mathrm{A}_{j1}, \mathrm{A}_{j2}, \dots, \mathrm{A}_{jn}\}$ *if* $i \preceq_2 j$ *in* $\mathcal{Q}$.

In a similar way as for one poset we can introduce the notions of a direct sum of matrix representations of a pair of posets, and the notion of an indecomposable matrix representation.

Lemma 5.1. (Kleiner's lemma, [11]) *Let* (X, Y) *be a pair of finite posets* X *and* Y*, where* $X = C_{m+1}$ *is a chain of length* $m+1$*. Then* (X, Y) *is of a finite representation type if and only if a poset* $C_m \sqcup Y$ *is of a finite representation type.*

Proof. Consider a cardinal sum of posets $C_m \sqcup Y$, where $C_m = \{x_1 \preceq_1 x_2 \preceq_1 \dots \preceq_1 x_m$ is a chain of length m, and $Y = \{y_1, y_2, \dots, y_n\}$ is a poset with partially order $\preceq_2$. Let R be a matrix representation of $C_m \sqcup Y$. Then it is partitioned into $m+n$ vertical strips:

$$\mathrm{R} = [\, \overbrace{\mathrm{A}_1 \mid \dots \mid \mathrm{A}_m}^{C_m} \,\|\, \overbrace{\mathrm{B}_1 \mid \dots \mid \mathrm{B}_n}^{Y} \,]$$

where matrices A_i corresponds to the elements x_i of the poset C_m, and matrices B_j corresponds to the elements y_j of the poset Y.

Considering the matrix $\mathrm{A} = \left[\ \mathrm{A}_1 \,|\, \dots \,|\, \mathrm{A}_m\ \right]$ as a representation of C_m we can reduce it by means of elementary row operations of the matrix R and elementary column operations inside each A_i, and by means of addition of the columns of matrices A_i to columns of matrices A_j if $i \preceq_1 j$ in C_m to the canonical form (3.5), by lemma 3.1. Then any matrix B_j is divided into $m+1$ horizontal strips, and we obtain the following form of R:

$$\mathrm{R} \simeq \begin{array}{c} \begin{array}{cc} \overbrace{\qquad\qquad\qquad\quad}^{C_m} & \overbrace{\qquad\qquad\qquad\qquad}^{Y} \end{array} \\ \left[\begin{array}{c|c|c|c||c|c|c} \mathrm{E} & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{B}_{11} & \cdots & \mathrm{B}_{1n} \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hline \mathrm{O} & \mathrm{O} & \cdots & \mathrm{E} & \mathrm{B}_{m1} & \cdots & \mathrm{B}_{mn} \\ \hline \mathrm{O} & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{B}_{m+1,1} & \cdots & \mathrm{B}_{m+1,n} \end{array}\right] \end{array} = \left[\begin{array}{c|c|c|c||c} \mathrm{E} & \mathrm{O} & \dots & \mathrm{O} & \mathrm{B}'_1 \\ \hline \vdots & \vdots & \ddots & \vdots & \vdots \\ \hline \mathrm{O} & \mathrm{O} & \cdots & \mathrm{E} & \mathrm{B}'_m \\ \hline \mathrm{O} & \mathrm{O} & \cdots & \mathrm{O} & \mathrm{B}'_{m+1} \end{array}\right]$$

And we can add the rows of a horizontal strip B'_i to the rows of a horizontal strip B'_j if $j \preceq_1 i$ in C_m. Therefore the matrix

$$\mathrm{B} = \left[\begin{array}{c|c|c} \mathrm{B}_{11} & \dots & \mathrm{B}_{1n} \\ \hline \dots & \ddots & \dots \\ \hline \mathrm{B}_{1m} & \cdots & \mathrm{B}_{nm} \\ \hline \mathrm{B}_{1m+1} & \cdots & \mathrm{B}_{nm+1} \end{array}\right]$$

can be considered as a representation of a pair of posets (Y, C_{m+1}). $\square$

6 The main construction

Before we introduce the main construction we give some examples which show in which way the representations of a finite poset are reduced to this construction.

