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STRONG ALGEBRABILITY AND RESIDUALITY ON CERTAIN SETS OF ANALYTIC FUNCTIONS

M.L. LOURENÇO AND D.M. VIEIRA

ABSTRACT. We show that the set of analytic functions from $\mathbb{C}^2$ into $\mathbb{C}^2$, which are not Lorch-analytic is spaceable and strongly $\mathfrak{c}$-algebrable, but is not residual in the space of entire functions from $\mathbb{C}^2$ into $\mathbb{C}^2$. We also show that the set of functions which belongs to the disk algebra but not a Dales-Davie algebra is strongly $\mathfrak{c}$-algebrable and is residual in the disk algebra.

1. Introduction. In the last two decades there has been increasing interest in the search of nice algebraic-topological structures within sets (mainly sets of functions or sequences) that do not themselves enjoy such structures. In this note, we study algebraic-topological structures in certain sets of analytic functions.

Now we fix the notation. The space of all analytic functions from $\mathbb{C}^2$ into $\mathbb{C}^2$, endowed with the compact open topology τ_0, will be denoted by $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$. We note that $(\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2), \tau_0)$ is a Fréchet algebra. Consider $\mathbb{C}^2$ as an algebra with the usual product. We denote the set of all (L)-analytic functions from $\mathbb{C}^2$ into $\mathbb{C}^2$ by $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$. The class of (L)-analytic mappings (cf. Definition 2.1) was introduced by E.R. Lorch in **[12]**. We call by $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$. If D denotes the open unit disk on the complex space, then for each specific sequence of positive numbers $M = (M_n)_{n\in\mathbb{N}}$, the set $\mathcal{D}(\bar{D}, M)$ introduced by Dales and Davie **[10]** is a subalgebra of the disk algebra $\mathcal{A}(D)$. As usual, we call these algebras *Dales-Davie algebras* and we write $\mathcal{H}(M) = \mathcal{A}(D) \setminus \mathcal{D}(\bar{D}, M)$.

There is extensive literature on these kind of functions; see for instance **[1, 2, 10, 11, 13]** for Dales-Davie algebras and **[12, 14, 15]** for (L)-analytic mappings.

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As we can see, these are different classes of analytic functions. In our work we are interested to see, in a linear/algebraic and topological sense, whether or not those differences are big. In this direction, our aim in this paper is to establish some structure in the sets $\mathcal{G}$ and $\mathcal{H}(M)$. Indeed, we show that $\mathcal{G}$ is spaceable and strongly $\mathfrak{c}$-algebrable, but is not topologically large, while $\mathcal{H}(M)$ is strongly $\mathfrak{c}$-algebrable and residual, that means it is topologically large. Research on the theme of describing spaceability, algebrability and residuality has been carried on in recent years, see among many others **[3, 4, 5, 6, 7, 8, 9, 13]**.

Next we recall some definitions. Let $\mathcal{B}$ be an algebra over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$. In this paper, the *dimension of* $\mathcal{B}$, denoted by $\dim \mathcal{B}$, will always refer to its dimension as a vector space.

Let $S=\{z_i : i\in I\}$ be a subset of an algebra $\mathcal{B}$. The *algebra generated by* S is the set $\mathcal{A}(S)=\left\{\sum_{j=1}^k \alpha_j\, z_i^j,\ \alpha_j\in\mathbb{K},\ z_i\in S,\ k\in\mathbb{N},\ i\in I\right\}$, and the set S is called a *system of generators of* $\mathcal{A}(S)$. A system of generators S is *minimal* if for every $i_0\in I$, $z_{i_0}\notin \mathcal{A}(S\setminus\{z_{i_0}\})$. Moreover, the set S is *free* or *algebraically independent* if $P(z_{i_1},\dots,z_{i_n})=0$ implies that $P=0$, for $P\in\mathbb{C}[z_1,\dots,z_n]$ and $z_{i_1}\dots,z_{i_n}\in S$.

