

Entropy production and heat transport in harmonic chains under time-dependent periodic drivings

Bruno A. N. Akasaki , Mário J. de Oliveira , and C. E. Fiore 

Universidade de São Paulo, Instituto de Física, Rua do Matão, 1371, 05508-090 São Paulo, SP, Brazil



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Using stochastic thermodynamics, the properties of interacting linear chains subject to periodic drivings are investigated. The systems are described by Fokker-Planck-Kramers equation and exact solutions are obtained as functions of the modulation frequency and strength constants. Analysis will be carried out for short and long chains. In the former case, explicit expressions are derived for a chain of two particles, in which the entropy production is written down as a bilinear function of thermodynamic forces and fluxes, whose associated Onsager coefficients are evaluated for distinct kinds of periodic drivings. The limit of long chains is analyzed by means of a protocol in which the intermediate temperatures are self-consistently chosen and the entropy production is decomposed as a sum of two individual contributions, one coming from real baths (placed at extremities of lattice) and other from self-consistent baths. Whenever the former dominates for short chains, the latter contribution prevails for long ones. The thermal reservoirs lead to a heat flux according to Fourier's law.

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I. INTRODUCTION

The description of thermodynamic quantities at the mesoscopic level gives rise to the stochastic thermodynamics [1–3], in which fluctuations in the thermodynamic fluxes become important. This theory not only allows to reproduce the fundamental concepts of thermodynamics of equilibrium systems but can also be extended for the more general case of nonequilibrium ones. In particular, it shows that stochastic fluxes satisfy general relations such as the Jarzynski equality [4,5] or/and it predicts the existence of general bounds among thermodynamic fluxes [6,7].

Entropy production has played a fundamental role in nonequilibrium statistical physics not only for typifying the irreversibility [8–10], but also for tackling general considerations about efficiency of heat engines [11], the analysis of (irreversible) phase transition portraits [12–14], thermodynamic uncertainties relations [6,7], and others. A fundamental relation for the entropy production comes from simple entropic arguments in which the system is coupled to a thermal reservoir. Its time entropy variation dS/dt is the difference of two terms

$$\frac{dS}{dt} = \Pi(t) - \Phi(t), \quad (1)$$

where $\Pi(t)$ is the entropy production rate and $\Phi(t)$ is the entropy flux rate from/to the system to/from the environment. Since the environment works as a subsystem in equilibrium, $\Pi(t)$ corresponds to the entropy produced inside the system. Equation (1) implies that all entropy spontaneously produced (by the system) has to be delivered to the environment in the steady state regime. When the system is in thermal equilibrium, it follows that $\Pi_s = \Phi_s = 0$, whereas $\Pi_s = \Phi_s > 0$ out of the equilibrium regime. Thereby entropy production discerns equilibrium and nonequilibrium systems, since it is

continuously produced in the latter case. In such case, the steady entropy production rate can alternatively be evaluated through the calculation of the steady entropy flux Φ_s .

The thermodynamic properties of Markovian systems have been extensively studied in the framework of master [8–10,15,16] and Fokker-Planck equations [17–25]. A special recent attention has been devoted to periodically driven systems [26–29]. In part because their thermodynamic properties can be experimentally accessible [26,30–47]. In addition, some of their remarkable features, such as a general description in the linear regime (Onsager coefficients and general reciprocal relations can be achieved), the existence of uncertainties constraints leading to existence of bounds among macroscopic averages and other features have been put under a firmer basis. In other cases, the probability distribution of work for systems described by Langevin equations with time dependent drivings has also been analyzed [24]. However, the more general case of interacting particles subject to time periodic drivings has not been studied thoroughly. In particular, the question of how the inclusion of interaction changes the nonequilibrium trademarks as well as the entropy production properties has not been addressed before.

In this paper, we fill this gap by investigating the thermodynamic properties of interacting chains of Brownian particles subject to (time dependent) periodic forces and temperature drivings. Exact expressions for the thermodynamic quantities, including the dissipated heat, entropy production, heat flux and others are obtained. The limits of short and long particle chains are thoroughly investigated. For the latter case, intermediate temperatures are self-consistently chosen through a protocol taking into account an inner entropy production source. This is meaningfully different from the original approach by Bosterli *et al.* [48], in which no heat flux is exchanged among the particles and self-consistent baths. Thus, our approach provides us not only to analyze the role

of two distinct sources of dissipation, but also establishing which contribution dominates in the limits of short and long chains. Two main findings can be drawn: For the case of two interacting particles, the entropy production is derived as a bilinear function of fluxes and forces for both drivings in forces and temperatures, whose associated Onsager coefficients depend on the interaction parameters and frequency driving. Remarkably, the effect of a phase difference (a lag) between external forces is investigated and the condition for maximum/minimum entropy production is found to depend only on the temperature reservoirs and frequency driving, irrespective of the interaction strength between particles. The entropy production of long particle chains can be split in two terms: one coming from the thermal reservoirs and the other from the self-consistent ones.

This paper is organized as follows. In Sec. II we describe the theoretical background and the exact solution for time dependent drivings is evaluated in Sec. III. In Secs. IV and V, the two and several particles cases are analyzed, respectively. Conclusions are discussed in Sec. VI.

II. FOKKER-PLANCK-KRAMERS EQUATION

We consider a set of N interacting particles with equal masses m , in which the i th particle evolves in time according to the following set of coupled Langevin equations:

$$\frac{dv_i}{dt} = f_i^* - \gamma v_i + \mathcal{F}_i(t), \quad (2)$$

and

$$\frac{dx_i}{dt} = v_i, \quad (3)$$

with x_i denoting its position with velocity $v_i = dx_i/dt$, respectively, whereas γ is the dissipation constant. Here, f_i^* stands for the force acting to the i th particle, which is assumed to be decomposed as the sum of a time dependent term $f_i^{\text{ext}}(t)$ plus a term f_i depending only on the positions. Thus, f_i can be written as the derivative of the potential energy V , $f_i = -\partial V/\partial x_i$. The stochastic force $\mathcal{F}_i(t)$ accounts for the interaction between particle i and the environment and satisfies the properties

$$\langle \mathcal{F}_i(t) \rangle = 0, \quad (4)$$

and

$$\langle \mathcal{F}_i(t) \mathcal{F}_j(t') \rangle = 2\gamma T_i \delta_{ij} \delta(t - t'), \quad (5)$$

respectively, where $T_i > 0$ is distinct for each particle. Let $P(x, v, t) \equiv P(x_1, \dots, x_N, v_1, \dots, v_N, t)$ be the joint probability distribution at time t , where x and v denote the collection of particle positions x_i and velocities v_i , respectively. Its time evolution is described by the Fokker-Planck-Kramers (FPK) equation [3,9,18]

$$\frac{\partial P}{\partial t} = - \sum_i \left(v_i \frac{\partial P}{\partial x_i} + [f_i + f_i^{\text{ext}}(t)] \frac{\partial P}{\partial v_i} + \frac{\partial J_i}{\partial v_i} \right), \quad (6)$$

where

$$J_i = -\gamma v_i P - \frac{\gamma k_B T_i}{m} \frac{\partial P}{\partial v_i}. \quad (7)$$

If the temperatures of all particles T_i are the same and the external forces are null, the probability distribution approaches for large times the Gibbs equilibrium distribution,

$$P^e(x, v) = \frac{1}{Z} e^{-E/k_B T}, \quad (8)$$

where $E = mv^2/2 + V$ is the energy of the system. This result shows that the FPK Eq. (6) indeed describes the contact of a system with a heat reservoir at a temperature T . On the other hand, this will not be the case of the system in contact with distinct reservoirs and/or when it is subject to time oscillating forces or temperatures. In such case, the system dissipates heat and continuously produce entropy.

