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Normalized solutions to a Schrödinger–Bopp–Podolsky system under Neumann boundary conditions

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In this paper, we study a Schrödinger–Bopp–Podolsky (SBP) system of partial differential equations in a bounded and smooth domain of $\mathbb{R}^3$ with a nonconstant coupling factor. Under a compatibility condition on the boundary data we deduce existence of solutions by means of the Ljusternik–Schnirelmann theory.

Keywords: Schrödinger–Bopp–Podolsky system; Krasnoselskii genus; Lagrange multipliers; weak solutions.

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1. Introduction

The Schrödinger–Newton equation consists of a nonlinear coupling of the Schrödinger equation with a gravitational potential of Newtonian form, representing the interaction of a particle with its own gravitational field.

In 1998, Benci and Fortunato [3] treated a similar problem, where the Schrödinger equation was coupled with Maxwell's equations. Such coupling represents the interaction of the particle with its own electromagnetic field. The coupling factor is a constant $q \neq 0$. In their paper, the authors consider standing waves

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solutions in the purely electrostatic field and this leads to the so-called Schrödinger–Poisson system. They imposed a Dirichlet boundary condition both on the matter field u and the electrostatic field ϕ and employed variational methods and critical point theory to develop a procedure that would become standard to treat other similar problems.

Later, Pisani and Siciliano [12] treated a Schrödinger–Poisson system with Neumann boundary conditions on the scalar field ϕ, and considered the case in which the interaction factor responsible for the coupling of the equations, namely q, is nonconstant and u satisfies the usual normalizing condition

$$\int_\Omega u^2 dx = 1. \tag{1.1}$$

More specifically, the second equation in the system they consider is

$$-\Delta\phi = qu^2 \quad \text{in } \Omega, \quad \frac{\partial\phi}{\partial\mathbf{n}} = h \quad \text{on } \partial\Omega.$$

Taking into account that in the physical model $\mathbf{E} = -\nabla\phi$ is the electric field, the boundary condition just means that the normal component of the electric field is assigned on the boundary. Indeed this is more natural than assigning the potential ϕ on the boundary, for the simply reason that the potential is usually defined only up to an additive constant and also because what is actually measurable is the electric field. As a consequence, the mathematical condition of compatibility for the above equation is

$$-\int_{\partial\Omega} h ds = \int_\Omega qu^2 dx, \tag{1.2}$$

which is interpreted by saying that the total flux $\mathfrak{F} := -\int_{\partial\Omega} h$ of the electric field through the boundary equals the total charge in the domain, which is nothing but the Gauss law. In case $q \neq 0$ were constant, from (1.1) and (1.2) one sees that the unique possibility to have solutions is when

$$\mathfrak{F} = q,$$

as proved in [11]. This strong restriction motivated the interest in considering a nonconstant charge density function q in Ω. Mathematically, it gives rise to interesting properties of the manifold of codimension two described by the conditions

$$\int_\Omega u^2 dx = 1 \quad \text{and} \quad \int_\Omega qu^2 dx = \int_{\partial\Omega} h ds,$$

as detailed in [12].

In this paper, we treat a modification of the problem studied in [12] consisting in the addition of a biharmonic term in the equation of the electrostatic potential and imposing appropriate boundary conditions. The problem studied can be interpreted as a coupling of the Schrödinger equation with the Bopp–Podolsky electrodynamics, where the Lagrangian of the electromagnetic field can be seen as a generalization of the classical Maxwell Lagrangian, implying then in more complicated equations for the electromagnetic field (for more details on this matter, see [5] and the references therein). However, here we focus on the mathematical aspects of the problem.

We point out that in the literature there are few papers concerning Schrödinger–Bopp–Podolsky systems. Besides [5], we cite here [4, 9], where the authors study the problem with a critical nonlinearity; [6], where solutions with *a priori* given interaction energy for the Schrödinger–Bopp–Podolsky system are found in presence of a positive nonlinearity; [8], where the problem has been addressed in the context of closed 3-dimensional manifolds both in the subcritical and critical case; [10] where the system is addressed in a bounded domain and a multiplicity result is obtained depending on the topology of the domain; [13], where the fibering method has been used to prove existence results depending on a parameter and also nonexistence.

Coming back to our problem, the aim here is to study the following system of partial differential equations in a bounded smooth domain $\Omega \subset \mathbb{R}^3$:

$$-\Delta u + q\phi u - \kappa|u|^{p-2}u = \omega u \quad \text{in } \Omega, \tag{1.3}$$

$$\Delta^2\phi - \Delta\phi = qu^2 \quad \text{in } \Omega \tag{1.4}$$

in the unknowns $u, \phi : \Omega \to \mathbb{R}$ and $\omega \in \mathbb{R}$. Here $\kappa \in \mathbb{R}$ and $q : \Omega \to \mathbb{R}$ are given. We assume the following boundary conditions:

$$u = 0 \quad \text{on } \partial\Omega, \tag{1.5}$$

$$\frac{\partial\phi}{\partial\mathbf{n}} = h_1 \quad \text{on } \partial\Omega, \tag{1.6}$$

$$\frac{\partial\Delta\phi}{\partial\mathbf{n}} = h_2 \quad \text{on } \partial\Omega \tag{1.7}$$

and for simplicity we assume h_1, h_2 continuous. The symbol $\mathbf{n}$ denotes the unit vector normal to $\partial\Omega$ pointing outwards. Since u represents physically the amplitude of the wave function of a particle confined in Ω, we assume the following normalizing condition:

$$\int_\Omega u^2 dx = 1. \tag{1.8}$$

We also assume that the coupling factor q is continuous on $\overline{\Omega}$:

$$q \in C(\overline{\Omega}). \tag{1.9}$$

Our main result is the following.

