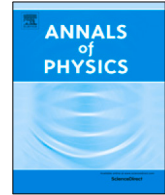




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Entanglement of classical and quantum short-range dynamics in mean-field systems

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ABSTRACT

The relationship between classical and quantum mechanics is usually understood via the limit $\hbar \rightarrow 0$. This is the underlying idea behind the quantization of classical objects. The apparent incompatibility of general relativity with quantum mechanics and quantum field theory has challenged for many decades this basic idea. We recently showed (Bru and de Siqueira Pedra, 0000; Bru and de Siqueira Pedra, 2021 [46,47]) the emergence of classical dynamics for very general quantum lattice systems with mean-field interactions, without (complete) suppression of its quantum features, in the infinite volume limit. This leads to a theoretical framework in which the classical and quantum worlds are entangled. Such an entanglement is noteworthy and is a consequence of the highly non-local character of mean-field interactions. Therefore, this phenomenon should not be restricted to systems with mean-field interactions only, but should also appear in presence of interactions that are sufficiently long-range, yielding effective, classical background fields, in the spirit of the Higgs mechanism of quantum field theory. In order to present the result in a less abstract way than in its original version, here we apply it to a concrete, physically relevant, example and discuss, by this means, various important aspects

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of our general approach. The model we consider is not exactly solvable and the particular results obtained are new.

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1. Introduction

The limit $\hbar \rightarrow 0$ of Planck's constant refers in mathematics to the semi-classical analysis, a well-developed and matured research field [1–6]. In physics, quantum systems are, in some cases, related to classical counterparts with $\hbar$ appearing as a small deformation parameter, like, for instance, in Weyl's quantization. See, e.g., [7, Chapter 13]. This is the common understanding¹ of the relationship between quantum and classical mechanics, which is formally seen as a limit case² of quantum mechanics, even if there exist physical features (such as the spin of quantum particles) which do not have a clear classical counterpart.

Nevertheless, classical mechanics does not only show up in the limit $\hbar \rightarrow 0$, but also in quantum systems with *mean-field* interactions, as explained in the present paper. Theoretical physicists are of course aware of this fact. See, e.g., [11]. This can be traced back, at least, down to the seminal paper [12], by Hepp and Lieb who made explicit, for the first time in 1973, the existence of Poisson brackets in some (commutative) algebra of functions, related to a *classical* effective dynamics. This research line was further developed by many other authors, at least until the nineties. See, e.g., [13–32].

We focus here on Bóna's approach, referring to his impressive series of papers, starting in 1975 with [13]. Based on his decisive progresses [17,19–21] on permutation-invariant quantum-spin systems with mean-field interactions, Bóna presents a full-fledged abstract theory³ in 1991 [25] and later in a mature textbook published in 2000 (revised in 2012) [34], named by him “extended quantum mechanics”. See also his new book [35] on the subject, published in 2020.

Following [34, Section 1.1-a], Bóna's original motivation was to “understand connections between quantum and classical mechanics more satisfactorily than via the limit $\hbar \rightarrow 0$.” His major conceptual contribution is to highlight the appearance of classical mechanics without necessarily the complete suppression of the quantum world, offering a general formal mathematical framework to understand physical phenomena with macroscopic quantum coherence.

Bóna's view point is different from recent approaches of theoretical physics like [36–41] (see also references therein), which propose a general formalism to get a consistent description of interactions between classical and quantum systems, having in mind chemical reactions, decoherence or the quantum measurement theory. The approaches [36–41] (see also references therein) refer to

¹ At least in many textbooks on quantum mechanics. See for instance [7, Section 12.4.2, end of the 4th paragraph of page 178]. In fact, the nowadays usual correspondence principle (which is, by the way, not precisely the original principle that Bohr had in mind [8, Section 4.2]) says that the classical world can appear for large quantum numbers via a statistical interpretation of quantum mechanics. Nonetheless, this does not necessarily mean that one has to perform the limit $\hbar \rightarrow 0$, as Bohr himself stressed. Quoting [9, p. 313]: “Edward M. Purcell informed me that Niels Bohr made a similar comment during a visit to the Physics Department at Harvard University in 1961. The place was Purcell's office where Purcell and others had taken Bohr for a few minutes of rest. They were in the midst of a general discussion when Bohr commented: *People say that classical mechanics is the limit of quantum mechanics when $\hbar$ goes to zero.* Then, Purcell recalled, Bohr shook his finger and walked to the blackboard on which he wrote $e^2/\hbar c$. As he made three strokes under $\hbar$, Bohr turned around and said, *you see $\hbar$ is in the denominator.*” A picture of the blackboard can be found in [9, p. 313]. See also [8,10] and references therein for an exhaustive discussions on relations between classical and quantum mechanics.

² This limit case $\hbar \rightarrow 0$ corresponds in fact to the so-called semiclassical mechanics, referring to “putting quantum flesh on classical bones” [8, Section 5.1].

³ The construction given in the recent paper [33] for a Hamiltonian flow associated with Schrödinger's dynamics of one quantum particle corresponds to a particular case of Bóna's theory. However, the author of [33] does not seem to be aware of Bóna's works.

quantum–classical hybrid theories for which the classical space exists by definition, in a *ad hoc* way, because of measuring instruments for instance. By contrast, in Bóna's view point, the classical world emerges *intrinsically* from macroscopic quantum systems, like in [42]. This is also similar to [43], which is however a much more elementary example⁴ referring to the Ehrenfest dynamics.

In [44] we revisit Bóna's conceptual lines, but propose a new method to mathematically implement them, with a much broader domain of applicability than his original version [34] (see also [10,35,45] and references therein). In fact, similar to Bóna, who constructs a general abstract theory [34] based on previous progress [17,19–21] on permutation-invariant quantum-spin systems with mean-field interactions, we also base our abstract theory [44] on our own (completely new) results [46–48] on the dynamical properties of (possibly non-permutation-invariant) quantum lattice systems with mean-field interactions. Our approach gives, in the infinite volume limit, an explicit representation of the full dynamics of such systems as entangled classical and quantum short-range dynamics. In particular, in contrast to Bóna's one, we highlight the relation between the phase space of the corresponding classical dynamics and the state space of the non-commutative algebra where the quantum short-range dynamics meanwhile runs, making meanwhile explicit the central role played by *self-consistency*. Our viewpoint is meanwhile different from recent rigorous studies within the C^* -algebraic approach to the classical limit in the context of mean-field quantum theories, like [49,50], which refer rather to semi-classical analysis.

The general theory can be found in [44], which is a rather long mathematical paper (72 pages). The aim of the current paper is thus to illustrate, in a simple manner, the entanglement of classical and quantum short-range dynamics, as well as important aspects of the general approach. This is done via the so-called strong-coupling BCS-Hubbard model, which serves here as a paradigm. From a technical viewpoint, the dynamical properties of this model are easy to study, albeit non-trivial, the model being not exactly solvable. From the physical point of view, this model is also interesting because it highlights the possible thermodynamic impact of the (screened) Coulomb repulsion on (*s*-wave) superconductivity, in the strong-coupling approximation. Its behavior at thermodynamical equilibrium is already rigorously known [51], but *not* its infinite volume dynamics. In fact, note that [52] is merely a concise introduction to this problem and the particular results presented here are new. The precise definition of the model and all its dynamical properties are explained in Section 2. The entanglement of classical and quantum (short-range) dynamics in this prototypical model is then made explicit in Section 3.

For convenience of the reader interested in the mathematical results [46,47] on the macroscopic dynamics of fermion and quantum-spin systems with mean-field interactions, we provide an appendix, since [46,47] are altogether about 126 pages long. Appendix A explains [46,47] in concise, albeit mathematically precise, terms. In Appendix A.4, note that we formulate the results in the special context of permutation-invariant models, making the link with the strong-coupling BCS-Hubbard model and previous results on permutation-invariant quantum-spin systems. Appendix A.4 contains *new* material that cannot be found in our previous papers [46–48,51,52] on the subject.

Remark 1. In all the paper, we focus on lattice-fermion systems, but all the results and discussions can be translated to quantum-spin systems via obvious modifications.

2. The strong-coupling BCS-hubbard model

2.1. Presentation of the model

The dynamics of the (reduced) BCS Hamiltonian can be *explicitly* computed by means of [46,47], but we prefer to consider here a BCS-type model including the Hubbard interaction. In fact, it is a much richer new example while the BCS Hamiltonian was already been extensively studied

⁴ It corresponds to a quantum systems with two species of particles in an extreme mass ratio limit: one species becomes, in this limit, infinitely more massive than the other one. In this limit, the massive species, like nuclei, becomes classical while the other one, like electrons, stays quantum.

in the literature, via various approaches. Observe, in particular, that the (usual, reduced) strong-coupling BCS model is exactly solvable, whereas its extension considered here is not. We call this new model the strong-coupling BCS-Hubbard Hamiltonian. Its equilibrium states were rigorously studied in [51], in order to understand the possible thermodynamic impact of the Coulomb repulsion on (s-wave) superconductivity. An interesting outcome of [51] is a mathematically rigorous proof of the existence of a superconductor-Mott insulator phase transition for the strong-coupling BCS-Hubbard Hamiltonian, like in cuprates, which must be doped in order to avoid the insulating (Mott) phase and become superconductors.

The results of [51] refer to an *exact* study of the phase diagram of the strong-coupling BCS-Hubbard model, whose Hamiltonian is defined in any cubic box $\Lambda_L := \{\mathbb{Z} \cap [-L, L]\}^d$ ($d \in \mathbb{N}$) of volume $|\Lambda_L|$, $L \in \mathbb{N}_0$, by

$$H_L := \sum_{x \in \Lambda_L} h_x - \frac{\gamma}{|\Lambda_L|} \sum_{x, y \in \Lambda_L} a_{x, \uparrow}^* a_{x, \downarrow}^* a_{y, \downarrow} a_{y, \uparrow} \quad (1)$$

for real parameters $\mu, h \in \mathbb{R}$ and $\lambda, \gamma \geq 0$, where, for all $x \in \mathbb{Z}^d$,

$$h_x := 2\lambda n_{x, \uparrow} n_{x, \downarrow} - \mu (n_{x, \uparrow} + n_{x, \downarrow}) - h (n_{x, \uparrow} - n_{x, \downarrow}). \quad (2)$$

Recall that the operator $a_{x, s}^*$ ($a_{x, s}$) creates (annihilates) a fermion with spin $s \in \{\uparrow, \downarrow\}$ at lattice position $x \in \mathbb{Z}^d$, $d = 1, 2, 3, \dots$, whereas $n_{x, s} := a_{x, s}^* a_{x, s}$ is the particle number operator at position x and spin s . They are linear operators acting on the fermion Fock space $\mathcal{F}_{\Lambda_L}$, where

$$\mathcal{F}_{\Lambda} := \bigwedge \mathbb{C}^{\Lambda \times \{\uparrow, \downarrow\}} \equiv \mathbb{C}^{2^{\Lambda \times \{\uparrow, \downarrow\}}} \quad (3)$$

for any $\Lambda \subseteq \mathbb{Z}^d$ and $d \in \mathbb{N}$.

The first term of the right-hand side of (2) represents the (screened) Coulomb repulsion as in the celebrated Hubbard model. The second term corresponds to the strong-coupling limit of the kinetic energy, also called “atomic limit” in the Hubbard model community, the real parameter μ being the so-called chemical potential. The third term is the interaction between spins and the external magnetic field h .

The last term in (1) is the (homogeneous) BCS interaction written in the position space (see, e.g., [51, Eq. (1.3)]). The *long-range* character of this interaction is apparent for it is an infinite-range hopping term (for fermion pairs). In fact, it is a mean-field interaction, since

$$\frac{1}{|\Lambda_L|} \sum_{x, y \in \Lambda_L} a_{x, \uparrow}^* a_{x, \downarrow}^* a_{y, \downarrow} a_{y, \uparrow} = \sum_{y \in \Lambda_L} \left(\frac{1}{|\Lambda_L|} \sum_{x \in \Lambda_L} a_{x, \uparrow}^* a_{x, \downarrow}^* \right) a_{y, \downarrow} a_{y, \uparrow}.$$

This is a simple example of the far more general case studied in [46, 47]. It is however a non-trivial and a very interesting mean-field model since, even when $\mu = h = \lambda = 0$, the Hamiltonian H_L qualitatively displays most of basic properties of real conventional type I superconductors. See, e.g. [53, Chapter VII, Section 4]. Note that the precise mediators leading to the effective BCS interaction are not relevant here, i.e., they could be phonons, as in conventional type I superconductors, or anything else.

2.2. Approximating Hamiltonians

The thermodynamic impact of the Coulomb repulsion on s-wave superconductors is analyzed in [51], via a rigorous study of equilibrium and ground states of the strong-coupling BCS-Hubbard Hamiltonian: At any $L_0 \in \mathbb{N}_0$ and inverse temperature $\beta > 0$, for any linear operator A acting on

the fermion Fock space⁵ $\mathcal{F}_{\Lambda_{L_0}}$ (3), we prove that

$$\lim_{L \rightarrow \infty} \omega^{(L)}(A) = \omega(A), \quad (4)$$

where, for $L \in \mathbb{N}_0$,

$$\omega^{(L)}(\cdot) := \text{Trace}_{\mathcal{F}_{\Lambda_L}} \left(\frac{e^{-\beta H_L}}{\text{Trace}_{\mathcal{F}_{\Lambda_L}}(e^{-\beta H_L})} \right) \quad (5)$$

is the Gibbs states associated with H_L , while ω is an *explicitly given* (infinite volume) equilibrium state, defined as being a (global, space-homogeneous) minimizer of the free energy density (i.e., free energy per unit volume). See [51, Section 6.2] for more details.

An important point in such an analysis is the study of an associate variational problem over complex numbers: By the so-called approximating Hamiltonian method [54–56], one defines an approximation of the Hamiltonian, which is, in the case of the strong-coupling BCS-Hubbard Hamiltonian, the c -dependent Hamiltonian

$$H_L(c) := \sum_{x \in \Lambda_L} \{h_x - \gamma (c a_{x,\uparrow}^* a_{x,\downarrow}^* + \bar{c} a_{x,\downarrow} a_{x,\uparrow})\} \quad (6)$$

where $c \in \mathbb{C}$. The main advantage of using this c -dependent Hamiltonian, in comparison with H_L , is the fact that it is a sum of shifts of the same on-site operator. For an appropriate choice of (order) parameter $c \in \mathbb{C}$, it leads to the exact pressure of the strong-coupling BCS-Hubbard model, in the limit $L \rightarrow \infty$: At inverse temperature $\beta > 0$,

$$\lim_{L \rightarrow \infty} p[H_L] = \sup_{c \in \mathbb{C}} \left\{ -\gamma |c|^2 + \lim_{L \rightarrow \infty} p[H_L(c)] \right\} \quad (7)$$

with $p[H]$ being the pressure

$$p[H] := \frac{1}{\beta |\Lambda_L|} \ln \text{Trace}_{\mathcal{F}_{\Lambda_L}}(e^{-\beta H}), \quad \beta > 0,$$

associated with any Hamiltonian H acting on the fermion Fock space $\mathcal{F}_{\Lambda_L}$. In fact, the (exact) Gibbs state $\omega^{(L)}$ converges⁶ to a convex combination of the thermodynamic limit $L \rightarrow \infty$ of the (approximating) Gibbs state $\omega^{(L,\mathfrak{d})}$ defined by

$$\omega^{(L,\mathfrak{d})}(\cdot) := \text{Trace}_{\mathcal{F}_{\Lambda_L}} \left(\frac{e^{-\beta H_L(\mathfrak{d})}}{\text{Trace}_{\mathcal{F}_{\Lambda_L}}(e^{-\beta H_L(\mathfrak{d})})} \right), \quad (8)$$

the complex number $\mathfrak{d} \in \mathbb{C}$ being a solution to the variational problem (7).

Since $\gamma \geq 0$, this can heuristically be seen from the inequality

$$\gamma |\Lambda_L| |c|^2 + H_L(c) - H_L = \gamma \left(c_0^* - \sqrt{|\Lambda_L|} \bar{c} \right) \left(c_0 - \sqrt{|\Lambda_L|} c \right) \geq 0,$$

where

$$c_0 := \frac{1}{\sqrt{|\Lambda_L|}} \sum_{x \in \Lambda_L} a_{x,\downarrow} a_{x,\uparrow} \quad (9)$$

⁵ For any finite subset $\Lambda \subseteq \mathbb{Z}^d$, the (finite-dimensional) C^* -algebra $\mathcal{B}(\mathcal{F}_\Lambda)$ of linear operators acting on the antisymmetric Fock space $\mathcal{F}_\Lambda$ is $*$ -isomorphic to the CAR (unital) C^* -algebra $\mathcal{U}_\Lambda$ generated by anticommuting elements $\{a_{x,s}\}_{x \in \Lambda, s \in S}$, which are identified with annihilation operators acting on $\mathcal{F}_\Lambda$, associated to the canonical basis of $\mathbb{C}^{\Lambda \times \{\uparrow, \downarrow\}}$. Therefore, by identifying the generators $\{a_{x,s}\}_{x \in \Lambda \cap \Lambda', s \in S}$ in two such C^* -algebras $\mathcal{U}_\Lambda$ and $\mathcal{U}_{\Lambda'}$, $\{\mathcal{B}(\mathcal{F}_\Lambda)\}_{\Lambda \subseteq \mathbb{Z}^d, |\Lambda| < \infty}$ is a net of C^* -algebras with respect to inclusion (of finite subsets of $\mathbb{Z}^d$).