Example 6.1. Consider the representations of a poset $\mathcal{P}$:

$$\begin{array}{ccc} 1 & 2 & 3 \\ \bullet & \bullet & \bullet \end{array}$$

and let $\mathcal{P}_1$ be a sub-poset of $\mathcal{P}$ with $w(\mathcal{P}_1) = 2$ of the form:

$$\begin{array}{cc} 1 & 2 \\ \bullet & \bullet \end{array}$$

The corresponding matrix representation of $\mathcal{P}$ has the following form

$$\mathbf{R} = [\ \overbrace{\mathbf{A}_1}^{1}\ \|\ \overbrace{\mathbf{A}_2}^{2}\ \|\ \overbrace{\mathbf{A}_3}^{3}\ ],$$

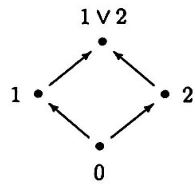
where the columns of $\mathbf{A}_i$ are assigns to elements i of the poset $\mathcal{P}$ for $i = 1, 2, 3$.

By means of elementary row operations on the matrix $\mathbf{R}$ and elementary column operations within each vertical strip $\mathbf{A}_1$ and $\mathbf{A}_2$ we can reduce the matrix $\mathbf{R}$ to the following form:

$$\mathbf{R} \simeq \begin{array}{c} \begin{array}{ccc|ccc|c} \multicolumn{3}{c}{\overbrace{\qquad\quad}^{1}} & \multicolumn{3}{c}{\overbrace{\qquad\quad}^{2}} & \overbrace{\quad}^{3} \end{array} \\ \begin{array}{|c|c|c||c|c|c||c|} \hline \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathbf{A}_{31} \\ \hline \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathbf{A}_{32} \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{E} & \mathrm{O} & \mathrm{O} & \mathbf{A}_{33} \\ \hline \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathrm{O} & \mathbf{A}_{34} \\ \hline \end{array} \end{array} \begin{array}{c} \\ 1 \vee 2 \\ 1 \\ 2 \\ 0 \end{array}$$

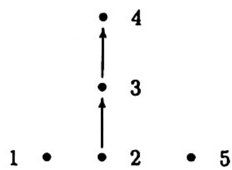
According to this reducing the matrix $\mathbf{A}_3$ is divided into 4 horizontal strips and the elementary operations between them are admissible with according the following partially ordered set:

$\mathcal{A}(\mathcal{P}_1)$:

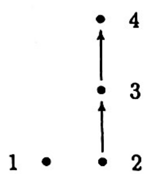


(6.1)

Example 6.2. Consider a poset $\mathcal{P}$ of the form



and its subposet $\mathcal{P}_1$ of the form



We have the corresponding matrix representation of $\mathcal{P}$:

$$\mathbf{R} = \begin{array}{ccccc} & & \curvearrowright & \curvearrowright & \\ \boxed{\mathbf{A}_1} & \boxed{\mathbf{A}_2} & \boxed{\mathbf{A}_3} & \boxed{\mathbf{A}_4} & \boxed{\mathbf{A}_5} \\ 1 & 2 & 3 & 4 & 5 \end{array}$$

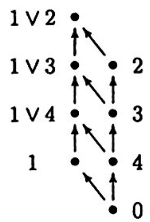
Reducing the poset $\mathcal{P}_1$, i.e., the first 4 matrices, we obtain the following equivalent representation:

$\mathbf{R} \simeq$

1	2	3	4	5	
E	E				$1 \vee 2$
E		E			$1 \vee 3$
E			E		$1 \vee 4$
	E				2
		E			3
			E		4
E					1
O	O	O	O		0

According to this reducing the matrix $\mathbf{A}_4$ is divided into 8 horizontal strips and the elementary operations between them are admissible with according the following partially ordered set:

$\mathcal{A}(\mathcal{P}_1):$



(6.2)

Now we can introduce the formal definitions.

