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BJÖRLING PROBLEM FOR MAXIMAL SURFACES IN THE LORENTZ-MINKOWSKI 4-DIMENSIONAL SPACE

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BJÖRLING PROBLEM FOR MAXIMAL SURFACES IN THE LORENTZ–MINKOWSKI 4–DIMENSIONAL SPACE

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ABSTRACT. In this paper, we extend and solve the Björling–type problem for maximal surfaces in the Lorentz–Minkowski 4–dimensional space. As an application we establish symmetry principles for the maximal surfaces in $\mathbb{L}^4$ and construct new examples.

1. INTRODUCTION

A maximal surface in the Lorentz–Minkowski n–dimensional space is a spacelike surface with zero mean curvature vector. It is well known that maximal surfaces in $\mathbb{L}^3$ represent locally a maximum for the area integral [16, 8] and also that they admit a Weierstrass type representation [22, 23]. But the spacelike surfaces with zero mean curvature vector in $\mathbb{L}^4$, represent locally the maximum(resp. minimum) for the area integral, if the normal variation is made in the timelike(resp. spacelike) direction [20]. For these surfaces we also have Weierstrass type representation [4, 14]. An important difference between the global theory of maximal surfaces in $\mathbb{L}^3$ and of the global theory of maximal surfaces in $\mathbb{L}^4$ is established by the so called *Calabi–Bernstein theorem.* It states that a complete maximal surface in $\mathbb{L}^3$ is a plane [8, 10]. However, this result cannot be extended to $\mathbb{L}^n$, $n \geq 4$ [13].

In the 3–dimensional Euclidian space $\mathbb{R}^3$, given a real analytic strip(see §3), the classical Björling problem [11, 17] was proposed by E. G. Björling [7] in 1844 and consists of the construction of a minimal surface in $\mathbb{R}^3$ containing the strip in the interior. The solution for this problem was given by H. A. Schwarz in [28] by means of a explicit formula in terms of the prescribed strip. This formula gives a beautiful method, besides the Weierstrass representation [27] , to construct minimal surfaces with interesting properties. For example, properties of symmetry.

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The equivalent problem in the Lorentz–Minkowski 3–dimensional space was proposed and solved, using a complex representation formula developed in [2]. The authors introduced the local theory of maximal surfaces in $\mathbb{L}^3$ in a different way of that given in [22, 23] through the Weierstrass representation. They constructed new examples of maximal surfaces, gave alternative proofs of the characterization of the maximal surfaces of revolution and the ruled surfaces in $\mathbb{L}^3$ and proved symmetry principles for those surfaces.

In Euclidian 4–dimensional space, the Björling problem for minimal surfaces was proposed and solved in [5], see also [3], from a complex representation formula. In that work the authors also recovered the symmetry principles of minimal surfaces in $\mathbb{R}^4$ obtained by Eisenhart [12].

In this paper, motivated by results and techniques of [2], [5] and [14], we introduce the local theory of maximal surfaces in $\mathbb{L}^4$, using a complex representation formula–see theorem 1 below– that describes the local geometry of these surfaces. This formula is used to solve the Björling problem in $\mathbb{L}^4$, which is illustred with two examples. As another consequence of theorem 1 we recover the representation formulae of the Bjöling problem for minimal surfaces in $\mathbb{R}^3$ and maximal surfaces in $\mathbb{L}^3$. We also recover the symmetry principles for these surfaces. Finally, we study the symmetry principles for the maximal surfaces in $\mathbb{L}^4$ and present new examples.

2. Preliminaries

Let $\mathbb{L}^4$ denote the 4–dimensional Lorentz–Minkowski space, that is, the Euclidian space $\mathbb{R}^4 := \{(x^1, x^2, x^3, x^4) : x^i \in \mathbb{R}\}$ endowed with the Lorentzian metric

$$\langle , \rangle := (dx^1)^2 + (dx^2)^2 + (dx^3)^2 - (dx^4)^2. \tag{1}$$

Given u, v, w in $\mathbb{L}^4$, we define the vector product $\boxtimes(u, v, w) \in \mathbb{L}^4$ by

$$\langle \boxtimes(u, v, w), x \rangle := -det(u, v, w, x), \tag{2}$$

which in coordinates takes the form

$$\boxtimes(u,v,w) = \left(\begin{vmatrix} u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \\ u^4 & v^4 & w^4 \end{vmatrix}, - \begin{vmatrix} u^1 & v^1 & w^1 \\ u^3 & v^3 & w^3 \\ u^4 & v^4 & w^4 \end{vmatrix}, \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^4 & v^4 & w^4 \end{vmatrix}, \begin{vmatrix} u^1 & v^1 & w^1 \\ u^2 & v^2 & w^2 \\ u^3 & v^3 & w^3 \end{vmatrix} \right).$$

Let $\{e_1, e_2, e_3, e_4\}$ be the canonical basis of $\mathbb{R}^4$. The proof of the following proposition is straightforward.

Proposition 2.1. *The vector product* $\boxtimes$ *has the following properties:*
1) $\langle\boxtimes(u,v,w),u\rangle = \langle\boxtimes(u,v,w),v\rangle = \langle\boxtimes(u,v,w),w\rangle = 0$;
2) $\boxtimes(u,v,e_4) = \hat{u}\times\hat{v}$, *where* $\hat{u} = (u^1,u^2,u^3,0), \hat{v} = (v^1,v^2,v^3,0) \in \mathbb{R}^3 \subset \mathbb{L}^4$;
3) $\boxtimes(u,v,e_2) = \check{u}\times\check{v}$, *where* $\check{u} = (u^1,0,u^3,u^4)$, $\check{v} = (v^1,0,v^3,v^4) \in \mathbb{L}^3 \subset \mathbb{L}^4$;
4) $\boxtimes(u,v,e_1) = -\check{u}\times\check{v}$ *and* $\boxtimes(u,v,e_3) = -\check{u}\times\check{v}$;
5) $\langle\boxtimes(u_1,u_2,u_3),\boxtimes(v_1,v_2,v_3)\rangle = -\det(\langle u_i,v_j\rangle)$, $1\le i;j\le 3$, $u_i,\ v_j \in \mathbb{L}^4$;
where $\times$ *is respectively the cross-product of* $\mathbb{R}^3$ *and* $\mathbb{L}^3$.