From the FK equation, the time variation of the energy $U = \langle E \rangle$ reads

$$\frac{dU}{dt} = - \sum_{i=1}^N (\Phi_q^{(i)} + \Phi_w^{(i)}), \quad (9)$$

where the heat flux $\Phi_q^{(i)}$ from the system to the environment (thermal bath) is expressed as [9,18]

$$\Phi_q^{(i)} = \gamma (m \langle v_i^2 \rangle - k_B T_i), \quad (10)$$

whose first and second terms can be understood as the heating power and the power of heat losses, respectively. The term $\Phi_w^{(i)}$ can be interpreted as the work per unity of time given by

$$\Phi_w^{(i)} = -m \langle v_i \rangle f_i^{\text{ext}}(t). \quad (11)$$

In the absence of external forces all heat flux comes from/goes to the thermal bath.

The entropy S of the system is determined from the Gibbs expression

$$S = -k_B \int P \ln P dx dv. \quad (12)$$

From the FPK equation, one finds that its time derivative has the form of Eq. (1), where the first is identified as the rate of entropy production given by [9,18]

$$\Pi = \frac{mk_B}{\gamma} \sum_{i=1}^N \frac{1}{T_i} \int \frac{J_i^2}{P} dx dv. \quad (13)$$

Note that $\Pi \geq 0$ (as expected). Conversely, the second term corresponds to the flux of entropy given by

$$\Phi = - \sum_{i=1}^N \frac{mk_B}{T_i} \int v_i J_i dx dv, \quad (14)$$

or even rewritten as

$$\Phi = k_B \sum_{i=1}^N \frac{\Phi_q^{(i)}}{T_i}. \quad (15)$$

As mentioned previously, Eq. (15) can be alternatively used for evaluated the steady production of entropy, since it depends only on averages $\langle v_i^2 \rangle$ and on the temperatures T_i .

III. EXACT SOLUTION FOR TIME DEPENDENT EXTERNAL FORCES

For simplifying matters, from now on we shall adopt $k_B = 1$. Except in Sec. IV B, all analyses will be restricted to the case of a chain of N particles interacting to its nearest neighbors by means of harmonic forces and also subject to individual and external forces. The expression for the force of i th particle f_i^* then reads

$$f_i^* = -\frac{k}{m}(x_i - x_{i+1}) - \frac{k^*}{m}x_i + f_i^{\text{ext}}(t), \quad (16)$$

$$f_i^* = -\frac{k}{m}(x_i - x_{i-1}) - \frac{k^*}{m}x_i + f_i^{\text{ext}}(t), \quad (17)$$

for particles placed at extremities, $i = 1$ and N , respectively, and

$$f_i^* = -\frac{k}{m}(2x_i - x_{i-1} - x_{i+1}) - \frac{k^*}{m}x_i + f_i^{\text{ext}}(t), \quad (18)$$

for the intermediate ones. Quantities k^* and k are spring constants characterizing individual harmonic forces and the coupling between neighboring particles, respectively. Above expressions can be conveniently rewritten as

$$f_i^* = -Kx_i + Lx_{i+1} + f_i^{\text{ext}}(t), \quad (19)$$

$$f_i^* = -Kx_i + Lx_{i-1} + f_i^{\text{ext}}(t), \quad (20)$$

and, for $i = 1$ and $i = N$, respectively, and

$$f_i^* = -(K + L)x_i + L(x_{i+1} + x_{i-1}) + f_i^{\text{ext}}(t), \quad (21)$$

respectively, where $L = k/m$ and $K = (k + k^*)/m$ for $i \neq 1$ and $i \neq N$.

The time evolution of a generic average of type $\langle g \rangle = \int g(x, v)P(x, v, t)dx dv$ is obtained through the expression

$$\frac{d}{dt}\langle g \rangle = \int g(x, v) \frac{\partial P}{\partial t} dx dv, \quad (22)$$

and by inserting Eq. (6) into Eq. (22) and performing appropriate partial integrations, an explicit equation for the time evolution of $\langle g \rangle$ is evaluated in terms of correlations associated to the positions and velocities. Due to the time dependence on the external forces, the evaluation of averages like $\langle g \rangle$ becomes cumbersome. However, the calculations become quite simpler by rewriting the motion equations in terms of their associate covariances. For instance, let us take for example a generic average $\langle g \rangle = \langle v_i^l x_j^m \rangle$ (with $l \geq 1$ and $m \geq 1$) with covariance given by $\langle v_i^l x_j^m \rangle_{cv} \equiv \langle v_i^l x_j^m \rangle - \langle v_i^l \rangle \langle x_j^m \rangle$. Unlike the time evolution of $\langle v_i^l x_j^m \rangle$, the time equation for $d\langle v_i^l x_j^m \rangle_{cv}/dt = d\langle v_i^l x_j^m \rangle/dt - \langle x_j^m \rangle d\langle v_i^l \rangle/dt - \langle v_i^l \rangle d\langle x_j^m \rangle/dt$ does not depend explicitly on t . Since the equations for all covariances are linear and time independent, the exact solution is possible for all system sizes N . Finally, having the covariances $\langle v_i^2 \rangle_{cv}$ and the averages $\langle v_i \rangle$, the entropy flux can be directly evaluated from the usage of Eqs. (10) and (15).

Below we derive explicit expressions for distinct covariances between the i -th and $i + 1$ -th particles for a generic

chain of N sites:

$$\frac{d}{dt}\langle x_i^2 \rangle_{cv} = 2\langle v_i x_i \rangle_{cv}, \quad (23)$$

$$\frac{d}{dt}\langle x_i x_{i+1} \rangle_{cv} = \langle v_i x_{i+1} \rangle_{cv} + \langle v_i x_{i+1} \rangle_{cv}, \quad (24)$$

$$\frac{d}{dt}\langle x_i v_i \rangle_{cv} = \langle v_i^2 \rangle_{cv} - K\langle x_i^2 \rangle_{cv} + L\langle x_i x_{i+1} \rangle_{cv} - \gamma\langle x_i v_i \rangle_{cv}, \quad (25)$$

$$\begin{aligned} \frac{d}{dt}\langle x_i v_{i+1} \rangle_{cv} &= \langle v_i v_{i+1} \rangle_{cv} - K\langle x_i x_{i+1} \rangle_{cv} + L\langle x_i^2 \rangle_{cv} \\ &\quad - \gamma\langle x_i v_{i+1} \rangle_{cv}, \end{aligned} \quad (26)$$

$$\frac{d}{dt}\langle v_i^2 \rangle_{cv} = -2K\langle v_i x_i \rangle_{cv} + 2L\langle x_{i+1} v_i \rangle_{cv} - 2\gamma\langle v_i^2 \rangle_{cv} + \Gamma_i, \quad (27)$$

$$\begin{aligned} \frac{d}{dt}\langle v_i v_{i+1} \rangle_{cv} &= -K\langle x_i v_{i+1} \rangle_{cv} + L\langle x_{i+1} v_{i+1} \rangle_{cv} \\ &\quad - K\langle x_{i+1} v_i \rangle_{cv} + L\langle x_i v_i \rangle_{cv} - 2\gamma\langle v_i v_{i+1} \rangle_{cv}. \end{aligned} \quad (28)$$

Here, we introduced the rescaled temperature Γ_i defined by $\Gamma_i = 2\gamma T_i/m$ and, thereby for fixed Γ_i 's, the achievement of $\langle v_i^2 \rangle_{cv}$'s reduces to systems of linear equations.