Theorem 1. *Let $\kappa > 0$, $p \in (2, 10/3)$ and*

$$\alpha := \int_{\partial\Omega} h_2 ds - \int_{\partial\Omega} h_1 ds. \tag{1.10}$$

Assume that $\inf_\Omega q < \alpha < \sup_\Omega q$ and that $|q^{-1}(\alpha)| = 0$. Then there exist infinitely many solutions $(u_n, \omega_n, \phi_n) \in H_0^1(\Omega) \times \mathbb{R} \times H^2(\Omega)$ to the problem (1.3) *and* (1.4) *under conditions* (1.5)–(1.9), *with*

$$\int_\Omega |\nabla u_n|^2 dx \to +\infty.$$

Moreover there is a positive ground state solution.

Our approach is variational, indeed the solutions will be found as critical points of an energy functional restricted to a suitable constraint. In this context, by a ground state solution we mean the solution with minimal energy. Moreover, as a byproduct of the proof, we also obtain that also the energy of these solutions is divergent.

Remark 1. The restriction on p is due to the fact that in this range we are able to show that the functional is bounded below on the unit sphere in L^2 and then the classical Ljusternick–Schnirelmann theory applies. It is easy to see that if $\kappa < 0$ the result holds with $p \in (2, 6)$, the functional being bounded blow on the unit sphere. For $\kappa = 0$, see [1].

Let us observe the analogies of the boundary conditions (1.6) and (1.7) with the Neumann condition imposed in [12]. Indeed, with the notation of Theorem 1 now we have

$$\alpha = \int_\Omega q u^2 dx,$$

where α has to be interpreted as the flux of the electric field in the Bopp–Podolsky electromagnetic theory, and then (1.4) is the analogous of the classical point-wise Gauss law. Moreover, due to the normalizing condition, as in the paper [11], if we had $q \neq 0$ constant, then the existence of solutions can be guaranteed only in the very special case $\alpha = q$. This motivated then in taking a nonconstant charge density in Ω.

Finally we observe the following fact which is a feature of the Neumann boundary conditions. The boundary operator $\frac{\partial \Delta}{\partial \mathbf{n}}$ is consistent with the classical theory of higher order boundary value problems: it satisfies the so-called *complementing condition*, an algebraic constraint which is crucial for obtaining *a priori* bounds. See [7, Sec. 2.3; 2, Sec. 10]. Moreover, if q does not vanishes, our system and boundary conditions can be written as

$$\begin{aligned} -\Delta u + q\widetilde{\phi} u - \kappa |u|^{p-2} u &= 0 \qquad \text{in } \Omega, \\ \Delta^2 \widetilde{\phi} - \Delta \widetilde{\phi} &= q u^2 \quad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \\ \frac{\partial \widetilde{\phi}}{\partial \mathbf{n}} &= h_1 \quad \text{on } \partial\Omega, \\ \frac{\partial \Delta \widetilde{\phi}}{\partial \mathbf{n}} &= h_2 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\widetilde{\phi} = \phi - \omega/q$ and our result gives the existence of infinitely many solutions $(u_n, \widetilde{\phi}_n)$ for this elliptic system. Moreover, the solutions $\{u_n\}$, which are in general sign-changing, are unbounded, namely a non-compactness behavior is observed. This fact should be compared with [8, Theorem 0.2] where the author shows, for a similar elliptic system, that the positive solutions exhibit a compactness behavior.

The paper is organized as follows.

In Sec. 2, we show how to obtain an equivalent system with homogeneous boundary conditions, which makes the mathematical treatment of the problem much easier. This is done by considering an auxiliary problem and making a change of variables.

In Sec. 3, we give some properties of the constraint M on which we will find the solution.

In Sec. 4, we introduce the energy functional and show that its critical points on M will give solutions of the problem.

In Sec. 5, by implementing the Ljusternick–Schnirelmann theory, we prove Theorem 1.

As a matter of notations, we use the letters $c, c', \ldots$ to denote positive constant whose value can change from line to line. We use $\| \cdot \|_p$ to denote the standard L^p-norm.

2. An Auxiliary Problem

Our aim is to define a functional whose critical points will be the weak solutions to the problem. In order to deal with homogeneous boundary conditions, that will permit to write the functional in a simpler form, we make a change of variables.

Consider the following auxiliary problem (where α is defined in (1.10)):

$$\Delta^2 \chi - \Delta \chi = \frac{\alpha}{|\Omega|} \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{\partial \chi}{\partial \mathbf{n}} = h_1 \quad \text{on } \partial\Omega, \tag{2.2}$$

$$\frac{\partial \Delta \chi}{\partial \mathbf{n}} = h_2 \quad \text{on } \partial\Omega, \tag{2.3}$$

$$\int_\Omega \chi dx = 0. \tag{2.4}$$

It is easy to see it has a unique solution. Indeed, let θ be the unique function satisfying

$$\Delta\theta - \theta = \frac{\alpha}{|\Omega|} \quad \text{in } \Omega,$$

$$\frac{\partial \theta}{\partial \mathbf{n}} = h_2 \quad \text{on } \partial\Omega,$$

$$\int_\Omega \theta dx = \int_{\partial\Omega} h_1 ds$$

and then let χ the unique function which satisfies

$$\Delta\chi = \theta, \quad \text{in } \Omega,$$
$$\frac{\partial\chi}{\partial\mathbf{n}} = h_1 \quad \text{in } \partial\Omega,$$
$$\int_\Omega \chi dx = 0,$$

see e.g. [15]. Then it is easy to see that by construction χ satisfies (2.1)–(2.4).