Let $\mathbb{C}_1^n$ be the n-dimensional complex vector space endowed with the *hermitian estruture*

$$\ll z,w \gg := \sum_{j=1}^{n-1} z^j\overline{w^j} - z^n\overline{w^n},$$

We will deal with the following subsets of the *complex projective space* $\mathbb{P}(\mathbb{C}_1^n)$ associated to $\mathbb{C}_1^n$ (see [15, 6, 24]):
1) $\mathbb{CP}_1^{n-1} := \{z\in\mathbb{C}^n\setminus\{0\} :\ll z,z\gg > 0\}/\mathbb{C}^*$;
2) $\mathbb{CH}^{n-1} := \{z\in\mathbb{C}^n\setminus\{0\} :\ll z,z\gg < 0\}/\mathbb{C}^*$;
3) $\partial\mathbb{CH}^{n-1} := \{z\in\mathbb{C}^n\setminus\{0\} :\ll z,z\gg = 0\}/\mathbb{C}^*$.

Denote by $G_{2,4}^+$ the Grassmannian of spacelike 2–planes of $\mathbb{L}^4$ with the induced orientation. Given $u,v\in\mathbb{L}^4$, with $\langle u,u\rangle = \langle v,v\rangle = \lambda^2 > 0$ and $\langle u,v\rangle = 0$, let $\Pi^2 = span[u,v] \in G_{2,4}^+$. We can identify $G_{2,4}^+$ with $Q_1^2 := \{[z]\in\mathbb{CP}_1^{n-1} :\ll z,\overline{z}\gg = 0\}$ through the mapping that sends each $\Pi^2\in G_{2,4}^+$ into $[z]\in Q_1^2$ where $z = u+iv$. Given $\Pi^2 = span[u,v] \in G_{2,4}^+$, let $\nu_0 := \boxtimes(u,v,e_4)$ and $\tau_0 := \boxtimes(u,v,\nu_0)$; then $\{\nu_0,\tau_0\}$ is a basis for $(\Pi^2)^\perp$.

Proposition 2.2. *Let* ν_0 *and* τ_0 *defined as above. We have:*
1) $\nu_0 = \hat{u}\times\hat{v}$, *where* $\times$ *is the cross-product in* $\mathbb{R}^3\subset\mathbb{L}^4$;
2) $\tau_0 = \lambda^2 e_4 + u^4 u + v^4 v$, *where* $\lambda^2 = \langle u,u\rangle$;
3) $\langle\nu_0,\nu_0\rangle = \lambda^2(\lambda^2+(u^4)^2+(v^4)^2)$ *and* $\langle\tau_0,\tau_0\rangle = -\lambda^2(\lambda^2+(u^4)^2+(v^4)^2)$;
4) *if* $\mu_0 := \sqrt{\lambda^2(\lambda^2+(u^4)^2+(v^4)^2)}$, $\tau := \frac{\tau_0}{\mu_0}$ *and* $\nu := \frac{\nu_0}{\mu_0}$, *then* $\{\frac{u}{\lambda},\frac{v}{\lambda},\nu,\tau\}$ *is a positively oriented orthonormal basis of* $\mathbb{L}^4$.

Denote by $G_{2,4}^-$ the Grassmannian of timelike 2–planes of $\mathbb{L}^4$ with the induced orientation. Given $\nu_1,\nu_2\in\mathbb{L}^4$, with $\langle\nu_1,\nu_1\rangle = -\langle\nu_2,\nu_2\rangle = \lambda^2 > 0$ and $\langle\nu_1,\nu_2\rangle = 0$, let $\Pi^2 = span[\nu_1,\nu_2]\in G_{2,4}^-$. We can identify, as above, $G_{2,4}^-$ with the real quadric QR, which is defined as the set of classes $[z]\in\partial\mathbb{CH}^{n-1}$ such that $\langle\mathfrak{Re}(z),\mathfrak{Im}(z)\rangle = 0$ and $\mathfrak{Re}(z),\mathfrak{Im}(z)$ are linearly independent.

Definition 2.3. A smooth immersion $X : M^2 \to \mathbf{L}^4$ of a 2-dimensional oriented connected manifold is called a *spacelike surface* S in $\mathbf{L}^4$ if the induced metric $ds^2 := X^*\langle , \rangle$ on M^2 is a Riemannian metric.

Let $(U, z = u + iv)$ be isothermal coordinates in a neighborhood of a point p in M^2, that is $\langle X_u, X_u \rangle = \langle X_v, X_v \rangle = \lambda^2$ and $\langle X_u, X_v \rangle = 0$. This induces a holomorphic estruture on M^2. We define an orthonormal basis $\{\nu, \tau\}$ of $(T_pS)^\perp$ by

$$\nu = \frac{\nu_0}{\mu_0} \text{ and } \tau = \frac{\tau_0}{\mu_0}, \tag{3}$$

where

$$\nu_0 = \boxtimes(X_u, X_v, e_4),\ \tau_0 = \boxtimes(X_u, X_v, \nu_0),\ \mu_0 = \sqrt{\lambda^2(\lambda^2 + (x_u^4)^2 + (x_v^4)^2)}.$$

Observe that ν and τ are respectively spacelike and timelike vector fields normal to the surface $S = X(M)$ and it is not hard to see that they are globally defined on S. Also, let $\beta = \{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the local orthonormal frame adapted to S, where

$$\partial_1 = \frac{X_u}{\lambda},\ \partial_2 = \frac{X_v}{\lambda},\ \partial_3 = \nu,\ \partial_4 = \tau. \tag{4}$$

As far as we know, the normal frame $\{\nu, \tau\}$ was introduced in [14], where spacelike surfaces in $\mathbf{L}^4$ are extensively studied.

Let $\overline{\nabla}$ and ∇ be the Levi–Civita connection of $\mathbf{L}^4$ and (M^2, ds^2), respectively. The *second fundamental form* of S is defined by $\alpha(V, W) := (\overline{\nabla}_V W)^\perp$ and the *mean curvature vector* by $H_p := \frac{1}{2} tr(\alpha_p)$ for all $p \in M^2$.

Proposition 2.4. *If* $S = X(M)$ *is a spacelike surface in* $\mathbf{L}^n$*, then* $\Delta_M X = 2H$.

Proof: See [14].

Definition 2.5. A spacelike surface S in $\mathbf{L}^4$ is *maximal* if $H = 0$.