The time evolution of single averages $\langle v_i \rangle$ and $\langle x_i \rangle$ are also required for obtaining $\langle v_i^2 \rangle$, whose expressions read

$$\begin{aligned} \frac{d}{dt}\langle v_i \rangle &= -(K + L)\langle x_i \rangle + L(\langle x_{i+1} \rangle + \langle x_{i-1} \rangle) \\ &\quad - \gamma\langle v_i \rangle + f_i^{\text{ext}}(t), \end{aligned} \quad (29)$$

for $i \neq 1, N$ and

$$\frac{d}{dt}\langle v_i \rangle = -K\langle x_i \rangle + L\langle x_{i+1} \rangle - \gamma\langle v_i \rangle + f_i^{\text{ext}}(t), \quad (30)$$

$$\frac{d}{dt}\langle v_i \rangle = -K\langle x_i \rangle + L\langle x_{i-1} \rangle - \gamma\langle v_i \rangle + f_i^{\text{ext}}(t), \quad (31)$$

for $i = 1$ and N , respectively and from Eq. (3), the time evolution of $\langle x_i \rangle$ reads

$$\frac{d}{dt}\langle x_i \rangle = \langle v_i \rangle. \quad (32)$$

Although the previous procedure does not depend on the shape of external forces, from now on we will restrict our analysis to harmonic external forces given by $f_i^{\text{ext}}(t) = f_{0i} \cos(\omega t + \phi)$ with ω and ϕ being its frequency and phase difference (lag), respectively. By assuming that each $\langle x_i \rangle$ has solution of type $\langle x_i \rangle = A_{0i} + A_{1i} \cos \omega t + A_{2i} \sin \omega t$, $\langle v_i \rangle$ becomes $\langle v_i \rangle = \omega[A_{2i} \cos(\omega t) - A_{1i} \sin(\omega t)]$. By inserting above solutions in Eqs. (29) [or Eqs. (30), (31)] and (32), the coefficients A_{1i} and A_{2i} are obtained. It is worth mentioning that coefficients vary with the number of particles and their expressions for two particles are listed in Appendix A.

IV. TWO PARTICLES CASE

A. Oscillating forces

In the first application we analyze a chain of two interacting particles subject to harmonic forces without phase

difference (lagless case, $\phi = 0$). From the solution of linear set of equations described in Sec. III, we obtain the following expressions for the covariances:

$$\langle v_1^2 \rangle_{cv} = \frac{\Gamma_1 + \Gamma_2}{4\gamma} + \frac{K\gamma(\Gamma_1 - \Gamma_2)}{4(L^2 + K\gamma^2)}, \quad (33)$$

and

$$\langle v_2^2 \rangle_{cv} = \frac{\Gamma_1 + \Gamma_2}{4\gamma} - \frac{K\gamma(\Gamma_1 - \Gamma_2)}{4(L^2 + K\gamma^2)}. \quad (34)$$

Taking into account that $\langle v_i^2 \rangle = \langle v_i^2 \rangle_{cv} + \langle v_i \rangle^2$ (for $i = 1$ and 2), we see that the entropy flux can be split in two parts,

$$\Phi(t) = \Phi_T + \Phi_f(t), \quad (35)$$

where Φ_T and $\Phi_f(t)$ read

$$\Phi_T = \frac{2\gamma^2}{\Gamma_1} \langle v_1^2 \rangle_{cv} + \frac{2\gamma^2}{\Gamma_2} \langle v_2^2 \rangle_{cv} - 2\gamma, \quad (36)$$

and

$$\Phi_f(t) = \frac{2\gamma^2}{\Gamma_1} \langle v_1 \rangle^2 + \frac{2\gamma^2}{\Gamma_2} \langle v_2 \rangle^2, \quad (37)$$

respectively. The former term can be identified as the entropy flux coming from the thermal reservoirs, whereas the latter

is associated to the entropy flux coming from the oscillating forces.

The above expressions can be simplified, acquiring the following form:

$$\Phi_T = \frac{\gamma L^2}{2(L^2 + K\gamma^2)} \frac{(\Gamma_1 - \Gamma_2)^2}{\Gamma_1 \Gamma_2}, \quad (38)$$

and

$$\Phi_f(t) = 2\gamma^2 \omega^2 \sum_i \left(\frac{(A_{2i} \cos \omega t - A_{1i} \sin \omega t)^2}{\Gamma_i} \right), \quad (39)$$

respectively, whose coefficients A_{1i} and A_{2i} are shown in Appendix A. Since we are interested in the steady state regime, we shall appeal to Eq. (1), referring to the first and second terms as the entropy production rates $\Pi_T \equiv \Phi_T$ and $\bar{\Pi} \equiv \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \Phi_f(t) dt$, respectively. Once again, $\Pi_T \equiv \Phi_T$ solely depends on the difference of temperatures and are similar to the case with no external forces [18], whereas $\bar{\Pi}$ is related to the time dependent forces averaged over a oscillation period and it is given by

$$\bar{\Pi} = \frac{\gamma^2 \omega^2 [\Gamma_1 (A_{12}^2 + A_{22}^2) + \Gamma_2 (A_{11}^2 + A_{21}^2)]}{\Gamma_1 \Gamma_2}. \quad (40)$$

By substituting the expressions for A_{1i} 's and A_{2i} 's we finally arrive at the following expression:

$$\bar{\Pi} = \frac{\gamma^2 \omega^2 \{ f_{01}^2 [\Gamma_2 (\gamma^2 \omega^2 + (K - \omega^2)^2) + L^2 \Gamma_1] + 2f_{01} f_{02} L (K - \omega^2) (\Gamma_1 + \Gamma_2) + f_{02}^2 [\Gamma_1 (\gamma^2 \omega^2 + (K - \omega^2)^2) + L^2 \Gamma_2] \}}{\Gamma_1 \Gamma_2 [\gamma^2 \omega^2 + (K + L - \omega^2)^2] [\gamma^2 \omega^2 + (-K + L + \omega^2)^2]}. \quad (41)$$

This is the one of main results of the paper, and we pause to make a few comments: First, from Eq. (40) it follows that $\bar{\Pi}$ is always greater than 0, vanishing when $f_{01} = f_{02} = 0$ and/or $\omega = 0$. Second, in the limit of slow or fast oscillations, $\omega \ll 1$ or $\omega \gg 1$, $\bar{\Pi}$ behaves as

$$\bar{\Pi} \sim \frac{\gamma^2 [\Gamma_1 (f_{01} K + f_{02} L)^2 + \Gamma_2 (f_{01} L + f_{02} K)^2] \omega^2}{\Gamma_1 \Gamma_2 (K^2 - L^2)^2}, \quad (42)$$

and

$$\bar{\Pi} \sim \left(\frac{f_{01}^2 \Gamma_2 + f_{02}^2 \Gamma_1}{\Gamma_1 \Gamma_2} \right) \frac{\gamma^2}{\gamma^2 + \omega^2}, \quad (43)$$

respectively, implying that $\bar{\Pi}$ vanishes as ω^2 and $1/\omega^2$ for low and large frequencies, respectively.