Let $\mathcal{P} = \{p_1, p_2, \ldots, p_n\}$ be a poset with ordering relation $\preceq$. Denote by $\mathcal{A}(\mathcal{P})$ the set of all antichains of $\mathcal{P}$ of length $l \geq 0$. We assume that the antichain of length 0 is an empty set, and we denote this antichain as 0. And we identify antichains of length 1 with the elements of $\mathcal{P}$ themselves. We define the order relation $\leq^+$ on $\mathcal{A}(\mathcal{P})$ as follows. When $X, Y \in \mathcal{A}(\mathcal{P})$, then $X \leq^+ Y$ if and only if for any $a \in X$ (resp. $b \in Y$) there exists $b \in Y$ (resp. $a \in X$) such that $b \preceq a$. We also assume that $0 <^+ X$ for all $X \neq 0$. Let $X \in \mathcal{A}(\mathcal{P})$. If $X = \{p_i\} \subset \mathcal{P}$, then we denote it in $\mathcal{A}(\mathcal{P})$ as p_i, and if $X = \{p_i, p_j\} \subset \mathcal{P}$, then we denote it in $\mathcal{A}(\mathcal{P})$ as $p_i \vee p_j$.

We denote by $\mathcal{P}^\circ$ the poset which is dual to $\mathcal{P}$, i.e. the set of elements of $\mathcal{P}^\circ$ is the same as $\mathcal{P}$ and the order relation is $\succeq$. Then from the definition of $\mathcal{A}(\mathcal{P})$ it follows that $\mathcal{A}(\mathcal{P}) \supseteq \mathcal{P}^\circ$.

For a poset $\mathcal{P}_1 = \mathcal{A}(\mathcal{P})$ in a similar way we can build $\mathcal{A}(\mathcal{P}_1)$. Then we shall denote $\mathcal{A}^2(\mathcal{P}) = \mathcal{A}(\mathcal{P}_1)$, and in the general case we denote

$$\mathcal{A}^n(\mathcal{P}) = \mathcal{A}(\mathcal{A}^{n-1}(\mathcal{P}))$$

From the definitions above it is easy to check the following statements.

Proposition 6.1. *If $\mathcal{P}$ is a poset, then $\mathcal{A}(\mathcal{P})$ is a poset with regard to order relation $\leq^+$ with least element 0 and greatest element which is the antichain containing all minimal elements of the poset $\mathcal{P}$.*

Proposition 6.2. *If $\mathcal{Q}$ is a subposet of a poset $\mathcal{P}$, then $\mathcal{A}^n(\mathcal{Q}) \subset \mathcal{A}^n(\mathcal{P})$ for all n.*

Example 6.3. Let $\mathcal{P}$ be a chain C_3:

$$\overset{1}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{3}{\bullet}$$

then $\mathcal{A}(\mathcal{P})$ is a chain C_4:

$$\overset{0}{\bullet} \longrightarrow \overset{3}{\bullet} \longrightarrow \overset{2}{\bullet} \longrightarrow \overset{1}{\bullet}$$

In the general case if $\mathcal{P}$ is a chain C_n, then $\mathcal{A}(\mathcal{P})$ is a chain C_{n+1}, i.e.,

$$\mathcal{A}(C_n) = C_{n+1}.$$

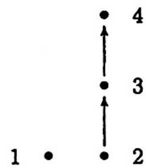
Example 6.4. Let $\mathcal{P}$ be an antichain A_2:

$$\overset{1}{\bullet} \qquad \overset{2}{\bullet}$$

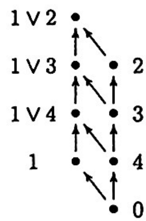
then $\mathcal{A}(\mathcal{P})$ has the following form:

$$\begin{tikzcd}
 & 1 \vee 2 & \\
1 \arrow[ur] & & 2 \arrow[ul] \\
 & 0 \arrow[ul] \arrow[ur] &
\end{tikzcd} \tag{6.4}$$

Example 6.5. Let $\mathcal{P}$ be a poset of the form:

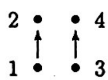


then $\mathcal{A}(\mathcal{P})$ has the following form:

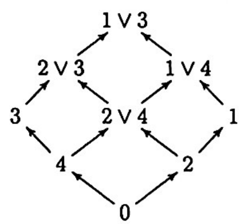


(6.5)

Example 6.6. Let $\mathcal{P}$ be a poset of the form:



then the diagram of $\mathcal{A}(\mathcal{P})$ is of the following form:



(6.6)

Since the elements 1, $2 \vee 4$ and 3 are incomparable, $w(\mathcal{A}(\mathcal{P}) = 3$.