Let $S = X(M)$ a spacelike surface in $\mathbf{L}^4$ defined in terms of local isothermal coordinates $(U, z = u + iv)$ of M^2, and define the complex functions

$$\varphi^k := \frac{\partial x^k}{\partial u} - i\frac{\partial x^k}{\partial v},\ k = 1, 2, 3, 4. \tag{5}$$

It is not hard see that

$$(\varphi^1)^2 + (\varphi^2)^2 + (\varphi^3)^2 - (\varphi^4)^2 = 0,\ |\varphi^1|^2 + |\varphi^2|^2 + |\varphi^3|^2 - |\varphi^4|^2 = 2\lambda^2 > 0.$$

The induced metric on M is $ds^2 = \lambda^2 |dz|^2$ and the complex 1–forms $\omega^k := \varphi^k dz$ are globally defined on M. Now if S is a maximal surface, it follows from Proposition 2.4 that ω^k is holomorphic. Thus, S can be represented as

$$X(z) = \mathfrak{Re}\int_{z_0}^{z} \omega + k_0, \quad \text{where } \omega = (\omega^1, \omega^2, \omega^3, \omega^4) \text{ and } z_0, z \in M. \tag{6}$$

The converse also holds.

Theorem 2.6. *Let M^2 be a connected Riemann surface and $\omega = (\omega^1, \omega^2, \omega^3, \omega^4)$ a holomorphic 1–form with values in $\mathbb{C}^4$ globally defined on M^2 satisfying*
1) $\ll \omega, \overline{\omega} \gg \equiv 0$,
2) $\ll \omega, \omega \gg > 0, \ \forall\, p \in M^2$,
3) $\mathfrak{Re}\int_\gamma \omega = 0$, *for all closed path γ on M^2.*
Then the map $X : M^2 \to \mathbb{L}^4$ given by the equation (6) defines a maximal surface in $\mathbb{L}^4$.

For the proof see [14].

The *Gauss map* $G : M^2 \to Q_1^2$ of a spacelike surface $S = X(M)$ in $\mathbf{L}^4$ is defined locally by $G(z) = [\overline{\Phi(z)}]$, with $X_z = \psi\Phi$ for some function $\psi : M^2 \to \mathbf{C}$ and $\Phi = (\phi^1, \phi^2, \phi^3, \phi^4)$, for more details see [19, 14]. Let $a(z), b(z)$ be the complex valued functions defined on M^2 by

$$a(z) := \frac{-\phi^3 + \phi^4}{\phi^1 - i\phi^2}, \quad b(z) := \frac{\phi^3 + \phi^4}{\phi^1 - i\phi^2}. \tag{7}$$

We have that

$$\Phi(z) = \mu(1 + ab, i(1 - ab), a - b, a + b). \tag{8}$$

It follows from (3) and (8) that

$$\tau(z) = \frac{1}{|1 - a\bar{b}|\sqrt{(1 + |a|^2)(1 + |b|^2)}} \begin{bmatrix} (1 + |b|^2)\mathfrak{Re}(a) + (1 + |a|^2)\mathfrak{Re}(b) \\ (1 + |b|^2)\mathfrak{Im}(a) + (1 + |a|^2)\mathfrak{Im}(b) \\ |a|^2 - |b|^2 \\ (1 + |a|^2)(1 + |b|^2) \end{bmatrix},$$

$$\nu(z) = \frac{1}{|1 - a\bar{b}|\sqrt{(1 + |a|^2)(1 + |b|^2)}} \begin{bmatrix} (1 + |b|^2)\mathfrak{Re}(a) - (1 + |a|^2)\mathfrak{Re}(b) \\ (1 + |b|^2)\mathfrak{Im}(a) - (1 + |a|^2)\mathfrak{Im}(b) \\ |a|^2|b|^2 - 1 \\ 0 \end{bmatrix}.$$

For more details see [14].

Let $A : M^2 \to \mathbb{C}^4$ be the complex map defined by

$$A(z) := \nu(z) + i\tau(z), \tag{9}$$

and observe that $[A(z)] \in QR$.

3. Main results

Now we are able to propose and solve the Björling problem for maximal surfaces in $\mathbf{L}^4$. Let $c : I$ ~~$\subseteq \mathbb{R} \to \mathbf{L}^4$ be a regular~~ real analytic spacelike curve in $\mathbf{L}^4$ and let $n : I \to \mathbf{C}^4$ be a real analytic vector field along c (that is, $\Re e(n), \Im m(n) : I \to \mathbf{L}^4$ are vector fields along c) such that $\langle c'(s), \Re e(n)\rangle = 0 = \langle c'(s), \Im m(n)\rangle$, $\langle \Re e(n), \Re e(n)\rangle = -\langle \Im m(n), \Im m(n)\rangle = 1$ and $\Im m(n)$ is future directed for all $s \in I$. In analogy with [2, 11], we call such a pair (c, n) a *analytical strip* in $\mathbf{L}^4$. The problem is then to find a maximal surface S defined by $X : \Omega \subseteq \mathbf{C} \to \mathbf{L}^4$ with $I \subset \Omega$, such that
1) $X(u, 0) = c(u)$,
2) $A(u, 0) = n(u), \quad \forall\, u \in I$.
It is easy to see that if $X : \Omega \subseteq \mathbf{C} \to \mathbf{L}^4$ is maximal surface in $\mathbf{L}^4$, then $c(u) := X(u, 0)$ and $n(u) := A(u, 0)$ satisfy the above data and, in particular, they are real analytic. Then there exist holomorphic extensions $c(z)$ and $n(z)$ and these extensions are unique by the *identity theorem* for analytic functions (see [21] pp. 87). In this situations, we can explicitly recover $X(z)$ from c and n by means of a unique complex representation formula.

Theorem 3.1. *Let S be a maximal surface in $\mathbf{L}^4$ given by $X : U \subseteq \mathbf{C} \to \mathbf{L}^4$. Define the curve $c(u) := X(u, 0)$ and the vector field $n(u) := A(u, 0)$ along c, on a real interval $I \subset U$. Choose any simply connected open set $\Omega \subseteq U$ containing I, over which we can define holomorphic extensions $c(z)$ and $n(z)$ of c and n. Then, for all $z \in \Omega$ it holds*

$$X(z) = \Re e \left(c(z) + i \int_{s_0}^{z} \boxtimes \left(\Re e(n(w)), \Im m(n(w)), c'(w)\right) dw \right), \tag{10}$$

where s_0 is a arbitrary fixed point of I and the integral is taken along an arbitrary path in Ω joining s_0 and z.