Third, there is an intermediate frequency ω^* in which $\bar{\Pi}$ is maximum. Although ω^* can be evaluated exactly, it displays an unwieldy dependence on the control parameters. For this reason, we split the analysis in four parts, by inspecting its dependence on the interaction parameters K and L , the dissipation constant γ and the ratio between external forces f_{02}/f_{01} , as depicted in Fig. 1. Whenever ω^* increases by raising K, L and the ratio f_{02}/f_{01} , it decreases when γ is increased. Fourth, when the interaction between particles is “weak”, $k \ll k^*$, $\bar{\Pi}$ reduces to the single forced harmonic oscillator expression

$$\bar{\Pi} \sim \frac{\gamma^2 \omega^2 (f_{01}^2 \Gamma_2 + f_{02}^2 \Gamma_1)}{\Gamma_1 \Gamma_2 [\gamma^2 \omega^2 + (K - \omega)^2]}, \quad (44)$$

acquiring the simpler form

$$\bar{\Pi} \sim \frac{\gamma^2 \omega^2 f_{01}^2}{\Gamma_1 [\gamma^2 \omega^2 + (K - \omega)^2]}, \quad (45)$$

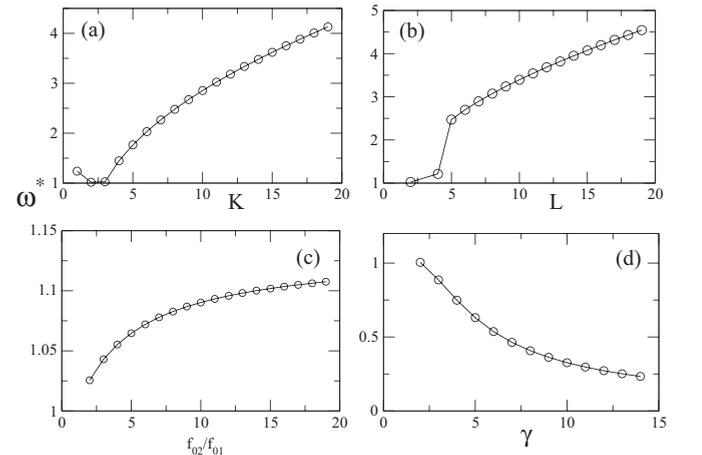


FIG. 1. The frequency ω^* in which $\bar{\Pi}$ is maximum versus distinct control parameters for $\Gamma_1 = 1$ and $\Gamma_2 = 10$. In (a), (b), (c), and (d) we take $\gamma = 1, L = 2$ and $f_{02}/f_{01} = 2$, $\gamma = 1, K = 3$ and $f_{02}/f_{01} = 2$, $\gamma = 1, K = 3$ and $L = 2$ and $K = 3, L = 2$ and $f_{02}/f_{01} = 2$, respectively.

as $\Gamma_1 = \Gamma_2$ and $f_{01} = f_{02}$. Fifth and last, in the strong coupling regime, $k \gg k^*$ and $k/m \gg \omega^2$ (or equivalently $L \approx K$ and $L \gg \omega^2$), $\bar{\Pi}$ becomes

$$\bar{\Pi} \sim \left(\frac{\gamma^2}{\gamma^2 + \omega^2} \right) \left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2} \right) (f_{01} + f_{02})^2, \quad (46)$$

which is independent on strength oscillator parameters K and L .

1. Bilinear form and Onsager coefficients

The shapes of Eqs. (38) and (41) show that the entropy production components can be written as flux-times-force

expressions $\Pi_T = \mathcal{J}_T f_T$ and

$$\bar{\Pi} = \mathcal{J}_1^f f_{01} + \mathcal{J}_2^f f_{02}, \quad (47)$$

respectively, where the forces $f_T = 1/\Gamma_1 - 1/\Gamma_2$ and $f_{0i(j)}$ have associated fluxes \mathcal{J}_T , \mathcal{J}_1^f , and \mathcal{J}_2^f given by

$$\mathcal{J}_T = \frac{\Gamma_1 \Gamma_2 \gamma L^2}{2(L^2 + K\gamma^2)} \left(\frac{1}{\Gamma_1} - \frac{1}{\Gamma_2} \right), \quad (48)$$

and

$$\mathcal{J}_1^f = L_{11} f_{01} + L_{12} f_{02} \quad \text{and} \quad \mathcal{J}_2^f = L_{21} f_{01} + L_{22} f_{02}, \quad (49)$$

respectively. The bilinear form for $\bar{\Pi}$ provides to identify the terms L_{11} and L_{12} as the associated Onsager coefficients given by

$$L_{11} = \frac{\gamma^2 \omega^2 [\Gamma_2 (\gamma^2 \omega^2 + (K - \omega^2)^2) + L^2 \Gamma_1]}{\Gamma_1 \Gamma_2 [\gamma^2 \omega^2 + (K + L - \omega^2)^2] [\gamma^2 \omega^2 + (-K + L + \omega^2)^2]}, \quad (50)$$

and

$$L_{12} = \frac{L(K - \omega^2)(\Gamma_1 + \Gamma_2)}{\Gamma_1 \Gamma_2 [\gamma^2 \omega^2 + (K + L - \omega^2)^2] [\gamma^2 \omega^2 + (-K + L + \omega^2)^2]}, \quad (51)$$

respectively. Analogous expressions are held valid for L_{21} and L_{22} by exchanging $1 \leftrightarrow 2$. Note that $L_{11} \geq 0$ and $L_{22} \geq 0$ (as expected). The non-negativity of the entropy production also requires that $4L_{11}L_{22} - (L_{12} + L_{21})^2 \geq 0$. To verify this, let us consider $\Gamma_2 = r\Gamma_1$ with r being an arbitrary (non-negative) real number. Such above inequality is always satisfied, since the term

$$\frac{[\gamma^2 \omega^2 + (K - \omega^2)^2 + L^2 r][r(\gamma^2 \omega^2 + (K - \omega^2)^2) + L^2]}{L^2(r + 1)^2(K - \omega^2)^2},$$

is greater than 1/4 for all values of r , K , ω , and γ .

