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FINITE DECIDABILITY AND POLYNOMIALS

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Abstract

We study here some aspects of *k-decidability* related with the following feature: Suppose that a germ is not k-decidable, is it true that are are two polynomials that shows this? We explicit and characterize the situations where the answer is yes (or no).

1 Introduction

When one speaks roughly, he can says that a germ $f:(\mathbb{R}^n;O) \longrightarrow (\mathbb{R};0)$ is k–decidable if its jet of order k assures that f has an extremum or a saddle (for precise definitions see the section 2). Therefore, if f is not k–decidable there is a germ $g:(\mathbb{R}^n;O) \longrightarrow (\mathbb{R};0)$ such that the jet of order k of f and g are equal and one of them has an extremum and another has a saddle.

The chief problem that we study here is *how simple can be g?*

Since f is k–decidable if, and only if, its jet of order k (which is a polynomial) has this propriety, the first tentative is try to find g in the class of polynomials.

In 1987, we had shown that there are situations where this is impossible, see [G0] or the section 4 below. Here we do a more accurate study of this problem.

In the section 2 we give the basic concepts of k–decidability that we will need in the text. The main reference for this section is the pioneer work of Barone-Netto, [B0], and a reader which is familiar with this text can be start the reading by the section 3, or 4.

The section 3 is a brief discussion about the relation between k–decidability and algebraic curves where we expose some results that we will use in the last two sections of the text. In this section we state some proprieties of algebraic curves and sets without the proof. We refer the reader to the classical, and excellent, books of Milnor ([M0]) and Walker ([W0]) for this.

In section 4 we define polynomially k–decidability, state same simple results about this and we present our example of 1987.

In section 5 we show a result that characterize the relation between k–decidability and polynomially k–decidability.

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2 Preliminaries

Let $f\colon(\mathbf{R}^n;0) \longrightarrow (\mathbf{R};0)$ be a germ, as usual we will use in this text, unless otherwise specified, the same symbol to design a germ or a function which represents it.

Definition 2.1 *We will say that f has* **punctual jet of order** *k if there is a polynomial P of degree less or equal to k*[1] *such that*

$$\lim_{x\to 0}\frac{\|f(x)-P(x)\|}{\|x\|^k}=0. \tag{1}$$

We will denote by $G(k;n)$ the set of germs of $(\mathbf{R}^n;0)$ in $(\mathbf{R};0)$ that has punctual jet of order k.

Remark 2.1 *It is easy to see that there is at most one polynomial of degree less or equal to k which obeys (1). When there exists such a polynomial, we will denote it by k-jet of f and we will represent it by $j^k f$.*

Remark 2.2 *It is clear that if f has Taylor polynomial of order k, P, then $f\in G(k;n)$ and $j^k f = P$.*

Remark 2.3 *Let s, k be integers such that $1\le s\le k$, we see easily that $G(k;n)\subset G(s;n)$ and, if $f\in G(k;n)$ then $j^s f = j^s(j^k f)$.*

Remark 2.4 *Consider $f\in G(k;n)$ and, for $1\le s\le k$, the polynomial $f_s = j^s f - j^{s-1} f$ (note that, by the remark 2.3 $f\in G(s;n)$). It is immediate that f_s is either the null polynomial, or an homogeneous polynomial of degree s. It follows from last remark that $j^k f = \sum_{s=1}^{k} f_s$, therefore, $f_s = j^s f$ if, and only if, $j^{s-1} f = 0$. Then if r are the order of the first non null jet of f (i.e., $j^r f \not\equiv 0$ and $j^{r-1} f \equiv 0$) it follows that $j^r f$ is an homogeneous polynomial of degree r.*

We will say that a germ $f\colon(\mathbf{R}^n;0) \longrightarrow (\mathbf{R};0)$ has a *strong maximum* (resp. a *strong minimum*) if there is a representative $\tilde f\colon\Omega \longrightarrow \mathbf{R}$ defined in the an open neighbourhood Ω of 0 such that $\tilde f(x) < 0$ (resp. $\tilde f(x) > 0$), for all $x\in\Omega\setminus\{0\}$. Analogously, we will say that the germ f has a *weak maximum* (resp. *weak minimum*) if there is a representative $\tilde f\colon\Omega \longrightarrow \mathbf{R}$ with $f(x)\le 0$ (resp. $f(x)\ge 0$) for $x\in\Omega$ and there is a sequence (x_j) in $\Omega\setminus\{0\}$ which tends to 0 such that $f(x_j)=0$.

It is obvious that if f has strong maximum (resp. strong minimum, weak maximum or weak minimum) then any representative of f has an local strong maximum (resp. local strong minimum, local weak maximum or local weak minimum) at the origin.

[1] We will adopt the convention that the null polynomial has degree $-\infty$.

We will say that the germ f has a extremum if either it has a maximum (strong or weak) or it has a minimum (strong or weak). When f has not a extremum then we will say that it has a saddle.

When two germs f and g has both strong maximum (resp. strong minimum, saddle, weak maximum, weak minimum) we will say that they *have the same behavior relatively to extremum*.

The concept of k–decidability is due to Barone-Netto, which introduces it in [B0], namely:

Definition 2.2 *Let $k \geq 1$ be a natural number and a germ $f\colon (\mathbb{R}^n;O) \longrightarrow (\mathbb{R};0)$. We will say that f* **is k–decidable** *if f obeys the following conditions:*

(I) $f \in G(k;n)$;

(II) For all $g \in G(k;n)$ such that $j^k f = j^k g$ then f and g has same behavior relatively to extremum.