Theorem 6.1. *Let $\mathcal{P}$ be a poset which is a cardinal sum of subposets $\mathcal{P}_1$ and $\mathcal{P}_2$, where $\mathcal{P}_1 = C_n$ and $\mathcal{P}_2$ is a poset of width 2. Then $\mathcal{P}$ is of a finite representation type if and only if the cardinal sum of posets $C_{n-1} \sqcup \mathcal{A}(\mathcal{P}_1)$ is of a finite representation type. In particular, if $n = 1$, then $\mathcal{P}$ is of a finite representation type if and only if the poset $\mathcal{A}(\mathcal{P}_1)$ is of a finite representation type. Moreover, if $\mathcal{A}^s(\mathcal{P})$ is a poset of width 2 for all $s < k$, then $\mathcal{P}$ is of finite representation type if and only if the cardinal sum of posets $C_{n-k} \sqcup \mathcal{A}^k(\mathcal{P}_1)$ is of a finite representation type.*

Proof. Let a poset $\mathcal{P} = \mathcal{P}_1 \sqcup \mathcal{P}_2 = C_n \sqcup \mathcal{P}_2$ be of a finite type. Consider the corresponding matrix representation R of $\mathcal{P}$:

$$\mathbf{R} = [\ \mathbf{X} \parallel \mathbf{Y}\ ]$$

where a matrix **X** corresponds to the chain C_n and a matrix **Y** corresponds to a subposet $\mathcal{P}_2 = \{\alpha_1, \ldots, \alpha_n\}$. Since $w(\mathcal{P}_2) = 2$, this poset is of finite representation type, by corollary 4.1. Moreover, all indecomposable representations of $\mathcal{P}_2$ are elementary, by this corollary. So we can reduce the matrix **Y** to the form which is a direct sum of elementary representations. We collect all the representations of the same form in the blocks. So we obtain the direct sum of representations of the following form:

1) zero representation of $\mathcal{P}_2$ with zero matrices with the same number of rows for each $\alpha_i \in \mathcal{P}_2$;

2) representations of $\mathcal{P}_2$ with one identical matrix in the place i which correspond to $\alpha_i \in \mathcal{P}_2$ and zero matrices on the other places, and all these matrices have the same number of rows;

3) representations of $\mathcal{P}_2$ with two identical matrix in the places i, j which corresponds to elements $\alpha_i, \alpha_j \in \mathcal{P}_2$ and zero matrices on the other places, and all these matrices have the same number of rows.

We denote the strip of the matrix X which corresponds to zero representation 1) by 0; the strip of the matrix X which corresponds to representation of the type 2) by i; and the strip of the matrix X which corresponds to representation of the type 3) by $i \vee j$. It is easy to see, that the representation of the type 3) contains in R if and only if the elements α_i and α_j are incomparable in the poset $\mathcal{P}_2$. Moreover, we can add the strip 0 to any strip, without any changing. We also can add any strip i to strip $i \vee j$, without any changing of the representation matrix of the poset $\mathcal{P}_2$, because we can kill the new appearing identical matrix in the strip $i \vee j$ by means of identical matrix in this strip.

Therefore according to this reduction the matrix **X** is divided into $|\mathcal{A}(\mathcal{P}_1)|$ vertical strips, and the elementary operations between these strips are admissible in accordance with order relation in the poset $\mathcal{A}(\mathcal{P}_1)$. Therefore we obtain the problem of the representation of a pair of finite posets $((n), \mathcal{A}(\mathcal{P}_1))$. By lemma 5.1, this pair of posets is of a finite type if and only if the cardinal sum $(n-1) \sqcup \mathcal{A}(\mathcal{P}_1)$ is of a finite type.

Continuing by induction, we obtain the last statement of the theorem.

□

7 Primitive posets of the infinite representation type

Recall that a finite partially ordered set $\mathcal{P}$ is a **primitive poset** (or **elementary poset**) if it is a cardinal sum of linearly ordered sets. In this section we prove that all critical primitive posets from theorem 1.2 are of infinite representation type.