If Y is a topological vector space, a subset A of Y is called: *lineable* if $A\cup\{0\}$ contains an infinite-dimensional vector space; *spaceable* if $A\cup\{0\}$ contains a closed infinite-dimensional vector space; *maximal lineable* if $A\cup\{0\}$ contains a vector subspace S of Y with $\dim S=\dim Y$; *dense-lineable* if $A\cup\{0\}$ contains a dense infinite-dimensional vector space. If Y is a function algebra, $A\subset Y$ is said to be:*algebrable* if there is an algebra $\mathcal{B}\subset A\cup\{0\}$, such that $\mathcal{B}$ has an infinite minimal system of generators; A is *strongly α-algebrable* if A admits a free system of generators S such that $\mathrm{card}(S)=\alpha$. We will write $\mathrm{card}(\mathbb{R})=\mathfrak{c}$. If Y is a Fréchet space, a set $A\subset Y$ is called *residual in* Y if $Y\setminus A=\bigcup_{n=1}^{\infty}F_n$, where the closure of each F_n has empty interior. So, by Baire's theorem, residual sets are topologically large. For background on above concepts we refer to **[6, 7, 8]**.

2. Strong algebrability and spaceability of $\mathcal{G}$. Lorch introduced in **[12]** a definition of analytic functions (see Definition 2.1) that have for their domains and ranges complex commutative Banach algebras with identity.

Definition 2.1. Let E be a commutative Banach algebra with identity over $\mathbb{C}$. A mapping $f : E \to E$ has a *derivative in the sense of Lorch* in $\omega \in E$ if there exists $\zeta \in E$ such that

$$\lim_{h\to 0} \frac{\|f(\omega+h) - f(\omega) - \zeta \cdot h\|}{\|h\|} = 0.$$

If f has a derivative in the sense of Lorch throughout a neighborhood of w, we say that f is *Lorch-analytic* (or (L)-*analytic*) in ω.

We say that f is (L)-*analytic* in E if f is (L)-*analytic* at every point of E. We denote the set of all (L)-analytic functions from E into E by $\mathcal{H}_L(E, E)$.

Remark 2.2. Let $f : E \to E$ be a (L)-analytic function at $\omega \in E$. So, the element $\zeta \in E$ given by Definition 2.1 is unique. It is called the (L)-*derivative* of f at ω and is denoted by $\zeta = f'(\omega)$.

It is clear that an (L)-analytic function is differentiable in the Fréchet sense and hence holomorphic. However, not every Fréchet-differentiable function on a commutative Banach algebra with identity is analytic in the Lorch sense. Accordingly, the Lorch theory is the richer.

The following example is well known but we include here without details for the sake of completeness. Consider in $\mathbb{C}^2$ the usual product $(z_1, w_1) \cdot (z_2, w_2) = (z_1\, z_2, w_1\, w_2)$, for all $(z_1, w_1), (z_2, w_2) \in \mathbb{C}^2$, and the norm $\|(z, w)\| = \max\{|z|, |w|\}$, for all $(z, w) \in \mathbb{C}^2$. Let $F : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $F(z, w) = (w, z)$, so F is analytic but it is not (L)-analytic. Thus the set $\mathcal{G} = \mathcal{H}(\mathbb{C}^2, \mathbb{C}^2) \setminus \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is not empty and $\mathcal{G}$ is not a vector space. Then it seems natural to study some algebraic structure inside $\mathcal{G}$.

Now we give a definition, which follows from [**6,** Section 7.5]. A function $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$ is called a *two-variable exponential-like function* if

$$\varphi(z, w) = \left(\sum_{j=1}^{m} a_j e^{b_j z}, \sum_{k=1}^{n} c_k e^{d_k w} \right)$$

for all $(z, w) \in \mathbb{C}^2$, $a_j, b_j, c_k, d_k \in \mathbb{C} \setminus \{0\}$, $j = 1, \ldots, m$ and $k = 1, \ldots, n$, such that $b_j's$ are distinct and $d_k's$ are distinct. We denote by $\mathcal{E}(\mathbb{C}^2, \mathbb{C}^2)$ the set of all two-variable exponential-like functions $\varphi : \mathbb{C}^2 \to \mathbb{C}^2$.

Using the function $F:\mathbb{C}^2\to\mathbb{C}^2$ given in the example above we present, in the next proposition, a family of functions which belong to $\mathcal{G}$, and that will be useful for our results.