The change of variables we make is

$$\varphi = \phi - \chi - \mu,$$

where

$$\mu = \frac{1}{|\Omega|}\int_\Omega \phi dx.$$

With the new variables $(u, \omega, \varphi, \mu)$ our problem becomes

$$-\Delta u + q(\chi + \varphi)u - \kappa|u|^{p-2}u = \omega u - \mu q u \quad \text{in } \Omega, \tag{2.5}$$
$$\Delta^2\varphi - \Delta\varphi = qu^2 - \frac{\alpha}{|\Omega|} \quad \text{in } \Omega, \tag{2.6}$$
$$u = 0 \quad \text{on } \partial\Omega, \tag{2.7}$$
$$\int_\Omega u^2 dx = 1, \tag{2.8}$$
$$\frac{\partial\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{2.9}$$
$$\frac{\partial\Delta\varphi}{\partial\mathbf{n}} = 0 \quad \text{on } \partial\Omega, \tag{2.10}$$
$$\int_\Omega \varphi dx = 0. \tag{2.11}$$

Notice that the compatibility condition between (2.6), (2.9) and (2.10) now reads as

$$\int_\Omega qu^2 dx = \alpha.$$

3. The Manifold M

Let us define the sets

$$S := \left\{u \in H_0^1(\Omega) : \int_\Omega u^2 dx = 1\right\},$$
$$N := \left\{u \in H_0^1(\Omega) : \int_\Omega qu^2 dx = \alpha\right\}, \quad M := S \cap N.$$

Recall that α depends on both the boundary conditions to the original problem.

If the problem has a solution, then of course $M \neq \emptyset$. Hence,

$$q_{\min} \leq \alpha \leq q_{\max}, \tag{3.1}$$

where

$$q_{\min} = \inf_{\Omega} q \quad \text{and} \quad q_{\max} = \sup_{\Omega} q.$$

Indeed, if $\alpha < q_{\min}$, then

$$\alpha = \int_{\Omega} qu^2 dx \geq q_{\min} > \alpha,$$

which is a contradiction. The case $\alpha > q_{\max}$ is analogous.

From (3.1) we deduce that $q^{-1}(\alpha)$ is not empty, and indeed its measure that will play a major role.

Suppose $\alpha = q_{\min}$ and $|q^{-1}(\alpha)| = 0$. Then

$$\int_{\Omega} qu^2 dx = \int_{\{x \in \Omega : q(x) > \alpha\}} qu^2 dx > \alpha,$$

so $M = \emptyset$. If $\alpha = q_{\max}$ and $|q^{-1}(\alpha)| = 0$ we proceed in an analogous manner to conclude that M is empty and so the problem has no solutions. Therefore, we arrive at the following necessary condition for the existence of solutions: either

$$q_{\min} < \alpha < q_{\max} \tag{3.2}$$

or

$$|q^{-1}(\alpha)| \neq 0. \tag{3.3}$$

We now state some properties of the set M, referring the reader to [12] for the omitted proofs.

We first note that M is symmetric with respect to the origin: if $u \in M$, then $-u \in M$. This follows trivially from the definition of M. We also note that M is weakly closed in $H_0^1(\Omega)$.

Now, we want to show that under condition (3.2) the set M is not empty. For this, we introduce the following notation.

Let $A \subset \Omega$ be an open subset and define

$$S_A := \left\{ u \in H_0^1(A) : \int_A u^2 dx = 1 \right\}$$

and

$$g_A : u \in S_A \mapsto \int_A qu^2 dx \in \mathbb{R}.$$

It is immediately seen that

$$g_A(S_A) \subset \left[\inf_A q, \sup_A q \right].$$

Lemma 1. *The following inclusion holds*:

$$\left(\inf_A q, \sup_A q\right) \subset \overline{g_A(S_A)}.$$

We can conclude the following.

Proposition 1. *Let $A \subset \Omega$ be an open subset. If $\alpha \in (\inf_A q, \sup_A q)$ then there exists $u \in H_0^1(A)$ such that*

$$\int_A u^2 dx = 1 \quad \textit{and} \quad \int_A q u^2 dx = \alpha.$$

In particular by taking $A = \Omega$ we get the following.

Theorem 2. *Assume that* $\inf_\Omega q < \alpha < \sup_\Omega q$. *Then M is not empty.*

Let us recall the definition of genus of Krasnoselki. Given A a closed and symmetric subset of some Banach space, with $0 \notin A$, the *genus* of A is denoted as $\gamma(A)$ and defined as the least integer k for which there exists a continuous and odd map $h : A \to \mathbb{R}^k \backslash \{0\}$. By definition it is $\gamma(\emptyset) = 0$ and if it is not possible to construct continuous odd maps from A to any $\mathbb{R}^k \backslash \{0\}$, it is set $\gamma(A) = +\infty$.

It is known that the genus is a topological invariant (by odd homeomorphism) and that the genus of the sphere in $\mathbb{R}^N$ is N.

The next result says that M has subsets of arbitrarily large genus.

Theorem 3. *Let $u_1, \ldots, u_k \in M$ be functions with disjoint supports and let*

$$V_k = \langle u_1, \ldots, u_k \rangle$$

be the space spanned by $u_1, \ldots, u_k$. Then $M \cap V_k$ is the $(k-1)$-dimensional sphere, hence $\gamma(M \cap V_k) = k$.

Now, it is natural if one raises the question of whether there exists such functions with disjoint supports for arbitrary k. The answer is affirmative.

Theorem 4. *If* (3.2) *holds then for every $k \geq 2$ there exist k functions $u_1, \ldots, u_k \in M$ with disjoint supports. Hence*, $\gamma(M) = +\infty$.