⁶ In the sense of (4), or, in the mathematical jargon, in the weak* topology.

(c_0^*) annihilates (creates) one Cooper pair within the condensate, i.e., in the zero-mode for fermion pairs. This suggests the (rigorously proven) fact [51, Theorem 3.1] that

$$|\vartheta|^2 = \lim_{L \rightarrow \infty} \frac{\omega^{(L)}(c_0^* c_0)}{|A_L|} \quad (10)$$

for any⁷ $\vartheta \in \mathbb{C}$ solution to the variational problem (7). The parameter $|\vartheta|^2$ is the condensate density of Cooper pairs and so, $\vartheta \neq 0$ corresponds to the existence of a superconducting phase, which is shown to exist for sufficiently large $\gamma \geq 0$. See also [51, Figs. 1,2,3].

2.3. Dynamical problem

As is usual, a Hamiltonian like the strong-coupling BCS-Hubbard Hamiltonian drives a dynamics in the Heisenberg picture of quantum mechanics: The corresponding time-evolution is, for $L \in \mathbb{N}_0$, a continuous group $(\tau_t^{(L)})_{t \in \mathbb{R}}$ of automorphisms of the algebra $\mathcal{B}(\mathcal{F}_{A_L})$ of linear operators acting on the fermion Fock space $\mathcal{F}_{A_L}$ (see (3)), defined by

$$\tau_t^{(L)}(A) := e^{itH_L} A e^{-itH_L} \quad (11)$$

for any $A \in \mathcal{B}(\mathcal{F}_{A_L})$ and $t \in \mathbb{R}$. The generator⁸ of this time evolution is the linear operator δ_L defined on $\mathcal{B}(\mathcal{F}_{A_L})$ by

$$\delta_L(A) := i[H_L, A] := i(H_L A - A H_L).$$

If $\gamma = 0$ then, for any time $t \in \mathbb{R}$ and linear operator A acting on the fermion Fock space $\mathcal{F}_{A_{L_0}}$ (3), $L_0 \in \mathbb{N}_0$,

$$\begin{aligned} \lim_{L \rightarrow \infty} \tau_t^{(L)}(A) &= \tau_t^{(L_0)}(A), \\ \lim_{L \rightarrow \infty} \delta_L(A) &= i[H_{L_0}, A], \end{aligned} \quad (12)$$

because $H_L|_{\gamma=0}$ is the sum of on-site terms. In particular, (12) uniquely defines an infinite volume dynamics in this case. Nonetheless, as soon as $\gamma > 0$, the thermodynamic limit (12) of the mean-field dynamics does *not* exist in general (even along subsequences).

One can try to approximate $\tau_t^{(L)}$ by $\tau_t^{(L,c)}$, where

$$\tau_t^{(L,c)}(A) := e^{itH_L(c)} A e^{-itH_L(c)} \quad (13)$$

for any $L \in \mathbb{N}_0$, $A \in \mathcal{B}(\mathcal{F}_{A_L})$ and some complex number $c \in \mathbb{C}$. In this case, the linear operator

$$\delta_{L,c}(\cdot) := i[H_L(c), \cdot] \quad (14)$$

on $\mathcal{B}(\mathcal{F}_{A_L})$ is the generator of the dynamics $(\tau_t^{(L,c)})_{t \in \mathbb{R}}$. In this case, since local Hamiltonians (6) are sums of on-site terms, for any $c \in \mathbb{C}$, $t \in \mathbb{R}$, $L_0 \in \mathbb{N}_0$ and $A \in \mathcal{F}_{A_{L_0}}$,

$$\begin{aligned} \lim_{L \rightarrow \infty} \tau_t^{(L,c)}(A) &= \tau_t^{(L_0)}(A), \\ \lim_{L \rightarrow \infty} \delta_{L,c}(A) &= i[H_{L_0}(c), A], \end{aligned} \quad (15)$$

like in the case $\gamma = 0$ with (12). In other words, there is an infinite volume dynamics for such approximating interactions.

A natural choice for $c \in \mathbb{C}$ would be a solution to the variational problem (7), but what about if the solution is not unique? Observe, moreover, that the variational problem (7) depends on the temperature whereas the time evolution (11) does not!

The validity of the approximation with respect to the primordial dynamics was an open question that Thirring and Wehrli [57,58] solve in 1967 for the special case $H_L|_{\mu=\lambda=\hbar=0}$, which is an exactly solvable permutation-invariant model for any $\gamma \in \mathbb{R}$. An attempt to generalize Thirring and Wehrli's results to a general class of fermionic models, including the BCS theory, has been done in 1978 [59],

⁷ This implies that any solution ϑ to the variational problem (7) must have the same absolute value.

⁸ That is, $\tau_t^{(L)}(A) = \exp(it\delta_L)$.

but at the cost of technical assumptions that are difficult to verify in practice. This research direction has been strongly developed by many authors until 1992, see [13–32]. All these papers study dynamical properties of *permutation-invariant* quantum-spin systems with mean-field interactions. Our results [46,47], summarized in [Appendix A](#), represent a significant generalization of such previous results to possibly non-permutation-invariant lattice-fermion or quantum-spin systems. In order to illustrate how our results [46,47] are used to control the infinite volume dynamics of mean-field Hamiltonians, we now come back to our pedagogical example, that is, the strong-coupling BCS-Hubbard model.

2.4. Dynamical self-consistency

Instead of considering the Heisenberg picture, let us consider the Schrödinger picture of quantum mechanics. In this case, recall that, at fixed $L \in \mathbb{N}_0$, a (finite volume) state $\rho^{(L)}$ is a positive and normalized functional acting on the algebra $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ of linear operators on the fermion Fock space $\mathcal{F}_{\Lambda_L}$. By finite dimensionality of $\mathcal{F}_{\Lambda_L}$,

$$\rho^{(L)}(\cdot) := \text{Trace}_{\mathcal{F}_{\Lambda_L}}(\cdot d^{(L)}),$$

for a uniquely defined positive operator $d^{(L)} \in \mathcal{B}(\mathcal{F}_{\Lambda_L})$ satisfying $\text{Trace}_{\mathcal{F}_{\Lambda_L}}(d^{(L)}) = 1$ and named the density matrix of $\rho^{(L)}$. Compare with (5) and (8). See also [Appendix A.2.1](#). At $L \in \mathbb{N}_0$, the expectation of any $A \in \mathcal{B}(\mathcal{F}_{\Lambda_L})$ at time $t \in \mathbb{R}$ is, as usual, equal to

$$\rho_t^{(L)}(A) := \text{Trace}_{\mathcal{F}_{\Lambda_L}}(e^{itH_L} A e^{-itH_L} d^{(L)}). \quad (16)$$

I.e., the time evolution of any finite volume state is

$$\rho_t^{(L)} := \rho^{(L)} \circ \tau_t^{(L)}, \quad t \in \mathbb{R}, \quad (17)$$

which corresponds to a time-dependent density matrix equal to $d_t^{(L)} = \tau_{-t}^{(L)}(d^{(L)})$. Compare with (77).

The thermodynamic limit of (16) for periodic states can be explicitly computed, as explained in [Appendix A.3.2](#). It refers to a *non*-autonomous state-dependent dynamics related to *self-consistency*: By (3) with $\Lambda = \Lambda_0 = \{0\}$, recall that

$$\mathcal{F}_{\{0\}} := \bigwedge \mathbb{C}^{\{0\} \times \{\uparrow, \downarrow\}} \equiv \mathbb{C}^4 \quad (18)$$

is the fermion Fock space associated with the lattice site $(0, \dots, 0) \in \mathbb{Z}^d$ and so, $\mathcal{B}(\mathcal{F}_{\{0\}})$ can be identified with the algebra $\text{Mat}(4, \mathbb{C})$ of complex 4×4 matrices, in some orthonormal basis⁹. For any continuous family $\omega := (\omega_t)_{t \in \mathbb{R}}$ of states acting on $\mathcal{B}(\mathcal{F}_{\{0\}})$, we define the (infinite volume) *non*-autonomous dynamics $(\tau_{t,s}^{(\omega)})_{s,t \in \mathbb{R}}$ by the Dyson–Phillips series¹⁰

$$\tau_{t,s}^{(\omega)} := \mathbf{1} + \sum_{k \in \mathbb{N}} \int_s^t dt_1 \cdots \int_s^{t_{k-1}} dt_k \delta^{\omega t_k} \circ \cdots \circ \delta^{\omega t_1}, \quad (19)$$

where

$$\delta^\rho := \lim_{L \rightarrow \infty} \delta_{L, \rho(a_{0,\uparrow} a_{0,\downarrow})}(\cdot) \quad (20)$$

is the generator of the infinite volume dynamics associated with the approximating Hamiltonian $H_L(c)$ for $c = \rho(a_{0,\uparrow} a_{0,\downarrow})$. See (6) and (15). Note that the precise definition of the generator δ^ρ – both acting on the CAR algebra $\mathcal{U}$ of the infinite lattice – is not necessary here to understand the action of the mappings (19)–(20) on local elements $A \in \mathcal{B}(\mathcal{F}_{\Lambda_{L_0}})$, $L_0 \in \mathbb{N}_0$, since in this case

$$\delta^\rho(A) = i[H_{L_0}(\rho(a_{0,\uparrow} a_{0,\downarrow})), A],$$

⁹ For instance, $(1, 0, 0, 0)$ is the vacuum; $(0, 1, 0, 0)$ and $(0, 0, 1, 0)$ correspond to one fermion with spin $\uparrow$ and $\downarrow$, respectively; $(0, 0, 0, 1)$ refers to two fermions with opposite spins.

¹⁰ That is, $\tau_{t,s}^{(\omega)}$ is a time-ordered exponential.

thanks to Equation of (15). In particular,

$$\tau_{t,s}^{(\omega)} \left(\mathcal{B}(\mathcal{F}_{\Lambda_{L_0}}) \right) \subseteq \mathcal{B}(\mathcal{F}_{\Lambda_{L_0}}), \quad L_0 \in \mathbb{N}_0.$$

Observe that the particular value $\rho(a_{0,\uparrow} a_{0,\downarrow}) \in \mathbb{C}$, which is taken here for the complex parameter c , is reminiscent of (9)–(10).

Now, by (75) and (101), for any fixed initial (even) state ρ_0 on $\mathcal{B}(\mathcal{F}_{\{0\}})$ at $t = 0$, there is a unique family $(\varpi(t; \rho_0))_{t \in \mathbb{R}}$ of on-site states acting on $\mathcal{B}(\mathcal{F}_{\{0\}})$ such that

$$\varpi(t; \rho_0) = \rho_0 \circ \tau_{t,0}^{\varpi(\cdot; \rho_0)}, \quad t \in \mathbb{R}. \quad (21)$$

This is a self-consistency equation on a finite-dimensional space, by (18).

2.5. Infinite volume dynamics for product states

For simplicity, at initial time $t = 0$, take a finite volume product¹¹ state $\otimes_{\Lambda_L} \rho$ associated with a fixed even¹² state ρ on $\mathcal{B}(\mathcal{F}_{\{0\}})$. An example of finite volume product states is given by the approximating Gibbs states (8). Then, in this case, as explained in Appendix A.4, for any $t \in \mathbb{R}$, the thermodynamic limit

$$\rho_t(A) := \lim_{L \rightarrow \infty} (\otimes_{\Lambda_L} \rho) \circ \tau_t^{(L)}(A) \quad (22)$$

of the expectation of any linear operator $A \in \mathcal{B}(\mathcal{F}_{\Lambda_{L_0}})$ for $L_0 \in \mathbb{N}_0$ exists and corresponds to the time-dependent product¹³ state

$$\rho_t = (\otimes_{\mathbb{Z}^d} \rho) \circ \tau_{t,0}^{\varpi(\cdot; \rho)} = \otimes_{\mathbb{Z}^d} \varpi(t; \rho_0), \quad (23)$$

where $\varpi(\cdot; \rho)$ is defined by (21). In other words, for any time $t \in \mathbb{R}$, the limit state is in this case completely determined by its restriction to the single lattice site $(0, \dots, 0) \in \mathbb{Z}^d$. Below, we give the explicit time evolution of the most important physical quantities related to this model, in this situation:

Proposition 2 (Infinite Volume Dynamics). *One has the following assertions:*

(i) *Electron density:*

$$d(\rho) := \rho(n_{0,\uparrow} + n_{0,\downarrow}) = \rho_{t=0}(n_{0,\uparrow} + n_{0,\downarrow}) = \rho_t(n_{0,\uparrow} + n_{0,\downarrow}) \in [0, 2].$$

(ii) *Magnetization density:*

$$m(\rho) := \rho(n_{0,\uparrow} - n_{0,\downarrow}) = \rho_{t=0}(n_{0,\uparrow} - n_{0,\downarrow}) = \rho_t(n_{0,\uparrow} - n_{0,\downarrow}) \in [-1, 1].$$

(iii) *Coulomb correlation density:*

$$w(\rho) := \rho(n_{0,\uparrow} n_{0,\downarrow}) = \rho_{t=0}(n_{0,\uparrow} n_{0,\downarrow}) = \rho_t(n_{0,\uparrow} n_{0,\downarrow}) \in [0, 1].$$

(iv) *Cooper-field and condensate densities:*

$$\rho_t(a_{0,\downarrow} a_{0,\uparrow}) = \sqrt{\kappa(\rho)} e^{i(t\nu(\rho) + \theta(\rho))}$$

with

$$\nu(\rho) := 2(\mu - \lambda) + \gamma(1 - d(\rho))$$

and $\kappa(\rho) \in [0, 1]$, $\theta(\rho) \in [-\pi, \pi]$ such that, at initial time, $\rho(a_{0,\downarrow} a_{0,\uparrow}) = \sqrt{\kappa(\rho)} e^{i\theta(\rho)}$.

¹¹ The product state $\rho^{(L)}$ is well-defined by $\rho^{(L)}(\alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n)) = \rho(A_1) \cdots \rho(A_n)$ for all $A_1, \dots, A_n \in \mathcal{B}(\mathcal{F}_{\{0\}})$ and all $x_1, \dots, x_n \in \Lambda_L$ such that $x_i \neq x_j$ for $i \neq j$, where $\alpha_{x_j}(A_j) \in \mathcal{B}(\mathcal{F}_{\{x_j\}})$ is the x_j -translated copy of A_j for all $j \in \{1, \dots, n\}$. See (43) and (95)–(96) for more details.

¹² Even means that the expectation value of any odd monomials in $\{a_{0,s}^*, a_{0,s}\}_{s \in \{\uparrow, \downarrow\}}$ with respect to the on-site state ρ is zero. Even states are the physically relevant ones.

¹³ For any even state $\tilde{\rho}$ on $\mathcal{B}(\mathcal{F}_{\{0\}})$, $\otimes_{\mathbb{Z}^d} \tilde{\rho}$ is a state acting on the CAR algebra of the infinite lattice, which includes all $\mathcal{B}(\mathcal{F}_{\Lambda_L})$, $L \in \mathbb{N}_0$. The restriction of $\otimes_{\mathbb{Z}^d} \tilde{\rho}$ to $\mathcal{B}(\mathcal{F}_{\Lambda_L})$ is of course equal to $\otimes_{\Lambda_L} \tilde{\rho}$.

Proof. Recall that $[A, B] := AB - BA$ is the commutator of A and B and, for any $s \in \{\uparrow, \downarrow\}$, $n_s := a_s^* a_s$ is the spin- s particle number operator on the lattice site 0 (with “0” being omitted in the notation for simplicity). We now prove Assertions (i)–(iv): By the canonical anti-commutation relations (CAR), for any $s \in \{\uparrow, \downarrow\}$,

$$[dh(\rho), n_s] = \gamma (\rho(a_\downarrow a_\uparrow) a_\uparrow^* a_\downarrow^* - \rho(a_\uparrow^* a_\downarrow^*) a_\downarrow a_\uparrow)$$

with $dh(\rho)$ defined from (2) by

$$dh(\rho) := h_0 - \gamma (a_\uparrow^* a_\downarrow^* \rho(a_\downarrow a_\uparrow) + \rho(a_\uparrow^* a_\downarrow^*) a_\downarrow a_\uparrow). \quad (24)$$

By (23), (i)–(ii) straightforwardly follow. (iii) is a direct consequence of the following computation:

$$[dh(\rho), n_\uparrow n_\downarrow] = \gamma a_\uparrow^* a_\downarrow^* \rho(a_\downarrow a_\uparrow) - \gamma \rho(a_\uparrow^* a_\downarrow^*) a_\downarrow a_\uparrow.$$

To obtain (iv), observe that

$$[dh(\rho), a_\downarrow a_\uparrow] = 2(\mu - \lambda) a_\downarrow a_\uparrow - \gamma \rho(a_\downarrow a_\uparrow) (n_\uparrow + n_\downarrow - 1),$$

using again the CAR. Then, by combining this with (i), one computes that the function $\mathfrak{z}_t := \rho_t(a_{0,\downarrow} a_{0,\uparrow})$, $t \in \mathbb{R}$, satisfies the elementary ODE

$$\partial_t \mathfrak{z}_t(\rho) = iv(\rho) \mathfrak{z}_t(\rho), \quad \mathfrak{z}_0(\rho) = \rho(a_\downarrow a_\uparrow),$$

from which (iv) directly follows. ■

In the special case $\lambda = 0$, i.e., without the Hubbard interaction, Proposition 2 reproduces the results of [20, Section A] on the strong-coupling BCS model, written in that paper as a permutation-invariant quantum-spin model. Observe also that $\kappa(\rho) := |\rho(a_\downarrow a_\uparrow)|^2$ is the Cooper-pair-condensate density, which, in this situation, stays constant for all times, by Proposition 2 (iv).