Proposition 2.3. *For each $\varphi\in\mathcal{E}(\mathbb{C}^2,\mathbb{C}^2)$, then $\varphi\circ F\in\mathcal{G}$, where $F(z,w)=(w,z)$, for all $(z,w)\in\mathbb{C}^2$.*

Proof. We observe that $f=\varphi\circ F$ is given by

$$f(z,w)=\left(\sum_{j=1}^{m}a_j e^{b_j w},\sum_{k=1}^{n}c_k e^{d_k z}\right),\ \forall (z,w)\in\mathbb{C}^2.$$

By Hartogs' theorem [**16,** Theorem 36.1], it follows that $f\in\mathcal{H}(\mathbb{C}^2,\mathbb{C}^2)$. It is enough to show that f is not (L)-differentiable at $\omega=(0,0)$. So let $\zeta=(z_1,z_2)$, and $h=(t,\lambda t)$, with $t>0$ and $0<|\lambda|\le 1$. Then $\|h\|=t$ and in this case if the limit

$$\lim_{t\to 0}\left\|\left(\frac{\sum_{j=1}^{m}a_j\left(e^{\lambda t b_j}-1\right)-tz_1}{t},\frac{\sum_{k=1}^{n}c_k\left(e^{td_k}-1\right)-\lambda t z_2}{t}\right)\right\|$$

is zero, then (z_1,z_2) would depend on λ, contradicting the fact that $\zeta=(z_1,z_2)$ is unique. Therefore f cannot be (L)-differentiable at the origin, hence not in $\mathbb{C}^2$. □

Let $f\in\mathcal{H}(\mathbb{C}^2,\mathbb{C}^2)$. For each $\alpha>0$ consider $f_\alpha(z,w):=f(\alpha(z,w))$, for all $(z,w)\in\mathbb{C}^2$. Then, for every $\alpha>0$, $f\in\mathcal{H}_L(\mathbb{C}^2,\mathbb{C}^2)$ if and only if $f_\alpha\in\mathcal{H}_L(\mathbb{C}^2,\mathbb{C}^2)$. This fact allows us to exhibit more elements of $\mathcal{G}$:

Lemma 2.4. *If $f\in\mathcal{G}$, then $f_\alpha\in\mathcal{G}$ for each $\alpha>0$.*

Proposition 2.5. *Let $f:\mathbb{C}^2\to\mathbb{C}^2$ be given by $f(z,w)=(e^w,e^z)$. Then $\{f_\alpha : \alpha>0\}$ is a linearly independent set in $\mathcal{H}(\mathbb{C}^2,\mathbb{C}^2)$ and $[f_\alpha : \alpha>0]\subset\mathcal{G}\cup\{0\}$.*

Proof. Let $\mathcal{C}=\{f_\alpha:\alpha>0\}$. Clearly

$$[\mathcal{C}]\subset\{\varphi\circ F,\ \varphi\in\mathcal{E}(\mathbb{C}^2,\mathbb{C}^2)\}\cup\{0\}$$

and hence, by Proposition 2.3, $\mathcal{C}\subset[\mathcal{C}]\subset\mathcal{G}\cup\{0\}$. Now, it is sufficient to prove that $\mathcal{C}$ is linearly independent. First suppose that $\sum_{k=1}^{n}\beta_k f_{\alpha_k}(z,w)=0$, for all $(z,w)\in\mathbb{C}^2$, where $\beta_k\in\mathbb{C}$, for $k=1,\dots,n$.

In particular, $\beta_1 e^{\alpha_1 x} + \beta_2 e^{\alpha_2 x} + \cdots + \beta_n e^{\alpha_n x} = 0$ for all $x \in \mathbb{R}$. Taking n derivatives and setting $x = 0$, we obtain $\sum_{k=1}^{n} \beta_k \alpha_k^j = 0$, $j = 1, \ldots, n$. Since the $\alpha_k' s$ are pairwise distinct, we have $\beta_1 = \beta_2 = \cdots = \beta_n = 0$. □

It follows from Proposition 2.5 that $\mathcal{G}$ is maximal lineable, since by Baire's category theorem, the dimension of any separable infinite-dimensional Fréchet space is $\mathfrak{c}$. Consequently, $\mathcal{G}$ is lineable. In general, lineability does not imply dense-lineability. But in this case $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$ is a separable Fréchet space and $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is a vector subspace of $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)$. Then [**6,** Theorem 7.3.3] shows that $\mathcal{G}$ is indeed dense-lineable. Maximal lineability also does not necessarily imply spaceability. However, in [**6,** Theorem 7.4.1] the authors showed a general theorem, which allowed us to prove that $\mathcal{G}$ is spaceable. To see this, we need to recall some definitions. Let M be a subspace of a vector space V, then each element $v + M$ of the quotient space V/M will be denoted by $\hat{v}$, for all $v \in V$. Recall that the *codimension* of M in V is the dimension of the quotient space V/M.