Lemma 7.1. *The poset $\mathcal{P}:\ \{\bullet\ \bullet\ \bullet\ \bullet\}$ is of infinite representation type.*

Proof. For this purpose we note that for any $\lambda \in k$ the matrix representation

$$\mathrm{A}^{(n,\lambda)} = \left|\begin{array}{c|c|c|c} J(n,\lambda) & \mathrm{E} & \mathrm{E} & \mathrm{O} \\ \mathrm{E} & \mathrm{E} & \mathrm{O} & \mathrm{E} \end{array}\right|$$

with $cdn(\mathrm{A}^{(n,\lambda)}) = (n,n,n,n,2n)$ is an indecomposable representation of $\mathcal{P}$, where E is the identity matrix in $M_n(k)$ and $J(n,\lambda)$ is a Jordan block of size $n \times n$. □

Lemma 7.2. *The poset $\mathcal{P} = (2) \sqcup (2) \sqcup (2)$:*

$$\begin{tikzcd} \bullet \arrow[d, dash] & \bullet \arrow[d, dash] & \bullet \arrow[d, dash] \\ \bullet & \bullet & \bullet \end{tikzcd} \tag{7.1}$$

is of a infinite representation type.

Proof. Let $\mathcal{P} = (2) \sqcup (2) \sqcup (2)$. Consider a subposet $\mathcal{Q} = (2) \sqcup (2)$ of $\mathcal{P}$ with diagram

$$\begin{tikzcd} 2\ \bullet & \bullet\ 4 \\ 1\ \bullet \arrow[u] & \bullet\ 3 \arrow[u] \end{tikzcd} \tag{7.2}$$

The matrix representation corresponding to $\mathcal{R}$ of a poset $\mathcal{P}$ has the following form:

$$\mathrm{R} = \begin{array}{|c||c||c||c||c||c|} \hline \mathrm{A}_1 & \mathrm{A}_2 & \mathrm{A}_3 & \mathrm{A}_4 & \mathrm{A}_5 & \mathrm{A}_6 \\ \hline \multicolumn{1}{c}{1} & \multicolumn{1}{c}{2} & \multicolumn{1}{c}{3} & \multicolumn{1}{c}{4} & \multicolumn{1}{c}{5} & \multicolumn{1}{c}{6} \end{array}$$

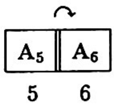
and the following operations are admissible with matrix R:

1) elementary row operations on the matrix R;

2) elementary column operations within each matrix A_i for $i = 1, \ldots, 6$;

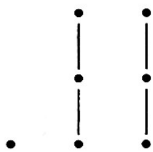
3) addition of columns of matrix A_1 to columns of matrix A_2; addition of columns of matrix A_3 to columns of matrix A_4; addition of columns of matrix A_5 to columns of matrix A_6.

By means of this operations we can reduce the first four matrices A_1, A_2, A_3, A_4. At the result the matrix

R = 

is divided into 9 horizontal strips which addition to each other are in the one-to-one correspondence with set $\mathcal{A}(\mathcal{Q})$ having form (6.6) which width is equal to 3. So we reduce our problem to describing the representations of a pair of poset $((2), \mathcal{A}(\mathcal{Q}))$. By Kleiner's lemma this problem is equivalent to describing the representations of the poset which is a cardinal sum $(1) \sqcup \mathcal{A}(\mathcal{Q})$. Since the width of $\mathcal{A}(\mathcal{Q})$ is equal 3, this cardinal sum contains a full subposet of width 4, and so, by lemma 7.1, it is of an infinite representation type. $\square$

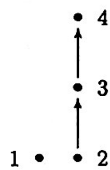
Lemma 7.3. *The poset* $\mathcal{P} = (1) \sqcup (3) \sqcup (3)$ *with diagram:*



(7.3)

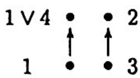
is of infinite representation type.