Let

$$G_1 : u \in H_0^1(\Omega) \mapsto \int_\Omega u^2 dx - 1 \in \mathbb{R},$$

$$G_2 : u \in H_0^1(\Omega) \mapsto \int_\Omega q u^2 dx - \alpha \in \mathbb{R}$$

and

$$G = (G_1, G_2).$$

Then

$$M = \{u \in H_0^1(\Omega) : G_1(u) = G_2(u) = 0\} = G^{-1}(0).$$

We note that G is of class C^1.

Let us show, for the reader convenience, that $G_1'(u)$ and $G_2'(u)$ are linearly independent, so G will be a submersion and M a submanifold of codimension 2.

Proposition 2. *Assume that M is not empty. The differentials $G_1'(u)$ and $G_2'(u)$ are linearly independent for every $u \in M$ if and only if*

$$|q^{-1}(\alpha)| = 0. \tag{3.4}$$

Proof. First, assume (3.4). We will show that $G_1'(u)$ and $G_2'(u)$ are linearly independent, for all $u \in M$. Suppose that there are $a, b \in \mathbb{R}$ such that

$$aG_1'(u) + bG_2'(u) = 0 \quad \text{in } H^{-1}(\Omega)$$

for some $u \in M$. Evaluating this expression in u, we find that $a + b\alpha = 0$. Then

$$aG_1'(u)[v] + bG_2'(u)[v] = b\left(-\alpha \int_\Omega uvdx + \int_\Omega quvdx\right) = 0, \quad \forall\, v \in H_0^1(\Omega),$$

that is,

$$b\int_\Omega (q - \alpha)uvdx = 0, \quad \forall\, v \in H_0^1(\Omega).$$

If $b \neq 0$ then we would have $(q - \alpha)u = 0$ a.e., and hence, in view of (3.4), $u = 0$, a contradiction. Thus, $G_1'(u)$ and $G_2'(u)$ are linearly independent for all $u \in M$.

Now, suppose (3.4) is not satisfied. Then $q^{-1}(\alpha)$ has not empty interior, hence there is some test function u with support in $q^{-1}(\alpha)$ such that $\|u\|_2 = 1$. It is immediately seen that $u \in M$ because $qu = \alpha u$ and hence

$$G_2'(u) = \alpha G_1'(u),$$

which completes the proof. □

4. Variational Setting

We now proceed to study the variational framework of the problem. Our aim is to construct a functional whose critical points will be the weak solutions of the problem.

Following [7], let

$$V = \left\{\xi \in H^2(\Omega) : \frac{\partial \xi}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega\right\}.$$

We remark that V is a closed subspace of $H^2(\Omega)$. Indeed, let $\{v_n\} \subset V$ such that $v_n \to v$ in V. Then $0 = \gamma_1(v_n) \to \gamma_1(v)$ and hence $\gamma_1(v) = 0$, where γ_1 denotes the trace operator which for smooth functions gives the directional derivative in the direction of the exterior normal on the boundary. Being a closed subspace, V inherits the Hilbert space structure of $H^2(\Omega)$.

Recall that

$$\varphi = \phi - \chi - \mu,$$

where

$$\mu = \frac{1}{|\Omega|}\int_\Omega \phi dx.$$

In this way, we have $\overline{\varphi} = 0$, where from now on, given a function f, we denote with $\overline{f}$ its mean in Ω. Consider then the following natural decomposition of V:

$$V = \widetilde{V} \oplus \mathbb{R}, \tag{4.1}$$

where

$$\widetilde{V} = \{\eta \in V : \overline{\eta} = 0\}.$$

On $\widetilde{V}$ we have the equivalent norm

$$\|\eta\|_{\widetilde{V}} = (\|\nabla\eta\|_2^2 + \|\Delta\eta\|_2^2)^{1/2}.$$

Consider the functional $F : H_0^1(\Omega) \times H^2(\Omega)$ defined as follows:

$$\begin{aligned} F(u,\varphi) = {} & \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega q(\varphi+\chi)u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\ & -\frac{1}{4}\int_\Omega (\Delta\varphi)^2 dx - \frac{1}{4}\int_\Omega |\nabla\varphi|^2 dx - \frac{\alpha}{2|\Omega|}\int_\Omega \varphi dx. \end{aligned}$$

It is easy to see that this functional is of class C^1 and that given $u \in H_0^1(\Omega)$ and $\varphi \in H^2(\Omega)$ we have the following partial derivatives:

$$\begin{aligned} F'_u(u,\varphi)[v] &= \int_\Omega \nabla u \nabla v dx + \int_\Omega q(\varphi+\chi)uv dx - \kappa\int_\Omega |u|^{p-2}uv dx, \\ F'_\varphi(u,\varphi)[\xi] &= \frac{1}{2}\int_\Omega q\xi u^2 dx - \frac{1}{2}\int_\Omega \Delta\varphi\Delta\xi dx - \frac{1}{2}\int_\Omega \nabla\varphi\nabla\xi dx - \frac{\alpha}{2|\Omega|}\int_\Omega \xi dx \end{aligned}$$

for every $v \in H_0^1(\Omega)$ and $\xi \in H^2(\Omega)$.

Then, $(u,\varphi,\omega,\mu) \in H_0^1(\Omega) \times H^2(\Omega) \times \mathbb{R} \times \mathbb{R}$ is a weak solution to (2.5)–(2.11) if, and only if,

$$\begin{aligned} & (u,\varphi) \in M \times \widetilde{V}, \\ & \forall\, v \in H_0^1(\Omega) : F'_u(u,\varphi)[v] = \omega\int_\Omega uv dx - \mu\int_\Omega quv dx, \\ & \forall\, \xi \in V : F'_\varphi(u,\varphi)[\xi] = 0. \end{aligned} \tag{4.2}$$

Theorem 5. *Let $(u,\varphi) \in H_0^1(\Omega) \times H^2(\Omega)$. Then there exist $\omega, \mu \in \mathbb{R}$ such that (u,φ,ω,μ) is a solution to (2.5)–(2.11) if and only if (u,φ) is a critical point of F constrained on $M \times \widetilde{V}$, in which case the real constants ω, μ are the two Lagrange multipliers with respect to F'_u.*

Proof. Indeed, (u,φ) is a critical point of F constrained on $M \times \widetilde{V}$ if, and only if

$$\forall\, v \in T_u M : F'_u(u,\varphi)[v] = 0,$$
$$\forall\, \xi \in \widetilde{V} : F'_\varphi(u,\varphi)[\xi] = 0.$$

Note that the tangent space to $\widetilde{V}$ at φ is $\widetilde{V}$ itself.