Proof: Since S is maximal, the complex function $\Psi : U \to \mathbf{C}^4$ defined by (5)

$$\Psi = 2\frac{\partial X}{\partial z} \quad \text{with} \quad \Psi = (\varphi^1, \varphi^2, \varphi^3, \varphi^4),$$

is holomorphic in U and by (6) we can write

$$X(z) = \Re e \int_{z_0}^{z} \Psi dz + k_0, \tag{11}$$

where $k_0 \in \mathbf{L}^4$ is a suitable constant such that $X(u, 0) = c(u)$ for all $u \in I$.

Let $\{\partial_1, \partial_2, \partial_3, \partial_4\}$ be the local orthonormal frame adapted to S given in (4). Now write $\boxtimes$ in this basis,

$$\boxtimes(\partial_3, \partial_4, \partial_1) = \langle \boxtimes(\partial_3, \partial_4, \partial_1), \partial_2\rangle \partial_2 = -\det(\partial_1, \partial_2, \partial_3, \partial_4)\partial_2 = -\partial_2,$$

and since $X_v = \lambda \partial_2$, we have

$$\Psi(z) = X_u(z) - iX_v(z) = X_u + i \boxtimes (\nu(z), \tau(z), X_u(z)) \qquad (12)$$

in isothermal coordinates $(U, z = u + iv)$. Restricting $\Psi(z)$ to I and using the definition of c, n we obtain

$$\begin{aligned}\Psi(u,0) &= X_u(u,0) + i \boxtimes (\nu(u,0), \tau(u,0), X_u(u,0)) = \\ &= c'(u) + i \boxtimes (\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u)).\end{aligned}$$

Since these funtions are real analytic, we can extend them to two holomorphic functions $\Psi(z)$, $c'(z) + i \boxtimes (\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z))$ on a simply connected open set $\Omega \subseteq U$ and they coincide on $I \subset \Omega$. Hence by the *identity theorem* for analytic functions it follows that

$$\Psi(z) = c'(z) + i \boxtimes (\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z)), \ \forall\, z \in \Omega.$$

Therefore

$$\Gamma(z) := c(z) + i \int_{s_0}^{z} \boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))dw, \ \forall\, z \in \Omega$$

is well defined on Ω and obviously is the primitive of the holomorphic mapping $\Psi(z)$. Thus, (11) yields

$$X(z) = \mathfrak{Re}\left(c(z) + i \int_{s_0}^{z} \boxtimes (\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\, dw\right).$$

This completes the proof of the Theorem. □

Remark 3.2. We can choose any $s_0 \in I$ in (10) and the values of $X(z)$ will remain the same, since $c'(z), \mathfrak{Re}(n(z)), \mathfrak{Im}(n(z))$ all take real values in $I \in \Omega$.

Using the complex representation formula given in (10), we now show that the Björling problem has a unique solution.

Theorem 3.3. *There exists a unique solution $X : \Omega \to \mathbb{L}^4$ to the Björling problem for maximal surfaces in $\mathbb{L}^4$, which is given by*

$$X(z) = \mathfrak{Re}\left(c(z) + i \int_{s_0}^{z} \boxtimes (\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w))\, dw\right), \qquad (13)$$

with $w = u + iv \in \Omega$, $s_0 \in I$, where Ω is a simply connected open subset of $\mathbb{C}$ containing the real interval I and for which c, n admit holomorphic extensions $c(z), n(z)$.

Proof: Define the holomorphic curve $\Psi : \Omega \subseteq \mathbf{C} \to \mathbf{C}^4$ by

$$\Psi(z) = c'(z) + i \boxtimes (\mathfrak{Re}(n(z)), \mathfrak{Im}(n(z)), c'(z)), \ \forall\, z \in \Omega. \tag{14}$$

where Ω is a simply connected open subset of $\mathbb{C}$ containing I on which the holomorphic extensions $c(z), n(z)$ exist. Since by Proposition 2.1, $c'(u)$ and $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ are orthogonal and have the same length, it follows that

$$(\varphi^1(u,0))^2 + (\varphi^2(u,0))^2 + (\varphi^3(u,0))^2 - (\varphi^4(u,0))^2 = 0, \ \forall\, u \in I.$$

We also have that

$$|\varphi^1(u,0)|^2 + |\varphi^2(u,0)|^2 + |\varphi^3(u,0)|^2 - |\varphi^4(u,0)|^2 = 2\langle c'(u), c'(u)\rangle > 0.$$

Thus

$$(\varphi^1(z))^2 + (\varphi^2(z))^2 + (\varphi^3(z))^2 - (\varphi^4(z))^2 = 0,$$
$$|\varphi^1(z)|^2 + |\varphi^2(z)|^2 + |\varphi^3(z)|^2 - |\varphi^4(z)|^2 > 0,$$

for all $z \in \Omega$. Moreover, the holomorphic curve Ψ has no real periods for Ω is simply connected. Therefore by Theorem 2.6, $X(z) = \mathfrak{Re} \int_{s_0}^{z} \Psi(w)dw$ defines a maximal surface $S = X(\Omega)$ in $\mathbf{L}^4$, where Ψ is given by (14) and $s_0 \in I$. Now we shall check that this surface satisfies the Björling conditions $X(u,0) = c(u)$ and $A(u,0) = n(u)$. The verification of the first condition is easy, since $\boxtimes(\mathfrak{Re}(n), \mathfrak{Im}(n), c')$ is real when restricted to I. To check the second condition, first recall that $\Psi = 2(\partial X/\partial z)$. So it follows from (14) that, restricted to I, we have

$$X_u(u,0) = c'(u) \text{ and } X_v(u,0) = -\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u)).$$

On the other hand, from (12) we have

$$X_v(u,0) = -\boxtimes(\nu(u,0), \tau(u,0), c'(u)).$$

Since $\mathfrak{Im}(n(u))$ is future directed it follows that $\mathfrak{Re}(n(u)) = \nu(u,0)$ and $\mathfrak{Im}(n(u)) = \tau(u,0)$.

At last we will prove the uniqueness, which is to be understood in the following sense: If $\widetilde{X}(u,v)$, $z = u + iv \in \widetilde{\Omega}$ is another solution, then $X(u,v) = \widetilde{X}(u,v)$ for $z = u + iv \in \Omega \cap \widetilde{\Omega}$. In fact, any pair of solutions $X, \widetilde{X}$ to the Björling problem coincide on a real interval $I \subset \Omega \cap \widetilde{\Omega}$, and since both are analytic they must coincide on $\Omega \cap \widetilde{\Omega}$. This completes the proof of the Theorem. □

Remark 3.4. Observe that the unicity in the above theorem is only referred to maximal surfaces $X : \Omega \subseteq \mathbf{C} \to \mathbf{L}^4$ satisfying $X(u,0) = c(u)$ and $A(u,0) = n(u)$. Actually a little more can be proven: given an analytic strip (c,n) in $\mathbf{L}^4$, there exists a unique maximal immersion $X : M^2 \to \mathbf{L}^4$ whose image contains $c(I)$ and A restricted to c is n.