Proposition 2 leads to the exact dynamics of the considered physical system prepared in a product state at initial time, driven by the strong-coupling BCS–Hubbard Hamiltonian, in the infinite volume limit. This set of states is still restrictive and our results [44,46,47], summarized in Appendix A, go far beyond this simple case, by allowing us to consider general periodic states as initial states, in contrast with all previous results on lattice-fermion or quantum-spin systems with mean-field interactions.

2.6. From product to periodic states as initial states

The strong-coupling BCS–Hubbard model is clearly permutation-invariant.¹⁴ First, take a permutation-invariant¹⁵ state ρ as initial state. As is explained in Appendix A.4.2, any permutation-invariant state can be written (or approximated, to be more precise) as a convex combination of product states (cf. the Størmer theorem). Thus, let $\rho_1, \dots, \rho_n$ be $n \in \mathbb{N}$ product states and $u_1, \dots, u_n \in [0, 1]$ such that $u_1 + \dots + u_n = 1$, and

$$\rho = \sum_{j=1}^n u_j \rho_j. \quad (25)$$

At fixed $L \in \mathbb{N}_0$, we take the restriction $\rho^{(L)}$ of ρ to $\mathcal{B}(\mathcal{F}_{\Lambda_L})$, which is thus a finite volume permutation-invariant state, like the Gibbs state (5) associated with the strong-coupling BCS–Hubbard model. Then, in this case, we infer from (22)–(23) that, for any time $t \in \mathbb{R}$,

$$\lim_{L \rightarrow \infty} \rho^{(L)} \circ \tau_t^{(L)} = \sum_{j=1}^n u_j \rho_j \circ \tau_{t,0}^{\varpi(\cdot; \rho_j)}, \quad (26)$$

¹⁴ It is invariant under the transformation $p_\pi : a_{x,s} \mapsto a_{\pi(x),s}$ with $x \in \mathbb{Z}^d$ and $s \in \{\uparrow, \downarrow\}$, for all bijective mappings $\pi : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ which leave all but finitely many elements invariant. See Appendix A.4.1.

¹⁵ I.e., $\rho \circ p_\pi = \rho$ for all bijective mappings $\pi : \mathbb{Z}^d \rightarrow \mathbb{Z}^d$ which leave all but finitely many elements invariant. See (93).

where by a slight abuse of notation, $\varpi(\cdot; \rho) = \varpi(\cdot; \rho|_{\mathcal{B}(\mathcal{F}_{[0]}})$. Note that all limits on states refer to the weak* topology, basically corresponding to apply all states of (26) on fixed elements of $\mathcal{B}(\mathcal{F}_{\Lambda_L})$, $L \in \mathbb{N}_0$, to perform the limit.

For general permutation-invariant states, one has to replace the finite sum (25) by an integral with respect to a probability measure μ_ρ on the set $E_\otimes$ of product states in order to generalize (26): Formally, for any time $t \in \mathbb{R}$,

$$\lim_{L \rightarrow \infty} \rho^{(L)} \circ \tau_t^{(L)} = \int_{E_\otimes} \hat{\rho} \circ \tau_{t,0}^{\varpi(\cdot; \hat{\rho})} d\mu_\rho(\hat{\rho}). \quad (27)$$

See, e.g., (103) for more details. As a consequence, by combining Proposition 2 with such a decomposition of permutation-invariant states into product states, we obtain all dynamical properties of the strong-coupling BCS-Hubbard model, in any permutation-invariant initial state.

For instance, taking the state (25) and combining (26) with Proposition 2 applied to the product states ρ_j , $j \in \{1, \dots, n\}$, we obtain the Cooper-field and condensate densities:

$$\rho_t(a_{0,\downarrow} a_{0,\uparrow}) = \sum_{j=1}^n u_j \sqrt{\kappa(\rho_j)} e^{i(t\nu(\rho_j) + \theta_{\rho_j})}. \quad (28)$$

In particular, the Cooper-pair-condensate density defined by $\kappa(\rho_t) := |\rho_t(a_{\downarrow} a_{\uparrow})|^2$ at time $t \in \mathbb{R}$ is *not anymore necessarily constant* and can have a complicated, highly non-trivial, time evolution, in particular when ρ is not a finite sum like (25), but only the barycenter of a general probability measure on the set of product states, see (27). Physically speaking, Eq. (28) expresses an *interference phenomenon* on the Cooper-field densities in each *pure state* ρ_j for $j \in \{1, \dots, n\}$.

The permutation-invariant case already applies to the (weak*) limit $\omega^{(\infty)}$ of the Gibbs state $\omega^{(L)}$ (5) which is proven to exist as a permutation-invariant state $\omega^{(\infty)}$ because, by [51, Theorem 6.5], away from the superconducting critical point, it is formally given by

$$\omega^{(\infty)} = \frac{1}{2\pi} \int_0^{2\pi} \omega^{(\infty, re^{i\theta})} d\theta \quad (29)$$

with $\{\mathfrak{d} = re^{i\theta}, \theta \in [0, 2\pi]\}$ being all solutions to the variational problem (7) and where the product state $\omega^{(\infty, \mathfrak{d})}$ is the thermodynamic limit $L \rightarrow \infty$ of the Gibbs state $\omega^{(L, \mathfrak{d})}(\cdot)$ defined by (8). In this case, by [51, Theorem 6.4 and previous discussions],

$$\omega^{(\infty, re^{i\theta})}(a_{\downarrow} a_{\uparrow}) = re^{i\theta} = \mathfrak{d}, \quad \theta \in [0, 2\pi], \quad (30)$$

and if one has a superconducting phase, i.e., $r > 0$, then, by [51, Eq. (3.3) and Theorem 6.4 (i)], one always has the equality

$$\omega^{(\infty, re^{i\theta})}(n_{\downarrow} + n_{\uparrow}) = 1 + 2\gamma^{-1}(\mu - \lambda) \quad (31)$$

for all $\theta \in [0, 2\pi]$. In fact, any equilibrium state is a state in the closed convex hull of $\{\omega^{(\infty, re^{i\theta})}, \theta \in [0, 2\pi]\}$. Eqs. (30)–(31) imply that, for any equilibrium state ω , like $\omega^{(\infty)}$, the frequency $\nu(\omega)$, defined in Proposition 2 (iv), vanishes, i.e., $\nu(\omega) = 0$. Hence, in this case, by Proposition 2, all densities are constant in time for any equilibrium state. The same property is also true at the superconducting critical point, by [51, Theorem 6.5 (ii)]. This is of course coherent with the well-known stationarity of equilibrium states. For more details on equilibrium states of mean-field models, see [48].

The results presented above could still have been deduced from Bóna's ones, as it is done in [20, Section A] for the strong-coupling BCS model, for $H_L|_{\lambda=\hbar=0}$ to be precise. Of course, in this case, one has to represent the lattice-fermion systems as a permutation-invariant quantum-spin system and a permutation-invariant state would again be required as initial state.

Using [46,47] one can easily extend this study of the strong-coupling BCS-Hubbard model to a much larger class of initial states: In fact, product states are only a particular case of so-called ergodic translation-invariant¹⁶ states and if the initial finite volume state $\rho^{(L)}$ is the restriction to

¹⁶ I.e., it is invariant, for any $x \in \mathbb{Z}^d$, under the transformation $a_{x,s} \mapsto a_{x+y,s}$, $y \in \mathbb{Z}^d$, $s \in \{\uparrow, \downarrow\}$. See Section A.2.3.

$\mathcal{B}(\mathcal{F}_{\Lambda_t})$ of an extreme or, equivalently, ergodic translation-invariant state¹⁷, then Eq. (103) also tells us that, for any time $t \in \mathbb{R}$,

$$\lim_{L \rightarrow \infty} \rho^{(L)} \circ \tau_t^{(L)} = \rho \circ \tau_{t,0}^{\varpi(\cdot; \rho)}$$

(in the weak* sense, as before), where, again by a slight abuse of notation, $\varpi(\cdot; \rho) = \varpi(\cdot; \rho|_{\mathcal{B}(\mathcal{F}_{\{0\}})})$. What is more, since

$$\tau_{t,0}^{\varpi(\cdot; \rho)}(\mathcal{B}(\mathcal{F}_{\{0\}})) \subseteq \mathcal{B}(\mathcal{F}_{\{0\}}),$$

because (6) is a sum of on-site terms, the time evolution of the electron, magnetization, Coulomb correlation, Cooper-pair-condensate and the Cooper-field densities can directly be deduced from Proposition 2, for extreme (ergodic), translation-invariant, initial states. Similar to (25)–(26), these quantities can be derived for general translation-invariant states, by using their decompositions (72) in terms of extreme (or ergodic) translation-invariant states.

All these outcomes can be extended to the case of general periodic initial states, via straightforward modifications: for any $(\ell_1, \dots, \ell_d) \in \mathbb{N}^d$ and initial $(\ell_1, \dots, \ell_d)$ -periodic¹⁸ state ρ , replace in all the above discussions on translation-invariant initial states terms like $\rho(a_{\downarrow} a_{\uparrow}) = \rho(a_{0,\downarrow} a_{0,\uparrow})$ by

$$\frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i - 1\}} \rho(a_{x,\downarrow} a_{x,\uparrow}). \quad (32)$$

Cf. (73)–(74). This goes far beyond all previous studies on lattice-fermion or quantum-spin systems with mean-field interactions.

3. Entanglement of classical and quantum dynamics

Quoting [8, p. 106], the “research in semiclassical mechanics, and especially in the subfield of quantum chaos, has revealed that the relationship between classical and quantum mechanics is much more subtle and intricate than the simple statement $\hbar \rightarrow 0$ might lead us believe.” In this section, we explicitly show an *intricate* combination of classical and quantum dynamics in mean-field systems. In order to illustrate this fact in a simple manner, we again use our pedagogical example, that is, the strong-coupling BCS-Hubbard model. We start by describing the classical part of the dynamics.

3.1. Emergence¹⁹ of Classical Mechanics

In the previous sections we rigorously derive the infinite volume dynamics of the BCS-Hubbard model, which is a model comprising mean-field interactions, and now one may ask how a classical dynamics appears in this scope. To unveil it, first observe from Proposition 2 that we recover the equation of a symmetric rotor: Fix an even on-site state ρ . For any $t \in \mathbb{R}$, define the 3D vector $(\Omega_1(t), \Omega_2(t), \Omega_3(t))$ by

$$\rho_t(a_{0,\downarrow} a_{0,\uparrow}) = \Omega_1(t) + i\Omega_2(t)$$

and

$$\Omega_3(t) := 2(\mu - \lambda) + \gamma(1 - \rho_t(n_{0,\uparrow} + n_{0,\downarrow})). \quad (33)$$

¹⁷ This state acts on the CAR algebra $\mathcal{U}$ of the lattice. See (38).

¹⁸ See Appendix A.2.3 for more details.

¹⁹ In physics, “emergence” may refer to different concepts, one possible meaning of this term being the very opposite of the so-called “reduction”. For more details, see, e.g., [60, Beginning of Chap. 10]. Here, with “emergence” we simply mean the process of appearance. Note however that the theory of entangled classical and quantum mechanics, described in this section, could be seen as an example of *asymptotically emergent* theory, in the sense of [60, Definition 10.1].

Then, this time-dependent 3D vector satisfies, for any time $t \in \mathbb{R}$, the following system of ODEs:

$$\begin{cases} \dot{\Omega}_1(t) = -\Omega_3(t) \Omega_2(t) , \\ \dot{\Omega}_2(t) = \Omega_3(t) \Omega_1(t) , \\ \dot{\Omega}_3(t) = 0 . \end{cases} \quad (34)$$

It describes the time evolution of the angular momentum of a symmetric rotor in *classical* mechanics. This is *not* accidental.

As a matter of fact, the equation governing the (infinite volume) mean-field dynamics can be written in terms of Poisson brackets, i.e., as some *Liouville's equation* of classical mechanics: In the algebraic approach to classical mechanics [7, Chapter 12], it is natural to consider real- or complex-valued functions on a *phase space* $\mathcal{P}$. Because of the self-consistency equation (21), we thus define a classical algebra of observables to be the real part of the (commutative C^* -)algebra $C(\mathcal{P}; \mathbb{C})$ of continuous functions on the space $\mathcal{P} \equiv E_{[0]}^+$ of all even states acting on $\mathcal{B}(\mathcal{F}_{[0]})$. The self-consistency equation leads to a group²⁰ $(V_t)_{t \in \mathbb{R}}$ of automorphisms of $C(\mathcal{P}; \mathbb{C})$ defined by

$$[V_t f](\rho) := f(\varpi(t; \rho)) \quad (35)$$

for any state $\rho \in \mathcal{P}$, function $f \in C(\mathcal{P}; \mathbb{C})$ and time $t \in \mathbb{R}$. The equation governing this dynamics can be written in terms of Poisson brackets:

Poisson bracket. Similar to (81), for any $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{B}(\mathcal{F}_{[0]})$ and $g \in C^1(\mathbb{R}^n, \mathbb{C})$, we define the function $\Gamma_g \in C(\mathcal{P}; \mathbb{C})$ by

$$\Gamma_g(\rho) := g(\rho(A_1), \dots, \rho(A_n)), \quad \rho \in \mathcal{P}.$$

A polynomial function in $C(\mathcal{P}; \mathbb{C})$ is a function f of the form Γ_g for some polynomial g of $n \in \mathbb{N}$ variables. Similar to (82), for such a function and any $\rho \in \mathcal{P}$, define

$$D\Gamma_g(\rho) := \sum_{j=1}^n (A_j - \rho(A_j) \mathbf{1}_{\mathcal{B}(\mathcal{F}_{[0]})}) \partial_{x_j} g(\rho(A_1), \dots, \rho(A_n)).$$

Note that $D\Gamma_g(\rho) \in \mathcal{B}(\mathcal{F}_{[0]})$. This definition comes from a notion of convex derivative introduced by us, as explained in [44, Section 3.4]. Then, for all functions of the form Γ_h and Γ_g with $g \in C^1(\mathbb{R}^n, \mathbb{C})$ and $h \in C^1(\mathbb{R}^m, \mathbb{C})$ ($n, m \in \mathbb{N}$), we define the continuous function $\{\Gamma_h, \Gamma_g\} \in C(\mathcal{P}; \mathbb{C})$ by

$$\{\Gamma_h, \Gamma_g\}(\rho) := \rho(i[D\Gamma_h(\rho), D\Gamma_g(\rho)])$$

for any $\rho \in \mathcal{P}$. This defines a Poisson bracket on the space of all (local) polynomial functions on $\mathcal{P}$. See [44, Proposition 3.11] for a general proof.

Liouville's equation. The *classical* Hamiltonian $h \in C(\mathcal{P}; \mathbb{C})$ related to the strong-coupling BCS-Hubbard model is a polynomial in $C(\mathcal{P}; \mathbb{C})$ defined in a very natural way by

$$h(\rho) := \rho(h) - \gamma |\rho(a_\uparrow a_\downarrow)|^2, \quad \rho \in \mathcal{P},$$

with $h \equiv h_0$ defined by (2) for $x = 0$, the 0 indices of operators acting on $\mathcal{F}_{[0]}$ having been omitted for notational simplicity. It leads to a state-dependent Hamiltonian equal to

$$Dh(\rho) = h - \gamma (a_\uparrow^* a_\downarrow^* (a_\downarrow a_\uparrow) + \rho(a_\uparrow^* a_\downarrow^*) a_\downarrow a_\uparrow) + (2\gamma |\rho(a_\downarrow a_\uparrow)|^2 - \rho(h)) \mathbf{1}_{\mathcal{B}(\mathcal{F}_{[0]})} \quad (36)$$

for any $\rho \in \mathcal{P}$. Compare with (24). Then, we can rigorously derive *Liouville's equation* (see, e.g., [61, Proposition 10.2.3]) for any polynomial f in $C(\mathcal{P}; \mathbb{C})$:

$$\partial_t V_t(f) = V_t(\{h, f\}) = \{h, V_t(f)\}, \quad t \in \mathbb{R}. \quad (37)$$

See Eq. (104). Liouville's equation is written here on a *finite-dimensional* phase space and can easily be studied analytically. Its solution at fixed initial state gives access to all dynamical properties

²⁰ The fact it is a group is not that obvious, a priori. See [44, Proposition 4.4] for a general proof.

of product states driven by the strong-coupling BCS-Hubbard model in the thermodynamic limit. In particular, it is straightforward to check the validity of [Proposition 2](#) from this equation: (i) $V_t(d) = d$; (ii) $V_t(m) = m$; (iii) $V_t(w) = w$; (iv) $V_t(z_0) = z_t$ with $z_t(\rho) := \rho_t(a_{0,\downarrow} a_{0,\uparrow})$ for $\rho \in \mathcal{P}$ and $t \in \mathbb{R}$.