Proof. Let $\mathcal{Q} = (1) \sqcup (3)$ be a sub-poset of $\mathcal{P}$ with diagram:



(7.4)

Then $\mathcal{P} = \mathcal{Q} \sqcup (3)$, and by theorem 6.1, it has the same representation type as the cardinal sum of posets $\mathcal{A}(\mathcal{Q}) \sqcup (2)$. Since $\mathcal{A}(\mathcal{Q})$ has the form (6.5), it contains a sub-poset



Therefore $\mathcal{A}(\mathcal{Q}) \sqcup (2)$ contains a subposet $(2) \sqcup (2) \sqcup (2)$, and so, by the previous lemma, $\mathcal{P}$ is of infinite representation type. $\square$

Lemma 7.4. *The poset $\mathcal{P} = (1) \sqcup (2) \sqcup (5)$ is of infinite representation type.*

Proof. Let $\mathcal{Q} = (1) \sqcup (5)$ be a subposet of $\mathcal{P}$ with diagram:

$$\begin{tikzcd} & 6 \\ & 5 \arrow[u] \\ & 4 \arrow[u] \\ & 3 \arrow[u] \\ 1 & 2 \arrow[u] \end{tikzcd} \tag{7.5}$$

Then $\mathcal{P} = \mathcal{Q} \sqcup (2)$, and by theorem 6.1, it has the same representation type as the cardinal sum of posets $\mathcal{A}(\mathcal{Q}) \sqcup (1)$. Since $\mathcal{A}(\mathcal{Q})$ has the following form

$$\begin{tikzcd} 1\vee 2 & \\ 1\vee 3 \arrow[u] & 2 \arrow[ul] \\ 1\vee 4 \arrow[u] & 3 \arrow[u] \arrow[ul] \\ 1\vee 5 \arrow[u] & 4 \arrow[u] \arrow[ul] \\ 1\vee 6 \arrow[u] & 5 \arrow[u] \arrow[ul] \\ 1 \arrow[u] & 6 \arrow[u] \arrow[ul] \\ & 0 \arrow[u] \arrow[ul] \end{tikzcd} \tag{7.6}$$

it contains a sub-poset

$$\begin{tikzcd} 1\vee 5 & 2 \\ 1\vee 6 \arrow[u] & 3 \arrow[u] \\ 1 \arrow[u] & 4 \arrow[u] \end{tikzcd}$$

Therefore $\mathcal{A}(\mathcal{Q}) \sqcup (1)$ contains a subposet $(3) \sqcup (3) \sqcup (1)$, and so, by the previous lemma, $\mathcal{P}$ is of infinite representation type. $\square$

8 Primitive posets of the finite representation type

In this section we prove that the following posets $(1) \sqcup (1) \sqcup (n)$, $(1) \sqcup (2) \sqcup (2)$, $(1) \sqcup (2) \sqcup (3)$ and $(1) \sqcup (2) \sqcup (4)$ are of a finite representation type.

Lemma 8.1. *If the width $w(\mathcal{P})$ of a poset $\mathcal{P}$ is greater than or equal to four, then $\mathcal{P}$ is of an infinite representation type.*

Proof. Suppose that $w(\mathcal{P}) \geq 4$. Then $\mathcal{P}$ contains a full subposet $\mathcal{R}$ consisting of four incomparable elements, which is of infinite representation type, by lemma 7.1.

From this lemma it follows that a poset of a finite representation type has width ≤ 3.

Lemma 8.2. *A finite poset $\mathcal{P} = (1) \sqcup (1) \sqcup (n)$ with diagram*

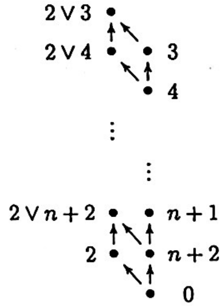
$$\begin{array}{cccl} & & \bullet & n+2 \\ & & | & \\ & & \bullet & n+1 \\ & & \vdots & \\ & & \bullet & 4 \\ 1 & 2 & | & \\ \bullet & \bullet & \bullet & 3 \end{array} \tag{8.1}$$

is of a finite representation type.

Proof. Really, consider a subposet $\mathcal{Q} = (1) \sqcup (n)$ with diagram:

$$\begin{array}{ccl} & \bullet & n+2 \\ & | & \\ & \bullet & n+1 \\ & \vdots & \\ & \bullet & 4 \\ 2 & | & \\ \bullet & \bullet & 3 \end{array}$$

We obtain that the diagram of $\mathcal{A}(\mathcal{Q})$ has the following form:

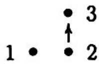


(8.2)

Consequently, the width of $\mathcal{A}(\mathcal{Q})$ equals two and so, by theorem 6.1 and corollary 4.1, $\mathcal{P} = \mathcal{Q} \sqcup (1)$ has a finite representation type. □