For the reader's convenience we repeat a statement from [**6**].

Theorem 2.6. [**6,** Theorem 7.4.1] *If Y is a closed vector subspace of a Fréchet space X, then $X \setminus Y$ is spaceable if and only if Y has infinite codimension.*

Proposition 2.7. *$\mathcal{G}$ is spaceable.*

Proof. Let $f : \mathbb{C}^2 \to \mathbb{C}^2$ be given by $f(z, w) = (e^w, e^z)$ and consider the set of classes $\mathcal{C} = \{\widehat{f_\alpha} : \alpha > 0\}$ contained in the quotient space $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)/\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$. Suppose that $\sum_{k=1}^{n} \beta_k \widehat{f_{\alpha_k}} = \widehat{0}$, where $\beta_k \in \mathbb{C}$, for $k = 1, \ldots, n$. This implies that $g = \sum_{k=1}^{n} \beta_k f_{\alpha_k} \in \mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ and $g \equiv 0$, because if $g \neq 0$, using Proposition 2.5 we have that $g \in \mathcal{G}$, which is a contradiction. Applying Proposition 2.5 again, we have that the family $\mathcal{C}$ is linearly independent, so $\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2)/\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$ is infinite-dimensional. Since $(\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2), \tau_b)$ is closed in $(\mathcal{H}(\mathbb{C}^2, \mathbb{C}^2), \tau_b)$ ([**14,** Proposition 2.4] and $\tau_0 = \tau_b$ in $\mathcal{H}_L(\mathbb{C}^2, \mathbb{C}^2)$, it follows by Theorem 2.6 that $\mathcal{G}$ is spaceable. □

In [**6**], the following criterion for strong algebrability is presented.

Theorem 2.8. [**6,** Theorem 7.5.1] *Let Ω be a nonempty set and let $\mathcal{F}$ be a family of functions $\Omega \to \mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Assume that there exists a function $f \in \mathcal{F}$ such that $f(\Omega)$ is uncountable and $\varphi \circ f \in \mathcal{F}$, for every exponential-like function $\varphi : \mathbb{C} \to \mathbb{C}$. Then $\mathcal{F}$ is strongly $\mathfrak{c}$-algebrable. More precisely, if $H \subset (0, +\infty)$ is a set with card$(H) = \mathfrak{c}$ and linearly independent over the field $\mathbb{Q}$ of rational numbers, then $\{\exp \circ (rf) : r \in H\}$ is a free system of generators of an algebra contained in $\mathcal{F} \cup \{0\}$.*

By adapting the proof of Theorem 2.8 to the two-variable case, it is possible to obtain the following result.

Theorem 2.9. *$\mathcal{G}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. For each $r \in \mathbb{R}$, consider $f_r(z, w) = (e^{rw}, e^{rz})$, for all $(z, w) \in \mathbb{C}^2$. If $H \subset (0, +\infty)$ is a set with $\operatorname{card}(H) = \mathfrak{c}$ and linearly independent over the field $\mathbb{Q}$ of the rational numbers, we will show that the set $\{f_r : r \in H\}$ is a free system of generators of an algebra contained in $\mathcal{G} \cup \{0\}$.

Let P be a nonzero polynomial in N complex variables without constant term, that is,

$$P(z_1, z_2, \ldots, z_N) = \sum_{j=1}^{m} a_j z_1^{k(j,1)} z_2^{k(j,2)} \cdots z_N^{k(j,N)},$$

where $a_1, \ldots, a_m \in \mathbb{C} \setminus \{0\}$ and the matrix $[k(j,l)]_{l=1,\ldots,N}^{j=1,\ldots,m}$ of nonnegative integers has distinct zero rows.