Then a weak solution, according to (4.2) and the Lagrange multipliers rule, is a constrained critical point.

Suppose on the contrary that (u,φ) is a constrained critical point. Then, again by the Lagrange multipliers rule, we have that there exist $\omega, \mu \in \mathbb{R}$ such that

$$\forall\, v \in H_0^1(\Omega) : F'_u(u,\varphi)[v] = \omega \int_\Omega uv dx - \mu \int_\Omega quv dx.$$

It remains to prove that $F'_\varphi(u,\varphi)[\xi] = 0$ for all $\xi \in V$. But this follows by the decomposition (4.1), noticing that $F'_\varphi(u,\varphi)[r] = 0$ for every constant $r \in \mathbb{R}$. Then (4.2) is satisfied and this concludes the proof. $\square$

The functional F constrained on $M \times \widetilde{V}$ is unbounded from above and from below. This issue has been addressed in [3] and in many subsequent papers. Their standard reduction argument goes as follows:

(i) For every fixed $u \in H_0^1(\Omega)$ there exists a unique $\Phi(u)$ such that $F'_\varphi(u, \Phi(u)) = 0$.
(ii) The map $u \mapsto \Phi(u)$ is of class C^1.
(iii) The graph of Φ is a manifold, and we are reduced to study the functional $J(u) = F(u, \Phi(u))$, possibly constrained.

However, the method sketched above fails in our situation, for two reasons. First, we see that $F'_\varphi(u,\varphi) = 0$ with $\varphi \in \widetilde{V}$ is just

$$\begin{aligned} \Delta^2\varphi - \Delta\varphi - qu^2 + \frac{\alpha}{|\Omega|} &= 0 \quad \text{in } \Omega,\\ \frac{\partial\varphi}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega,\\ \frac{\partial\Delta\varphi}{\partial\mathbf{n}} &= 0 \quad \text{on } \partial\Omega,\\ \int_\Omega \varphi dx &= 0. \end{aligned}$$

The problem has a unique solution if and only if $u \in N$, due to the compatibility condition. Moreover, since N is not a manifold (unless $\alpha \neq 0$) we cannot require the map $\Phi : u \mapsto \Phi(u)$ to be of class C^1 in N. We shall then extend such a map Φ.

Proposition 3. *For every $w \in L^{6/5}(\Omega)$ there exists a unique $L(w) \in \widetilde{V}$ solution of*

$$\begin{aligned} \Delta^2\varphi - \Delta\varphi - w + \overline{w} &= 0 \quad \text{in } \Omega, \\ \frac{\partial \varphi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \\ \frac{\partial \Delta\varphi}{\partial \mathbf{n}} &= 0 \quad \text{on } \partial\Omega, \\ \int_\Omega \varphi dx &= 0. \end{aligned}$$

The map $L : L^{6/5}(\Omega) \to \widetilde{V}$ is linear and continuous, hence of class C^∞.

Proof. The weak solutions to the problem are functions φ in the Hilbert space $\widetilde{V}$ such that

$$\int_\Omega \Delta\varphi\Delta v dx + \int_\Omega \nabla\varphi\nabla v dx - \int_\Omega wv dx = 0, \quad \forall\, v \in \widetilde{V},$$

so the result follows by applying the Riesz theorem, since the bilinear form $b : \widetilde{V} \times \widetilde{V} \to \mathbb{R}$ given by

$$b(\varphi, v) = \int_\Omega \Delta\varphi\Delta v dx + \int_\Omega \nabla\varphi\nabla v dx$$

is just the scalar product in $\widetilde{V}$. □

The following proposition follows from well-known properties of Nemytsky operators.

Proposition 4. *The map*

$$u \in L^6(\Omega) \mapsto qu^2 \in L^{6/5}(\Omega)$$

is of class C^1.

As a consequence of the previous propositions, we can define the following map:

$$\Phi : u \in H_0^1(\Omega) \mapsto L(qu^2) \in \widetilde{V}.$$

It is clear that

$$\Phi(u) = \Phi(-u) = \Phi(|u|).$$

Moreover, for every $(u, \varphi) \in H_0^1(\Omega) \times \widetilde{V}$ we have that $\varphi = \Phi(u)$ if and only if for every $\eta \in \widetilde{V}$,

$$\int_\Omega \Delta\varphi\Delta\eta dx + \int_\Omega \nabla\varphi\nabla\eta dx = \int_\Omega qu^2\eta dx.$$

Taking $\eta = \Phi(u)$ we have in particular the important relation

$$\int_\Omega (\Delta\Phi(u))^2 dx + \int_\Omega |\nabla\Phi(u)|^2 dx = \int_\Omega qu^2\Phi(u)dx. \tag{4.3}$$

The right-hand side above is the *interaction energy* term. Then we infer

$$\begin{aligned}\|\Phi(u)\|_{\widetilde{V}}^2 &\le \|q\|_\infty \int_\Omega u^2\Phi(u)dx\\ &\le c\|u\|_4^2\|\Phi(u)\|_2\\ &\le c\|\nabla u\|_2^2\|\nabla\Phi(u)\|_2\\ &\le c\|\nabla u\|_2^2\|\Phi(u)\|_{\widetilde{V}}\end{aligned}$$

and hence

$$\|\Phi(u)\|_{\widetilde{V}} \le c\|\nabla u\|_2^2, \tag{4.4}$$

that is, Φ is bounded on bounded sets. We have

Lemma 2. *If $u_n \rightharpoonup u$ in $H_0^1(\Omega)$, then*

$$\int_\Omega qu_n^2\Phi(u_n)dx \to \int_\Omega qu^2\Phi(u)dx.$$

Moreover, the map Φ is compact.