Remark 2.5 *It is immediate that f is k–decidable if, and only if, $j^k f$ is k–decidable.*

Remark 2.6 *It is immediate that in the real line ($n = 1$), $f \in G(k;1)$ is k–decidable if, and only if, $j^k f \not\equiv 0$.*

Remark 2.7 *Let $1 \leq s \leq k$ be natural numbers. If $f \in G(k;n)$ and it is s–decidable then f is k–decidable.*

Remark 2.8 *One see easily that f is k–decidable if, and only if, for all $h \in G(k;n)$ such that $j^k h \equiv 0$ we have that f and $f + h$(or, by the precedent remark, $j^k f$ and $j^k f + h$) have the same behavior with respect to extremum.*

Remark 2.9 *If $f\colon (\mathbb{R}^n;O) \longrightarrow (\mathbb{R};0)$ is k–decidable and $T\colon (\mathbb{R}^n;O) \longrightarrow (\mathbb{R}^n;O)$ is a germ of a C^∞ change of coordinates then $f \circ T$ is k–decidable. This follows immediately of the following elementary facts:*

(i) If $f \in G(k;n)$ then $(f \circ T) \in G(k;n)$;

(ii) If $h \in G(k;n)$, with $j^k h \equiv 0$, then $j^k(h \circ T) \equiv 0$;

(iii) The remark 2.8.

Remark 2.10 *If f is k–decidable then $j^k f$ has a saddle of a strong extremum. In fact, if $j^k f$ has a weak extremum then $g = j^k f + \|x\|^{2k}$ has a behavior with respect to extremum distinct than $j^k f$ and $j^k f = j^k g$.*

The reciprocal of this remark is not true, as we see in the following example.

Example 2.1 *Consider $f = j^4 f = (y - x^2)^2 + y^4$.*

It is clear that $j^4 f$ has a strong minimum (at the origin), but f is not 4-decidable, in fact, f is not 7-decidable, as we see by taking $g = (y - x^2)^2 + y^4 - 2x^8 = j^7 f - 2x^8$.

We have that $j^7 f = j^7 g$ but g has a saddle, since $g(x;x^2) = -x^8$ and $g(0;y) = y^2 + y^4$.$\diamond$

However, if $j^k f$ is an homogeneous polynomial, the reciprocal of the remark 2.10 is true. In order to prove this we will state a useful particular case, namely, the case when $j^k f$ is the the first non null jet of f.

Lemma 2.1 *Suppose that $f \in G(k;n)$, $j^k f \not\equiv 0$ and $j^{k-1} f \equiv 0$. Then there are equivalent:*

(i) f is k-decidable.

(ii) $j^k f$ has a strong extremum or a saddle (i.e. $j^k f$ has not a weak extremum).

Proof: *(i) $\Rightarrow$ (ii)* It is the remark 2.10.

(ii) $\Rightarrow$ (i) In this situation $j^k f$ is an homogeneous polynomial of degree k.

We will consider two cases:

- $j^k f$ has a saddle:

  Then there is two points $\overline{x}$ and $\overline{y}$ such that $f(\overline{x}) = a > 0$ and $f(\overline{y}) = b < 0$.

  Since $j^k f$ is homogeneous, $j^k f(\lambda \overline{x}) = a\lambda^k$ and $j^k f(\lambda \overline{y}) = b\lambda^k$.

  Now, consider $h \in G(k;n)$ such that $j^k h \equiv 0$. It is clear that $\lim\limits_{\lambda \to 0} \frac{h(\lambda \overline{x})}{\lambda^k} = \lim\limits_{\lambda \to 0} \frac{h(\lambda \overline{y})}{\lambda^k} = 0$.

  Then, there is an $\varepsilon > 0$ such that, for $0 < \lambda < \varepsilon$, we have $h(\lambda \overline{x}) > 0$ and $h(\lambda \overline{y}) < 0$. Then $j^k f$ and $j^k f + h$ has a saddle and this shows that f is k-decidable, by remark 2.8.

•• $j^k f$ has a strong extremum:

  Since $j^k f$ is homogeneous it follows that, in this case, the origin is a global strong extremum point of $j^k f$ (there is a unique polynomial representative of this germ defined in $\mathbf{R}^n$). Suppose, without loss of generality, that $j^k f$ has a strong minimum at origin and consider $m = \min \{f(x) : \|x\| = 1\}$.

  It is clear that $m > 0$ and, by using the homogeneity of $j^k f$, it follows that, for all $x \in \mathbf{R}^n \setminus \{O\}$, $j^k f(x) = \|x\|^k j^k f(\frac{x}{\|x\|}) \geq m\|x\|^k$.

  Consider $h \in G(k;n)$ with $j^k h \equiv 0$, we have, by (1), that $\lim\limits_{x \to 0} \frac{f(x)}{\|x\|^k} = 0$.

  It follows that, there is a $\varepsilon > 0$ such that, $j^k f(x) + h(x) \geq \frac{m}{2}\|x\|^k$, for all x with $\|x\| \leq \varepsilon$. Then $j^k f + h$ has a strong minimum and this ends the proof. ■

Fact 2.1 *Consider $f: (\mathbf{R}^n;O) \longrightarrow (\mathbf{R};0)$, such that $f \in G(k;n)$ and $j^k f$ is homogeneous. Then f is k-decidable if, and only if, $j^k f$ has an strong extremum or a saddle.*

Proof: ($\Rightarrow$) If $j^k f$ has a strong extremum or a saddle, then $j^k f \not\equiv 0$. Then as, by hypotheses, $j^k f$ is homogeneous, if s is the degree of $j^k f$, by the remark 2.4, $j^k f = j^s f$, and $j^{s-1} f \equiv 0$. Then the result follows, by the remark 2.7 and the lemma 2.1.
($\Leftarrow$) It is the remark 2.10. ■

We will see now that the first non null jet of a germ determines, at least partially, the kind of the behavior of the germ with respect to extremum.