2. Phase difference between harmonic forces

Here, we extend the results from the previous subsection but taking into account a phase difference between external forces $f_1^{\text{ext}}(t)$ and $f_2^{\text{ext}}(t)$. More specifically, $f_1^{\text{ext}}(t)$ has the same expression as previously, but $f_2^{\text{ext}}(t)$ now reads $f_2^{\text{ext}}(t) = f_{02} \cos(\omega t + \phi)$. By repeating afore-described procedures, we assume that $\langle v_i \rangle = w(C_{2i} \cos \omega t - C_{1i} \sin \omega t)$, whose coefficients C_{1i} and C_{2i} are decomposed in two parts: $C_{1i} = A_{1i} + B_{1i}(\phi)$, whose A_{1i} and A_{2i} are the same as Eqs. (A1) and (A2) and the dependence on the phase difference appears only in B_{1i} and B_{2i} , whose explicit coefficients are listed in Appendix B. We then arrive at the following expression for the steady entropy production $\bar{\Pi}$:

$$\bar{\Pi} = \frac{\gamma^2 \omega^2 [\Gamma_1 (C_{12}^2 + C_{22}^2) + \Gamma_2 (C_{11}^2 + C_{21}^2)]}{\Gamma_1 \Gamma_2}, \quad (52)$$

which is quite similar to Eq. (40). As in the lagless case, it has three terms with first and third terms being identical to Eq. (41) and the phase difference dependence appearing only in middle term reading

$$\left[\frac{2f_{01}f_{02}L[\gamma\omega(\Gamma_2 - \Gamma_1)\sin\phi + (K - \omega^2)(\Gamma_1 + \Gamma_2)\cos\phi]}{\Gamma_1\Gamma_2(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)} \right]. \quad (53)$$

Note that the it reduces to the middle term from Eq. (41) when $\phi = 0$. The position of the maximum and minimum in $\bar{\Pi}$ fulfills the above relation

$$\phi = \tan^{-1} \left[\frac{\gamma\omega(\Gamma_2 - \Gamma_1)}{(K - \omega^2)(\Gamma_1 + \Gamma_2)} \right]. \quad (54)$$

Note that ϕ depends only on the signs of both $\Gamma_2 - \Gamma_1$ and $K - \omega^2$ and it is independent on L . In particular, in

the regime of $\Gamma_2 \gg \Gamma_1$ ($\Gamma_2 \ll \Gamma_1$), ϕ is independent on Γ_i 's, reading $\pm\gamma\omega/(K - \omega^2)$. Conversely, for fast and slow oscillations, it approaches to zero as $\gamma(\Gamma_1 - \Gamma_2)/K\omega(\Gamma_2 + \Gamma_1)$ and $\gamma\omega(\Gamma_2 - \Gamma_1)/K(\Gamma_2 + \Gamma_1)$, respectively. Figure 2 plots $\bar{\Pi}$ versus ϕ for distinct set of values of ω and Γ_i 's. Note that the maxima of mean entropy production yields at $\phi \sim 0(\pi)$ for small (large) values of ω and $\phi \rightarrow \pi/2$ when $\omega \rightarrow \sqrt{K}$. The dependence of extremes clearly follows theoretical predictions from Eq. (54) (see, e.g., panels (a) and (b) in Fig. 2).

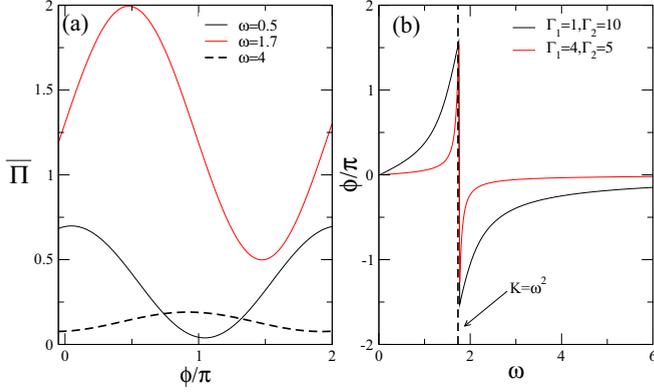


FIG. 2. For distinct frequencies ω 's, (a) depicts the average entropy production $\bar{\Pi}$ versus the phase difference ϕ for $\Gamma_1 = 1$, $\Gamma_2 = 10$, $K = 3$, and $\gamma = 1$. For $K = 3$ and $\gamma = 1$ and distinct sets of Γ_1 and Γ_2 , (b) shows the positions ϕ of maximum/minimum of entropy production $\bar{\Pi}$ versus ω .

B. Oscillating temperatures

The investigation of systems under oscillating temperature has been reported in several works [30–47], which provides a way of measuring the heat capacity experimentally. Here, we intend to verify the nonequilibrium trademarks of a chain of two interacting particles under time oscillating temperatures. For simplicity, we consider external forces absent. The entropy production can also be evaluated straightforwardly from Eq. (15), but instead the temperature $\Gamma_i(t)$ of the each reservoir is now time dependent $\Gamma_i(t) = \Gamma_{0i} + F_{Ti} \cos(\omega t)$ with Γ_{0i} and F_{Ti} being the reference temperature and the strength of temperature driving, respectively.

Although such a problem is exactly solvable [see, e.g., Fig. 3(b)] and reduces to the findings from Ref. [19] when $\Gamma_{01} = \Gamma_{02}$ and $L = 0$, the expression for $\bar{\Pi}$ is much more complex than previous cases and involves many terms related to distinct powers of interaction parameters K, L and driving frequency ω . For this reason, our analysis will be carried out close to equilibrium regime, in which a linear treatment

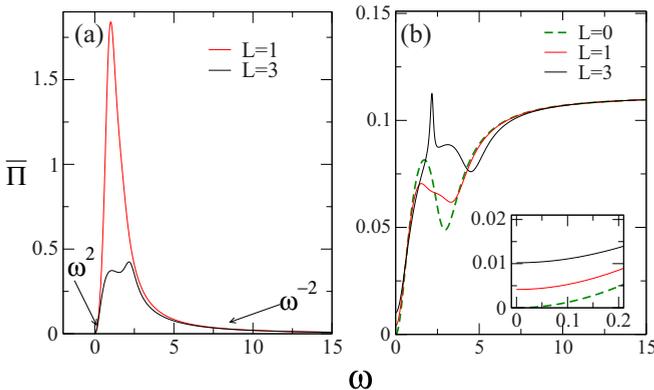


FIG. 3. (a) and (b) depict the steady entropy production rate $\bar{\Pi}$ versus frequency driving ω for time dependent oscillating forces and temperatures, respectively. In all cases we take $\gamma = 1$, $K = 2$, $\Gamma_1 = \Gamma_2 = 10$, and $f_{02} = 2f_{01} = 4$ [(a)] and $F_{T2} = 2F_{T1} = 4$ [(b)]. Inset: The steady $\bar{\Pi}$ for distinct L 's for low ω .

can be performed. More specifically, we take both reference temperatures to be equal $\Gamma_{01} = \Gamma_{02} = \Gamma_0$ and the driving strengths are sufficiently low $F_{Ti} \ll \Gamma_0$. In such case, the entropy production $\bar{\Pi}$ can also be written down in the bilinear form $\bar{\Pi} = \mathcal{J}_{T_1} F_{T_1} + \mathcal{J}_{T_2} F_{T_2}$, where the fluxes \mathcal{J}_{T_1} and \mathcal{J}_{T_2} read

$$\mathcal{J}_{T_1} = L_{T_1, T_1} F_{T_1} + L_{T_1, T_2} F_{T_2}, \quad (55)$$

and

$$\mathcal{J}_{T_2} = L_{T_2, T_1} F_{T_1} + L_{T_2, T_2} F_{T_2}, \quad (56)$$

respectively, where L_{T_i, T_j} are the associated Onsager coefficients given by

$$L_{T_1, T_1} = \frac{\gamma}{2\Gamma_0^2} \left(\frac{\sum_{\ell=0}^9 B_{\ell} \omega^{2\ell}}{\sum_{\ell=0}^9 G_{\ell} \omega^{2\ell}} \right), \quad (57)$$

and

$$L_{T_1, T_2} = \frac{-4\gamma^3 L^2}{\Gamma_0^2} \left(\frac{\sum_{\ell=0}^6 A_{\ell} \omega^{2\ell}}{\sum_{\ell=0}^9 G_{\ell} \omega^{2\ell}} \right), \quad (58)$$

respectively, where $L_{T_1, T_1} = L_{T_2, T_2}$ and $L_{T_1, T_2} = L_{T_2, T_1}$ and coefficients A_{ℓ} 's, B_{ℓ} 's, and G_{ℓ} 's solely depend on the parameters γ and $L = 2K$ and are listed in Appendix C.