The time evolution $V_t(p_n)$ of the *non-affine* polynomials

$$p_n(\rho) := |\rho(a_{\downarrow} a_{\uparrow})|^{2n} = \frac{1}{4^n} (\rho(a_{\downarrow} a_{\uparrow} + a_{\uparrow}^* a_{\downarrow}^*)^2 + \rho(i a_{\downarrow} a_{\uparrow} - i a_{\uparrow}^* a_{\downarrow}^*)^2)^n, \quad \rho \in \mathcal{P},$$

for any integer $n \geq 1$ can be obtained by using [\(37\)](#). In particular, for $n = 1$, since the (convex) derivative of p_1 at $\rho \in \mathcal{P}$ equals

$$Dp_1(\rho) = a_{\uparrow}^* a_{\downarrow}^* \rho(a_{\downarrow} a_{\uparrow}) + \rho(a_{\uparrow}^* a_{\downarrow}^*) a_{\downarrow} a_{\uparrow} - 2 |\rho(a_{\downarrow} a_{\uparrow})|^2 \mathbf{1}_{\mathcal{B}(\mathcal{F}_{[0]})},$$

one directly recovers from Liouville's equation, combined with the CAR and [\(36\)](#), that the Cooper-pair-condensate density is static. Compare with [Proposition 2](#) (iv). Moreover, by considering complex-valued polynomials g in the space

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\} \times [2(\mu - \lambda) - \gamma, 2(\mu - \lambda) + \gamma],$$

of $(\Omega_1, \Omega_2, \Omega_3)$ -coordinates one can recover the classical dynamics of a symmetric rotor, as stated in [\(34\)](#). In fact, one can write another Liouville's equation on a convenient reduced (or effective) phase space. The real and imaginary parts of $\rho(a_{\downarrow} a_{\uparrow})$ (Cooper-field densities), respectively Ω_1 and Ω_2 , and the shifted particle density Ω_3 [\(33\)](#) represent three physical quantities that can be seen as *macroscopic* in the fermionic system under consideration. See, e.g., [\(10\)](#).

To conclude this subsection, recall that the classical dynamics presented above has the space $\mathcal{P} \equiv E_{[0]}^+$ of all even states acting on $\mathcal{B}(\mathcal{F}_{[0]})$ as phase space, i.e., this dynamics is defined on $C(\mathcal{P}; \mathbb{C})$. Taking a broader perspective, a classical dynamics can also be defined on $C(E_{\Pi}; \mathbb{C})$, with E_{Π} being the space of permutation-invariant states on $\mathcal{U}$, the CAR algebra of the infinite lattice. For more details, see [Appendix A.4](#). In this case, the classical dynamics constructed on $C(E_{\Pi}; \mathcal{U})$ can be pushed forward, through the restriction mapping $E_{\Pi} \rightarrow \mathcal{P}$, from $C(E_{\Pi}; \mathbb{C})$ to $C(\mathcal{P}; \mathbb{C})$. For an even more general definition of classical dynamics, which can be extended to periodic states, see [Appendix A.3.3](#).

3.2. Classical versus quantum pictures

For product states at initial time, in the case of the strong-coupling BCS-Hubbard model, it is natural to restrict the quantum observables to the algebra $\mathcal{B}(\mathcal{F}_{[0]})$ of linear operators on the fermion Fock space $\mathcal{F}_{[0]}$. This keeps things simple. In this case, for any even state ρ on $\mathcal{B}(\mathcal{F}_{[0]})$, we can define a non-autonomous quantum dynamics by the continuous evolution²¹ family $(\tau_{t,s}^{(\varpi(\cdot;\rho))})_{s,t \in \mathbb{R}}$ of automorphisms of $\mathcal{B}(\mathcal{F}_{[0]})$, defined by [\(19\)](#) for $\omega = \varpi(\cdot;\rho)$. The physical relevance of this dynamics comes from Eqs. [\(23\)](#). Therefore, for initial product states and on on-site observables, the mean-field dynamics can be seen either as a classical one on $C(\mathcal{P}; \mathbb{C})$ or as a non-autonomous quantum dynamics on $\mathcal{B}(\mathcal{F}_{[0]})$. The classical dynamics uniquely defines the quantum dynamics and conversely.

For initial states that are not product states, the situation is more involved, but also much more interesting, since interference phenomena on macroscopic quantities may take place. See, e.g., [\(28\)](#).

Let us consider general permutation-invariant states (i.e., not necessarily product states) as initial states. In this case, the quantum world refers to *all* local observables of the infinite lattice and we thus have to consider the CAR C^* -algebra

$$\mathcal{U} \supsetneq \bigcup_{L \in \mathbb{N}} \mathcal{B}(\mathcal{F}_{A_L}) \supsetneq \mathcal{B}(\mathcal{F}_{[0]}), \quad (38)$$

²¹ It satisfies the reverse cocycle property: $\tau_{t,s}^{(\varpi(\cdot;\rho))} = \tau_{r,s}^{(\varpi(\cdot;\rho))} \circ \tau_{t,r}^{(\varpi(\cdot;\rho))}$ for any $s, r, t \in \mathbb{R}$.

which is the C^* -algebra generated by all finite volume quantum observables for fermions in the lattice. See [Appendix A.1.1](#). This algebra corresponds to what we call the “*primordial*” quantum algebra in our general abstract setting, introduced in [\[44\]](#)

Still in relation to the terminology we introduce in [\[44\]](#), the *secondary* quantum algebra corresponds here to the C^* -algebra $C(E_\Pi; \mathcal{U})$ of all continuous $\mathcal{U}$ -valued functions on the space E_Π of permutation-invariant states on $\mathcal{U}$. This is nothing else²² than the following tensor product:

$$C(E_\Pi; \mathcal{U}) \equiv C(E_\Pi; \mathbb{C}) \otimes \mathcal{U}. \quad (39)$$

Having in mind that a classical dynamics can be defined on $C(E_\Pi; \mathbb{C})$, this is similar to quantum–classical hybrid theories of theoretical physics, described for instance in [\[36–41\]](#). With this definition we naturally have the inclusions $\mathcal{U} \subseteq C(E_\Pi; \mathcal{U})$ and $C(E_\Pi; \mathbb{C}) \subseteq C(E_\Pi; \mathcal{U})$, by identifying elements of $\mathcal{U}$ with constant functions and elements of $C(E_\Pi; \mathbb{C})$ with functions whose values are scalar multiples of the unit of the primordial algebra $\mathcal{U}$.

The quantum (short-range) dynamics on the secondary quantum algebra $C(E_\Pi; \mathcal{U})$ refers to the continuous evolution²³ family $(\mathfrak{T}_{t,s})_{s,t \in \mathbb{R}}$ of $*$ -automorphisms²⁴ of $C(E_\Pi; \mathcal{U})$ defined by

$$[\mathfrak{T}_t(f)](\rho) := \tau_{t,0}^{\varpi(\cdot;\rho)}(f(\rho)), \quad \rho \in E_\Pi,$$

for any function $f \in C(E_\Pi; \mathcal{U})$ and time $t \in \mathbb{R}$, where, again by a slight abuse of notation, $\varpi(\cdot; \rho) = \varpi(\cdot; \rho|_{\mathcal{B}(\mathcal{F}_{[0]})})$. This state-dependent dynamics lets every element of the classical algebra $C(E_\Pi; \mathbb{C})$ invariant, i.e., $\mathfrak{T}_t(f) = f$ for any classical function $f \in C(E_\Pi; \mathbb{C})$. In other words, the classical algebra $C(E_\Pi; \mathbb{C})$ is a subalgebra of the so-called fix point algebra of the family $(\mathfrak{T}_t)_{t \in \mathbb{R}}$ of $*$ -automorphisms of $C(E_\Pi; \mathcal{U})$.

The physical relevance of the above mathematical structure comes from the fact that, for each time $t \in \mathbb{R}$, permutation-invariant state $\rho \in E_\Pi$ and any element $A \in \mathcal{U} \subseteq C(E_\Pi; \mathcal{U})$,

$$\lim_{L \rightarrow \infty} \rho^{(L)} \circ \tau_t^{(L)}(A) = \int_{E_\otimes} \hat{\rho} \circ \tau_{t,0}^{\varpi(\cdot;\hat{\rho})}(A) d\mu_\rho(\hat{\rho}) = \rho \circ \mathfrak{T}_t(A),$$

by [Eq. \(27\)](#), where in the last equality ρ is seen as a state of $C(E_\Pi; \mathcal{U})$ via the definition

$$\rho(f) \doteq \int_{E_\otimes} \hat{\rho}(f(\hat{\rho})) d\mu_\rho(\hat{\rho})$$

for any $f \in C(E_\Pi; \mathcal{U})$. See [\[47, Theorem 4.3\]](#) for the general mathematical statement. The classical part of the full mean-field dynamics explicitly appears in the time evolution of product states $\hat{\rho} \in E_\otimes$ (cf. the Størmer theorem) and is related to a Liouville’s equation in the classical (i.e., commutative) algebra of continuous functions $C(\mathcal{P}; \mathbb{C})$, as explained in the previous subsection.

In the theoretical framework we present here, the classical and quantum worlds are intrinsically interdependent, in the following manner:

- The quantum (short-range) dynamics on $C(E_\Pi; \mathcal{U})$ yields a well-defined classical dynamics on $C(\mathcal{P}; \mathbb{C})$, by restriction on product states.
- Conversely, the classical dynamics on $C(\mathcal{P}; \mathbb{C})$ uniquely defines a quantum (short-range) dynamics on $C(E_\Pi; \mathcal{U})$.

This is a mathematical fact proven for general quantum systems in [\[44, Sections 4.2–4.3 and 5.2\]](#).

On the one hand, the classical world, represented by the commutative (sub)algebra $C(E_\Pi; \mathbb{C})$, is embedded in the quantum world, as mathematically expressed by the above inclusion $C(E_\Pi; \mathbb{C}) \subseteq C(E_\Pi; \mathcal{U})$. On the other hand, our theory entangles the quantum and classical worlds through self-consistency. As a result, (effective) non-autonomous short-range dynamics can emerge. Seeing both entangled worlds, quantum and classical, as “two sides of the same coin” looks like an

²² Up to an isomorphism. See [\[44, Section 1\]](#) for very general mathematical arguments proving that fact.

²³ It satisfies the reverse cocycle property: $\mathfrak{T}_{t,s} = \mathfrak{T}_{r,s} \circ \mathfrak{T}_{t,r}$ for any $s, r, t \in \mathbb{R}$.

²⁴ The mathematical fact that it is a continuous evolution family of automorphisms is not obvious, a priori. The proof uses that E_Π is a metrizable weak*-compact space, by separability of $\mathcal{U}$. See [\[44, Lemma 5.2\]](#) for a general proof.

oxymoron, but there is no contradiction there, for everything refers to a *primordial* quantum world mathematically encoded in the structure of $\mathcal{U}$. For instance, the phase space $\mathcal{P}$ and state space $E_{\mathcal{I}}$ are imprints left by $\mathcal{U} \supsetneq \mathcal{B}(\mathcal{F}_{\{0\}})$ in the classical world, see (39).

Note that if $\mathcal{U}$ was a commutative algebra, the corresponding Poisson bracket and, hence, the dynamics would have been trivial. Observe also that, if the primordial algebra would be $\mathcal{B}(\mathcal{F}_{\{0\}})$, instead of $\mathcal{U}$, then all the above construction would be still relevant for the case of initial states being product states. In this situation, the introduction of the secondary quantum algebra is superfluous to derive the mean-field dynamics, whereas it becomes essential when the initial state is permutation-invariant, but *not* a product state.

All the above construction can be extended to periodic states and general lattice-fermion or quantum spin systems. For more details, see [Appendix A.3.4](#).

4. Conclusions

The dynamics of the strong-coupling BCS-Hubbard model has been exactly derived, in the infinite volume limit. It explicitly determines, among other things, the dynamical impact of the (screened) Coulomb repulsion on (s-wave) superconductivity. For non-pure phases, we also prove that the Cooper-pair-condensate density is not anymore necessarily constant in time and can have a complicated time evolution, as a consequence of interference phenomena.

Much more importantly, this model illustrates the general behavior of mean-field dynamics at infinite volume, as rigorously explained in [Appendix A](#). We demonstrate via this example that a classical mechanics does not only appear in the limit $\hbar \rightarrow 0$, as explained for instance in [10,11]. This was already observed by various mathematical physicists. In particular, Bóna's major conceptual contribution [34] is to highlight the emergence of classical mechanics without necessarily the disappearance of the quantum world. However, we propose here a *new* method to mathematically implement it, with a *broader* domain of applicability than Bóna's original version [34] (see also [10, 35,45] and references therein). For detailed explanations, see [44, Section 3].

In contrast with *all* previous approaches, including those of theoretical physics (see, e.g., [11,36–42]), in ours the classical and quantum worlds are *entangled*, with *backreaction*²⁵ (that is, feedbacks), as expected. Differently from Bóna's setting, our perspective has the advantage to highlight inherent *self-consistency* aspects, which are absolutely not exploited in [34], as well as in quantum–classical hybrid theories of physics described, for instance, in [36–43].

Remark that the theoretical construction presented here is not useful when the macroscopic²⁶ time evolution in the Heisenberg picture is *not* state-dependent, as in quantum lattice systems with *short-range* interactions. Nevertheless, quantum many-body systems in the continuum are expected to have, in general, only a state-dependent Heisenberg dynamics, as explained for instance in [64, Section 6.3]. Additionally, we show that such a mathematical framework is generally *imperative* to describe the macroscopic dynamics of quantum many-body systems with mean-field interactions, because of the necessity of coupled quantum–classical evolution equations, implementing self-consistency *when long-range order take place*. The phenomenological aspects of quantum dynamics in presence of mean-field interactions discussed here and that are highlighted by our original approach to this problem, are very likely not restricted to mean-field case only, but should also appear in presence of interactions that are sufficiently long-range²⁷ to yield non-vanishing²⁸

²⁵ We do not mean here the so-called *quantum backreaction*, commonly used in physics, which refers to the backreaction effect of quantum fluctuations on the classical degrees of freedom. Note further that the phase spaces we consider are, generally, much more complex than those related to the position and momentum of simple classical particles.

²⁶ “Macroscopic” can still mean short (even atomic) length scales. For lattice systems, it should quantitatively be measured in terms of lattice units (l.u.), which is typically a few ångströms. For instance, a length $L \simeq 1000$ refers to a few hundreds of nanometers, only. One thousand is a priori a large number, but everything depends of course on the rate of convergence of microscopic dynamics in the thermodynamic limit $L \rightarrow \infty$. In general, this may be an exponential rate (with respect to the volume $|\Lambda_L|$), similar to what is proven in [62,63] for electric current densities in non-interacting lattice fermions with disorder.

²⁷ In fact, the existence of long-range order in quantum systems with sufficiently long-range interactions can be mathematically proven by using the celebrated Bishop–Phelps theorem.

²⁸ In a given representation of the observable algebra, which is fixed by the initial state.

background fields, in the spirit of the Higgs mechanism of quantum field theory. We therefore think that our mathematical framework for long-range dynamics, outlined here by means of a pedagogical explicit example, opens new theoretical perspectives²⁹ in the understanding of the classical world within the quantum one.

CRediT authorship contribution statement

J.-B. Bru: Idea, Computations, Editorial works. **W. de Siqueira Pedra:** Idea, Computations, Editorial works.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Mathematical foundations

The entanglement of classical and quantum short-range dynamics in mean-field systems refers to results obtained in [46,47] on the dynamics of quantum lattice systems with mean-field interactions. They are far more general than previous ones because the invariance under permutations of lattice sites is *not* required anymore:

- The short-range part of the corresponding Hamiltonian is very general since only a sufficiently strong polynomial decay of its interactions and a translation invariance are necessary.
- The mean-field part is also very general, being an infinite sum (over n) of mean-field terms of order $n \in \mathbb{N}$. In fact, even for permutation-invariant systems, the class of mean-field interactions we are able to handle is much larger than what was previously studied.
- The initial state is only required to be periodic. By [46, Proposition 2.3], observe that the set of all such initial states is dense within the set of all even states, the physically relevant ones.

The papers [46,47] are altogether about 126 pages long. Therefore, the aim of the appendix is to present, in a concise way, their key points, being meanwhile mathematically rigorous. Note, however, that the contents of Appendix A.4 are new and cannot be found in [46–48,51,52].

²⁹ Even after a few centuries, the Newtonian gravitational constant is still *not* accurately known, in comparison with all other fundamental constants. See [65] for an account of recent experiment. It is also very difficult to detect gravity at scales below micrometers, still a macroscopic scale as compared with atomic ones. On the other hand, interference phenomena for gravitational waves appear (2017 Nobel Prize in Physics). Gravitation looks like a macroscopic background (Higgs-like) field (cf. Bogoliubov approximation), similar to the Cooper-field densities in the strong-coupling BCS-Hubbard model on which classical mechanics applies.