If $r_1, r_2, \ldots, r_N \in H$, the function $\Psi : \mathbb{C}^2 \to \mathbb{C}^2$ given by $\Psi = P \circ (f_{r_1}, f_{r_2}, \ldots, f_{r_N})$ is of the form

$$\begin{aligned} P(f_{r_1}, f_{r_2}, \ldots, f_{r_N})(z, w) &= \sum_{j=1}^{m} a_j (f_{r_1}(z, w))^{k(j,1)} (f_{r_2}(z, w))^{k(j,2)} \cdots (f_{r_N}(z, w))^{k(j,N)} \\ &= \sum_{j=1}^{m} a_j \big(e^{(k(j,1) r_1 + \cdots + k(j,N) r_N) w}, e^{(k(j,1) r_1 + \cdots + k(j,N) r_N) z}\big). \end{aligned}$$

Thus, the numbers $b_j := \sum_{l=1}^{N} r_l k(j,l)$ are distinct and nonzero, due to the $\mathbb{Q}$-independence of H. Hence the function $\varphi(z, w) =$

$\left(\sum_{j=1}^m a_j e^{b_j z}, \sum_{k=1}^m a_k e^{b_k w}\right)$ belongs to $\mathcal{E}(\mathbb{C}^2,\mathbb{C}^2)$ and $\Psi = \varphi \circ F \in \mathcal{G}$ by Proposition 2.3.

If $\Psi \equiv (0,0)$ then $\varphi \equiv (0,0)$, since $F(\mathbb{C}^2)=\mathbb{C}^2$. But $\varphi = (\varphi_1,\varphi_2)$, where $\varphi_1(w) = \sum_{j=1}^m a_j e^{b_j w} = 0$ for all $w \in \mathbb{C}$ and $\varphi_2(z) = \sum_{k=1}^m a_k e^{b_k z} = 0$ for all $z \in \mathbb{C}$. Hence $\varphi \equiv (0,0)$ would contradict the fact that φ_1 and φ_2 have each one at most a countable number of zeros (see the first part of the proof of [**6,** Theorem 7.5.1]). Consequently $\Psi \not\equiv 0$ and, by Theorem 2.3, $\Psi \in \mathcal{G}$. $\square$

Remark 2.10. As $\mathcal{H}_L(\mathbb{C}^2,\mathbb{C}^2)$ is of the second category, the set $\mathcal{G}$ is not residual in $\mathcal{H}(\mathbb{C}^2,\mathbb{C}^2)$. Indeed, $\mathcal{H}(\mathbb{C}^2,\mathbb{C}^2) \setminus \mathcal{G} = \mathcal{H}_L(\mathbb{C}^2,\mathbb{C}^2)$, and $\mathcal{H}_L(\mathbb{C}^2,\mathbb{C}^2)$ is Fréchet space. So $\mathcal{G}$ is not topologically large.

3. Maximal lineability and Residuality of $\mathcal{H}$. Let $D \subset \mathbb{C}$ denote the open unit disk, that is, $D = \{z \in \mathbb{C} : |z| < 1\}$. The Banach algebra of all continuous functions on $\overline{D}$ that are analytic on D with the *sup* norm is denoted by $\mathcal{A}(D)$. As usual we call $\mathcal{A}(D)$ the *disk algebra*.

Let $X \subset \mathbb{C}$ be a perfect and compact set. A complex valued function $f : X \to \mathbb{C}$ is *differentiable at a point* $z_0 \in X$ if the following limit exists:

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0}.$$

A complex-valued function f is *differentiable on* X if it is differentiable at every point of X. The algebra of all functions on X with continuous n-th derivative is denoted by $\mathcal{D}^n(X)$, and $\mathcal{D}^\infty(X)$ denotes the algebra of functions on X with derivative of all orders. We denote by $f^{(n)}$ the n-th derivative of f and $\|f\|_X = \sup_{z\in X} |f(z)|$. We denote by $R_0(X)$ the algebra of all rational functions with poles off X.

Let $(M_n)_{n\in\mathbb{N}}$ be a sequence of positive numbers such that $M_0 = 1$, and for each $n \geq 1$,

$$\frac{M_n}{M_k\, M_{n-k}} \geq \binom{n}{k} \quad (0 \leq k \leq n).$$

The sequence $M = (M_n)_{n\in\mathbb{N}}$ is an *algebra sequence* if it satisfies the above conditions.

The *Dales-Davie algebras* on X are defined by

$$\mathcal{D}(X,M)=\left\{f\in D^{\infty}(X)\,:\,\sum_{n=0}^{\infty}\frac{\|f^{(n)}\|_X}{M_n}<+\infty\right\}.$$

The norm on $\mathcal{D}(X,M)$ is defined by $\|f\|=\sum_{n=0}^{\infty}\|f^{(n)}\|_X/M_n$. These algebras were introduced and studied by Dales and Davie in 1973 [**10**], and have been investigated by Abtahi and Honary in ([**1**], [**2**] and [**11**]).