The existence part of this statement follows from Theorem 3.3. For the unicity part, we refer to corollary 3.4 of [2]. There unicity is proven for analytic strips in $\mathbb{L}^3$ and maximal surfaces in $\mathbb{L}^3$, but their arguments work in our case as well.

Example 3.5. Consider

$$\begin{cases} c(s) = (s - s^3, 0, s^2, 0) \in \mathbb{L}^4, \\ n(s) = \frac{1}{(1-2s^2+9s^4)^{1/2}}(2s, -2\sqrt{2}si, -(1-3s^2), (1+3s^2)i) \in \mathbb{C}^4, \end{cases}$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(s)), \mathfrak{Im}(n(s)), c'(s)) = (0, 1+3s^2, 0, -2\sqrt{2}s),$$

whose holomorphic extension is

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, 1+3w^2, 0, -2\sqrt{2}w).$$

Thus

$$X(z) := \mathfrak{Re}((z-z^3, 0, z^2, 0)) - \mathfrak{Im}((0, z+z^3-(s_0+s_0^3), 0, -\sqrt{2}z^2+\sqrt{2}s_0^2))$$

and therefore, the solution of the Björling problem for the given strip is

$$X(z) = (u + 3uv^2 - u^3, -v - 3u^2v + v^3, u^2 - v^2, 2\sqrt{2}uv),$$

with $z = u + iv \in \mathbb{C}$.

Example 3.6. Consider

$$\begin{cases} c(s) = (1 + \cos(s), 0, \sin(s), 2\sin(s/2)) \in \mathbb{L}^4, \\ n(s) = (\cos(s), 0, \sin(s), 0) + i(-1 - \cos(s), 0, \cos(s)\cot(s/2), csc(s/2)) \in \mathbb{C}^4, \end{cases}$$

for all $s \in (0, 2\pi)$. By similar calculations,

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, \sin(z/2), 0, 0).$$

Then

$$X(z) := \mathfrak{Re}((1+\cos(z), 0, \sin(z), 2\sin(z/2))) - \mathfrak{Im}(-2\cos(z/2) + 2\cos(s_0/2))$$

and therefore, the solution of the Björling problem for the given strip is

$$X(z) = (1+\cos(u)\cosh(v), -2\sin(u/2)\sinh(v/2), \cosh(v)\sin(u), 2\cosh(v/2)\sin(u/2)),$$

with $z = u + iv$, where $u \in (0, 2\pi)$ and $v \in \mathbb{R}$.

As consequences of Theorem 3.3 we recover the classical Björling problem for $\mathbb{R}^3$ and also the Björling problem for $\mathbb{L}^3$, see [2].

Corollary 3.7. *Let $c : I \to \mathbb{R}^3$, $\mathbb{R}^3 \cong \{x^4 = 0\} \subset \mathbb{L}^4$, be a regular real analytic curve and let $n : I \to \mathbb{C}^4$ be a real analytic vector field along c such that $n(s) = \xi(s) + ie_4$, where $\xi(s) \in \mathbb{R}^3$ is a unitary vector field satisfying $\langle c'(s), \xi(s)\rangle = 0$ for all $s \in I$. Then there exists a unique solution to the Björling problem for minimal surfaces in $\mathbb{R}^3$, which is given by*

$$X(z) = \Re\left\{c(z) - i\int_{s_0}^{z}(\xi(w) \times c'(w))dw\right\}, \tag{15}$$

where $w = u + iv \in \Omega$, $s_0 \in I$, Ω is a simply connected open set of $\mathbb{C}$ containing I and $\times$ is the cross-product of $\mathbb{R}^3$.

Proof: From Theorem 3.3 it follows that the solution to the Björling problem is given by

$$\begin{aligned} X(z) &= \Re\left(c(z) + i\int_{s_0}^{z} \boxtimes(\xi(w), e_4, c'(w))\, dw\right) \\ &= \Re\left(c(z) - i\int_{s_0}^{z} \boxtimes(\xi(w), c'(w), e_4)\, dw\right). \end{aligned}$$

Hence, from Proposition 2.1 item 2 we have

$$\begin{aligned} X(z) &= \Re\left(c(z) - i\int_{s_0}^{z} \widehat{\xi}(w) \times \widehat{c'}(w)dw\right) \\ &= \Re\left(c(z) - i\int_{s_0}^{z} \xi(w) \times c'(w)dw\right). \end{aligned}$$

□

Corollary 3.8. *Let $c : I \to \mathbb{L}^3$, $\mathbb{L}^3 \cong \{x^2 = 0\} \subset \mathbb{L}^4$, be a regular real analytic spacelike curve and let $n : I \to \mathbb{C}^4$ be a real analytic vector field along c of the form $n(s) = e_2 + iV(s)$, where $V(s) \in \mathbb{L}^3$ is a future directed, timelike unitary vector field such that $\langle c'(s), V(s)\rangle = 0$ for all $s \in I$. Then there exists a unique solution to the Björling problem for maximal surfaces in $\mathbb{L}^3$, which is given by*

$$X(z) = \Re\left\{c(z) + i\int_{s_0}^{z}(V(w) \times c'(w))dw\right\}, \tag{16}$$

where $w = u + iv \in \Omega$, $s_0 \in I$, Ω is a simply connected open set of $\mathbb{C}$ containing I and $\times$ is the cross-product in $\mathbb{L}^3$.