We pause again to make some few comments: First, in the limit of slow and fast frequencies, $\bar{\Pi}$ approaches to the following expressions:

$$\bar{\Pi} \sim \frac{4\gamma^3 L^2 A_0}{\Gamma_0^2 G_0} (F_{T_1} - F_{T_2})^2, \quad (59)$$

and

$$\bar{\Pi} \sim \frac{\gamma}{2\Gamma_0^2} (F_{T_1}^2 + F_{T_2}^2), \quad (60)$$

respectively. They contrast with the oscillating forced case, since are independent on ω and different from zero in both extreme cases. Whenever it depends on L for low oscillations, the entropy production is independent on the coupling for fast oscillations. Finally, for strong interaction strength, $L \gg 1$ and $L \gg \omega^2$, $\bar{\Pi}$ reads

$$\bar{\Pi} \sim \frac{\gamma}{4\Gamma_0^2(\gamma^2 + \omega^2)} [\gamma^2 (F_{T_1} - F_{T_2})^2 + 2\omega^2 (F_{T_1}^2 + F_{T_2}^2)], \quad (61)$$

which is also independent on L . We close this section by comparing, in Fig. 3(a) and 3(b) the steady entropy production behaviors versus the frequency driving ω for both oscillating temperature and forces (obtained from the exact solution). They exhibit meaningfully different dependence on ω , even for extreme ω . Whenever $\bar{\Pi}$ vanishes for $\omega \ll 1$ and $\omega \gg 1$ in the case of time oscillating forces, it reaches constant values for temperature drivings, in accordance with asymptotic expressions Eqs. (59) and (60), respectively, obtained from the linear regime approximation.

V. MORE THAN TWO PARTICLES

In this section we present the main results for long chains of oscillators. In real systems, due to the lattice imperfections and impurities, the difference of temperature between particles placed at extremities is responsible for a transport of heat following Fourier's law. More concretely, it states that the heat

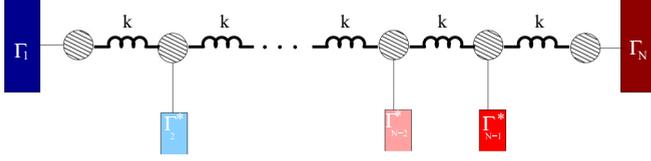


FIG. 4. Schematic diagram consisting of a one-dimensional chain of N harmonic oscillators with individual and coupling interactions k^* and k , respectively. Γ_1 and Γ_N denote the cold and hot bath temperatures, respectively, whereas the self-consistent reservoirs range from Γ_2^* to Γ_{N-1}^* . Each intermediate particle is also subject to an external force $f_i^{\text{ext}}(t)$. The usage of our protocol leads to intermediate temperatures changing linearly from Γ_1 to Γ_N , consistent to a flux of heat along the chain, symbolized by the color gradient from the red to the blue reservoirs.

current is proportional to the inverse of the length of the chain given by

$$\mathcal{J}_T = -\kappa \frac{dT}{dx}, \quad (62)$$

where κ is the heat conductivity. In the case of a finite difference of temperatures ΔT , it follows that $\mathcal{J}_T \sim 1/N$ and thereby the heat flux is proportional to the inverse of the system size. Obtaining Fourier's law from microscopic models have attracted great interest in the last years [49–52]. In principle, one could suppose that a linear chain of particles interacting through harmonic forces in contact with two temperature reservoirs placed at extremities would lead to a heat flux obeying Fourier's law. However, this is not the case [53]. Among the distinct approaches aimed at obtaining a heat flux inversely proportional to the system chain, we mention the self-consistent protocol proposed by Bosterli *et al.* [48]. It consists of baths acting on all sites, but intermediate temperatures are chosen self-consistently in such a way that they do not exchange any heat with the system in the steady state, ensuring that heat flux is only due to particles placed at extremities. Here, we take a similar approach by Bosterli *et al.*, in which each intermediate temperature is chosen so that it equals to the variance $\langle v_i^2 \rangle_{cv}$, $\Gamma_i^* = 2\gamma \langle v_i^2 \rangle_{cv}$. Although it reduces to the original protocol when external forces are absent, here all self-consistent reservoirs are expected to produce entropy coming from external forces. Figure 4 illustrates a linear chain in the presence of thermal and self-consistent baths.

In order to compare the distinct sources of dissipation, thermal and time oscillating forces, we will consider that particles placed at extremities are not subjected to external forces. Thereby, under the above choice of the intermediate Γ_i^* 's, the flux of entropy becomes

$$\Phi(t) = \Phi_T + \Phi_f(t), \quad (63)$$

where Φ_T read

$$\Phi_T = \frac{2\gamma^2}{\Gamma_1} \langle v_1^2 \rangle_{cv} + \frac{2\gamma^2}{\Gamma_N} \langle v_N^2 \rangle_{cv} - 2\gamma, \quad (64)$$

and $\Phi_f(t)$ is a sum of individual contributions

$$\Phi_f(t) = 2\gamma^2 \left(\frac{\langle v_1 \rangle^2}{\Gamma_1} + \sum_{i=2}^{N-1} \frac{\langle v_i \rangle^2}{\Gamma_i^*} + \frac{\langle v_N \rangle^2}{\Gamma_N} \right). \quad (65)$$

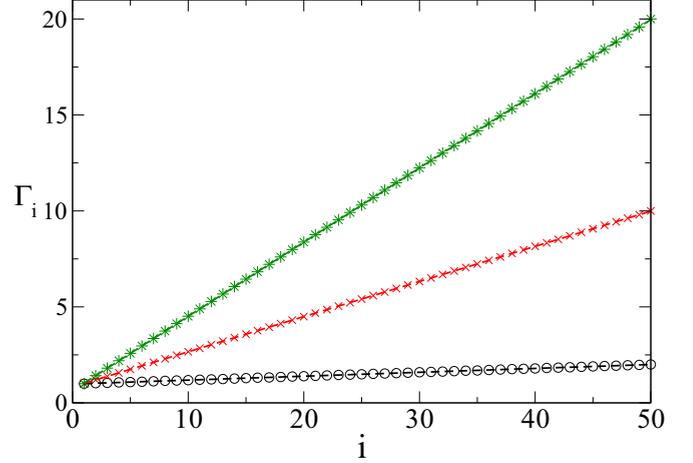


FIG. 5. For a chain of $N = 50$ particles with $K = 2$, $L = 1$, $\gamma = 1$, and $\omega = 1$, the rescaled temperatures Γ_i 's versus the position of the i th site for three set of temperatures (Γ_1, Γ_N) . The intermediate temperatures are calculated according to the prescription $\Gamma_i^* = 2\gamma \langle v_i^2 \rangle_{cv}$.