For each sequence $M=(M_n)_{n\in\mathbb{N}}$ of positive numbers, $\mathcal{D}(X,M)$ is a normed vector space. When $M=(M_n)_{n\in\mathbb{N}}$ is an algebra sequence, then $\mathcal{D}(X,M)$ is a normed algebra.

When $X=\bar{D}$ it follows that $\mathcal{D}(\bar{D},M)$ is a subalgebra of $\mathcal{A}(D)$. However, $\mathcal{H}(M)=\mathcal{A}(D)\setminus\mathcal{D}(\bar{D},M)$ is not a vector space, hence is not an algebra. In [**13**] we have shown that $\mathcal{H}(M)$ is algebrable and spaceable, for several algebra sequences $M=(M_n)_{n\in\mathbb{N}}$. In this note we show that the set $\mathcal{H}(M)$ is residual, strongly $\mathfrak{c}$-algebrable and we determine a linearly independent set in $\mathcal{H}(M)$, giving us another way to see that $\mathcal{H}(M)$ is maximal lineable.

Let α be a real number such that $0<\alpha<1$. For each $f\in\mathcal{A}(D)$, we define $f_\alpha:\bar{D}\to\mathbb{C}$ by $f_\alpha(z)=f(\alpha z)$, for all $z\in\bar{D}$. Then it is clear that $f_\alpha\in\mathcal{A}(D)$. We set $D_\alpha=\{z\in\mathbb{C}\,:\,|z|\leq\alpha\}$.

The following lemma was inspired by [**7,** Lemma 4].

Lemma 3.1. *Let $f\in\mathcal{A}(D)$ such that f is not a polynomial. Then the family $\{f_\alpha\,:\,0<\alpha<1\}$ is linearly independent.*

Proof. Let $c_1,c_2,\ldots,c_N\in\mathbb{C}$, $0<\alpha_1,\alpha_2,\ldots,\alpha_N<1$ and suppose that $\sum_{k=1}^{N}c_kf_{\alpha_k}=0$. We can assume that $N\geq 2$ and $\alpha_1<\alpha_2<\cdots<\alpha_N$.

Since $f\in\mathcal{A}(D)$, then $f(z)=\sum_{n=0}^{\infty}a_nz^n$, uniformly on δD, where $0<\delta<1$. Then we have that

$$\sum_{k=1}^{N}c_kf_{\alpha_k}(z)=\sum_{k=1}^{N}c_kf(\alpha_kz)=\sum_{k=1}^{N}c_k\sum_{n=0}^{\infty}a_n\,\alpha_k^n\,z^n=0,\ \text{for all } z\in\delta D.$$

Since the series converges uniformly, it follows that $a_n(c_1\alpha_1^n+\cdots+c_N\alpha_N^n)=0,$ for all $n\in\mathbb{N}$.

As f is not a polynomial, then there exists an increasing sequence $(n_j)_{j\in\mathbb{N}}$ such that $a_{n_j}\neq 0$, for all $j\in\mathbb{N}$. Then in particular we have that: $c_1\alpha_1^{n_j}+\cdots+c_N\alpha_N^{n_j}=0$, for all $j\in\mathbb{N}$. If $c_N\neq 0$ then

$$-1=\frac{c_1\alpha_1^{n_j}+\cdots+c_{N-1}\alpha_{N-1}^{n_j}}{c_N\alpha_N^{n_j}}=\sum_{k=1}^{N-1}\frac{c_k}{c_N}\left(\frac{\alpha_k}{\alpha_N}\right)^{n_j}.$$

By taking $j\to\infty$ we find a contradiction, so that $c_N=0$, and inductively we have $c_1=c_2=\cdots=c_{N-1}=c_N=0$, and then the family $\{f_\alpha : 0<\alpha<1\}$ is linearly independent. □

Our main goal is to display a vector space in $\mathcal{H}(M)$ for some M, which has a uncountable system of generators. The following fact, which was observed by Dales and Davie in **[10]**, can be used to find algebra sequences $M=(M_n)_{n\in\mathbb{N}}$ such that $\mathcal{H}(M)\neq\varnothing$.

Theorem 3.2. [**1,** Theorem 2.3] *Let $X\subset\mathbb{C}$ be a perfect compact set. Then $\mathcal{R}_0(X)\subseteq\mathcal{D}(X,M)$ if and only if $\lim_{n\to\infty}(n!/M_n)^{1/n}=0$.*

If we take $M_n=n!$, for all $n\in\mathbb{N}$, then we have $\lim_{n\to\infty}(n!/M_n)^{1/n}\neq 0$. So $\mathcal{R}_0(X)\nsubseteq\mathcal{D}(X,M)$. In this case we use $\mathcal{H}$ instead of $\mathcal{H}(M)$.