Fact 2.2 *Suppose that $f \in G(k;n)$ and $j^k f$ is the first non null jet of f. Then, if $j^k f$ has a weak minimum (resp. weak maximum) then either f has a saddle or f has an minimum (resp. maximum), in the last case the minimum (resp. maximum) could be strong or weak.*

Proof: Consider $\overline{x}$ such that $j^k f(\overline{x}) > 0$ and proceed as in the proof of the lemma 2.1. ■

A simple consequence of this is:

Fact 2.3 *Suppose that $f \in G(k;n)$ and f is not k-decidable then there is a germ $g \in G(k;n)$ such that $j^k f = j^k g$ and one of the germs $\left\{j^k f, g\right\}$ has a extremum, while the other has a saddle.*

Proof: If $j^k f$ has a weak extremum or a saddle the result is immediate.

So, suppose that $j^k f$ a strong extremum. Without loss of generality, we will assume that $j^k f$ has a strong minimum, and consider $\overline{g} \in G(k;n)$ such that f and $\overline{g}$ have distinct behavior with respect to extremum and $j^k f = j^k \overline{g}$.

It follows, by the fact 2.2, that $\overline{g}$ has a saddle or a weak minimum.

In both cases the germ $g = \overline{g} - \|x\|^{2k}$ has a saddle and $j^k f = j^k g$. ■

3 Finite Decidability and Algebraic Curves

In his 1984's paper, [B0], Barone-Netto has introduced to study k-decidability the following concept for a polynomial $P: \mathbf{R}^n \longrightarrow \mathbf{R}$.

Definition 3.1 *Let $P: \mathbf{R}^n \longrightarrow \mathbf{R}$ be a polynomial, we define the* **radial set**[2] *of P as*

$$V(P) = \left\{ x \in \mathbf{R}^n : rk \begin{bmatrix} x \\ \nabla P(x) \end{bmatrix} \leq 1 \right\}. \quad (2)$$

Remark 3.1 *This set is an algebraic subset of $\mathbf{R}^n$ and $O \in V(P)$.*

Remark 3.2 *If $S_r = \{x \in \mathbf{R}^n : \|x\| = r\}$ then $V(P) \cap S_r$ is the set of the critical points of $P|_{S_r}$.*

[2] This definition is from Barone's work, and it was denoted by this name by the first time in this text, with the permission of the creator of the idea.

Remark 3.3 *Since $V(P)$ is an algebraic subset of $\mathbf{R}^n$ it follows that $V(P)\setminus\{O\}$ has a finite number of connected components each one of then is a analytic sub-manifold of $\mathbf{R}^n$ and there is a $\varepsilon > 0$ such that each connected component of $V(P)$ is transversal to S_r, for $0 < r < \varepsilon$. The reader can be found a demonstration of this in [B0].*

The importance of this set is related to the following functions associate to P.

Definition 3.2 *Let $P: \mathbf{R}^n \longrightarrow \mathbf{R}$ be a polynomial. We will define the functions P^+ and P^- of $[0; +\infty[$ in $\mathbf{R}$ as*

$$P^+(r) = \max P|_{S_r} \ \textit{and}\ P^-(r) = \min P|_{S_r},\ \forall r \geq 0. \tag{3}$$

Remark 3.4 *By the remark 3.2 we see that for all $r \geq 0$ there is points x_r^+ and x_r^- in $V(P) \cap S_r$ such that $P^+(r) = P(x_r^+)$ and $P^-(r) = P(x_r^-)$.*

Remark 3.5 *By using the remark 3.3, the isotope theorem of Thom (see [BM0]), and the curve selection lemma (see [M0]) we can prove that there are two germs of algebraic curves, γ^+ and γ^- defined in $([0; +\infty[;0)$ in $(\mathbf{R}^n;0)$ such that there are $\varepsilon > 0$ and representatives $\tilde{\gamma}^+$ and $\tilde{\gamma}^-$ of γ^+ and γ^-, respectively, defined in $[0;\varepsilon]$ that obeys $P(\tilde{\gamma}^+(t)) = P^+(\|\tilde{\gamma}^+(t)\|)$ and $P(\tilde{\gamma}^-(t)) = P^-(\|\tilde{\gamma}^+(t)\|)$, for $0 \leq t \leq \varepsilon$. We will call this germs respectively by* **curve of maximum of** P *and* **curve of minimum of** P. *For a proof if this see, again, ([B0]).*

In order to study k-decidability we will use this concepts applied to the polynomial $j^k f$.

In the future, we will need the following classical result about algebraic curves (see, by example, [W0])

Lemma 3.1 *If $\alpha: [0;\varepsilon] \longrightarrow \mathbf{R}^n$ is an algebraic curve such that $\alpha(0) = 0$ then α can be parameterized as $\alpha(t) = (\alpha_1(t);\alpha_2(t);\ldots;\alpha_n(t))$, where $\alpha_j(t) = \sum_{k=1}^{\infty} a_k^j t^k$, with a_k^j are real numbers.*

We will be interested especially in the following corollary:

Corollary 3.1 *If $\alpha: [0;\varepsilon] \longrightarrow \mathbf{R}^n$ is an algebraic curve with $\alpha(0) = 0$ and P is a polynomial of $\mathbf{R}^n$ in $\mathbf{R}$, with $P(O) = 0$ and $(P \circ \alpha) \not\equiv 0$, then there is an unique $\mu \geq 1$ such that*

$$\lim_{t \downarrow 0} \frac{P(\alpha(t))}{\|\alpha(t)\|^\mu} \in \mathbf{R} \setminus \{0\} \tag{4}$$

Proof: Observe that the propriety (4) is invariant by re-parameterizations of α, then we choose the parameterization given by the lemma 3.1.