Proof: From Theorem 3.3 if follows that the solution to the Björling problem is given by

$$\begin{aligned} X(z) &= \Re e\left(c(z) + i\int_{s_0}^{z} \boxtimes\left(e_2, V(w), c'(w)\right) dw\right) \\ &= \Re e\left(c(z) + i\int_{s_0}^{z} \boxtimes\left(V(w), c'(w), e_2\right) dw\right). \end{aligned}$$

Hence, from Proposition 2.1 item 3 we have

$$\begin{aligned} X(z) &= \Re e\left(c(z) + i\int_{s_0}^{z} \check{V}(w) \times \check{c}'(w) dw\right) \\ &= \Re e\left(c(z) + i\int_{s_0}^{z} V(w) \times c'(w) dw\right). \end{aligned}$$

□

4. Symmetries

Now, we will study the symmetries of the maximal surfaces in $\mathbf{L}^4$ via the complex representation formula of the Björling problem for maximal surfaces. In order to do so we fix the following notation. Let $f(z) = x(z) + iy(z)$, where $x(z), y(z)$ are real–valued functions defined on the open set Ω of $\mathbb{C}$. If $x(z)$ is harmonic and $f(z)$ is holomorphic in Ω, then $x(\overline{z})$ is harmonic and $\overline{f(\overline{z})}$ is holomorphic as a function of z in the open set $\Omega^* := \{\overline{z} : z \in \Omega\}$. Note that, Ω is symmetric if only if $\Omega = \Omega^*$. We also have that, if $I \subset \Omega$, f is holomorphic in Ω and f restrict to I take only real values, then $f(z) = \overline{f(\overline{z})}$ on $I \subset \Omega \cap \Omega^*$. Therefore, $f(z)$ can be holomorphically extended to $\Omega \cup \Omega^*$.

Proposition 4.1. *Let $X : \Omega \subseteq \mathbb{C} \to \mathbf{L}^4$ be the solution of the Björling problem, for a given strip (c, n) in $\mathbf{L}^4$, where Ω is a symmetric simply connected open set containing the real interval I and for which c and n admit holomorphic extensions $c(z)$ and $n(z)$, where $z = u + iv \in \Omega$. Then for all $z \in \Omega$ we have*

$$X(\overline{z}) = \Re e\left\{c(z) - i\int_{s_0}^{z} \boxtimes(\Re e(n(w)), \Im m(n(w)), c'(w)) dw\right\}. \tag{17}$$

Proof: The surface $\widetilde{S} = \widetilde{X}(\Omega)$ given by $\widetilde{X}(u,v) := X(u,-v)$, clearly satisfies $\widetilde{X}_{uu}(u,v) = X_{uu}(u,-v)$, $\widetilde{X}_{vv}(u,v) = X_{vv}(u,-v)$ and still is a maximal surface in $\mathbf{L}^4$. Associated to $\widetilde{X}$, let $\widetilde{A}(u,v) := \widetilde{\nu}(u,v) + i\widetilde{\tau}(u,v)$. From Proposition 2.2 and the definition of $\boxtimes$, we have that

$$\widetilde{\tau_0}(u,v) = (\lambda^2 e_4 + x_u^4 X_u + x_v^4 X_v)(u,-v),$$

$$\widetilde{\nu_0}(u,v) = -\boxtimes(e_4, X_u(u,-v), X_v(u,-v)),$$

and hence $\widetilde{\tau}(u,v) = \tau(u,-v)$ and $\widetilde{\nu}(u,v) = -\nu(u,-v)$. Therefore,

$$\widetilde{A}(u,v) = -\overline{A(u,-v)}. \tag{18}$$

This implies that $\widetilde{A}(u,0) = -\overline{A(u,0)} = -\overline{n(u)}$ and $\widetilde{X}(u,0) = X(u,0) = c(u)$. Hence $\widetilde{X}$ is a solution of the Björling problem for $\tilde{c} = c$, $\tilde{n} = -\overline{n}$ and then $\widetilde{X}(z) = \Re e \int_{s_0}^{z} \widetilde{\Psi}(w)dw$, where $\widetilde{\Psi}(z) = \widetilde{X}_u + i\boxtimes(\widetilde{\nu}(z), \widetilde{\tau}(z), \widetilde{X}_u(z))$, see (13). Restricting $\widetilde{\Psi}(z)$ to I and using (18) we obtain

$$\begin{aligned}\widetilde{\Psi}(u,0) &= X_u(u,0) + i\boxtimes(-\nu(u,0), \tau(u,0), X_u(u,0)) \\ &= c'(u) - i\boxtimes(\Re e(n(u)), \Im m(n(u)), c'(u)).\end{aligned}$$

When we extend these functions to Ω^*, the result follows. □

The proofs of the following corollaries are analogous to those of corollaries 3.7 and 3.8.

Corollary 4.2. *Under the hypothesis of Proposition* 4.1, *if* $S = X(\Omega) \subset \mathbb{R}^3 \cong \{x^4 = 0\}$ *and* n *is of the form* $n(s) = \xi(s) + ie_4$, *with* $\xi(s) \in \mathbb{R}^3$ *unitary such that* $\langle c'(s), \xi(s)\rangle = 0$ *for all* $s \in I$, *then*

$$X(\overline{z}) = \Re e\left\{c(z) + i\int_{s_0}^{z}(\xi(w) \times c'(w))dw\right\}, \text{ for all } z \in \Omega. \tag{19}$$

Corollary 4.3. *Under the hypothesis of Proposition* 4.1, *if* $S = X(\Omega) \subset \mathbf{L}^3 \cong \{x^3 = 0\}$ *and* n *is of the form* $n(s) = e_2 + iV(s)$, *with* $V(s) \in \mathbf{L}^3$ *unitary, future directed, timelike and such that* $\langle c'(s), V(s)\rangle = 0$ *for all* $s \in I$, *then*

$$X(\overline{z}) = \Re e\left\{c(z) - i\int_{s_0}^{z}(V(w) \times c'(w))dw\right\}, \text{ for all } z \in \Omega. \tag{20}$$

Remark 4.4. Using the formulae (15) and (19), it is not difficult to recover the two symmetry principles discovered by Schwarz for minimal surfaces in $\mathbb{R}^3$(see [11] p. 123). Also, by using (16) and (20), we can recover the two symmetry principles for maximal surfaces in $\mathbf{L}^3$ given in [2], Theorem 3.10.

Now using (13) and (17) we will derive three symmetry principles for maximal surfaces in $\mathbf{L}^4$. They were motivated by the works of Schwarz and [2] above mentioned. Before going to it, we have the following definitions.

Definition 4.5. Let Π^k be a k–plane in $\mathbf{L}^4$. Assume that Π^k is spacelike if $k = 1$; Π^k is spacelike, timelike or degenerate if $k = 2$; Π^k is timelike if $k = 3$. Under those conditions, we say that Π^k is a k–plane of symmetry of a spacelike surface $X : M^2 \to \mathbf{L}^4$ if for all $p \in M^2$ there exists a certain $q \in M^2$ such that $X(p), X(q)$ are symmetric with respect to Π^k, that is, such that $(X(q)+X(p))/2 \in \Pi^k$ and $X(q)-X(p)$ is perpendicular to Π^k.