Proposition 3.3. *Let $f(z)=\dfrac{1}{z-\frac{3}{2}}$. Then $[f_\alpha : \frac{3}{4}<\alpha<1]\subset\mathcal{H}\cup\{0\}$.*

Proof. Let us first observe that each $f_\alpha\in\mathcal{H}$. By a simple calculation of the derivatives of f_α, we get

$$\|(f_\alpha)^{(n)}\|_{\bar D}=\alpha^n\|(f^{(n)})_\alpha\|_{\bar D}=\alpha^n\, n!\,\|f\|_{D_\alpha}^{n+1}=n!\,\frac{2}{3-2\alpha}\left(\frac{2\alpha}{3-2\alpha}\right)^n.$$

Since $\alpha>\frac{3}{4}$, it follows that $\sum_{n=0}^{\infty}\|(f_\alpha)^{(n)}\|_{\bar D}/n!=+\infty$ and hence $f_\alpha\in\mathcal{H}$.

Let $g\in[f_\alpha : \frac{3}{4}<\alpha<1]$. Then $g=\sum_{j=1}^{k}\beta_j\, f_{\alpha_j}$, for $\beta_j\in\mathbb{C}\setminus\{0\}$ and $\frac{3}{4}<\alpha_j<1$, for $j=1,\ldots,k$. Then

$$g^{(n)}(z)=\sum_{j=1}^{k}\beta_j\, f_{\alpha_j}^{(n)}(z)=\sum_{j=1}^{k}\beta_j\,\alpha_j^n\, n!(-1)^n\left(\frac{1}{\alpha_j z-\frac{3}{2}}\right)^{n+1}.$$

To prove that $g\in\mathcal{H}$, we show that $\lim\limits_{n\to\infty}|g^{(n)}(1)/n!|=+\infty$. We write

$$\frac{g^{(n)}(1)}{n!} = \sum_{j=1}^{k}(-1)^n D_j\, C_j^n, \quad \text{where } C_j = \frac{2\alpha_j}{2\alpha_j - 3} \text{ and } D_j = \frac{2\beta_j}{2\alpha_j - 3}.$$

We suppose without loss of generality that $C_1 > C_2 > \cdots > C_k$. Then

$$\frac{g^{(n)}(1)}{n!} = C_1^n(-1)^n \left(D_1 + D_2\Big(\frac{C_2}{C_1}\Big)^n + D_3\Big(\frac{C_3}{C_1}\Big)^n + \cdots + D_k\Big(\frac{C_k}{C_1}\Big)^n\right).$$

Now

$$\lim_{n\to\infty} \left(D_1 + D_2\Big(\frac{C_2}{C_1}\Big)^n + D_3\Big(\frac{C_3}{C_1}\Big)^n + \cdots + D_k\Big(\frac{C_k}{C_1}\Big)^n\right) = D_1,$$

and since $|C_1|>1$ we have $\lim_{n\to\infty} |C_1^n(-1)^n| = +\infty$. Then $\lim_{n\to\infty} \left|\frac{g^{(n)}(1)}{n!}\right| = +\infty$, and the result follows. □

Remark 3.4. In [**13,** Proposition 2.3] we showed that $\mathcal{H}$ is spaceable, then $\mathcal{H}$ is maximal lineable. Now, since every element of $\mathcal{H}$ is not a polynomial, we can apply Lemma 3.1 for $\frac{3}{4} < \alpha < 1$ and get that $\{f_\alpha : \frac{3}{4} < \alpha < 1\}$ is linearly independent, and by Proposition 3.3 $[f_\alpha : \frac{3}{4} < \alpha < 1] \subset \mathcal{H}$. Hence $\mathcal{H}$ is maximal lineable.

In [**13**] we showed that $\mathcal{H}$ is algebrable. Now, using Theorem 2.8 it is possible to get a better result.