Considering that $\alpha_j(t) = \sum_{k=1}^{\infty} a_k^j t^k$, denote for $1 \leq j \leq n$, $k_j = \min\left\{k: a_k^j \neq 0\right\}$, when this set in not empty and put $k_j = 0$ otherwise. Observe that, since $\alpha \not\equiv 0$, at least one of the k_j is not zero. Then, choose j_0 such that k_{j_0} is the minimum of $\{k_j: k_j \neq 0\}$.

It follows that there is a $\sigma > 0$ such that $\lim_{t \downarrow 0} \frac{\|\alpha(t)\|}{t^{k_{j_0}}} = \sigma$.

Let ℓ be the degree of P and set $P(x) = \sum_\beta a_\beta x_1^{\beta_1} x_2^{\beta_2} \ldots x_n^{\beta_n}$, with $a_\beta \neq 0$, $\beta = (\beta_1;\beta_2;\ldots;\beta_n) \in \mathbb{N}^n$ and $\sum_{j=1}^{n} \beta_j \leq \ell$.

Then, note that $P(\alpha(t)) = \sum_{k=1}^{\infty} b_k t^k$ where, at least one b_k is different from zero, since $(P \circ \alpha) \not\equiv 0$.

It is clear then that, if $\bar{k} = \min\{k : b_k \neq 0\}$, we have $\bar{k} \geq k_j$, for all $j \in \{1;\ldots;n\}$.

In particular, $\bar{k} \geq k_{j_0}$ and the result follows. ■

Remark 3.6 *We call the number μ determined in the corollary 3.1 by* the order of P in *α. If, for an algebraic curve α, $P \circ \alpha \equiv 0$, we will say that the order of P in α is $+\infty$.*

Remark 3.7 *If P is a polynomial with $P(O) = 0$, we can have more than one curve of minimum of P (by example, consider $P(q) = \|q\|^2$), but it follows from the remark 3.5 that, if γ_1 and γ_2 are curves of minimum of P, then the order of P in γ_1 is equal to the order of P in γ_2. An analogous observation is true for curves of maximum.*

Now, we can state:

Fact 3.1 *Suppose that $f \in G(k;n)$ and consider γ^+ and γ^- curves of maximum and minimum of $j^k f$. Then f is k-decidable if, and only if, the order of $j^k f$ in γ^+ and in γ^- are less or equal to k.*

Proof: ($\Rightarrow$) Suppose that the order of $j^k f$ in this two curves are less or equal to k. Then $\left(j^k f\right) \circ \gamma^+ \not\equiv 0$ and $\left(j^k f\right) \circ \gamma^- \not\equiv 0$.

So, by the definition of order, we can assure that $\left(j^k f\right) \circ \gamma^+$ and $\left(j^k f\right) \circ \gamma^-$ have a strong extremum (i.e. any representative of these germs has a strong extremum at $0+$).

We have two situations to consider:

- $\left(j^k f\right) \circ \gamma^+$ has a minimum and $\left(j^k f\right) \circ \gamma^-$ has a maximum, i.e., $j^k f$ has a saddle.

  In this situation is obvious that, if $h \in G(k;n)$ with $j^k h \equiv 0$ we have that (remember that $k \geq \mu$ and $\lim_{x \to 0} \frac{j^k f(x)}{\|x\|^k} = 0$)

$$\lim_{t \downarrow 0} \frac{\left((j^k f) + h\right)(\gamma^+(t))}{\|\gamma^+(t)\|^\mu} = \lim_{t \downarrow 0} \frac{j^k f}{\|\gamma^+(t)\|^\mu} > 0$$

  and

$$\lim_{t \downarrow 0} \frac{\left((j^k f) + h\right)(\gamma^-(t))}{\|\gamma^-(t)\|^\mu} = \lim_{t \downarrow 0} \frac{j^k f}{\|\gamma^-(t)\|^\mu} < 0.$$

  This shows that $j^k f + h$ has a saddle, then by remark 2.8 f is k-decidable.

•• $\left(j^k f\right) \circ \gamma^+$ and $\left(j^k f\right) \circ \gamma^-$ have a strong extremum of same kind, i.e., $j^k f$ has an strong extremum.

We can assume, without loss of generality that $j^k f$ has a strong minimum.

Now, suppose, by *redutio ad absurdum*, that f is not k-decidable, then, by the fact 2.3 and remark 2.5 there is a germ $h \in G(k;n)$ with $j^k h \equiv 0$ and $g = f + h$ has a saddle.

Then, there is a sequence (x_j) of points in $\mathbf{R}^n \setminus \{O\}$ such that $x_j \to 0$ and $g(x_j) < 0$. So, we have $\liminf_{j\to+\infty} \frac{g(x_j)}{\|x_j\|^\mu} \leq 0$ (it could be equal to $-\infty$).

As we have that $j^k h = 0$ and $k \geq \mu$, we see that

$$\liminf_{j\to+\infty} \frac{g(x_j)}{\|x_j\|^\mu} = \liminf_{j\to+\infty} \frac{j^k f(x_j)}{\|x_j\|^\mu} \geq \liminf_{j\to+\infty} \frac{(j^k f)^-(\|x_j\|)}{\|x_j\|^\mu} > 0,$$

which is a contradiction with our previous inequality and holds the result.

($\Leftarrow$) Suppose, without loss of generality, that the order of $j^k f$ in γ^- is μ with $\mu > k$.

If $\left(j^k f\right) \circ \gamma^- \equiv 0$ then $j^k f$ has a weak minimum and by the remark 2.10 f is not k-decidable.

Then, either $\left(j^k f\right) \circ \gamma^-$ has a minimum (and $j^k f$ has a strong minimum), or $\left(j^k f\right) \circ \gamma^-$ has a maximum (and $j^k f$ has a saddle).