Theorem 4.6. *Let S be a maximal surface in L^4, given by $X : U \subseteq \mathbf{C} \to \mathbf{L}^4$. Then we have:*
1) Every spacelike straight line contained in S is an axis of symmetry of S;
2) If S intersects any timelike or spacelike 2–plane Π^2, orthogonally along a curve regular of S, then Π^2 is a plane of symmetry of S;
3) If S intersects any timelike 3–space Π^3, orthogonally along a curve regular of S, then Π^3 is a 3–plane of symmetry of S.

Before going through the proof, it is convenient to make the following observation. Suppose for instance that the maximal surface S contains a segment of line L, which, we may assume is a portion of the x^1–axis. Then it is possible to define isothermal coordinates $z = u + iv$ in a neighborhood of L so that $X(u,0)$ parametrizes L, see [18]. Analogous observations are in place in case S intersects orthogonally the x^1, x^4–plane, or the x^1, x^2–plane or the 3–space $\{x^3 = 0\}$.

Whith this in mind, it is not difficult to see that Theorem 4.6 is now a consequence of the following

Lemma 4.7. *Let S be a maximal surface in $\mathbf{L}^4$, given by $X : \Omega \subseteq \mathbf{C} \to \mathbf{L}^4$, with Ω is symmetric and simply connected.*
1) If, for all $u \in I$, the curve $c(u) = X(u,0)$, is contained in the x^1–axis, then

$$X(u,-v) = (x^1(u,v), -x^2(u,v), -x^3(u,v), -x^4(u,v)). \qquad (21)$$

2) If, for all $u \in I$, the curve $c(u) = X(u,0)$, is contained in the timelike x^1, x^4– plane Π^2, and if the surface S intersects Π^2 orthogonally along c, then

$$X(u,-v) = (x^1(u,v), -x^2(u,v), -x^3(u,v), x^4(u,v)). \qquad (22)$$

3) If, for all $u \in I$, the curve $c(u) = X(u,0)$, is contained in the spacelike x^1, x^2– plane Π^2, and if the surface S intersects Π^2 orthogonally along c, then

$$X(u,-v) = (x^1(u,v), x^2(u,v), -x^3(u,v), -x^4(u,v)). \qquad (23)$$

4) If, for all $u \in I$, the curve $c(u) = X(u,0)$, is contained in the timelike 3–space $\Pi^3 = \{x^2 = 0\}$, and if the surface S intersects Π^3 orthogonally along c, then

$$X(u,-v) = (x^1(u,v), -x^2(u,v), x^3(u,v), x^4(u,v)). \qquad (24)$$

Proof: (1) Set $c(u) := X(u,0)$ and $n(u) := A(u,0)$. By hypothesis, it follows that $c(u) = (c^1(u), 0, 0, 0)$, $\mathfrak{Re}(n(u)) = (0, \nu^2(u,0), \nu^3(u,0), \nu^4(u,0))$ and $\mathfrak{Im}(n(u)) = (0, \tau^2(u,0), \tau^3(u,0), \tau^4(u,0))$. Hence, by a straightforward calculation we have that $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2(u), \boxtimes^3(u), \boxtimes^4(u))$. On account of (13), (17) it follows respectively that

$$X(z) = \left(\mathfrak{Re}(c^1(z)), -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^2(w)dw, -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^3(w)dw, -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)dw\right),$$

$$X(\bar{z}) = \left(\mathfrak{Re}(c^1(z)), \mathfrak{Im}\int_{s_0}^{z} \boxtimes^2(w)dw, \mathfrak{Im}\int_{s_0}^{z} \boxtimes^3(w)dw, \mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)dw\right),$$

which proves (21).
(2) Since by hypothesis, S intersects $\Pi^2 = \{x^2 = 0, x^3 = 0\}$ orthogonally at $c(u) := X(u,0)$, it follows that $c(u) = (c^1(u), 0, 0, c^4(u))$. Now recall that the 2–plane P^2 generated by $\mathfrak{Re}(n(u))$ and $\mathfrak{Im}(n(u))$ is orthogonal to $T_{c(u)}S$ along c. It follows that $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2(u), \boxtimes^3(u), 0)$. On account of (13) and (17) we then arrive at the formula (22).
(3) The proof is analogous to item (2).
(4) The hypothesis implies that $c(u) = (c^1(u), 0, c^3(u), c^4(u))$. Since S intersects Π^3 orthogonally, we have that $X_v(u,0) \in (\Pi^3)^\perp$ and therefore $X_v(u,0)$ is parallel to the unitary vector e_2 which is normal to Π^3. Then $\mathfrak{Re}(n(u))$ and $\mathfrak{Im}(n(u))$ lie in Π^3, which implies that the second component of both vectors are equal to zero. Hence $\boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(0, \boxtimes^2, 0, 0)$. Therefore, in conjunction with (13) and (17) we obtain (24).

□

If in Theorem 4.1 Π^2 is a degenerate two plane, we have:

Proposition 4.8. *Let $X : \Omega \subseteq \mathbb{C} \to \mathbb{L}^4$ be a maximal surface, with Ω symmetric, simply connected and assume that $S = X(\Omega)$ intersects the degenerate 2–plane $\Pi^2 = [e_1 + e_4, e_2]$ orthogonally along the curve $c(u) = X(u,0)$. Then S is contained in the degenerate 3–space $\Pi^3 = [e_1 + e_4, e_2, e_3]$. Moreover Π^2 is a plane of symmetry for S if and only if $X_v(u,0)$ is a multiple of e_3.*

Proof: Consider the basis $\mathcal{F}=\{\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4\}$ of $\mathbb{L}^4$, where $\epsilon_1 = \frac{\sqrt{2}}{2}(e_1 + e_4)$, $\epsilon_2 = \frac{\sqrt{2}}{2}(e_1 - e_4)$, $\epsilon_3 = e_2$, $\epsilon_4 = e_3$ and observe that $\Pi^2 = [\epsilon_1, \epsilon_3]$. It is clear that $c(u) = X(u,0)$ is of the form $c(s) = (c^1(s), c^2(s), 0, c^1(s))$. Since the 2–plane $P^2 = [\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u))]$ is orthogonal to $T_{c(u)}S$ along c, is follows that $-X_v(u,0) = \boxtimes(\mathfrak{Re}(n(u)), \mathfrak{Im}(n(u)), c'(u))$ is of the form $(\boxtimes^1(u), 0, \boxtimes^3(u), \boxtimes^1(u))$. By the same arguments as before, we obtain that

$$X(z) = \left(\mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw, \mathfrak{Re}(c^2(z)), -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^3(w)dw,\right.$$

$$\left.\mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw\right),$$

$$X(\overline{z}) = \left(\mathfrak{Re}(c^1(z)) + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw, \mathfrak{Re}(c^2(z)), \mathfrak{Im}\int_{s_0}^{z} \boxtimes^3(w)dw,\right.$$

$$\left.\mathfrak{Re}(c^1(z)) + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw\right),$$

which written in the basis $\mathcal{F}$ gives respectively

$$X(z) = \left(\mathfrak{Re}(c^1(z)) - \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw, 0, \mathfrak{Re}(c^3(z)), -\mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)dw,\right)_{\mathcal{F}},$$

$$X(\overline{z}) = \left(\mathfrak{Re}(c^1(z)) + \mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw, 0, \mathfrak{Re}(c^3(z)), \mathfrak{Im}\int_{s_0}^{z} \boxtimes^4(w)dw,\right)_{\mathcal{F}}.$$