Lemma 8.3. *Finite posets $\mathcal{P}_1 = (1) \sqcup (2) \sqcup (2)$, $\mathcal{P}_2 = (1) \sqcup (2) \sqcup (3)$, $\mathcal{P}_3 = (1) \sqcup (2) \sqcup (4)$ are of a finite representation type.*

Proof. Consider the poset $\mathcal{Q} = (1) \sqcup (2)$ with diagram



(8.3)

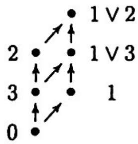
which is a sub-poset of all posets $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3$:

$$\mathcal{P}_1 = \mathcal{Q} \sqcup (2),$$

$$\mathcal{P}_2 = \mathcal{Q} \sqcup (3),$$

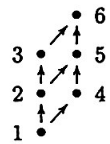
$$\mathcal{P}_3 = \mathcal{Q} \sqcup (4)$$

We obtain that a poset $\mathcal{A}(\mathcal{Q})$ has the diagram

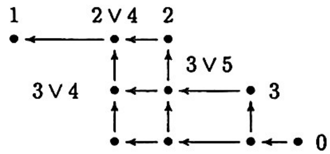


(8.4)

Therefore $\mathcal{A}(\mathcal{Q})$ has width 2, and so, by corollary 4.1, it is of a finite representation type. The diagram (8.4) after renumbering has the following form



Consider the next poset $\mathcal{A}^2(\mathcal{Q})$. In $\mathcal{A}^2(\mathcal{Q})$ new elements are $2 \vee 4$, $3 \vee 4$, $3 \vee 5$ and 0. Then a poset $\mathcal{A}^2(\mathcal{Q})$ has the following diagram:

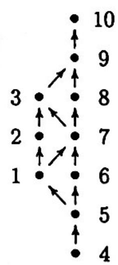


(8.5)

Therefore $\mathcal{A}^2(\mathcal{Q})$ has width 2, and so, by corollary 4.1, it is of a finite representation type.

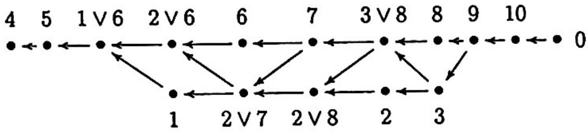
We obtain after renumbering:

$\mathcal{A}^2(\mathcal{Q}) =$



In $\mathcal{A}^3(\mathcal{Q})$ new elements are $1 \vee 6$, $2 \vee 6$, $2 \vee 7$, $2 \vee 8$, $3 \vee 8$ and 0.

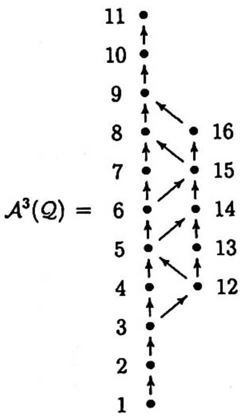
The diagram of the poset $\mathcal{A}^3(\mathcal{Q})$ has the following form:



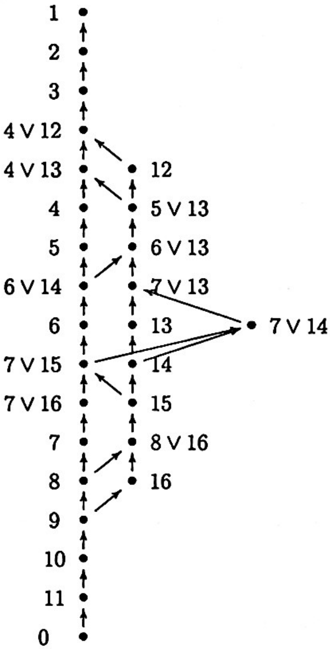
(8.6)

Therefore $\mathcal{A}^3(\mathcal{Q})$ has width 2, and so, by corollary 4.1, it is of finite representation type.