A.1. C^* -Algebraic setting

A.1.1. CAR algebra of lattices

Let $\mathbb{Z}^d$ be the d -dimensional cubic lattice and $\mathcal{P}_f \subseteq 2^{\mathbb{Z}^d}$ the set of all non-empty finite subsets of $\mathbb{Z}^d$. In order to define the thermodynamic limit, for simplicity, we use cubic boxes

$$\Lambda_L := \{\mathbb{Z} \cap [-L, L]\}^d, \quad L \in \mathbb{N}_0. \quad (40)$$

Let S be a fixed (once and for all) finite set (of orthonormal spin modes). For any $\Lambda \in \mathcal{P}_f \cup \{\mathbb{Z}^d\}$, $\mathcal{U}_\Lambda$ is the universal unital C^* -algebra³⁰ generated by the elements $\{a_{x,s}\}_{x \in \Lambda, s \in S}$ satisfying the canonical anti-commutation relations (CAR): for any $x, y \in \mathbb{Z}^d$ and $s, t \in S$,

$$a_{x,s}a_{y,t} + a_{y,t}a_{x,s} = 0, \quad a_{x,s}a_{y,t}^* + a_{y,t}^*a_{x,s} = \delta_{s,t}\delta_{x,y}1. \quad (41)$$

Here, $\delta_{k,l}$ is the Kronecker delta, that is, the function of two variables defined by $\delta_{k,l} := 1$ if $k = l$ and $\delta_{k,l} = 0$ otherwise. By identifying the generators $\{a_{x,s}\}_{x \in \Lambda \cap \Lambda', s \in S}$ of both C^* -algebras $\mathcal{U}_\Lambda$ and $\mathcal{U}_{\Lambda'}$, $\{\mathcal{U}_\Lambda\}_{\Lambda \in 2^{\mathbb{Z}^d}}$ is a net of C^* -algebras with respect to inclusion: For all pairs of subsets $\Lambda, \Lambda' \subseteq \mathbb{Z}^d$ with $\Lambda \subseteq \Lambda'$, one has $\mathcal{U}_\Lambda \subseteq \mathcal{U}_{\Lambda'}$.³¹ Note that we use the notation $\mathcal{U} \equiv \mathcal{U}_{\mathbb{Z}^d}$ and define

$$\mathcal{U}_0 := \bigcup_{\Lambda \in \mathcal{P}_f} \mathcal{U}_\Lambda, \quad (42)$$

which is a dense normed $*$ -subalgebra of $\mathcal{U}$. In particular, $\mathcal{U}$ is separable, since, for every finite region $\Lambda \in \mathcal{P}_f$, $\mathcal{U}_\Lambda$ has finite dimension. Elements of $\mathcal{U}_0$ are called local elements. The (real) Banach subspace of all self-adjoint elements of $\mathcal{U}$ is denoted by $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

Translations are represented by a group homomorphism $x \mapsto \alpha_x$ from $\mathbb{Z}^d$ to the group of $*$ -automorphisms of $\mathcal{U}$, which is uniquely defined by the condition

$$\alpha_x(a_{y,s}) = a_{y+x,s}, \quad y \in \mathbb{Z}^d, \quad s \in S. \quad (43)$$

The mapping $x \mapsto \alpha_x$ is used below to define symmetry groups of states as well as translation-invariant interactions of lattice-fermion systems.

The results presented in the current paper also hold true in the context of quantum-spin systems, but we focus on lattice-fermion systems which are, from a technical point of view, slightly more difficult. In fact, the additional difficulty in Fermi systems is that, for any disjoint $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$ and elements $B_1 \in \mathcal{U}_{\Lambda^{(1)}}$, $B_2 \in \mathcal{U}_{\Lambda^{(2)}}$, the commutator

$$[B_1, B_2] := B_1B_2 - B_2B_1$$

may be non-zero, in general. For instance, the CAR (41) trivially yield $[a_{x,s}, a_{y,t}] = 2a_{x,s}a_{y,t} \neq 0$ for any $x, y \in \mathbb{Z}$ and $s, t \in S$, $(x, s) \neq (y, t)$. Because of the CAR (41), the commutation property of disjoint lattice regions is satisfied for all *even* elements, which are defined as follows: The condition

$$\sigma(a_{x,s}) = -a_{x,s}, \quad x \in \Lambda, \quad s \in S, \quad (44)$$

defines a unique $*$ -automorphism σ of the C^* -algebra $\mathcal{U}$. The subspace

$$\mathcal{U}^+ := \{A \in \mathcal{U} : A = \sigma(A)\} \quad (45)$$

is the C^* -subalgebra of so-called even elements of $\mathcal{U}$. Then, for any disjoint $\Lambda^{(1)}, \Lambda^{(2)} \in \mathcal{P}_f$,

$$[B_1, B_2] = 0, \quad B_1 \in \mathcal{U}_{\Lambda^{(1)}} \cap \mathcal{U}^+, \quad B_2 \in \mathcal{U}_{\Lambda^{(2)}}.$$

This last condition is the expression of the principle of locality in quantum field theory. Using well-known constructions,³² the C^* -algebra $\mathcal{U}$, generated by anticommuting elements, can be

³⁰ $\mathcal{U}_\Lambda \equiv \mathcal{B}(\mathbb{C}^{2^{\Lambda \times S}})$ is equivalent to the algebra of $2^{|\Lambda \times S|} \times 2^{|\Lambda \times S|}$ complex matrices, when $\Lambda \in \mathcal{P}_f$.

³¹ In fact, strictly speaking, there is a canonical injective $*$ -homomorphism $\mathcal{U}_\Lambda \rightarrow \mathcal{U}_{\Lambda'}$ (identifying generators) that allows one to see $\mathcal{U}_\Lambda$ as being a C^* -subalgebra of $\mathcal{U}_{\Lambda'}$.

³² More precisely, the so-called sector theory of quantum field theory.

recovered from $\mathcal{U}^+$. As a consequence, the C^* -algebra $\mathcal{U}^+$ should thus be seen as more fundamental than $\mathcal{U}$ in physics. In fact, $\mathcal{U}$ corresponds in this context to the so-called local field algebra. See, e.g., [66, Sections 4.8 and 6].

The principle of locality of quantum field theory, usually invoked to see the algebra $\mathcal{U}^+$ as being more fundamental than $\mathcal{U}$, does not prevent us from considering mean-field interactions as possibly fundamental interactions, as explained in [47, Section 1]. The choice of $\mathcal{U}^+$ only refers to the fact that measurable physical quantities (observables) are local. By contrast, the full energy of lattice-fermion systems with short-range and mean-field interactions are generally inaccessible to measurements, in infinite volume. In fact, the mean-field part yields possibly non-vanishing (effective) background fields, in the spirit of the Higgs mechanism of quantum field theory, in a given representation of the observable algebra, which is fixed by the initial state.

A.1.2. Short-range interactions

A (complex) *interaction* is a mapping $\Phi : \mathcal{P}_f \rightarrow \mathcal{U}^+$ such that $\Phi_\Lambda \in \mathcal{U}_\Lambda$ for all $\Lambda \in \mathcal{P}_f$. The set of all interactions can be naturally endowed with the structure of a complex vector space. By using the norm

$$\|\Phi\|_{\mathcal{W}} := \sup_{x, y \in \mathbb{Z}^d} \sum_{\Lambda \in \mathcal{P}_f, \Lambda \supseteq \{x, y\}} \frac{\|\Phi_\Lambda\|_{\mathcal{U}}}{\mathbf{F}(x, y)}, \quad (46)$$

where, given some fixed parameters $\epsilon, \varsigma > 0$, for any $x, y \in \mathbb{Z}^d$,

$$\mathbf{F}(x, y) := e^{2\varsigma|x-y|} (1 + |x - y|)^{(d+\epsilon)}. \quad (47)$$

We then define a separable Banach space $\mathcal{W}$ of short-range interactions, which are, by definition, those interactions that have finite norm. Here, $|\cdot - \cdot|$ stands for the Euclidean metric. The particular choice of function (47) defining the norm (46) is made only for simplicity and a much more general class of space decays could be considered, as discussed in [46, Section 3.1]. We use in the sequel three important properties of short-range interactions:

(i) *Self-adjointness*: There is a natural involution $\Phi \mapsto \Phi^* := (\Phi_\Lambda^*)_{\Lambda \in \mathcal{P}_f}$ defined on the Banach space $\mathcal{W}$ of short-range interactions. Self-adjoint interactions are, by definition, interactions Φ satisfying $\Phi = \Phi^*$. The (real) Banach subspace of all self-adjoint short-range interactions is denoted by $\mathcal{W}^{\mathbb{R}} \subsetneq \mathcal{W}$, similar to $\mathcal{U}^{\mathbb{R}} \subsetneq \mathcal{U}$.

(ii) *Translation invariance*: By definition, the interaction Φ is translation-invariant if

$$\Phi_{\Lambda+x} = \alpha_x(\Phi_\Lambda), \quad x \in \mathbb{Z}^d, \Lambda \in \mathcal{P}_f,$$

where

$$\Lambda + x := \{y + x \in \mathbb{Z}^d : y \in \Lambda\}.$$

Here, $\{\alpha_x\}_{x \in \mathbb{Z}^d}$ is the family of (translation) $*$ -automorphisms of $\mathcal{U}$ defined by (43). We then denote by $\mathcal{W}_1 \subsetneq \mathcal{W}$ the (separable) Banach subspace of translation-invariant, short-range interactions on $\mathbb{Z}^d$.

(iii) *Finite range*: For any $R \in [0, \infty)$, we define the closed subspace

$$\mathcal{W}^R := \left\{ \Phi \in \mathcal{W}_1 : \Phi_\Lambda = 0 \text{ for } \Lambda \in \mathcal{P}_f \text{ such that } \max_{x, y \in \Lambda} |x - y| > R \right\} \quad (48)$$

of finite-range, translation-invariant interactions. For $R = 0$, we obtain the space $\mathcal{W}_\Pi := \mathcal{W}^0$ of permutation-invariant interactions described in Appendix A.4.

Short-range interactions define sequences of local (complex) energy elements: For any $\Phi \in \mathcal{W}$ and $L \in \mathbb{N}_0$,

$$U_L^\Phi := \sum_{\Lambda \subseteq \Lambda_L} \Phi_\Lambda \in \mathcal{U}_{\Lambda_L} \cap \mathcal{U}^+, \quad (49)$$

where we recall that $\Lambda_L := \{\mathbb{Z} \cap (-L, L)\}^d$ is the cubic box used to define the thermodynamic limit (see (40)). The energy elements U_L^Φ , $L \in \mathbb{N}_0$, refer to an extensive quantity since their norm are

proportional to the volume of the region they correspond to: In fact, for any $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}$,

$$\|U_L^\Phi\|_{\mathcal{U}} \leq \mathbf{C} |A_L| \|\Phi\|_{\mathcal{W}}, \quad (50)$$

where

$$\mathbf{C} := \sum_{x \in \mathbb{Z}^d} \frac{1}{(1 + |x|)^{d+\epsilon}} < \infty. \quad (51)$$

Moreover, for any self-adjoint interaction $\Phi \in \mathcal{W}^{\mathbb{R}}$, $U_L^\Phi \in \mathcal{U}^{\mathbb{R}}$ (i.e., $U_L^\Phi = (U_L^\Phi)^*$), $L \in \mathbb{N}_0$, is a sequence of local Hamiltonians.

Each local Hamiltonian associated with $\Phi \in \mathcal{W}^{\mathbb{R}}$ leads to a local dynamics on the full C^* -algebra $\mathcal{U}$ via the group $(\tau_t^{(L,\Phi)})_{t \in \mathbb{R}}$ of $*$ -automorphisms of $\mathcal{U}$ defined by

$$\tau_t^{(L,\Phi)}(A) = e^{itU_L^\Phi} A e^{-itU_L^\Phi}, \quad A \in \mathcal{U}. \quad (52)$$

It is the continuous group which is the solution to the evolution equation

$$\forall t \in \mathbb{R} : \quad \partial_t \tau_t^{(L,\Phi)} = \tau_t^{(L,\Phi)} \circ \delta_L^\Phi,$$

where $\tau_0^{(L,\Phi)} = \mathbf{1}_{\mathcal{U}}$ is the identity mapping on $\mathcal{U}$. Here, at each $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}^{\mathbb{R}}$, δ_L^Φ is defined on $\mathcal{U}$ by

$$\delta_L^\Phi(A) := i[U_L^\Phi, A] := i(U_L^\Phi A - A U_L^\Phi)$$

for any $A \in \mathcal{U}$. This corresponds to a quantum dynamics in the Heisenberg picture. Note that, for every $L \in \mathbb{N}_0$ and $\Phi \in \mathcal{W}^{\mathbb{R}}$, δ_L^Φ is a so-called symmetric derivation which belongs to the Banach space $\mathcal{B}(\mathcal{U})$ of bounded operators acting on the C^* -algebra $\mathcal{U}$, see, e.g., [46, Section 3.3].

More generally, for possibly time-dependent interactions, the (generally non-autonomous) local dynamics is defined, for any continuous function $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$ and $L \in \mathbb{N}_0$, as the unique (fundamental) solution $(\tau_{t,s}^{(L,\Psi)})_{s,t \in \mathbb{R}}$ in the Banach space $\mathcal{B}(\mathcal{U})$ to the (finite volume) non-autonomous evolution equation³³

$$\forall s, t \in \mathbb{R} : \quad \partial_t \tau_{t,s}^{(L,\Psi)} = \tau_{t,s}^{(L,\Psi)} \circ \delta_L^{\Psi(t)} \quad (53)$$

with $\tau_{s,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}}$. The solution to (53) can be explicitly written as a Dyson–Phillips series: For any $s, t \in \mathbb{R}$,

$$\tau_{t,s}^{(L,\Psi)} = \mathbf{1}_{\mathcal{U}} + \sum_{k \in \mathbb{N}} \int_s^t dt_1 \cdots \int_s^{t_{k-1}} dt_k \delta_L^{\Psi(t_k)} \circ \cdots \circ \delta_L^{\Psi(t_1)}. \quad (54)$$

By [67, Corollary 5.2], in the thermodynamic limit $L \rightarrow \infty$, for any $\Psi \in C(\mathbb{R}; \mathcal{W}^{\mathbb{R}})$, the group $(\tau_{t,s}^{(L,\Psi)})_{s,t \in \mathbb{R}}$, $L \in \mathbb{N}_0$, strongly converges, at any fixed s, t , to a strongly continuous two-parameter family $(\tau_{t,s}^\Psi)_{s,t \in \mathbb{R}}$ of $*$ -automorphisms of $\mathcal{U}$:

$$\lim_{L \rightarrow \infty} \tau_{t,s}^{(L,\Psi)}(A) =: \tau_{t,s}^\Psi(A), \quad A \in \mathcal{U}, \quad s, t \in \mathbb{R}. \quad (55)$$

In other words, (time-dependent) self-adjoint interactions lead to an infinite volume (possibly non-autonomous) dynamics on the CAR algebra of the lattice.

A.1.3. Mean-field models

We start with some preliminary definitions: Let $\mathbb{S}$ be the unit sphere of $\mathcal{W}_1$. For any $n \in \mathbb{N}$ and signed Borel measure of finite variation α on the Cartesian product $\mathbb{S}^n$ (endowed with its product

³³ Let $\mathcal{H}$ be some Hilbert space and $(H_t)_{t \in \mathbb{R}}$ some continuous family of bounded Hamiltonians acting on $\mathcal{H}$. The corresponding Schrödinger equation with $\hbar = 1$ reads $i\partial_t \varphi_t = H_t \varphi_t$ and so, $\varphi_t = U_{t,s} \varphi_s$ with $U_{t,s}$ being the solution to $\partial_t U_{t,s} = H_t U_{t,s}$. Then, in the Heisenberg picture, the time-evolution of any (bounded) observable B acting on $\mathcal{H}$ at initial time $t = s \in \mathbb{R}$ is $B_t = \tau_{t,s}(B_s) \doteq U_{t,s}^* B_s U_{t,s}$ for $s, t \in \mathbb{R}$, which yields $\partial_t \tau_{t,s} = \tau_{t,s} \circ \delta_t$ with $\delta_t(A) \doteq i[H_t, A]$.

topology), we define the signed Borel measure (of finite variation) α^* to be the pushforward of α through the self-homeomorphism

$$(\psi^{(1)}, \dots, \psi^{(n)}) \mapsto ((\psi^{(n)})^*, \dots, (\psi^{(1)})^*) \in \mathbb{S}^n \quad (56)$$

of $\mathbb{S}^n$. A finite signed Borel measure α on $\mathbb{S}^n$ is, by definition, *self-adjoint* whenever $\alpha^* = \alpha$. For any $n \in \mathbb{N}$, the real Banach space of self-adjoint signed Borel measures α of finite variation on $\mathbb{S}^n$ with the norm of total variation

$$\|\alpha\|_{\mathcal{S}(\mathbb{S}^n)} := |\alpha|(\mathbb{S}^n), \quad n \in \mathbb{N},$$

is denoted by $\mathcal{S}(\mathbb{S}^n)$. We define a norm for sequences $\alpha \equiv (\alpha_n)_{n \in \mathbb{N}}$ of finite signed Borel measures $\alpha_n \in \mathcal{S}(\mathbb{S}^n)$ as follows:

$$\|\alpha\|_{\mathcal{S}} := \sum_{n \in \mathbb{N}} n^2 \mathbf{C}^{n-1} \|\alpha_n\|_{\mathcal{S}(\mathbb{S}^n)}, \quad \alpha \equiv (\alpha_n)_{n \in \mathbb{N}} \in \mathcal{S}, \quad (57)$$

where the constant $\mathbf{C} > 0$ is defined by (51). The sequences which are finite in this norm form a (real) Banach space that we denote by $\mathcal{S}$.