Proof. Let $u_n \rightharpoonup u$ in $H_0^1(\Omega)$ and define $B_n, B : \widetilde{V} \longrightarrow \mathbb{R}$ by

$$B_n(\eta) := \int_\Omega qu_n^2\eta dx, \quad B(\eta) := \int_\Omega qu^2\eta dx.$$

Such operators are continuous due to the Hölder's inequality. For example:

$$\left|\int_\Omega qu^2\eta dx\right| \le \|q\|_\infty\|u\|_4^2\|\eta\|_2 \le c\|\nabla\eta\|_2 \le c\|\eta\|_{\widetilde{V}}$$

(where here c depends on u).

Due to the compact embedding of $H_0^1(\Omega)$ into $L^p(\Omega)$ for $p \in [1,6)$, we get $u_n^2 \to u^2$ in $L^{6/5}(\Omega)$ and then

$$\begin{aligned}|B_n(\eta) - B(\eta)| &\le \|q\|_\infty\|u_n^2 - u^2\|_{6/5}\|\eta\|_6\\ &\le c\|q\|_\infty\|u_n^2 - u^2\|_{6/5}\|\eta\|_{\widetilde{V}}.\end{aligned}$$

Hence,

$$\|B_n - B\| \le \sup_{\eta\neq 0}\frac{c\|u_n^2 - u^2\|_{6/5}\|\eta\|_{\widetilde{V}}}{\|\eta\|_{\widetilde{V}}} \to 0,$$

namely $B_n \to B$ as operators in $\widetilde{V}$.

On the other hand, we have that $\Phi(u_n) \rightharpoonup \Phi(u)$ in $\widetilde{V}$. Indeed, let $g \in \widetilde{V}'$. Then there is some $v_g \in \widetilde{V}$ such that

$$g(\Phi(u_n)) = \int_\Omega \nabla\Phi(u_n)\nabla v_g dx + \int_\Omega \Delta\Phi(u)\Delta v_g dx = \int_\Omega q u_n^2 v_g dx.$$

But then

$$\begin{aligned} g(\Phi(u_n)) - g(\Phi(u)) &= \int_\Omega q(u_n^2 - u^2) v_g dx \\ &\leq \|q\|_\infty \|u_n^2 - u^2\|_2 \|v_g\|_2 \to 0 \end{aligned}$$

since $u_n^2 \to u^2$ in $L^2(\Omega)$ as well.

We then conclude that

$$\int_\Omega q u_n^2 \Phi(u_n) dx \to \int_\Omega q u^2 \Phi(u) dx$$

and by (4.3) that $\|\Phi(u_n)\|_{\widetilde{V}} \to \|\Phi(u)\|_{\widetilde{V}}$. Consequently $\Phi(u_n) \to \Phi(u)$ in $\widetilde{V}$. $\square$

Note that for every $u \in N$ we have that $F'_\varphi(u, \Phi(u)) = 0$. Indeed, $\Phi(u)$ is the unique solution to the problem in Proposition 3 with $w = qu^2$.

We now define the reduced functional of a single variable:

$$\begin{aligned} J : H_0^1(\Omega) &\to \mathbb{R}, \\ u &\mapsto F(u, \Phi(u)). \end{aligned}$$

With the notation $\varphi_u := \Phi(u)$ the functionl J is explicitly given by (recall (4.3))

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{2}\int_\Omega q\varphi_u u^2 dx + \frac{1}{2}\int_\Omega q\chi u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\ &\quad - \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx - \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx - \frac{\alpha}{2|\Omega|}\int_\Omega \varphi_u dx \\ &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx \\ &\quad + \int_\Omega q\chi u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx. \end{aligned}$$

We note that J is of class C^1 on $H_0^1(\Omega)$ and even. Moreover, for every $u \in M$ we have that

$$J'(u)[v] = F'_u(u, \varphi_u)[v] + F'_\varphi(u, \varphi_u)[\Phi'(u)[v]] = F'_u(u, \varphi_u)[v], \quad \forall\, v \in H_0^1(\Omega)$$

and hence we deduce the following.

Theorem 6. *The pair $(u, \varphi) \in M \times \widetilde{V}$ is a critical point of F constrained on $M \times \widetilde{V}$ if and only if u is a critical point of $J|_M$ and $\varphi = \Phi(u)$.*

5. Proof of the Main Result

The next lemma will be useful.

Lemma 3. *Let D be a regular domain of $\mathbb{R}^N$ and*

$$1 \leq s \leq N,$$

$$s < p < s^* = \frac{Ns}{N-s}$$

and

$$0 < r \leq N\left(1 - \frac{p}{s^*}\right).$$

Then there exists a constant $C > 0$ such that for every $u \in W^{1,s}(D)$ it holds that

$$\|u\|_p^p \leq C\|u\|_{W^{1,s}}^{p-r}\|u\|_s^r.$$

Proof. See [11, Lemma 3.1]. □

Remark 2. If D is bounded, then the conclusion of the lemma is true also in the case $p \in [1, s]$ with $r < p$. Also, if D is bounded and $u \in W_0^{1,p}(D)$, then, by Poincaré inequality

$$\|u\|_p^p \leq C\|\nabla u\|_s^{p-r}\|u\|_s^r.$$

The following lemma gives the existence of solutions to our modified problem.