Despite the absence of external forces for extreme particles, the averages $\langle v_1 \rangle$ and $\langle v_N \rangle$ present oscillating behavior coming from couplings with neighboring particles [see, e.g., Eqs. (30) and (31)].

In all cases, $\Pi_T = \Phi_T$ can be written as for the two particles case

$$\Pi_T = \mathcal{J}_T f_T, \quad (66)$$

where the thermodynamic force f_T and its associate flux \mathcal{J}_T read $f_T = 1/\Gamma_1 - 1/\Gamma_N$ and $\mathcal{J}_T = \kappa(\Gamma_N - \Gamma_1)/N$, respectively. Thereby, the expression for Π_T becomes

$$\Pi_T = \frac{\kappa}{N} \frac{(\Gamma_N - \Gamma_1)^2}{\Gamma_1 \Gamma_N}. \quad (67)$$

Since the thermal conduction coefficient κ is finite (it depends only on parameters Γ_1 , Γ_N , K and L), the entropy production Π_T decays as N^{-1} [see, e.g., Fig. 6(b)].

A. Three particles

Here, we derive explicit results for a chain of $N = 3$ particles. In such case, Eq. (65) becomes

$$\Phi_f(t) = \frac{2\gamma^2}{\Gamma_1} \langle v_1 \rangle^2 + \frac{2\gamma^2}{\Gamma_2^*} \langle v_2 \rangle^2 + \frac{2\gamma^2}{\Gamma_3} \langle v_3 \rangle^2, \quad (68)$$

and the entropy production Π_T due to thermal reservoirs has the shape of Eq. (66) with \mathcal{J}_T given by

$$\mathcal{J}_T = \frac{\Gamma_1 \Gamma_3 \gamma L^2 (2\gamma^2 K + L^2)}{2[L^2 + \gamma^2(4K - 2L)][L^2 + \gamma^2(K + L)]} f_T. \quad (69)$$

Once again, $\Pi_T \geq 0$, since $4K - 2L = 2(k + 2k^*)/m$. Using the motion equations we arrive at the following expression

for $\bar{\Pi}$:

$$\bar{\Pi} = \frac{f_{02}^2 \gamma^2 \omega^2 [\Gamma_1 \Gamma_3 (\gamma^2 \omega^2 + (K - \omega^2)^2) + L^2 \Gamma_2^* (\Gamma_1 + \Gamma_3)]}{\Gamma_1 \Gamma_2^* \Gamma_3 [\gamma^2 \omega^2 + (K + 2L - \omega^2)^2] [\gamma^2 \omega^2 + (-K + L + \omega^2)^2]}, \quad (70)$$

which is strictly positive and vanishes when f_{02} and/or ω are equal to zero. Also, in the regime of slow and fast oscillations, $\bar{\Pi}$ exhibit similar dependencies on ω to the two particles case

$$\bar{\Pi} \sim \left(\frac{K^2}{\Gamma_2^*} + \frac{L^2 (\Gamma_1 + \Gamma_3)}{\Gamma_1 \Gamma_3} \right) \frac{\gamma^2 \omega^2 f_{02}^2}{(K + 2L)^2 (-K + L)^2}, \quad (71)$$

for $\omega \ll 1$ and

$$\bar{\Pi} \sim \frac{\gamma^2 f_{02}^2}{\Gamma_2^*} \frac{1}{\omega^2}, \quad (72)$$

for $\omega \gg 1$, respectively, implying that for such latter limit the entropy production is independent on extreme temperatures. For strong couplings between particles, $L \approx K \gg \omega^2$, $\bar{\Pi}$ approaches to

$$\bar{\Pi} \sim \frac{\gamma^2}{\gamma^2 + \omega^2} \left(\frac{1}{\Gamma_1} + \frac{1}{\Gamma_2^*} + \frac{1}{\Gamma_3} \right) f_{02}^2, \quad (73)$$

which is quite similar to Eq. (46) [for $N = 2$] and it is independent on the interaction strengths.

B. The limit of long particle chains

All results obtained for $N = 3$ particles can be straightforwardly extended for long chains. However, it becomes very cumbersome to obtain simplified expressions for $\bar{\Pi}$ in such cases. For this reason, we will restrict the next analysis for specific values of control parameters. Figure 5 shows, for a chain of $N = 50$ particles and three sets of temperatures (Γ_1, Γ_N) , the temperature profiles calculated from the self-consistent protocol. In all cases, the set of intermediate temperatures changes linearly from Γ_1 to Γ_N , consistent to a flux of heat along the chain from the hot to the cold reservoirs.

Figure 6(a) compares the individual entropy production contributions for distinct system sizes for $f_{02} = f_{03} = \dots = f_{0N-1}$. Since external forces are equally presented in all intermediate particles, the entropy production associated to

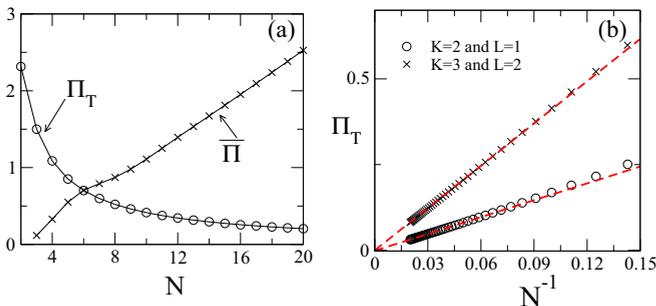


FIG. 6. For $\Gamma_1 = 1$ and $\Gamma_N = 10$, (a) depicts the individual entropy production contributions from the thermal and self-consistent baths versus N for $K = 3$, $L = 2$, and $\omega = 1.5$. In (b) the behavior of entropy production from the thermal reservoirs Π_T vs N^{-1} .

self-consistent baths increases linearly with N . Also, panel (a) depicts the existence of two regimes. For small chains the thermal reservoir contribution Π_T dominates over the self-consistent ones $\bar{\Pi}$, whereas $\bar{\Pi}$ wins over Π_T upon N is increased. In the limit $N \rightarrow \infty$ [see e.g., panel (b)], only the contributions from self-consistent reservoirs prevail, in consonance with Fourier's law [Eq. (67)]. Finally, it is worth emphasizing two distinct linear behaviors of $\bar{\Pi}$. It arises from the particles closer to the thermal reservoirs providing more contribution for the entropy production for small chains than for large ones.

VI. CONCLUSIONS

The nonequilibrium properties of linear chains of Brownian particles were analyzed via stochastic thermodynamics. Expressions for the heat flux, entropy production, and allied quantities were exactly obtained. The regimes of short and long chains were detailed inspected. In the former case the entropy production was derived as bilinear functions of fluxes and forces, from which the associated Onsager coefficients depend on interaction couplings and frequency drivings. Reciprocal relations were also obtained. The limit of long chains was studied by means of a self-consistent protocol for choosing intermediate temperatures. The entropy production is a sum of two terms: one coming from the real baths and the other from the self-consistent reservoirs. Whenever the former dominates for short chains, the latter contribution prevails for long ones. The contribution from the thermal reservoirs is responsible to heat flux according to Fourier's law.