In the first case, consider $g = j^k f - \|x\|^{\frac{\mu+k}{2}}$. It is clear that $j^k g = j^k f$ and $g \circ \gamma^-$ has a maximum. Then f and g have different behavior with respect to extremum and this shows that f is not k-decidable.

If $\left(j^k f\right) \circ \gamma^-$ has a maximum then choose $g = j^k f + \|x\|^{\frac{\mu+k}{2}}$. Obviously, $j^k f = j^k g$. We claim that g has a strong minimum.

In order to prove this, it is enough to show that $g \circ \gamma^-$ has a strong minimum, because $\|x\|^{\frac{\mu+k}{2}}$ is constant in the spheres S_r, $r \geq 0$.

Since the order of $j^k f$ in γ^- is μ and $\mu > k$, it follows that $\lim_{t\downarrow 0} \frac{g(\gamma^-(t))}{\|\gamma^-(t)\|^{\frac{\mu+k}{2}}} = 1$, which shows the claim and ends the proof. ■

Remark 3.8 *The fact 3.1 shows that a germ $f \in G(k;n)$ is k-decidable if, and only if, $\lim_{t\downarrow 0} \frac{(j^k f)^+(t)}{t^k}$ and $\lim_{t\downarrow 0} \frac{(j^k f)^-(t)}{t^k}$ are both in $(\mathbf{R} \setminus \{0\}) \cup \{-\infty; +\infty\}$.*

Corollary 3.2 *Suppose that $P: \mathbf{R}^n \longrightarrow \mathbf{R}$ is a polynomial with $P(0) = 0$ and P has a saddle or a strong extremum at the origin. Then there is a natural $k \geq 1$ such that P is k-decidable.*

Proof: Consider γ^+ and γ^-, respectively, curves of maximum and minimum of P. Since P has not a weak extremum, we have that $P \circ \gamma^+ \not\equiv 0$ and $P \circ \gamma^- \not\equiv 0$. Then the order of $P \circ \gamma^+$ and $P \circ \gamma^-$ are real numbers μ^+ and μ^-.

It follows from the fact 3.1 that, if $k = \max\{\lceil\mu^+\rceil, \lceil\mu^-\rceil\}$, then P is k-decidable. ■

The fact 3.1, or the remark 3.8, shows that, in order to study the k-decidability of f it is *enough* to study the behavior of two germs of one variable, namely, $\left(j^k f\right) \circ \gamma^+$ and $\left(j^k f\right) \circ \gamma^-$, or $(j^k)^+$ and $(j^k)^-$. This is the main result of [B0].

Remark 3.9 *In the proof of fact 3.1 we exhibit, when $f \in G(k;n)$ is not k-decidable, a germ g with $j^k f = j^k g$ such that f and g have different behavior with respect to extrema. The germ g is equal to $j^k f + \|x\|^\nu$, with ν a real number that belongs to the interval $(k;\mu)$, where μ is the maximum of the orders of $\left(j^k f\right) \circ \gamma^+$ and $\left(j^k f\right) \circ \gamma^-$.*

Observe that when $\mu > k+2$, we can choose ν as the even number of $\{k+1; k+2\}$ and, in this case, g is a polynomial.

In the next two sections we will study this remark in detail.

4 Polynomial Decidability

Definition 4.1 *We will say that $f: (\mathrm{R}^n;0) \longrightarrow (\mathrm{R};0)$ is* **polynomially k–decidable** *if*

(i) $f \in G(k;n)$;

(ii) For all **polynomial** *g, such that $j^k g = j^k f$, f and g has the same behavior with respect to extremum.*

Remark 4.1 *As in k–decidability, f is polynomially k–decidable if, and only if $j^k f$ is polynomially k–decidable.*

Remark 4.2 *It is clear that if f is k-decidable then it is polynomially k-decidable. We are interested in the reciprocal of this statement.*

Remark 4.3 *Since $j^k f + \|x\|^{2k}$ is a polynomial, the remark 2.10 is true in the context of polynomial k-decidability, too. If $j^k f$ has a weak extremum then f is not polynomially k-decidable.*

This remark and the fact 2.1 show imply the following result.

Fact 4.1 *If $j^k f$ is an homogeneous polynomial, $j^k f \not\equiv 0$. Then f is k-decidable if, and only if, f is polynomially k-decidable.*

Proof: Immediate. ■

Corollary 4.1 *If $f \in G(2;n)$ is polynomially 2–decidable then f is 2-decidable.*

Proof: If $j^1 f \not\equiv 0$, then it is clear that f is 1–decidable, since this jet is homogeneous and has a saddle.

Otherwise, we must have $j^2 f$ an homogeneous polynomial and the the result follows from the fact 4.1. ∎

Remark 4.4 *Another case where k–decidability is equivalent to the polynomially k–decidability is when $n = 1$, since in* R, *by the remark 2.6, the first non null jet of f always decides the question.*

We will state now a less trivial result.

Fact 4.2 *Suppose that $f \in G(k;n)$ and that $j^2 f$ is a quadratic form with rank $n-1$. Then are equivalent*

(i) f is k–decidable.

(ii) f is polynomially k–decidable.

Proof: ((*i*) ⇒ (*ii*)) It is the remark 4.1.
((*ii*) ⇒ (*i*)) Since $j^2 f$ is quadratic form, this is an homogeneous polynomial.

Then, if $j^2 f$ has a saddle, the lemma 2.1 shows that f is two–decidable and we are done.

So, we can assume that $j^2 f$ is a quadratic form semi–defined. Without loss of generality, we will suppose that it is positive semi–defined.