The first part is clear and S is symmetric with respect to Π^2 if and only if $\mathfrak{Im}\int_{s_0}^{z} \boxtimes^1(w)dw = 0$, that is, $\int_{s_0}^{z} \boxtimes^1(w)dw \equiv 0$ and the last claim follows.

□

Remark 4.9. 1)It is not difficult to see that Lemma 4.7 and Proposition 4.8 hold without the simply connectivity assumption.
2) Observe that if Π^3 is spacelike or degenerate, then there is no spacelike vector orthogonal to Π^3 in $\mathbb{L}^4$. Therefore the symmetry problem of maximal surfaces in not defined is these cases.

Example 4.10. Consider

$$\begin{cases} c(s) = (0, s, 0, 0) \in \mathbb{L}^4, \\ n(s) = \left(\frac{e^{-s}}{\sqrt{4+e^{-2s}}}, 0, -\frac{2}{\sqrt{4+e^{-2s}}}, 0\right) + i\left(-\frac{e^{-s}}{\sqrt{4+e^{-2s}}}, 0, -\frac{e^{-2s}}{2\sqrt{4+e^{-2s}}}, \frac{\sqrt{4+e^{-2s}}}{2}\right) \in \mathbb{C}^4, \end{cases}$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (-1, 0, -\frac{e^{-w}}{2}, \frac{e^{-w}}{2}).$$

Therefore, the solution of the Björling problem for the given strip is

$$X(z) = (v, u, \frac{1}{2}e^{-u}\sin(v), -\frac{1}{2}e^{-u}\sin(v)),$$

with $z = u + iv \in \mathbb{C}$. Note that x^2 is an axis of symmetry of the complete maximal surface $S = X(\mathbb{C})$.

Example 4.11. Consider

$$\begin{cases} c(s) = (\sinh(s), 0, 0, \cosh(s)) \in \mathbf{L}^4, \\ n(s) = (0, \cos(s), \sin(s), 0) + i(\sinh(s), 0, 0, \cosh(s)) \in \mathbb{C}^4, \end{cases}$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, -\sin(w), \cos(w), 0).$$

Therefore, the solution of the Björling problem for the given strip is the complete maximal surface

$$X(z) = \begin{bmatrix} \cosh(u) & 0 & 0 & \sinh(u) \\ 0 & \cos(u) & -\sin(u) & 0 \\ 0 & \sin(u) & \cos(u) & 0 \\ \sinh(u) & 0 & 0 & \cosh(u) \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ -\sinh(v) \\ \cos(v) \end{bmatrix},$$

with $z = u + iv \in \mathbb{C}$. Note that $\Pi^2 = [e_1, e_4]$ is a timelike 2–plane of symmetry of the surface $S = X(\mathbb{C})$.

Example 4.12. Consider

$$\begin{cases} c(s) = (\cos(s), \sin(s), 0, 0) \in \mathbf{L}^4, \\ n(s) = (\cos(s), \sin(s), 0, 0) + i(0, 0, \sinh(s), \cosh(s)) \in \mathbb{C}^4, \end{cases}$$

for all $s \in \mathbb{R}$. By a straightforward calculation, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (0, 0, -\cosh(w), -\sinh(w), 0).$$

Therefore, the solution of the Björling problem for the given strip is the complete maximal surface

$$X(z) = \begin{bmatrix} \cos(u) & -\sin(u) & 0 & 0 \\ \sin(u) & \cos(u) & 0 & 0 \\ 0 & 0 & \cosh(u) & \sinh(u) \\ 0 & 0 & \sinh(u) & \cosh(u) \end{bmatrix} \begin{bmatrix} \cosh(v) \\ 0 \\ \sin(v) \\ 0 \end{bmatrix},$$

with $z = u + iv \in \mathbb{C}$. Note that $\Pi^2 = [e_1, e_2]$ is a spacelike 2–plane of symmetry of the surface $S = X(\mathbb{C})$.

Example 4.13. Consider

$$\begin{cases} c(s) = (s^2, s, 0, s^2) \in \mathbf{L}^4, \\ n(s) = \frac{1}{\sqrt{2+4s^2}}\{(1, -2s, -1, 0) + i(1 + 4s^2, 2s, 1, 2 + 4s^2)\} \in \mathbb{C}^4, \end{cases}$$

for all $s \in \mathbb{R}$. Calculating as above, we obtain that

$$\boxtimes(\mathfrak{Re}(n(w)), \mathfrak{Im}(n(w)), c'(w)) = (-1, 0, -1, -1).$$

The solution of the Björling problem for the given strip is the complete maximal surface

$$X(z) = (u^2 - v^2 + v, u, v, u^2 - v^2 + v),$$

with $z = u + iv \in \mathbb{C}$. This surface intersects the degenerate 2–plane $\Pi^2 = [e_1 + e_4, e_2]$ orthogonally along $X(u, 0) = c(u)$, but Π^2 is not a plane of symmetry of S. On the other hand, if we take again the curve $c(s) = (s^2, s, 0, s^2)$, but take

$$n(s) = \frac{1}{\sqrt{1+4s^2}}\{(1, -2s, 0, 0) + i(4s^2, 2s, 0, 1 + 4s^2)\},$$

this time, we obtain

$$\boxtimes(\Re e(n(w)), \Im m(n(w)), c'(w)) = (0, 0, -1, 0)$$

and

$$X(z) = (u^2 - v^2, u, v, u^2 - v^2),$$

which is symmetric with respect to the 2–plane Π^2.

Example 4.14. The timelike 3–space $\Pi^3 = \{x^2 = 0\}$ is a 3–space of symmetry of maximal surface S given in the example 3.6.

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