After renumbering diagram (8.6) we obtain:



Now consider the poset $\mathcal{A}^4(\mathcal{Q})$. In $\mathcal{A}^4(\mathcal{Q})$ new elements are $4 \vee 12$, $4 \vee 13$, $5 \vee 13$, $6 \vee 13$, $6 \vee 14$, $7 \vee 13$, $7 \vee 14$, $7 \vee 15$, $7 \vee 16$, $8 \vee 16$ and 0. Consequently, $\mathcal{A}^4(\mathcal{Q})$ contains 27 elements and $\mathcal{A}^4(\mathcal{Q})$ has the following diagram:



(8.7)

So the poset $\mathcal{A}^4(\mathcal{Q})$ has width 3. We shall show that it is of a finite representation type. For this aim we use two times the Trichotomy Lemma. As a poset Y we take the left chain consisting of 17 elements

$$\{0, 11, 10, 9, 8, 7, 7 \vee 16, 7 \vee 15, 6, 6 \vee 14, 5, 4, 4 \vee 13, 4 \vee 12, 3, 2, 1\}.$$

And as a poset X we take a subset $\{16 \prec 8 \vee 16 \prec 15 \prec 14\}$ and $Z = \{13, 7 \vee 14, 7 \vee 13, 6 \vee 13, 5 \vee 13, 12\}$. All indecomposable representations have support either in $X \cup Y$ or in $Y \cup Z$. The poset $X \cup Y$ has a width equal to 2 and so it is of a finite representation type. So we must only consider the poset $Y \cup Z$. We consider two subsets $X_1 = \{13, 7 \vee 14\}$ and $Z_1 = \{7 \vee 13 \prec 6 \vee 13 \prec 5 \vee 13 \prec 12\}$ in the poset Z. The poset Y is the same as above. It is clear that $Y \cup Z_1$ is a poset of width 2 and so it is of a finite type. The poset $X_1 \cup Y$ is of the form $(1, 1, 17)$ and so it is of a finite type, by lemma 8.2. Therefore the poset $\mathcal{A}^4(\mathcal{Q})$ is of a finite representation type.

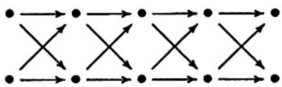
I. Since $\mathcal{P}_1 = \mathcal{Q} \sqcup (2)$, and $\mathcal{A}(\mathcal{Q})$, $\mathcal{A}^2(\mathcal{Q})$ are posets of width 2, we obtain, by theorem 6.1, that $\mathcal{P}_1$ is a poset of a finite representation type.

II. In a similar way, since $\mathcal{P}_2 = \mathcal{Q} \sqcup (3)$, and $\mathcal{A}(\mathcal{Q})$, $\mathcal{A}^2(\mathcal{Q})$, $\mathcal{A}^3(\mathcal{Q})$ are posets of width 2, we obtain, by theorem 6.1, that $\mathcal{P}_2$ is a poset of a finite representation type.

III. In a similar way, since $\mathcal{P}_3 = \mathcal{Q} \sqcup (4)$, and $\mathcal{A}^i(\mathcal{Q})$ for $i < 4$ are posets of width 2, and $\mathcal{A}^4(\mathcal{Q})$ is of a finite representation type, we obtain, by theorem 6.1, that $\mathcal{P}_3$ is a poset of a finite representation type. $\square$

Definition 8.1. *A poset $\mathcal{P}$ of width 2 is called a* garland *if for any $x \in \mathcal{P}$ there is at most one element $y \in \mathcal{P}$ such that x and y are incomparable in $\mathcal{P}$.*

Example 8.1. A poset with diagram



is a garland.

Corollary 8.1. *Let $\mathcal{P}$ be a poset of width 2 and let $\mathcal{A}^m(\mathcal{P})$ be a poset of width 2 for all $m > 1$. Then $\mathcal{P}$ is a garland.*

Proof. Let $\mathcal{Q} = (1) \sqcup (2)$. Then, by lemma 5.2, $\mathcal{A}^4(\mathcal{Q})$ has width 3. If $\mathcal{Q}$ is a subposet of $\mathcal{P}$, then, by proposition 6.2, $\mathcal{A}^4(\mathcal{Q}) \subseteq \mathcal{A}^4(\mathcal{P})$, and so $w(\mathcal{A}^4(\mathcal{P})) \geq 3$. Therefore, if $\mathcal{P}$ is not a garland, then there exists n such that $w(\mathcal{A}^n(\mathcal{P})) \geq 3$. So, by the transposition low, we obtain the required statement. $\square$

Acknowledgments

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