The (separable) Banach space of mean-field models is then defined by

$$\mathcal{M} := \mathcal{W}^{\mathbb{R}} \times \mathcal{S} \quad (58)$$

with the norm of $\mathcal{M}$ being defined from (46) and (57) by

$$\|\mathbf{m}\|_{\mathcal{M}} := \|\Phi\|_{\mathcal{W}} + \|\alpha\|_{\mathcal{S}}, \quad \mathbf{m} := (\Phi, \alpha) \in \mathcal{M}. \quad (59)$$

The spaces $\mathcal{W}^{\mathbb{R}}$ and $\mathcal{S}$ are seen as subspaces of $\mathcal{M}$. In particular, $\Phi \equiv (\Phi, 0) \in \mathcal{M}$ for $\Phi \in \mathcal{W}^{\mathbb{R}}$. Using the subspace $\mathcal{W}^{\mathbb{R}}$ of finite-range interactions defined by (48) for $R \in [0, \infty)$, we introduce the subspace

$$\mathcal{S}^{\infty} := \bigcup_{R \in [0, \infty)} \{(\alpha_n)_{n \in \mathbb{N}} \in \mathcal{S} : \forall n \in \mathbb{N}, |\alpha_n|(\mathbb{S}^n) = |\alpha_n|((\mathbb{S} \cap \mathcal{W}^{\mathbb{R}})^n)\}. \quad (60)$$

Long-range dynamics are studied for models in the following two subspaces

$$\mathcal{M}^{\infty} := \mathcal{W}^{\mathbb{R}} \times \mathcal{S}^{\infty}, \quad \mathcal{M}_1^{\infty} := (\mathcal{W}^{\mathbb{R}} \cap \mathcal{W}_1) \times \mathcal{S}^{\infty}. \quad (61)$$

Clearly, $\mathcal{W}^{\mathbb{R}} \subseteq \mathcal{M}^{\infty} \subseteq \mathcal{M}$ and $(\mathcal{W}^{\mathbb{R}} \cap \mathcal{W}_1) \subseteq \mathcal{M}_1^{\infty} \subseteq \mathcal{M}^{\infty}$.

Similar to self-adjoint short-range interactions, each mean-field model leads to a sequence of local Hamiltonians: For any $L \in \mathbb{N}_0$ and $\mathbf{m} \in \mathcal{M}$,

$$U_L^{\mathbf{m}} := U_L^{\Phi} + \sum_{n \in \mathbb{N}} \frac{1}{|\Lambda_L|^{n-1}} \int_{\mathbb{S}^n} U_L^{\psi^{(1)}} \dots U_L^{\psi^{(n)}} \alpha_n(d\psi^{(1)}, \dots, d\psi^{(n)}) \quad (62)$$

with U_L^{Φ} and $U_L^{\psi^{(k)}}$ been defined by (49). Note that $U_L^{\mathbf{m}} = (U_L^{\mathbf{m}})^*$ and straightforward estimates yield the upper bound

$$\|U_L^{\mathbf{m}}\|_{\mathcal{U}} \leq \mathbf{C} |\Lambda_L| \|\mathbf{m}\|_{\mathcal{M}}, \quad L \in \mathbb{N}_0. \quad (63)$$

Compare with (50).

Similar to (52), each model $\mathbf{m} \in \mathcal{M}$ leads to finite volume dynamics defined, for any $L \in \mathbb{N}_0$, by

$$\tau_t^{(L, \mathbf{m})}(A) = e^{itU_L^{\mathbf{m}}} A e^{-itU_L^{\mathbf{m}}}, \quad A \in \mathcal{U}. \quad (64)$$

In contrast with short range interactions (see (55)), for any fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the thermodynamic limit $L \rightarrow \infty$ of $\tau_t^{(L, \mathbf{m})}(A)$ does *not* necessarily exist in the C^* -algebra $\mathcal{U}$. However, by [47, Theorem 4.3], for any $\mathbf{m} \in \mathcal{M}_1^{\infty}$, it converges in the σ -weak topology within some representation of $\mathcal{U}$. This is reminiscent of the fact that the energy-density observable $U_L^{\Phi}/|\Lambda_L|$ does not converge in $\mathcal{U}$, as $L \rightarrow \infty$, but its expectation value with respect to any periodic state does. See Appendix A.3 for more details.

A.2. State spaces

A.2.1. Finite volume state space

For any (non-empty) finite subset $\Lambda \subseteq \mathbb{Z}^d$, i.e., $\Lambda \in \mathcal{P}_f$, let $\mathcal{U}_\Lambda^*$ be the dual space of the local C^* -algebra $\mathcal{U}_\Lambda$. For every $\Lambda \in \mathcal{P}_f$, we denote by

$$E_\Lambda := \{\rho_\Lambda \in \mathcal{U}_\Lambda^* : \rho_\Lambda \geq 0, \rho_\Lambda(1) = 1\} \quad (65)$$

the space of all states on $\mathcal{U}_\Lambda$. For all $\Lambda \in \mathcal{P}_f$, the space E_Λ is a norm-compact convex subset of the dual space $\mathcal{U}_\Lambda^*$ and, for any $\rho_\Lambda \in E_\Lambda$, there is a unique, positive, trace-one operator $d_\Lambda \in \mathcal{B}(\mathcal{F}_{\Lambda_L})$ satisfying

$$\rho_\Lambda(A) := \text{Trace}_{\mathcal{F}_{\Lambda_L}}(d_\Lambda A), \quad A \in \mathcal{U}_\Lambda, \quad (66)$$

named the density matrix of ρ_Λ . In fact, E_Λ is affinely equivalent to the set of all states on the algebra of $2^{|\Lambda| \times |S|} \times 2^{|\Lambda| \times |S|}$ complex matrices. The structure of states for infinite systems is more subtle, as demonstrated in [44,68].

Note that the physically relevant finite volume states ρ_Λ , $\Lambda \in \mathcal{P}_f$, are *even*, i.e., $\rho_\Lambda \circ \sigma|_{\mathcal{U}_\Lambda} = \rho_\Lambda$ with $\sigma|_{\mathcal{U}_\Lambda}$ being the restriction to $\mathcal{U}_\Lambda$ of the unique $*$ -automorphism σ of $\mathcal{U}$ satisfying (44). It means that ρ_Λ vanishes on all *odd* monomials in $\{a_{x,S}, a_{x,S}^*\}_{x \in \Lambda, S \in \mathcal{S}}$. For any $\Lambda \in \mathcal{P}_f$, we define

$$E_\Lambda^+ := \{\rho_\Lambda \in E_\Lambda : \rho_\Lambda \circ \sigma|_{\mathcal{U}_\Lambda} = \rho_\Lambda\} \subseteq E_\Lambda \quad (67)$$

as being the space of all finite volume even states.

A.2.2. Infinite volume state spaces

For the infinite system, let $\mathcal{U}^* \equiv \mathcal{U}_{\mathbb{Z}^d}^*$ be the dual space of $\mathcal{U} \equiv \mathcal{U}_{\mathbb{Z}^d}$. In contrast with $\mathcal{U}_\Lambda$, $\Lambda \in \mathcal{P}_f$, $\mathcal{U}$ has infinite dimension and the natural topology on $\mathcal{U}^*$ is the weak* topology³⁴ (and not the norm topology). Thus, the topology of $\mathcal{U}^*$ considered here is always the weak* topology and, in this case, $\mathcal{U}^*$ is a locally convex space, by [69, Theorem 3.10].

Similar to (65), the state space on $\mathcal{U}$ is defined by

$$E \equiv E_{\mathbb{Z}^d} := \{\rho \in \mathcal{U}^* : \rho \geq 0, \rho(1) = 1\}. \quad (68)$$

It is a metrizable, convex and compact subset of $\mathcal{U}^*$, by [69, Theorems 3.15 and 3.16]. It is also the state space of the classical dynamics we define in [44]. By the Krein–Milman theorem [69, Theorem 3.23], E is the closure of the convex hull of the (non-empty) set of its extreme points, which are meanwhile dense in E , by [46, Eq. (13)].

As explained below Eq. (45), recall that the C^* -algebra $\mathcal{U}^+$ should be considered as more fundamental than $\mathcal{U}$ in physics, because of the principle of locality in quantum field theory. As a consequence, states on the C^* -algebra $\mathcal{U}^+$ should be seen as being the physically relevant ones. The set of states on $\mathcal{U}^+$ is canonically identified with the metrizable, convex and compact set of *even* states defined by

$$E^+ \equiv E_{\mathbb{Z}^d}^+ := \{\rho \in \mathcal{U}^* : \rho \geq 0, \rho(1) = 1, \rho \circ \sigma = \rho\}, \quad (69)$$

σ being the unique $*$ -automorphism of $\mathcal{U}$ satisfying (44). This space and the full state space, E , are equivalent convex topological spaces, i.e., there is an affine homeomorphism between them. In particular, E^+ is the closure of the convex hull of the (non-empty) set of its extreme points, which are dense in E^+ . See [46, Proposition 2.1] and its proof.

Note that the spaces E and E^+ , having a dense set of extreme points – i.e., having a dense extreme boundary – has a much more peculiar structure than the finite volume state space E_Λ for $\Lambda \in \mathcal{P}_f$. At first glance, it may look very strange for a non-expert on convex analysis, but it should not be that surprising: For instance, the unit ball of any infinite-dimensional Hilbert space has clearly a dense extreme boundary in the weak topology. In fact, the existence of convex compact sets with

³⁴ Recall that the weak* topology of $\mathcal{U}^*$ is the coarsest topology on $\mathcal{U}^*$ that makes the mapping $\rho \mapsto \rho(A)$ continuous for every $A \in \mathcal{U}$. See [69, Sections 3.8, 3.10, 3.14] for more details.

dense extreme boundary is *not an accident* in infinite-dimensional spaces, but rather generic, in the topological sense. This has been first proven [70] in 1959 for convex norm-compact sets within a separable Banach space. Recently, in [44, Section 2.3] and more generally in [68], the property of having dense extreme boundary is proven to be generic for weak*-compact convex sets within the dual space of an infinite-dimensional topological space. As a matter of fact, all state spaces of infinite volume systems one is going to encounter in the current paper have a dense extreme boundary, except the set of permutation-invariant states described in Appendix A.4, because these can be reduced to states of finite-dimensional matrix algebras, via de Finetti-type results.

A.2.3. Periodic state spaces

Consider the subgroups $(\mathbb{Z}_{\vec{\ell}}^d, +) \subseteq (\mathbb{Z}^d, +)$, $\vec{\ell} \in \mathbb{N}^d$, where

$$\mathbb{Z}_{\vec{\ell}}^d := \ell_1 \mathbb{Z} \times \cdots \times \ell_d \mathbb{Z}.$$

At fixed $\vec{\ell} \in \mathbb{N}^d$, a state $\rho \in E$ satisfying $\rho \circ \alpha_x = \rho$ for all $x \in \mathbb{Z}_{\vec{\ell}}^d$ is called $\mathbb{Z}_{\vec{\ell}}^d$ -invariant on $\mathcal{U}$ or $\vec{\ell}$ -periodic, α_x being the unique $*$ -automorphism of $\mathcal{U}$ satisfying (43). Translation-invariant states refer to $(1, \dots, 1)$ -periodic states. For any $\vec{\ell} \in \mathbb{N}^d$, let

$$E_{\vec{\ell}} := \left\{ \rho \in E : \rho \circ \alpha_x = \rho \quad \text{for all } x \in \mathbb{Z}_{\vec{\ell}}^d \right\} \quad (70)$$

be the space of $\vec{\ell}$ -periodic states. By [71, Lemma 1.8], periodic states are always even and, by [46, Proposition 2.3], the set of all periodic states

$$E_p := \bigcup_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}} \subseteq E^+ \quad (71)$$

is dense in the space E^+ of even states.

For each $\vec{\ell} \in \mathbb{N}^d$, $E_{\vec{\ell}}$ is metrizable, convex and compact and, by the Krein–Milman theorem [69, Theorem 3.23], it is the closure of the convex hull of the (non-empty) set $\mathcal{E}_{\vec{\ell}}$ of its extreme points, which is a G_δ subset of $E_{\vec{\ell}}$ (in particular it is Borel measurable). In fact, by [71, Theorem 1.9] (which uses the Choquet theorem [72, p. 14]), for any $\rho \in E_{\vec{\ell}}$, there is a unique probability measure μ_ρ on $E_{\vec{\ell}}$ with support in $\mathcal{E}_{\vec{\ell}}$ such that³⁵

$$\rho = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho} \, d\mu_\rho(\hat{\rho}). \quad (72)$$

The set $\mathcal{E}_{\vec{\ell}}$ can be characterized by an ergodicity property of states, see [71, Theorem 1.16]. Moreover, $\mathcal{E}_{\vec{\ell}}$ is dense in $E_{\vec{\ell}}$, by [71, Corollary 4.6]. In other words, like the sets E and E^+ , $E_{\vec{\ell}}$ has dense extreme boundary for any $\vec{\ell} \in \mathbb{N}^d$.

A.3. Long-range dynamics

A.3.1. Self-consistency equations

Generically, as already discussed in the main part of the paper, mean-field dynamics in infinite volume are intricate combinations of a classical and quantum dynamics. Similar to [44, Theorem 4.1], both dynamics are related the existence of a solution to a (dynamical) *self-consistency equation*. In order to present such equations we need some preliminary definitions: For $\vec{\ell} \in \mathbb{N}^d$, $\mathfrak{m} = (\Phi, \mathfrak{a}) \in \mathcal{M}$ and $\rho \in E$, we define the approximating (self-adjoint, short-range) interaction $\Phi^{(\mathfrak{m}, \rho)} \in \mathcal{W}^{\mathbb{R}}$ by

$$\Phi^{(\mathfrak{m}, \rho)} := \Phi + \sum_{n \in \mathbb{N}} \int_{\mathbb{S}^n} \mathfrak{a}_n (d\psi^{(1)}, \dots, d\psi^{(n)}) \sum_{m=1}^n \psi^{(m)} \prod_{j \in \{1, \dots, n\}, j \neq m} \rho(\mathfrak{e}_{\psi^{(j)}, \vec{\ell}}), \quad (73)$$

³⁵ The integral in (72) means that $\rho \in E_{\vec{\ell}}$ is the (unique) barycenter of the normalized positive Borel regular measure μ_ρ on $E_{\vec{\ell}}$. See, e.g., [71, Definition 10.15, Theorem 10.16, and also Lemma 10.17].

where

$$\mathfrak{e}_{\phi, \vec{\ell}} := \frac{1}{\ell_1 \cdots \ell_d} \sum_{x=(x_1, \dots, x_d), x_i \in \{0, \dots, \ell_i - 1\}} \sum_{\Lambda \in \mathcal{P}_f, \Lambda \ni x} \frac{\Phi_\Lambda}{|\Lambda|}. \quad (74)$$

Recall meanwhile that $\mathcal{M}^\infty := \mathcal{W}^\mathbb{R} \times \mathcal{S}^\infty$, see (60)–(61). Then, by [46, Theorem 6.5], for any $\mathfrak{m} \in \mathcal{M}^\infty$, there is a unique continuous³⁶ mapping $\varpi^\mathfrak{m}$ from $\mathbb{R}$ to the space of automorphisms³⁷ (or self-homeomorphisms) of E such that

$$\varpi^\mathfrak{m}(t; \rho) = \rho \circ \tau_{t,0}^{\psi^{(\mathfrak{m}, \rho)}}, \quad t \in \mathbb{R}, \rho \in E, \quad (75)$$

with $\psi^{(\mathfrak{m}, \rho)} \in C(\mathbb{R}; \mathcal{W}^\mathbb{R})$, $\rho \in E$, defined by

$$\psi^{(\mathfrak{m}, \rho)}(t) := \phi^{(\mathfrak{m}, \varpi^\mathfrak{m}(t; \rho))}, \quad t \in \mathbb{R}, \quad (76)$$

and where the strongly continuous two-parameter family $(\tau_{t,s}^{\psi^{(\mathfrak{m}, \rho)}})_{s,t \in \mathbb{R}}$ is the strong limit, at any fixed $s, t \in \mathbb{R}$, of the local dynamics $(\tau_{t,s}^{(L, \psi^{(\mathfrak{m}, \rho)})})_{s,t \in \mathbb{R}}$ defined by (53) for $\Psi = \psi^{(\mathfrak{m}, \rho)}$, see (55) and [67, Corollary 5.2]. Eq. (75) is named here the (dynamical) self-consistency equation.

A.3.2. Quantum part of mean-field dynamics

Recall that any model $\mathfrak{m} \in \mathcal{M}$ leads to finite volume dynamics $(\tau_t^{(L, \mathfrak{m})})_{t \in \mathbb{R}}$, $L \in \mathbb{N}_0$, defined by (64). Therefore, at $L \in \mathbb{N}_0$, the time-evolution $(\rho_t^{(L)})_{t \in \mathbb{R}}$ of any finite volume state $\rho^{(L)} \in E_{\Lambda_L}$ is given by

$$\rho_t^{(L)} := \rho^{(L)} \circ \tau_t^{(L, \mathfrak{m})}. \quad (77)$$

The corresponding time-dependent density matrix is $d_t^{(L)} = \tau_{-t}^{(L, \mathfrak{m})}(d^{(L)})$. Eq. (77) refers to the Schrödinger picture of quantum mechanics.