Then, by the *splitting lemma* (see [B0] for a version of this on our context) we can do a convenient change of coordinates T of class C^∞ such that, in the news coordinates, $(u_1;\ldots;u_{n-1};u_n) = (\overline{u};u_n)$, we have that

$$(f \circ T)(\overline{u};u_n) = \sum_{j=1}^{n-1} u_j^2 + g(u_n),$$

with $j^2 g \equiv 0$.

Then it is clear that $j^k(f \circ T) = \sum_{j=1}^{n-1} u_j^2 + j^k g(u_n)$.

Since f is polynomially k–decidable, it follows by the remark 4.3 that $j^k f$ has not a weak minimum. Then, by the corollary 3.2, there exists an natural ℓ such that $j^k f$ is ℓ–decidable.

Therefore, the remark 2.9 implies that $j^k(f \circ T)$ is ℓ–decidable and this shows that $j^k g \not\equiv 0$, otherwise $j^k(f \circ T)$ would have a weak minimum against our conclusion.

Let s be the order of the first non null jet of g, then $j^s(f \circ T) = \sum_{j=1}^{n-1} u_j^2 + \alpha u_n^s$, with $\alpha \neq 0$ and $3 \leq s \leq k$.

A direct calculus shows that the germ of the radial set of $j^s(f \circ T)$ at the origin is

$$(V;0) = \left(\{(\overline{u};u_n) \in \mathbf{R}^n : u_n = 0\} \cup \{(\overline{u};u_n) \in \mathbf{R}^n : \overline{u} = O\};O\right).$$

Then, it follows immediately, that the order of $j^s(f \circ T)$ in the curves of maximum and minimum are respectively 2 and s.

The fact 3.1 holds that $f \circ T$ is s–decidable and, by the remark 2.9, this assures that f is s–decidable.

Since $s \leq k$ this ends the proof. ■

Corollary 4.2 *Let f be a germ of $(\mathbf{R}^2;0)$ in $(\mathbf{R};0)$. If $f \in G(k;2)$, with $k \geq 2$, and $j^2 f \not\equiv 0$ then f is k–decidable if, and only if, f is polynomial k–decidable.*

Proof: It follows immediately from the facts 4.1 and 4.2. ■

As we will see, if the rank of $j^2 f$ is less or equal than $n-2$, the fact 4.2 can be false (see the remark 5.1).

Now, we will show that polynomial k–decidability does not imply k–decidability.

Example 4.1 *(see [G0]) Consider $f(x;y) = y^4 - x^4 y + x^{5.2}$. We will prove that:*

(A) If $g \in G(6;2)$ and $j^5 g = j^5 f$ then g has saddle.

(B) f has a strong minimum at the origin.

It is clear that *(A)* shows that f is polynomially 5–decidable (really, *it shows that f is 5-decidable in the class of the germs that are in* $G(6;2)$) and these two statements show that f is not 5–decidable.

Proof of (A): As $g \in G(6;2)$ and $j^5 g = j^5 f$, we have that $g = j^6 g + R = j^5 f + g_6 + R$, where $j^6 R \equiv 0$ and either g_6 is an homogeneous polynomial of degree 6 or g_6 is the null polynomial (see remark 2.4).

Consider γ the germ of the algebraic curve $y = x^{1.4}$, $x \geq 0$, at the origin and note that $j^6 g(x;x^{1.4}) = x^{5.6} - x^{5.4} + g_6(x;x^{1.4})$.

It follows from the definition of g_6 that $\lim_{x \to 0} \frac{g_6(x;x^{1.4})}{x^{5.4}} = 0$.

Moreover, since $j^6 R \equiv 0$, it is clear that $\lim_{x \to 0} \frac{R(x;x^{1.4})}{x^{5.4}} = 0$.

Then, we have

$$\lim_{x \to 0} \frac{g(x;x^{1.4})}{x^{5.4}} = -1.$$

This shows that the order of $g \circ \gamma$ is 5.4 and that $g \circ \gamma$ has a strong maximum.

Since $g(0;y)$ has a strong minimum, we have showed *(A)*. ■

Proof of (B): Consider $(x_o;y_o) \in \mathbf{R}^2 \setminus \{O\}$ such that $f(x_o;y_o) = 0$.

We must have $y_o > 0$, $x_o \neq 0$ and

$$y_o^4 - x_o^4 y_o = y_o(y_o^3 - x_o^4) = -x_o^{5.2} < 0. \qquad (5)$$

Therefore

$$0 < y_o < x_o^{\frac{4}{3}}. \qquad (6)$$

Now, choose $\varepsilon > 0$ such that $x^{5.2} \geq x^{\frac{16}{3}}$, for all $x \in \mathbf{R}$ with $|x| < \varepsilon$.

By and (5) and (6), we see that $|x_o| \geq \varepsilon$, because, if this fails, we would have

$$y_o^4 = x_o^4 y_o - x_o^{5.2} < x_o^{\frac{16}{3}} - x_o^{5.2} \leq 0,$$

which is a contradiction.

Then $f(x;y) \neq 0$, for all $(x;y) \in \mathbb{R}^2 \setminus \{O\}$, with $|x| < \varepsilon$. As $f(0;y) = y^4$, it follows that f has a strong minimum at origin. ■

In an analogous way, we will see that:

Example 4.2 *Consider the polynomial $P(x;y) = y^8 - x^7y^3$.*

(C) If Q is a polynomial with $j^{10}Q = j^{10}P$, then Q has a saddle at the origin (i.e., P is polynomial 10–decidable).

(D) The function $f(x;y) = P(x;y) + x^{11.02}$ has a strong maximum at the origin (This is, P is not 11–decidable).

Proof of (C): Consider the curve $y^5 = \frac{1}{2}x^7$ (i.e. $\gamma(x) = (x;\frac{1}{2}x^{\frac{7}{5}})$).

We see that $P(\gamma(x)) = -\frac{1}{2}x^{\frac{56}{5}}$ and $\|\gamma(x)\| = |x|\sqrt{1 + x^{\frac{4}{5}}}$.