Proposition 3.5. *$\mathcal{H}$ is strongly $\mathfrak{c}$-algebrable.*

Proof. Let $f(z) = 1/(z+2)$, for all $z \in D$. Using Lemma 2.2 of [**13**], we have $f \in \mathcal{H}$. Consider $\varphi : \mathbb{C} \to \mathbb{C}$ an exponential-like function given by $\varphi(z) = \sum_{j=}^{m} a_j e^{b_j z}$, for all $z \in \mathbb{C}$ and for some $m \in \mathbb{N}$, where $a_1, \ldots, a_m \in \mathbb{C}\setminus\{0\}$ and some distinct $b_1, \ldots, b_m \in \mathbb{C}\setminus\{0\}$. One can see that the proof of Theorem 3.3 of [**13**] works the same for this general φ, since the $b_j's$ are distinct. Then $\varphi \circ f \in \mathcal{H}$. As $f(D)$ is uncountable, then it follows by Theorem 2.8 that $\mathcal{H}$ is strongly $\mathfrak{c}$-algebrable. □

We finish this section by studying the residuality of $\mathcal{H}$ in $\mathcal{A}(D)$.

Theorem 3.6. *$\mathcal{H}$ is residual in $\mathcal{A}(D)$.*

Proof. We show that $\mathcal{H} = \bigcap_{n=1}^{\infty} S_n$, where the S_n are open and dense sets in $\mathcal{A}(D)$. We define

$$S_{m,n} = \left\{ f \in \mathcal{A}(D) : \sum_{k=0}^{m} \frac{\|f^{(k)}\|_{\overline{D}}}{k!} > n \right\} \quad \text{and} \quad S_n = \bigcup_{m \geq n} S_{m,n}.$$

Let $f \in \mathcal{H}$. Since $\sum_{k=0}^{\infty} \|f^{(k)}\|_{\overline{D}}/k! = +\infty$, given $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $f \in S_{m,n}$ and $m > n$. Therefore $\mathcal{H} = \bigcap_{n=1}^{\infty} S_n$.

Since it is clear that each S_n is open let us finish the proof by showing that each S_n is dense in $\mathcal{A}(D)$. Given $n \in \mathbb{N}$, $g \in \mathcal{A}(D)$ and $\varepsilon > 0$, we want to show that $B(g, \varepsilon) \cap S_n \neq \varnothing$. If $g \in \mathcal{H}$, then it is clear. If $g \notin \mathcal{H}$, then $\sum_{k=0}^{\infty} \|g^{(k)}\|_{\overline{D}}/k! = r$. Let

$$f(z) = \frac{\varepsilon}{2} \frac{1}{z-2}.$$

Then $f \in \mathcal{H}$ and $\|f\|_{\overline{D}} < \varepsilon$. Let us show that $f + g \in S_n$. We know that there exists $m \in \mathbb{N}$ such that $m \geq n$ and $\sum_{k=0}^{m} \|f^{(k)}\|_{\overline{D}}/k! > r + n$. Then

$$\sum_{k=0}^{m} \frac{\|f^{(k)} + g^{(k)}\|_{\overline{D}}}{k!} \geq \sum_{k=0}^{m} \frac{\|f^{(k)}\|_{\overline{D}}}{k!} - \sum_{k=0}^{m} \frac{\|g^{(k)}\|_{\overline{D}}}{k!} > r + n - r = n,$$

which shows that $f + g \in S_n$. □

Similarly to **[13]**, in the next corollary, we show that not only $\mathcal{H}$ is maximal lineable, strongly $\mathfrak{c}$-algebrable and residual, but actually there is an infinite collection of algebra sequences $(M_n)_{n\in\mathbb{N}}$ such that $\mathcal{H}(M)$ also have these properties.

Corollary 3.7. *Let $(M_n)_{n\in\mathbb{N}}$ be an algebra sequence such that $M_n \leq n!$, for all $n \in \mathbb{N}$. Then $\mathcal{H}(M)$ is maximal lineable, strongly $\mathfrak{c}$-algebrable and residual.*

Proof. If $M_n \leq n!$, then $\mathcal{H} \subseteq \mathcal{H}(M)$. The proof of Theorem 3.6 is also valid in this case. □

Remark 3.8. If $0 < \alpha \leq 1$, let $M_n := \alpha^n \, n!$ for all $n \in \mathbb{N}$. Then $(M_n)_{n\in\mathbb{N}}$ is an algebra sequence such that $M_n \leq n!$, for all $n \in \mathbb{N}$.

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Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, CEP 05508-090, São Paulo, SP, Brazil

Email address: mllouren@ime.usp.br

Email address: danim@ime.usp.br