Lemma 4. *The functional J on M is weakly lower semicontinuous and coercive. In particular, it has a minimizer $u \in M$ and it can be assumed positive.*

Proof. We have

$$\begin{aligned}
J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2 dx + \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx \\
&\quad + \int_\Omega q\chi u^2 dx - \frac{\kappa}{p}\int_\Omega |u|^p dx \\
&\geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \|q\|_\infty\|\chi\|_\infty - \frac{\kappa}{p}\int_\Omega |u|^p dx.
\end{aligned}$$

Finally, we apply Lemma 3 with $s = 2$ and $N = 3$. Since $p \in (2, 10/3)$ it holds that

$$p - 2 < 3\left(1 - \frac{p}{6}\right) < 2$$

and we can choose

$$p - 2 < r < 3\left(1 - \frac{p}{6}\right),$$

so that by the lemma it follows that

$$\frac{\kappa}{p}\int_\Omega |u|^p dx \leq c\|\nabla u\|_2^{p-r}.$$

Hence,

$$J(u) \geq \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \|q\|_\infty \|\chi\|_\infty - c'\|\nabla u\|_2^{p-r}$$

and thus J is coercive and bounded from below on M.

Now, let $\{u_n\} \subset M$ such that $u \rightharpoonup u$. Since M is weakly closed, $u \in M$. By Lemma 2 we know that

$$\frac{1}{4}\int_\Omega (\Delta\varphi_{u_n})^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_{u_n}|^2 dx \to \frac{1}{4}\int_\Omega (\Delta\varphi_u)^2 dx + \frac{1}{4}\int_\Omega |\nabla\varphi_u|^2 dx.$$

We also know that $u_n^2 \to u^2$ in $L^{6/5}(\Omega)$ so

$$\begin{aligned}\int_\Omega q\chi(u_n^2 - u^2)dx &\leq c\int_\Omega u_n^2 - u^2 dx \\ &\leq c\|u_n - u\|_{6/5} \to 0.\end{aligned}$$

Finally, the first and last terms are the norms of u in $H_0^1(\Omega)$ and $L^p(\Omega)$ (up to constants), so they are weakly lower semicontinuous.

Thus, J is weakly lower semicontinuous and the existence of the minimum follows by standard results. Note that $J(u) = J(|u|)$, the minimum can be assumed to be positive. □

We will use a deformation argument to show that there are infinitely many solutions. A crucial point is that the functional satisfies the Palais–Smale condition. We recall that in general, it is said that the C^1 functional $\mathcal{I}$ satisfies the Palais–Smale condition on the manifold $\mathcal{M}$ if any sequence $\{u_n\} \subset \mathcal{M}$ such that $\{\mathcal{I}(u_n)\}$ is bounded and $\mathcal{I}'(u_n) \to 0$ in the tangent bundle admits a convergent subsequence to an element $u \in \mathcal{M}$.

Proposition 5. *The functional J satisfies the Palais–Smale condition on M.*

Proof. Let $\{u_n\} \subset M$ be such that

$$\{J(u_n)\} \text{ is bounded}$$

and

$$J|'_M(u_n) \to 0. \tag{5.1}$$

By (5.1) there exist two sequences of real numbers $\{\lambda_n)\}, \{\beta_n\}$ and a sequence $\{v_n\} \subset H^{-1}$ such that $v_n \to 0$ and

$$-\Delta u_n + q(\varphi_n + \chi)u_n - \kappa|u_n|^{p-2}u_n = \lambda_n u_n + \beta_n q u_n + v_n, \tag{5.2}$$

where $\varphi_n := \varphi_{u_n}$.

Since J is coercive and $\{J(u_n)\}$ is bounded, we know that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Hence, there exists $u \in H_0^1(\Omega)$ such that $u_n \rightharpoonup u$, up to a subsequence. By the compact embeddings and Lemma 2 we know that

$$u_n \to u \quad \text{in } L^p(\Omega), \quad \varphi_n \to \varphi_u \quad \text{in } H^2(\Omega). \tag{5.3}$$

Also, since M is weakly closed, we know that $u \in M$. It only remains to show that $u_n \to u$ in $H_0^1(\Omega)$.

By (5.2) we have that

$$\begin{aligned}\frac{1}{2}\int_\Omega |\nabla u_n|^2 dx + \frac{1}{2}\int_\Omega q(\varphi_n+\chi)u_n^2 dx - \frac{\kappa}{p}\int_\Omega |u_n|^p dx - \langle v_n, u_n\rangle \\ = \lambda_n + \alpha\beta_n.\end{aligned} \tag{5.4}$$

By (5.3) we infer

$$\begin{aligned}&\left|\int_\Omega (q(\varphi_n+\chi)u_n^2 - q(\varphi_u+\chi)u^2)dx\right| \\ &\quad \le c\int_\Omega |\varphi_n+\chi||u_n^2-u^2|dx + \int_\Omega u^2|\varphi_n-\varphi|dx \\ &\quad = o_n(1),\end{aligned}$$

where we are denoting with $o_n(1)$ a vanishing sequence. Then the right-hand side of (5.4) is bounded and we can assume that

$$\lambda_n + \alpha\beta_n = \xi + o_n(1)$$

with $\xi \in \mathbb{R}$. Then (5.2) becomes

$$-\Delta u_n + q(\varphi_n+\chi)u_n - \kappa|u_n|^{p-2}u_n - v_n = (\xi + o(1))u_n - \beta_n(q-\alpha)u_n. \tag{5.5}$$