Then the order of $P \circ \gamma$ is $\mu = \frac{56}{5} = 11.2$ and $P \circ \gamma$ has a strong maximum (in 0).

We must observe here that γ is defined for $x \in \mathbb{R}$, but for our proposal it is convenient consider two distinct arcs, $\gamma_1(x) = \gamma(x)$, $x \geq 0$ and $\gamma_2(x) = \gamma(x)$, $x \leq 0$.

Then $P \circ \gamma_1$ (resp. $P \circ \gamma_2$) has a strong maximum at $x = 0+$ (resp. at $x = 0-$) of order μ.

Now consider a germ $g: (\mathbb{R}^2;O) \longrightarrow (\mathbb{R};0)$ such that $g \in G(12;2)$ and $j^{10}g = j^{10}P = P$.

Since $P(0;y) = y^8$ it follows that $g(0;y)$ has a minimum (at $y = 0$).

Moreover, $g = P + g_{11} + g_{12} + R$, where $j^{12}R \equiv 0$ and, either $g_k \equiv 0$, or g_k is an homogeneous polynomial of degree k, for $k \in \{11,12\}$.

Then, we can write $g_{11}(x;y) = a_0x^{11} + Q(x;y)$, where $Q(x;y) = \sum_{k=1}^{11} a_kx^{11-k}y^k$ and a_k are real numbers, $0 \leq k \leq 11$.

A direct calculus shows that the order of $Q(x;\frac{1}{2}x^{\frac{7}{5}})$ is greater or equal than $\frac{57}{5} = 11.4 > \mu$. Therefore,

$$\lim_{x\to 0} \frac{(Q + g_{12} + R)(x;x^{\frac{7}{5}})}{\|(x;x^{\frac{7}{5}})\|^{\mu}} = 0$$

Then, if $a_0 = 0$, it is clear that $g \circ \gamma$ has a strong maximum. So, g has a saddle and we have finished the proof, in this case.

At least, if $a_0 \neq 0$ we see immediately that $P(x;y) + a_0x^{11}$ has a strong strong maximum of order 11 in one of the curves $\gamma_j, j = 1,2$ (and has a strong minimum of order 11 in the other).

Then, in all the cases, g has a saddle. ■

Proof of (D): It is analogous to the proof of *(B)*. ■

Remark 4.5 *Observe that in the proof of (C) we have showed that P is 12–decidable, since j^8P is homogeneous of degree 8 and has a weak minimum and $P \circ \gamma$ has a maximum of order $\mu = 11.2 < 12$.*

The example 4.2 is more *degenerate* than the first one, since the difference of the degree of polynomial decidability and the degree of decidability is 2, as we will see in the next section, this is the maximum value of this difference.

To finish this section, we present another example, in order to see that the distinction between k–decidability and polynomial k–decidability is not a *honor* of the polynomials that have a saddle.

Example 4.3 *(see [G0]) Consider $P(x;y) = (x^4 - y^3)^2 + x^{12}y^2$.*

(E) P is polynomially 14–decidable.

(F) P is not 14–decidable.

Proof of (E) and (F): A direct calculus shows that the radial set of $P = j^{14}P$ is given by

$$V(P) = \left\{(x;y) \in \mathbf{R}^2 \colon 2xy\left[(x^4 - y^3)(4x^2 + 3y) + x^{10}(6y^2 - x^2)\right] = 0\right\}.$$

Then $V(P)$ is the union of the coordinated axes and the algebraic set described by the equation $(x^4 - y^3)(4x^2 + 3y) + x^{10}(6y^2 - x^2) = 0$.

By using the Newton polygon (see [W0]) for this equation we see that this set is an algebraic curve γ which is tangent to the x–axe and it can be parameterized, in a neighbourhood of 0 as $y = y(x) = x^{\frac{4}{3}}h(x)$ where h is a C^1 function defined in an open interval of $\mathbf{R}$ containing zero and $h(0) = 1$.

Then, $P \circ \gamma = P(x;y(x))$ and a simple verification shows that the order of $P \circ \gamma$ is $\mu = \frac{44}{3}$ and $P \circ \gamma$ has minimum.

It follows that γ is the curve of minimum of P (since $P(x;0) = x^8$ and $P(0;y) = y^6$) and, by the fact 3.1 that f is 15–decidable and not 14–decidable. So, we have finished the proof of *(F)*.

Moreover, if $g \in G(15;2)$ and $j^{14}g = j^{14}P = P$, then $g = P + g_{15} + R$, where $j^{15}R \equiv 0$ and g_{15} is either an homogeneous polynomial of degree 15, or $g_{15} \equiv 0$.

Then it is easy to see that, if $q_j = (x_j;y_j)$ is a sequence in $\mathbf{R}^2$ which converges to the origin, then

$$\lim_{j \to +\infty} \frac{(g_{15} + R)(q_j)}{\|q_j\|^{\mu}} = 0$$

and this shows that g has a minimum at origin.

In particular, this proves *(E)*. ∎

5 Polynomial k–Decidability $\Rightarrow \ldots$ Decidability

We will show in this section the following results:

Theorem 5.1 *Suppose that $f:(\mathbf{R}^n;O) \longrightarrow (\mathbf{R};0)$ is polynomially k-decidable and $j^k f$ has a strong extremum. Then f is $(k+1)$-decidable. Moreover, it is possible to exhibit an f that obeys this hypothesis and f is not k-decidable.*

Theorem 5.2 *Suppose that $f:(\mathbf{R}^n;O) \longrightarrow (\mathbf{R};0)$ is polynomially k-decidable, k is odd and $j^k f$ has a saddle. Then f is $(k+1)$-decidable. Moreover, it is possible to exhibit an f that obeys this hypothesis and f is not k-decidable.*

Theorem 5.3 *Suppose that $f:(\mathbf{R}^n;O) \longrightarrow (\mathbf{R};0)$ is polynomially k-decidable, k is even and $j^k f$ has a saddle. Then f is $(k+2)$-decidable. Moreover, it is possible to exhibit an f that obeys this hypothesis and f is not $(k+1)$-decidable.*

This shows all the possibles relations between polynomial k-decidability and k–decidability.