As already mentioned, for any fixed $A \in \mathcal{U}$ and $t \in \mathbb{R}$, the thermodynamic limit $L \rightarrow \infty$ of $\tau_t^{(L, \mathfrak{m})}(A)$ does not necessarily exist in $\mathcal{U}$, but the limit $L \rightarrow \infty$ of $\rho_t^{(L)}$ can still make sense: Fix once and for all a translation-invariant model $\mathfrak{m} \in \mathcal{M}_1^\infty$, see (61). Take $\vec{\ell} \in \mathbb{N}^d$ and recall that $E_{\vec{\ell}}$ is the space of $\vec{\ell}$ -periodic states defined by (70), whose set of extreme points is denoted $\mathcal{E}_{\vec{\ell}}$. Recall also (72), i.e., that, for any $\rho \in E_{\vec{\ell}}$, there is a unique probability measure μ_ρ on $E_{\vec{\ell}}$ with support in $\mathcal{E}_{\vec{\ell}}$ such that

$$\rho = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho} \, d\mu_\rho(\hat{\rho}).$$

From the fact that the set $\mathcal{E}_{\vec{\ell}}$ is characterized by an ergodicity property (see [71, Theorem 1.16]), one can prove that, for any $A \in \mathcal{U}$,

$$\lim_{L \rightarrow \infty} \rho \circ \tau_t^{(L, \mathfrak{m})}(A) = \int_{\mathcal{E}_{\vec{\ell}}} \varpi^\mathfrak{m}(t; \hat{\rho})(A) \, d\mu_\rho(\hat{\rho}) = \int_{\mathcal{E}_{\vec{\ell}}} \hat{\rho} \circ \tau_{t,0}^{\psi^{(\mathfrak{m}, \hat{\rho})}}(A) \, d\mu_\rho(\hat{\rho}), \quad (78)$$

where $\varpi^\mathfrak{m}$ is the solution to the self-consistency equation (75). See [47, Proposition 4.2, Theorem 4.3]. Using in particular, for any $L \in \mathbb{N}_0$, the restriction $\rho^{(L)} := \rho|_{\mathcal{U}_{\Lambda_L}}$ of a state $\rho \in E_{\vec{\ell}}$ to $\mathcal{U}_{\Lambda_L}$ then (78) can also be seen as the thermodynamic limit $L \rightarrow \infty$ of the expectation value $\rho_t^{(L)}(A)$ of any local element $A \in \mathcal{U}_0$, the time-dependent state $\rho_t^{(L)}$ been defined by (77).

Eq. (78) means in fact that the thermodynamic limit $L \rightarrow \infty$ of $\tau_t^{(L, \mathfrak{m})}(A)$ exists in the GNS representation³⁸ $(\mathcal{H}_\rho, \pi_\rho, \Omega_\rho)$ of $\mathcal{U}$ associated with the initial state ρ . More precisely, one obtains

³⁶ We endow the set $C(E; E)$ of continuous functions from E to itself with the topology of uniform convergence. See [46, Eq. (100)] for more details.

³⁷ I.e., elements of $C(E; E)$ with continuous inverse.

³⁸ Recall that $\mathcal{H}_\rho$ is an Hilbert space, $\pi_\rho : \mathcal{U} \rightarrow \mathcal{B}(\mathcal{H}_\rho)$ is a representation of $\mathcal{U}$ and $\Omega_\rho \in \mathcal{H}_\rho$ is a cyclic vector for $\pi_\rho(\mathcal{U})$.

a dynamics $(T_t^m)_{t \in \mathbb{R}}$ defined by

$$T_t^m \circ \pi_\rho(A) = \lim_{L \rightarrow \infty} \pi_\rho \circ \tau_t^{(L,m)}(A), \quad A \in \mathcal{U},$$

on the (von Neumann) subalgebra $\pi_\rho(\mathcal{U})''$ of the algebra $\mathcal{B}(\mathcal{H}_\rho)$ of bounded operators on the Hilbert space $\mathcal{H}_\rho$. The above limit has to be understood in the σ -weak topology within $\mathcal{B}(\mathcal{H}_\rho)$ (and in many cases one could even prove strong convergence). This refers to the quantum part of the mean-field dynamics (in some representation), which is generally *non-autonomous*, although the primordial local dynamics is autonomous.

A.3.3. Classical part of mean-field dynamics

For any $\vec{\ell} \in \mathbb{N}^d$, the infinite volume mean-field dynamics of $\vec{\ell}$ -periodic states, as given by (78), involves the knowledge of a continuous flow³⁹ on $\mathcal{E}_{\vec{\ell}}$. Seeing $\mathcal{E}_{\vec{\ell}}$ or $E_{\vec{\ell}} = \overline{\mathcal{E}_{\vec{\ell}}}$ as a (classical) phase space, it becomes natural to study the classical Hamiltonian dynamics associated with this flow, as is usual in classical mechanics. Note that, for a (possibly non-translation-invariant) model $m \in \mathcal{M}^\infty$, for any $t \in \mathbb{R}$, $\varpi^m(t; \cdot)$ preserves the space E^+ of even states defined by (69), but not necessarily $E_{\vec{\ell}}$. If $m \in \mathcal{M}_1^\infty$ then, for any $\vec{\ell} \in \mathbb{N}^d$, the flow lets the sets $\mathcal{E}_{\vec{\ell}}$ and $E_{\vec{\ell}}$ invariant. See (87). Here, we adopt a broader perspective by taking the full state space E , defined by (68), because the classical dynamics described below can be easily pushed forward, through the restriction map, from $C(E; \mathbb{C})$ to $C(E^+; \mathbb{C})$ for general $m \in \mathcal{M}^\infty$, and also to $C(E_{\vec{\ell}}; \mathbb{C})$ for any $\vec{\ell} \in \mathbb{N}^d$, when $m \in \mathcal{M}_1^\infty$ is translation-invariant.

Note that $C(E; \mathbb{C})$, $C(E^+; \mathbb{C})$ and $C(E_{\vec{\ell}}; \mathbb{C})$, endowed with the point-wise operations and complex conjugation as well as the supremum norm, are unital commutative C^* -algebras. For any model $m \in \mathcal{M}^\infty$, the mapping ϖ^m , the solution to the self-consistency equation (75), yields a family $(V_t^m)_{t \in \mathbb{R}}$ of $*$ -automorphisms on $C(E; \mathbb{C})$ defined by

$$V_t^m(f) := f \circ \varpi^m(t), \quad f \in C(E; \mathbb{C}), \quad t \in \mathbb{R}. \quad (79)$$

It is a Feller group: $(V_t^m)_{t \in \mathbb{R}}$ is a strongly continuous group of $*$ -automorphisms of $C(E; \mathbb{C})$, which is obviously positivity preserving and has operator norm equal to one. See [46, Proposition 6.8]. When restricted to the dense subspace $E_p \subseteq E^+$ (71) of all periodic states, the ones we are interested in (cf. (78)), for any translation-invariant model $m \in \mathcal{M}_1^\infty$, the one-parameter group $(V_t^m)_{t \in \mathbb{R}}$ is generated by a Poissonian symmetric derivation:

(i) *Local polynomials*: Elements of the C^* -algebra $\mathcal{U}$ naturally define continuous and affine functions $\hat{A} \in C(E; \mathbb{C})$ by

$$\hat{A}(\rho) := \rho(A), \quad \rho \in E, \quad A \in \mathcal{U}.$$

This is the well-known Gelfand transform. Recall that $\mathcal{U}_0$ is the normed $*$ -algebra of local elements of $\mathcal{U}$ defined by (42). We denote by

$$\mathbb{P} := \mathbb{C}[\{\hat{A} : A \in \mathcal{U}_0\}] \subseteq C(E; \mathbb{C}) \quad (80)$$

the subspace of (local) polynomials in the elements of $\{\hat{A} : A \in \mathcal{U}_0\}$, with complex coefficients.

(ii) *Poisson structure*: For any $n \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{U}$ and $g \in C^1(\mathbb{R}^n, \mathbb{C})$ we define the function $\Gamma_g \in C(E; \mathbb{C})$ by

$$\Gamma_g(\rho) := g(\rho(A_1), \dots, \rho(A_n)), \quad \rho \in E. \quad (81)$$

Functions of this type are known in the literature as cylindrical functions. For such a function and any $\rho \in E$, define

$$D\Gamma_g(\rho) := \sum_{j=1}^n (A_j - \rho(A_j) \mathbf{1}) \partial_{x_j} g(\rho(A_1), \dots, \rho(A_n)) \quad (82)$$

³⁹ That is, the continuous mapping ϖ^m from $\mathbb{R}$ to the space of automorphisms (or self-homeomorphisms) of E defined by (75).

for any $\rho \in E$. This definition comes from a notion, introduced by us, of a convex weak*-continuous Gateaux derivative, as explained in [46, Section 5.2]. Then, for any $n, m \in \mathbb{N}$, $A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{U}$, $g \in C^1(\mathbb{R}^n, \mathbb{C})$ and $h \in C^1(\mathbb{R}^m, \mathbb{C})$, we define the continuous function $\{\Gamma_h, \Gamma_g\} \in C(E; \mathbb{C})$ by

$$\{\Gamma_h, \Gamma_g\}(\rho) := \rho(i[D\Gamma_h(\rho), D\Gamma_g(\rho)]) \quad (83)$$

for any $\rho \in E$, where $A_1, \dots, A_n \in \mathcal{U}$ and $B_1, \dots, B_m$ respectively determine Γ_g and Γ_h via (81). This defines a Poisson bracket on the space $\mathbb{P}$ of all (local) polynomial functions acting on E . By construction, for any $\ell \in \mathbb{N}^d$,

$$\begin{aligned} \{\Gamma_h|_{E^+}, \Gamma_g|_{E^+}\} &:= \{\Gamma_h, \Gamma_g\}|_{E^+} \\ \{\Gamma_h|_{E_\ell}, \Gamma_g|_{E_\ell}\} &:= \{\Gamma_h, \Gamma_g\}|_{E_\ell} \\ \{\Gamma_h|_{\mathcal{E}_\ell}, \Gamma_g|_{\mathcal{E}_\ell}\} &:= \{\Gamma_h, \Gamma_g\}|_{\mathcal{E}_\ell} \end{aligned} \quad (84)$$

also define a Poisson bracket on polynomials of $C(E^+; \mathbb{C})$, $C(E_\ell; \mathbb{C})$ and $C(\mathcal{E}_\ell; \mathbb{C})$, respectively. This definition can be extended to the space

$$\mathfrak{Y} \equiv C^1(E; \mathbb{C}) \subseteq C(E; \mathbb{C})$$

of continuously differentiable functions. See [46, Section 5.2] and [44, Section 3] for a more detailed construction of such Poisson structures.

(iii) *Liouville's equation*: Local classical energy functions [46, Definition 6.9] associated with $m \in \mathcal{M}$ are defined, for any $L \in \mathbb{N}_0$, by

$$h_L^m := \widehat{U_L^\phi} + \sum_{n \in \mathbb{N}} \frac{1}{|A_L|^{n-1}} \int_{S^n} \widehat{U_L^{\psi^{(1)}}} \cdots \widehat{U_L^{\psi^{(n)}}} a_n(d\psi^{(1)}, \dots, d\psi^{(n)}). \quad (85)$$

Note that $h_L^m \in C^1(E; \mathbb{C})$. Compare with the local Hamiltonian U_L^m defined by (62). Then, by [46, Corollary 6.12], for each translation-invariant model $m \in \mathcal{M}_1^\infty$, any time $t \in \mathbb{R}$ and all local polynomials $f \in \mathbb{P}$, one has $V_t^m(f) \in C^1(E; \mathbb{C})$ and

$$\partial_t V_t^m(f) = V_t^m\left(\lim_{L \rightarrow \infty} \{h_L^m, f\}\right) = \lim_{L \rightarrow \infty} \{h_L^m, V_t^m(f)\}, \quad (86)$$

where all limits have to be understood point-wise on the dense subspace $E_p \subseteq E^+$ of all periodic states. We thus obtain the usual (autonomous) dynamics of classical mechanics written in terms of Poisson brackets. See, e.g., [61, Proposition 10.2.3]. This corresponds to *Liouville's equation*.

By [46, p. 34, e.g., Eq. (114)], observe additionally that, for any $m \in \mathcal{M}_1^\infty$ and $\ell \in \mathbb{N}^d$, the flow preserves the sets E^+ , E_ℓ and $\mathcal{E}_\ell$, i.e.,

$$\begin{aligned} \bigcup_{t \in \mathbb{R}} \varpi^m(t; E^+) &\subseteq E^+ \\ \bigcup_{t \in \mathbb{R}} \varpi^m(t; E_\ell) &\subseteq E_\ell \\ \bigcup_{t \in \mathbb{R}} \varpi^m(t; \mathcal{E}_\ell) &\subseteq \mathcal{E}_\ell. \end{aligned} \quad (87)$$

Therefore, $V_{t,s}^m$ can be seen as a mapping from $C(E^+; \mathbb{C})$, $C(E_\ell; \mathbb{C})$ or $C(\mathcal{E}_\ell; \mathbb{C})$ to itself:

$$\begin{aligned} V_t^m(f|_{E^+}) &:= (V_t^m f)|_{E^+} \\ V_t^m(f|_{E_\ell}) &:= (V_t^m f)|_{E_\ell} \\ V_t^m(f|_{\mathcal{E}_\ell}) &:= (V_t^m f)|_{\mathcal{E}_\ell} \end{aligned} \quad (88)$$

for any $t \in \mathbb{R}$, $f \in C(E; \mathbb{C})$, $m \in \mathcal{M}_1^\infty$ and $\vec{\ell} \in \mathbb{N}^d$. By using the Poisson brackets (84), Liouville's equation (86) can be written on $C(E^+; \mathbb{C})$, $C(E_\ell; \mathbb{C})$ or $C(\mathcal{E}_\ell; \mathbb{C})$ for any $m \in \mathcal{M}_1^\infty$ and $\vec{\ell} \in \mathbb{N}^d$.

Remark 3. The mathematically rigorous derivation of Liouville's equation (86) is non-trivial and results from Lieb–Robinson bounds for multi-commutators [67], first derived in 2017.

A.3.4. Entanglement of quantum and classical dynamics

In the thermodynamic limit, the “primordial” algebra is the separable unital C^* -algebra $\mathcal{U}$, generated by fermionic annihilation and creation operators satisfying the canonical anti-commutation relations, as explained in [Appendix A.1.1](#). Fix once and for all $\mathfrak{m} \in \mathcal{M}_1^\infty$. Let $K = E, E^+$ or $E_{\vec{\ell}} = \bar{E}_{\vec{\ell}}$ for some $\vec{\ell} \in \mathbb{N}^d$, which is, in each case, a metrizable, convex (weak*-) compact subset of the dual space $\mathcal{U}^*$.

(i) Classical dynamics. The classical (i.e., commutative) unital C^* -algebra is the algebra $C(K; \mathbb{C})$ of continuous and complex-valued functions on K . The mapping $\varpi^{\mathfrak{m}}$, the solution to the self-consistency equation (75), yields a strongly continuous group $(V_t^{\mathfrak{m}})_{t \in \mathbb{R}}$ of $*$ -automorphisms of $C(K; \mathbb{C})$, satisfying Liouville’s equation as previously explained.

(ii) Quantum dynamics. Similar to quantum–classical hybrid theories of theoretical physics, described for instance in [36–41], consider now a secondary quantum algebra $C(K; \mathbb{C}) \otimes \mathcal{U}$, which is nothing else (up to isomorphism) than the C^* -algebra $C(K, \mathcal{U})$ of all (weak*) continuous $\mathcal{U}$ -valued functions on states. By [46, Proposition 6.2] and (87), the mapping $\varpi^{\mathfrak{m}}$ from $\mathbb{R}$ to the space of automorphisms (or self-homeomorphisms) of K leads to a (state-dependent) quantum dynamics $\mathfrak{T}^{\mathfrak{m}} := (\mathfrak{T}_t^{\mathfrak{m}})_{t \in \mathbb{R}}$ on

$$C(K, \mathcal{U}) \equiv C(K; \mathbb{C}) \otimes \mathcal{U},$$

via the strongly continuous, state-dependent two-parameter family $(\tau_{t,s}^{\psi^{(\mathfrak{m}, \rho)}})_{s,t \in \mathbb{R}}$ with $\psi^{(\mathfrak{m}, \rho)}$ defined by (76):

$$[\mathfrak{T}_t^{\mathfrak{m}}(f)](\rho) := \tau_{t,0}^{\psi^{(\mathfrak{m}, \rho)}}(f(\rho)) \quad , \quad \rho \in K,$$

for any function $f \in C(K, \mathcal{U})$ and time $t \in \mathbb{R}$.

(iii) Quantum–classical dynamical entanglement. By following arguments of [44, End of Section 5.2], any (state-dependent) quantum dynamics on $C(K, \mathcal{U})$ letting every single element of $C(K; \mathbb{C}_1) \subseteq C(K, \mathcal{U})$ invariant yields a classical dynamics, which, in the case of $\mathfrak{T}^{\mathfrak{m}}$, is exactly $(V_t^{\mathfrak{m}})_{t \in \mathbb{R}}$. More interestingly, as we remark in [44, Section 4.2], each *classical* Hamiltonian, i.e., a continuously differentiable function of $C(K; \mathbb{R})$, leads to a state-dependent quantum dynamics. If the classical Hamiltonian equals (85) then the limit quantum dynamics, when $L \rightarrow \infty$, is precisely $\mathfrak{T}^{\mathfrak{m}}$. In other words, one can recover the classical dynamics from the quantum one, and vice versa. The classical and quantum systems are completely interdependent, i.e., *entangled*. This view point is very different from the common understanding⁴⁰ of the relation between quantum and classical mechanics, which is widely seen as a limiting case of quantum mechanics, even if there exist physical features (such as the spin of quantum particles) which do not have a clear classical counterpart.