Now, since $u \in M$ we know that $\|u\|_2 = 1$. This, together with the assumption $|q^{-1}(\alpha)| = 0$ implies that $(q-\alpha)u$ is not identically zero. Then there exists a test function $w \in C_0^\infty(\Omega)$ such that

$$\int_\Omega (q-\alpha)uw dx \neq 0.$$

Evaluating (5.5) on this w we get

$$\begin{aligned}&\int_\Omega \nabla u_n \nabla w dx + \int_\Omega q(\varphi_n+\chi)u_n w dx - \kappa\int_\Omega |u_n|^{p-2}u_n w dx \\ &\qquad - \langle v_n, w\rangle - (\lambda + o_n(1))\int_\Omega u_n w dx \\ &\quad = \beta_n \int_\Omega (q-\alpha)u_n w dx\end{aligned} \tag{5.6}$$

and using again (5.3) we see that every term in the left-hand side converges. Also, by the weak convergence of $\{u_n\}$,

$$\int_\Omega (q-\alpha)u_n w dx \to \int_\Omega (q-\alpha)uw dx.$$

This implies, coming back to (5.6), that $\{\beta_n\}$ is bounded, which in turn implies that $\{\lambda_n\}$ is bounded.

Applying (5.5) to $u_n - u$ we get

$$\begin{aligned}
&\int_\Omega \nabla u_n \nabla (u_n - u)dx + \int_\Omega q(\varphi_n + \chi)u_n(u_n - u)dx \\
&\qquad - \kappa \int_\Omega |u_n|^{p-2}u_n(u_n - u) - \langle v_n, u_n - u\rangle \\
&\qquad = (\lambda + o(1)) \int_\Omega u_n(u_n - u)dx + \beta_n \int_\Omega (q - \alpha)u_n(u_n - u)dx. \qquad (5.7)
\end{aligned}$$

Since (again by (5.3)) we have

$$\int_\Omega q(\varphi_n + \chi)u_n(u_n - u)dx \to 0, \quad \langle v_n, u_n - u\rangle \to 0,$$

$$\int_\Omega |u_n|^{p-2}u_n(u_n - u)dx \to 0, \quad (\lambda + o(1)) \int_\Omega u_n(u_n - u)dx \to 0$$

and

$$\beta_n \int_\Omega (q - \alpha)u_n(u_n - u)dx \to 0,$$

we conclude by (5.7) that $\|\nabla u_n\|_2 \to \|\nabla u\|_2$ and so $u_n \to u$ in $H_0^1(\Omega)$. □

Now we can give the proof of Theorem 1.

By Theorem 3, M has compact, symmetric subsets of genus k for every $k \in \mathbb{N}$.

Let us recall now a classical result in critical point theory. We give the proof for the reader convenience.

Lemma 5. *For any $b \in \mathbb{R}$ the sublevel*

$$J^b = \{u \in M : J(u) \le b\}$$

has finite genus.

Proof. We argue by contradiction. Suppose that

$$D = \{b \in \mathbb{R} : \gamma(J^b) = \infty\} \neq \emptyset.$$

Since $J|_M$ is bounded from below, then D is bounded from below. Then

$$-\infty < \overline{b} = \inf D < \infty.$$

Moreover, since $J|_M$ satisfies the Palais–Smale condition, the set

$$Z = \{u \in M : J(u) = \overline{b}, J|'_M(u) = 0\}$$

is compact. Hence, there exists a closed symmetric neighborhood U_Z of Z such that $\gamma(U_Z) < \infty$. By the Deformation Lemma, there exists an $\varepsilon > 0$ such that $J^{\overline{b}-\varepsilon}$ includes a deformation retract of $J^{\overline{b}+\varepsilon}\backslash U_Z$. Then, by the properties of the genus

$$\gamma(J^{\overline{b}+\varepsilon}) \le \gamma(J^{\overline{b}+\varepsilon} \setminus U_Z) + \gamma(U_Z) \le \gamma(J^{\overline{b}-\varepsilon}) + \gamma(U_Z) < \infty,$$

a contradiction. □

Let $n \in \mathbb{N}$. By Lemma 5 there exists some $k \in \mathbb{N}$ depending on n such that

$$\gamma(J^n) = k.$$

Let

$$A_{k+1} = \{A \subset M : A = -A, \overline{A} = A, \gamma(A) = k+1\}$$

that we know is not empty by Theorem 3.

By the monotonicity property of the genus, any $A \in A_{k+1}$ is not contained in J^n, then $\sup_A J > n$ and therefore

$$c_n = \inf_{A \in A_{k+1}} \sup_{u \in A} J(u) \geq n.$$

Well known results (see e.g. Szulkin [14]) say that c_n are critical levels for $J|_M$ and then there is a sequence $\{u_n\}$ of critical points such that

$$J(u_n) = c_n \to +\infty.$$

The critical points give rise to Lagrange multipliers ω_n, μ_n and then, recalling the decomposition $\varphi = \phi - \chi - \mu$, to solutions $(u_n, \omega_n, \phi_n) \in H_0^1(\Omega) \times \mathbb{R} \times H^2(\Omega)$ of the original problem.

We show that $\|\nabla u_n\|_2 \to +\infty$. Since

$$\int_\Omega q\chi u_n^2 dx \leq \|q\chi\|_\infty$$

and by (4.4) it is

$$\|\varphi_n\|_{\widetilde{V}} = \int_\Omega (\Delta\varphi_n)^2 dx + \int_\Omega |\nabla\varphi_n|^2 dx \leq c\|\nabla u_n\|_2^2,$$

we see that

$$|J(u_n)| \leq (1+c)\|\nabla u_n\|^2 + c'\|\nabla u_n\|_2^p + \|q\chi\|_\infty$$

and then $\{u_n\}$ cannot be bounded.

This concludes the proof of Theorem 1.

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