We end the text with a simple fact showing that polynomial k–decidability implies, by example, C^∞-k-decidability, namely, we will prove the following:

Corollary 5.1 *Let be f a germ in $G(k;n)$ such that f is polynomial k-decidable, then, for all $g \in G(k+2;n)$ with $j^k f = j^k g$, we have that f and g has same behavior with respect to extremum.*

Now, we will do the proofs of the theorems.

Proof of the theorem 5.1: Observe that the last part of the statement is proved in the example 4.3.

Since $j^k f$ has an strong extremum, we can suppose without loss of generality, that $j^k f$ has a strong minimum.

If f is k-decidable then the thesis holds immediately.

Then, we will suppose that f is not k-decidable. By the fact 3.1, this implies that, if γ^+ and γ^- are, respectively, the curves of maximum and minimum of $j^k f$, or the order of $f \circ \gamma^+$ is greater than k, or the order of $f \circ \gamma^-$ is greater than k.

Let s be the order of the first non null jet of f. Obviously, $s \leq k$.

Since f is not s–decidable, $j^s f$ has a weak extremum. Then $s < k$ and, as $j^k f$ has a strong minimum, it follows, by the fact 2.2, that $j^s f$ has a weak minimum.

Therefore the order of $\left(j^k f\right) \circ \gamma^+$ is s. In fact, as $j^{s-1} f \equiv 0$ it is clear that this order must be greater or equal to s. Now, choose a point $\overline{x} \in \mathbf{R}^n$ such that $j^s f(\overline{x}) > 0$ to see that in the curve $t\overline{x}$, $t \geq 0$, the order of $j^k f$ is s (and $j^k f$ has a minimum at $t = 0$ in this curve). Since $(j^k f)^+(t\|\overline{x}\|) \geq j^k f(t\overline{x})$, we have that the order of $f \circ \gamma^+$ is less or equal to s and our claim is proved.

So, the order of $\left(j^k f\right) \circ \gamma^-$ is $\mu > k$.

In order to finish the proof we must prove that $\mu \leq k+1$.

Suppose, by *redutio ad absurdum* that $\mu > k+1$ and consider $u \in \mathbb{R}^n \setminus \{O\}$ the versor of γ^- at the origin (u exists because γ^- is algebraic).

Then consider the polynomial $g(x) = j^k f(x) - (\, x \mid u \,)^{k+1}$.

Obviously g is a polynomial of degree $k+1$ and $j^k g = j^k f$.

Now, a simple calculus show that

$$\lim_{t \downarrow 0} \frac{g \circ \gamma^-(t)}{\|\gamma^-(t)\|^{k-1}} = -1 (= -\|u\|^2).$$

Then $g \circ \gamma^-$ has order $k+1$ and this germ has a maximum, what contradicts our assumption that f is polynomial k–decidable. ■

Proof of the theorem 5.2: The example 4.1 proves the last part of the theorem.

Suppose that f is polynomially k–decidable and that $j^k f$ has a saddle, with k an odd natural number.

If f is k–decidable, there is nothing to do. Then we will assume that f is not k–decidable.

As in the previous demonstration, if s is the order of the first non null jet of f, our assumption that f is not k–decidable implies that $s < k$ and $j^s f$ has an weak extremum. We will suppose, without loss of generality, that $j^s f$ has a weak minimum.

Also as in the demonstration of the theorem 5.1, if γ^+ is a curve of maximum of $j^k f$ we have that the order of $j^k \circ \gamma^+$ is s.

Then, if γ^- is a curve of minimum of $j^k f$, we must have, by the fact 3.1 that the order of $\left(j^k f\right) \circ \gamma^-$ is $\mu > k$ (in this case $\left(j^k f\right) \circ \gamma^-$ has a maximum, since $j^k f$ has a saddle and $j^s f$ has a weak minimum).

Then, we must prove that $\mu \leq k+1$. Suppose, by *redutio ad absurdum* that $\mu > k+1$.

Observe that, as k is odd, $k+1$ is even and consider the polynomial

$$g(x) = j^k f(x) + \|x\|^{k+1}.$$

Since the order of $\left(j^k f\right) \circ \gamma^-$ is $\mu > k+1$ we have that $g \circ \gamma^-$ has a strong minimum and since $\|x\|^{k+1}$ is constant in S_r this shows that g has a strong minimum.

Then we have a contradiction with our hypothesis about f. ■

Proof of the theorem 5.3: The example 4.2 shows the last part of the statement.

The rest of the proof is the same of the theorem 5.2 changing $k+1$ by $k+2$. ■

Proof of the corollary 5.1: It is immediate from the previous theorems of this section.■

Remark 5.1 *We have seen in the fact 4.2 that, if the rank of $j^2 f$ is greater or equal $n-1$ then polynomial decidability is equivalent to decidability. This is false if the rank of $j^2 f$ is less or equal to $n-2$ as we see considering, by example, for $n \geq 3$, $f(x_1;x_2;\overline{x}) = x_2^4 - x_1^4 x_2 + \|\overline{x}\|^2$, where $\overline{x} = (x_3;\ldots;x_n)$. It is clear that $j^2 f = \|\overline{x}\|^2$ and, by the example 4.1, f is 5–polynomial decidable but not 5–decidable.*

6 References

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