As a final comment, it is worth to discuss future extensions of the present study. The inclusion of both temperature and external force drivings in harmonic chains should be interesting, in order to compare not only the structure of entropy production but also the Onsager coefficients. Also, the investigation of other kinds of drivings, such as the time discrete drivings should also be interesting in order to compare with sinusoidally time dependent ones.

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APPENDIX A: OBTAINING THE COEFFICIENTS A_{ij} FOR $N = 2$ PARTICLES CASE WITH NO PHASE DIFFERENCE

Here, we show explicit expressions for the coefficients A_{ij} 's for the two particles case subject to oscillating forces.

The index i stands for the i th particle ($i = 1, 2$):

$$A_{1i} = \frac{f_{0i}(K - \omega^2)[\gamma^2\omega^2 + (K - \omega^2)^2 - L^2] + f_{0j}L[-\omega^2(\gamma^2 + 2K) + K^2 - L^2 + \omega^4]}{[\gamma^2\omega^2 + (K + L - \omega^2)^2][\gamma^2\omega^2 + (-K + L + \omega^2)^2]} \quad (\text{A1})$$

and

$$A_{2i} = \frac{\gamma\omega[f_{0i}(\gamma^2\omega^2 + (K - \omega^2)^2 + L^2) + 2f_{0j}L(K - \omega^2)]}{(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)}. \quad (\text{A2})$$

Having the A_{ij} 's, the steady entropy production $\bar{\Pi}$ is straightforwardly evaluated.

APPENDIX B: OBTAINING THE COEFFICIENTS B_{ij} FOR $N = 2$ PARTICLES CASE AND PHASE DIFFERENCE

Here, we show explicit expressions for the coefficients B_{ij} 's for the two particles subject to phase difference between oscillating forces:

$$B_{11} = \frac{f_{02}L((\cos\phi - 1)(-\omega^2(\gamma^2 + 2K) + K^2 - L^2 + \omega^4) + 2\gamma\omega(K - \omega^2)\sin\phi)}{(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)}, \quad (\text{B1})$$

$$B_{21} = \frac{f_{02}L(\sin\phi(\gamma^2\omega^2 - (K - \omega^2)^2 + L^2) + 2\gamma\omega(K - \omega^2)(\cos\phi - 1))}{(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)}, \quad (\text{B2})$$

$$B_{12} = \frac{f_{02}\gamma\omega\sin\phi(\gamma^2\omega^2 + (K - \omega^2)^2 + L^2) + f_{02}(K - \omega^2)(\cos\phi - 1)(\gamma^2\omega^2 + (K - \omega^2)^2 - L^2)}{(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)}, \quad (\text{B3})$$

and

$$B_{22} = \frac{f_{02}(\omega^2 - K)\sin\phi(\gamma^2\omega^2 + (K - \omega^2)^2 - L^2) + f_{02}\gamma\omega(\cos\phi - 1)(\gamma^2\omega^2 + (K - \omega^2)^2 + L^2)}{(\gamma^2\omega^2 + (K + L - \omega^2)^2)(\gamma^2\omega^2 + (-K + L + \omega^2)^2)}, \quad (\text{B4})$$

respectively. Note that all of them vanish as $\phi = 0$, restoring the expressions (A1) and (A2), respectively.

APPENDIX C: OBTAINING THE COEFFICIENTS A_i 's, B_i 's, AND G_i 's FOR $N = 2$ PARTICLES CASE AND OSCILLATING TEMPERATURE

Here, we show explicit expressions for the coefficients A_i 's, B_i 's, and G_i 's for the two particles case and time oscillating temperatures:

$$A_0 = 2304L^5(2\gamma^2 + L),$$

$$A_1 = -128L^3(-7\gamma^4 + 58\gamma^2L + 123L^2),$$

$$A_2 = 16L(8\gamma^6 - 50\gamma^4L + 34\gamma^2L^2 + 931L^3),$$

$$A_3 = 4(-11\gamma^6 + 78\gamma^4L + 319\gamma^2L^2 - 1606L^3),$$

$$A_4 = -3(17\gamma^4 + 82\gamma^2L - 548L^2),$$

$$A_5 = 3\gamma^2 - 214L,$$

$$A_6 = +10,$$

$$B_0 = 36864\gamma^4L^7 + 18432\gamma^2L^8,$$

$$B_1 = 31744\gamma^6L^5 + 112640\gamma^4L^6 + 27648\gamma^2L^7 + 36864L^8,$$

$$B_2 = 4608\gamma^8L^3 + 18176\gamma^6L^4 - 77056\gamma^4L^5 + 203904\gamma^2L^6 - 172032L^7,$$

$$B_3 = 128\gamma^{10}L + 608\gamma^8L^2 - 6592\gamma^6L^3 + 85920\gamma^4L^4 - 230720\gamma^2L^5 + 269824L^6,$$

$$B_4 = 16\gamma^{10} - 64\gamma^8L + 4536\gamma^6L^2 - 34864\gamma^4L^3 + 125488\gamma^2L^4 - 170496L^5,$$

$$B_5 = 56\gamma^8 - 792\gamma^6L + 8112\gamma^4L^2 - 34928\gamma^2L^3 + 54288L^4,$$

$$B_6 = 73\gamma^6 - 928\gamma^4L + 5120\gamma^2L^2 - 9536L^3,$$

$$B_7 = 43\gamma^4 - 376\gamma^2L + 936L^2,$$

$$B_8 = 11\gamma^2 - 48L,$$

$$B_9 = 1,$$

$$G_0 = 147456\gamma^6L^6 + 147456\gamma^4L^7 + 36864\gamma^2L^8,$$

$$G_1 = 50176\gamma^8L^4 - 94208\gamma^6L^5 + 262144\gamma^4L^6 - 24576\gamma^2L^7 + 36864L^8,$$

$$G_2 = 3584\gamma^{10}L^2 - 23552\gamma^8L^3 + 166400\gamma^6L^4 - 323584\gamma^4L^5 + 384512\gamma^2L^6 - 172032L^7,$$

$$\begin{aligned}
G_3 &= 64\gamma^{12} - 768\gamma^{10}L + 14720\gamma^8L^2 - 77312\gamma^6L^3 \\
&\quad + 262528\gamma^4L^4 - 399872\gamma^2L^5 + 269824L^6, \\
G_4 &= 240\gamma^{10} - 2688\gamma^8L + 24672\gamma^6L^2 - 96960\gamma^4L^3 \\
&\quad + 200592\gamma^2L^4 - 170496L^5,
\end{aligned}$$

$$\begin{aligned}
G_5 &= 348\gamma^8 - 3504\gamma^6L + 20024\gamma^4L^2 \\
&\quad - 52736\gamma^2L^3 + 54288L^4, \\
G_6 &= 245\gamma^6 - 2064\gamma^4L + 7424\gamma^2L^2 - 9536L^3, \\
G_7 &= 87\gamma^4 - 528\gamma^2L + 936L^2, \\
G_8 &= 15\gamma^2 - 48L, \\
G_9 &= 1.
\end{aligned}$$

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