The physical relevance of the mathematical framework we present here comes from the fact that it is able to encode the infinite volume dynamics of very general mean-field models, for initial states which are only required to be periodic in space. In fact, the classical part of the mean-field dynamics explicitly appears in the time evolution of extreme periodic states in (78), while the quantum part corresponds to the last integral over extreme states of (78). The fact that the initial state must be a periodic state does not represent a serious constraint since any initial even state ρ can be approximated by a periodic state constructed⁴¹ from its restriction $\rho|_{\mathcal{U}_{\Lambda_l}}$ to $\mathcal{U}_{\Lambda_l}$ for sufficiently large $l \in \mathbb{N}_0$. See, e.g., [46, Proof of Proposition 2.3]. Since $l \in \mathbb{N}_0$ is arbitrarily large, hence there is no real physical restriction in assuming that the initial state is a periodic one, noting that the physical states of fermion systems are always even⁴².

⁴⁰ At least in many textbooks on quantum mechanics. See for instance [7, Section 12.4.2, end of the 4th paragraph of page 178].

⁴¹ This is possible because of [73, Theorem 11.2].

⁴² If the initial state is not even, we cannot a priori construct a periodic state from its restriction $\rho|_{\mathcal{U}_{\Lambda}}$ for any $\Lambda \in \mathcal{P}_f$.

A.4. Permutation-invariant lattice fermi systems

A.4.1. Permutation-invariant mean-field models

Recall that $\mathcal{W}_\Pi := \mathcal{W}^0$ is the space of permutation-invariant (or on-site) interactions, defined by Eq. (48) for $R = 0$. Define

$$\mathcal{M}_\Pi := (\mathcal{W}^\mathbb{R} \cap \mathcal{W}_\Pi) \times \mathcal{S}^0. \quad (89)$$

We name it the space of permutation-invariant mean-field models, because all associated local Hamiltonians are invariant under permutations: Let Π be the set of all bijective mappings from $\mathbb{Z}^d$ to itself which leave all but finitely many elements invariant. It is a group with respect to the composition of mappings. The condition

$$p_\pi : a_{x,s} \mapsto a_{\pi(x),s}, \quad x \in \mathbb{Z}^d, s \in S, \quad (90)$$

defines a group homomorphism $\pi \mapsto p_\pi$ from Π to the group of $*$ -automorphisms of the C^* -algebra $\mathcal{U}$. Then, for any $m \in \mathcal{M}_\Pi$ and $L \in \mathbb{N}_0$, the local Hamiltonian U_L^m defined by (62) is permutation-invariant, that is,

$$p_\pi(U_L^m) = U_L^m, \quad \pi \in \Pi, \quad \pi(\Lambda_L) = \Lambda_L. \quad (91)$$

An example of permutation-invariant model is given by the strong-coupling BCS-Hubbard model: Fix $S = \{\uparrow, \downarrow\}$. Let $\Phi^{Hubb}, \Psi^{BCS} \in \mathcal{W}_\Pi \cap \mathcal{W}^\mathbb{R}$ be defined by

$$\begin{aligned} \Phi_{\{x\}}^{Hubb} &:= -\mu(n_{x,\uparrow} + n_{x,\downarrow}) - h(n_{x,\uparrow} - n_{x,\downarrow}) + 2\lambda n_{x,\uparrow} n_{x,\downarrow} \\ \Psi_{\{x\}}^{BCS} &:= a_{x,\downarrow} a_{x,\uparrow} \end{aligned}$$

for $x \in \mathbb{Z}^d$ and $\Phi_\Lambda^{Hubb} := 0 =: \Psi_\Lambda^{BCS}$ otherwise. Let $\alpha^{BCS} \in \mathcal{S}^0$ be defined, for all Borel subset $\mathfrak{B} \subseteq \mathcal{S}$, by

$$\alpha^{BCS}(\mathfrak{B}) := -\gamma \mathbf{1}[\Psi^{BCS} \in \mathfrak{B}]. \quad (92)$$

for some $\gamma \geq 0$, with $\mathbf{1}[\cdot]$ being the indicator function⁴³. Then,

$$m_0 := (\Phi^{Hubb}, \alpha^{BCS}) \in \mathcal{M}_\Pi$$

is the strong-coupling BCS-Hubbard model since, in this case, the local Hamiltonian $U_L^{m_0}$ is equal to the strong-coupling BCS-Hubbard Hamiltonian H_L defined by (1).

A.4.2. Permutation-invariant state space

The set of all permutation-invariant states is defined by

$$E_\Pi := \{\rho \in E : \rho = \rho \circ p_\pi \text{ for all } \pi \in \Pi\}, \quad (93)$$

p_π being the unique $*$ -automorphism of $\mathcal{U}$ satisfying (90). Obviously,

$$E_\Pi \subseteq \bigcap_{\vec{\ell} \in \mathbb{N}^d} E_{\vec{\ell}} \subseteq E^+.$$

Furthermore, E_Π is metrizable, convex and compact and, by [71, Theorem 5.3], for any $\rho \in E_\Pi$, there is a unique probability measure μ_ρ on E_Π with support in the (non-empty) set $\mathcal{E}_\Pi$ of its extreme points such that

$$\rho = \int_{\mathcal{E}_\Pi} \hat{\rho} \, d\mu_\rho(\hat{\rho}). \quad (94)$$

The set $\mathcal{E}_\Pi$ can be characterized by a version of the Størmer theorem for permutation-invariant states on the C^* -algebra $\mathcal{U}$. This theorem is a non-commutative version of the celebrated de Finetti theorem of (classical) probability theory. It is proven in the case of quantum-spin systems in [74]

⁴³ $\mathbf{1}[p] = 1$ if the proposition p holds true and $\mathbf{1}[p] = 0$ otherwise.

and for the fermion algebra $\mathcal{U}$ in [51, Lemmata 6.6–6.8]. It asserts that extreme permutation-invariant states $\rho \in \mathcal{E}_\Pi$ are product states defined as follows: First recall that the space E_Λ^+ of finite volume even states is defined by (67) for any $\Lambda \in \mathcal{P}_f$. Then, via [73, Theorem 11.2], for any $\rho_0 \in E_{\{0\}}^+$, there is a unique even state

$$\rho := \otimes_{\mathbb{Z}^d} \rho_0 \in E^+ \quad (95)$$

satisfying

$$\rho(\alpha_{x_1}(A_1) \cdots \alpha_{x_n}(A_n)) = \rho_0(A_1) \cdots \rho_0(A_n) \quad (96)$$

for all $A_1 \dots A_n \in \mathcal{U}_{\{0\}}$ and all $x_1, \dots, x_n \in \mathbb{Z}^d$ such that $x_i \neq x_j$ for $i \neq j$. Recall that $\alpha_x, x \in \mathbb{Z}^d$, defined by (43), are the $*$ -automorphisms of $\mathcal{U}$ that represent translations. The set of all states of the form (95), called *product states*, is denoted by $E_\otimes$. It is nothing else but the set $\mathcal{E}_\Pi$ of extreme points of E_Π , i.e.,

$$E_\otimes = \mathcal{E}_\Pi. \quad (97)$$

This identity refers to the Størmer theorem, see, e.g., [71, Theorem 5.2].

Since product states are particular extreme states⁴⁴ of $E_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$, it follows from (97) that

$$\mathcal{E}_\Pi = E_\otimes \subseteq \bigcap_{\vec{\ell} \in \mathbb{N}^d} \mathcal{E}_{\vec{\ell}} \quad (98)$$

and the set $E_\Pi \subseteq E_{\vec{\ell}}$ is thus a closed metrizable face⁴⁵ of $E_{\vec{\ell}}$. For a more thorough exposition on this subject, see [71, Section 5.1]. By (97), the extreme boundary $\mathcal{E}_\Pi$ of E_Π is also closed and, in contrast with E, E^+ and $E_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$, $\mathcal{E}_\Pi$ is *not* a dense subset of E_Π . This is not surprising since states of $\mathcal{E}_\Pi = E_\otimes$ are in one-to-one correspondence with even states on the finite-dimensional C^* -algebra $\mathcal{U}_{\{0\}}$.

A.4.3. Quantum part of permutation-invariant mean-field dynamics

Fix once and for all $m \in \mathcal{M}_\Pi$. If $\rho \in E_1 := E_{(1, \dots, 1)}$, i.e., it is translation-invariant, then the approximating interaction (73) satisfies

$$\Phi^{(m, \rho)} = \Phi^{(m, \rho|_{\mathcal{U}_{\{0\}}})} \in \mathcal{W}_\Pi \cap \mathcal{W}^{\mathbb{R}} \quad (99)$$

and the infinite volume dynamics constructed from this interaction, as defined by (55), preserves the local C^* -algebra $\mathcal{U}_\Lambda$ for any $\Lambda \in \mathcal{P}_f$. By (53)–(55) and (75)–(76), it also follows that

$$\begin{aligned} \bigcup_{t \in \mathbb{R}} \omega^m(t; E_\Pi) &\subseteq E_\Pi \subseteq E_1 \\ \bigcup_{t \in \mathbb{R}} \omega^m(t; E_\otimes) &\subseteq E_\otimes \subseteq E_\Pi \end{aligned} \quad (100)$$

(compare with (87)) and, for any $\Lambda \in \mathcal{P}_f$, $t \in \mathbb{R}$ and translation-invariant state $\rho \in E_1 \supseteq E_\Pi$,

$$\omega^m(t; \rho)|_{\mathcal{U}_\Lambda} = \omega^m(t; \rho|_{\mathcal{U}_\Lambda})|_{\mathcal{U}_\Lambda} \in E_\Lambda^+ \quad (101)$$

with E_Λ^+ being the space of finite volume even states defined by (67) for any $\Lambda \in \mathcal{P}_f$.

If the initial state $\rho \in E_\Pi$ is permutation-invariant, then, by (78), (94) and (98), there is a unique probability measure μ_ρ on E_Π with support in $\mathcal{E}_\Pi = E_\otimes$ such that, for any $A \in \mathcal{U}$,

$$\lim_{L \rightarrow \infty} \rho \circ \tau_t^{(L, m)}(A) = \int_{E_\otimes} \omega^m(t; \hat{\rho})(A) \, d\mu_\rho(\hat{\rho}) = \int_{E_\otimes} \hat{\rho} \circ \tau_{t,0}^{\psi(m, \hat{\rho})}(A) \, d\mu_\rho(\hat{\rho}) \quad (102)$$

⁴⁴ By [71, Theorem 5.2], all product states are strongly mixing, which means [71, Eq. (1.10)]. They are, in particular, strongly clustering and thus ergodic with respect to any sub-groups $(\mathbb{Z}_{\vec{\ell}}^d, +) \subseteq (\mathbb{Z}^d, +)$, where $\vec{\ell} \in \mathbb{N}^d$. By [71, Theorem 1.16], all product states belong to $\mathcal{E}_{\vec{\ell}}$ for any $\vec{\ell} \in \mathbb{N}^d$.

⁴⁵ A face F of a convex set K is defined to be a subset of K with the property that, if $\rho = \lambda_1 \rho_1 + \dots + \lambda_n \rho_n \in F$ with $\rho_1, \dots, \rho_n \in K$, $\lambda_1, \dots, \lambda_n \in (0, 1)$ and $\lambda_1 + \dots + \lambda_n = 1$, then $\rho_1, \dots, \rho_n \in F$.

with ϖ^m being the solution to the self-consistency equation (75). In particular, by (100), the time-evolution of a permutation-invariant state is uniquely determined by its restriction to the finite-dimensional subalgebra $\mathcal{U}_{\{0\}}$ (dimension $2^{2|S|}$).

If the initial state $\rho \in E_1 \supseteq E_\Pi$ is translation-invariant, then Eq. (78) restricted to the finite-dimensional C^* -algebra $\mathcal{U}_\Lambda$ with $\Lambda \in \mathcal{P}_f$ reads⁴⁶

$$\lim_{L \rightarrow \infty} \rho|_{\mathcal{U}_\Lambda} \circ \tau_t^{(L,m)}(A) = \int_{E_1^+} \varpi^m(t; \hat{\rho})(A) \, d\mu_\rho(\hat{\rho}) \quad (103)$$

for any $A \in \mathcal{U}_\Lambda$. For each fixed $\Lambda \in \mathcal{P}_f$, this gives now a family of equations on the finite-dimensional algebra $\mathcal{U}_\Lambda$ (dimension $2^{2|\Lambda| \times |S|}$). These equations completely determine the time-evolution of a translation-invariant initial states.

For any $\vec{\ell}$ -periodic state $\rho \in E_{\vec{\ell}}$ ($\vec{\ell} \in \mathbb{N}^d$), the approximating interaction (73) also belongs to $\mathcal{W}_\Pi \cap \mathcal{W}^{\mathbb{R}}$. The only difference with respect to translation-invariant states is that the on-site state $\rho|_{\mathcal{U}_{\{0\}}}$ in (99) has to be replaced with the finite volume state $\rho|_{\mathcal{U}_{\mathbb{Z}_{\vec{\ell}}}}$, where, for $\vec{\ell} = (\ell_1, \dots, \ell_d) \in \mathbb{N}^d$,

$$\mathbb{Z}_{\vec{\ell}} := \{(x_1, \dots, x_d) \in \mathbb{Z}^d : x_i \in \{0, \dots, \ell_i - 1\}\}.$$

Compare, as an example, with (32). Hence, if the initial state is periodic then Eq. (78) leads again to a family of equations on the finite-dimensional algebra $\mathcal{U}_\Lambda$ (dimension $2^{2|\Lambda| \times |S|}$) for each $\Lambda \in \mathcal{P}_f$ such that⁴⁷ $\Lambda \supseteq \mathbb{Z}_{\vec{\ell}}$. These equations again determine the time-evolution of a periodic initial state.

A.4.4. Classical part of permutation-invariant mean-field dynamics

Fix again once and for all $m \in \mathcal{M}_\Pi$. By (100), the strongly continuous group $(V_t^m)_{t \in \mathbb{R}}$ of $*$ -automorphisms defined by (79) can be restricted to the unital C^* -algebra $C(E_\otimes; \mathbb{C})$ of continuous functions on the compact space $E_\otimes$ of product states. See also [44, Section 5.4 with $\mathcal{B} = \mathcal{U}_{\{0\}}$]. Without any risk of confusion, we denote the restriction of $(V_t^m)_{t \in \mathbb{R}}$ to $E_\otimes$ again by $(V_t^m)_{t \in \mathbb{R}}$.

Using (95)–(97) we identify $E_\otimes$ with the space $E_{\{0\}}^+$ of on-site even states and see now $(V_t^m)_{t \in \mathbb{R}}$ as acting on the algebra $C(E_{\{0\}}^+; \mathbb{C})$. Similar to (80), the set of polynomials in this space of functions is denoted by

$$\mathbb{P}_{\{0\}} := \mathbb{C}[\{\hat{A}|_{E_{\{0\}}^+} : A \in \mathcal{U}_{\{0\}}\}] \subseteq C(E_{\{0\}}^+; \mathbb{C}).$$

Local classical energy functions [46, Definition 6.9] on $\mathcal{U}_{\{0\}}$ are defined by $h_0^m|_{E_{\{0\}}^+}$, where, by (49) and (85),

$$h_0^m = \widehat{\Phi}_{\{0\}} + \sum_{n \in \mathbb{N}} \int_{\mathbb{S}^n} \widehat{\Psi}_{\{0\}}^{(1)} \dots \widehat{\Psi}_{\{0\}}^{(n)} a_n(d\psi^{(1)}, \dots, d\psi^{(n)}).$$

Then, for any time $t \in \mathbb{R}$ and polynomials $f \in \mathbb{P}_{\{0\}}$, Liouville's equation (86) restricted to the algebra $C(E_{\{0\}}^+; \mathbb{C})$ equals

$$\partial_t V_t^m(f) = V_t^m(\{h_0^m, f\}) = \{h_0^m, V_t^m(f)\}, \quad (104)$$

where, for any $n, m \in \mathbb{N}$, $A_1, \dots, A_n \in \mathcal{U}$, $B_1, \dots, B_m \in \mathcal{U}$, $g \in C^1(\mathbb{R}^n; \mathbb{C})$ and $h \in C^1(\mathbb{R}^m; \mathbb{C})$,

$$\{\Gamma_h|_{\mathcal{U}_{\{0\}}}, \Gamma_g|_{\mathcal{U}_{\{0\}}}\} := \{\Gamma_h, \Gamma_g\}|_{\mathcal{U}_{\{0\}}} \in C(E_{\{0\}}^+; \mathbb{C}) \quad (105)$$

defines again a Poisson bracket, which can be extended to the space $C^1(E_{\{0\}}^+; \mathbb{C})$ of continuously differentiable functions. Similar to (102), Liouville's equation (104) is now written on the finite-dimensional algebra $\mathcal{U}_{\{0\}}$ (dimension $2^{2|S|}$) and completely determines a continuous flow on the compact space $E_\otimes$ of product states.

⁴⁶ Note that μ_ρ in (78) is a probability measure on $E_1 \subseteq E^+$, but since the restriction mapping $\rho \mapsto \rho|_{\mathcal{U}_\Lambda}$ is continuous for any $\Lambda \in \mathcal{P}_f$, μ_ρ can be pushed forward to a probability measure on E_Λ^+ , which we also denote μ_ρ .

⁴⁷ The restriction $\Lambda \supseteq \mathbb{Z}_{\vec{\ell}}$ can also be easily understood by seeing $\vec{\ell}$ -periodic states as a translation-invariant state on the CAR C^* -algebra with new spin set $\mathbb{Z}_{\vec{\ell}} \times S$.

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