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AN EXACT FEM GEOMETRIC NON-LINEAR ANALYSIS OF FRAMES BASED ON POSITION DESCRIPTION

Humberto Breves Coda
Escola de Engenharia de São Carlos da Universidade de São Paulo
Av. Trabalhador Sãocarlense 400, São Carlos São Paulo CEP 13566-900
hbcoda@sc.usp.br

Abstract. *This work presents a simple formulation to treat large deflections by the Finite Element Method. The present formulation does not use the concept of displacement and deformation function; it considers position as the real variable of the problem. The strain determination is done directly from the proposed position concept. A non-dimensional space is created and relative curvature and fibers length are calculated for both reference and deformed configurations and used to directly calculate the strain energy at general points. This point of view is very precise as demonstrated in the example section. The technique is applied for 2D frame problems*

Keywords. *frame, geometrical non-linearity, finite element method, positions*

1. Introduction

The analysis of structures that exhibit large deflections is of great importance in nowadays engineering. The crescent search for economy and optimal material application leads to the conception of very flexible structures. As a consequence, the equilibrium analysis in the non-deformed position is no more acceptable for most of applications. In this sense, a lot of important works has been presented in this area. For example, the analytical solution of slender bars and simple composition of them has been developed by various authors during the last years (Bisshop and Drucker, 1945; Mattiason, 1981; Chuchkeepsakull et all., 1995; Simitses et all, 1981; Pai and Palazotto, 1996; Goto et all, Eliaz ,1986, Frish-fay, 1962; Barten, 1944; Barten, 1945; Jenkins et all, 1966 and Kerr, 1964(. This approach is quite complicate and not general because the superposition of effects is not valid as it is for linear applications.

In order to create automatic, general and reliable tools for the analysis of largely deflected structures, various researchers have presented important contributions along time regarding finite element procedures (James et all, 1974; Oran and Kassimali, 1976; Mondkar and Powell, 1977; Belitschko et all, 1977; Argyris et all, 1978; Bathe and Bolourchi, 1979; Argyris et all, 1979; Riks, 1979; Argyris et all, 1984; Argyris et all, 1986; Petersson and Petersson, 1985; Bathe et all, 1975; Crisfield, 1991; Gattass and Abel, 1987; Bathe, 1996; Crisfield, 1981; Wriggers and Simo, 1990; Kleiber, 1989; Bathe, 1996; Izzuddin and Elnashai, 1993; Petrolito and Legge, 1996; Crisfield, 1990; Wong and Tinloi,1990; The and Clarke, 1998; Bergan et all, 1978; Behdinan et all, 1998; Schulz and Filippou, 2001). These works are also very important to the development of the human knowledge on the subject, clarifying and opening the understanding of nowadays researchers. It is difficult to put together all works in this area, identifying them regarding their approach and classifying them by their importance. However the authors would like to make a brief summary of consulted works that helped the understanding and inspired the present formulation development.

For frame geometrical non-linear analysis it is usual to classify the approaches regarding the adopted displacement representation. Following this parameter of classification one can find in literature valuable works based on total Lagrangean description, i.e., displacements referred to a fixed co-ordinate system during all analysis (James et all, 1974; Mondkar and Powell, 1977; Schulz and Filippou, 2001). Updated Lagrangean description can be found, for example, in references (Petersson and Petersson, 1985; Crisfield, 1991; Gattass and Abel, 1987). Few works use both above representations and compare one to another as, for example reference (Bathe, 1996). One kind of updated description, slight different from the above mentioned is the co-rotational one (Crisfield, 1990; The and Clarke, 1998; Behdinan et all, 1998). Another kind of displacement approach, adopted for solid analysis, is the so called natural co-ordinates (Argyris et all, 1979; Argyris et all, 1984; Argyris et all), in this case the usual Cartesian system of reference is not adopted. Some works use the Eurelian description, as references (Oran and Kassimali, 1976; Bathe et all, 1975), where the co-ordinate system is fixed upon the structure. Other works are better classified by their way in controlling strain, for example, references (Crisfield, 1981; Wong and Tinloi,1990) are of the arch-length type.

This work proposes a formulation based on the minimum principle of potential energy. Two main novelties may be pointed out. The first is the identification that the definition of body's kinematics necessarily, for all consulted references, passes trough the explicit definition of displacement. In the present formulation the word displacement is not mentioned, i.e., to define the kinematics of the body only the concept of position is assumed. The strain determination is done directly from the proposed position concept.

The second difference is that in all consulted works the deformation function is achieved by differentiating the deformed configuration regarding the reference one. In this work a non-dimensional space is created and no deformation function is used, but relative curvature and fibers length are calculated for both reference and deformed configurations and used to calculate the strain energy at general points.

As a secondary consequence of these considerations one does not necessarily think about concepts as: increment, linearization, prevision, correction, tangent matrix etc., largely used in literature. Of course, that these concepts are right and important, but their use is not necessary for the development of the proposed formulation. In the end of the work various examples are shown in order to demonstrate the precision of the proposed formulation.

2. General description

To state the proposed methodology one should start from the Minimum Principle of Total Potential Energy, stated from position considerations (not displacements):

$$\Pi = U_e - P \tag{1}$$

where Π is the total potential energy, U_e is the strain energy and P is the potential energy of applied forces.

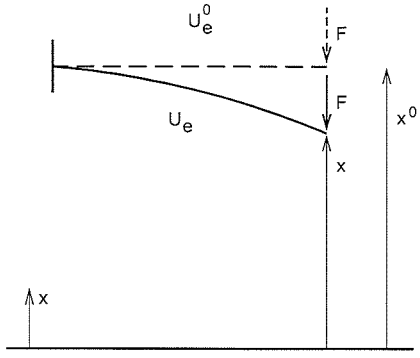

Figure 1 – Total Potential Energy written for a body in two different positions.

Adopting linear elasticity the strain energy is written for the volume V as:

$$U_e = \int_V \frac{1}{2}\sigma\varepsilon dV \tag{2}$$

Where stresses are evaluated in the Cauchy sense and strain are given by linear measurement (conjugate). The strain energy is assumed to be zero in a reference position, called non-deformed. The potential energy of applied forces is written as:

$$P = \sum FX \tag{3}$$

where X is the set of positions, independent from each other, that a point of the body can occupy. It is important noting that the potential energy of applied forces can be non-zero in the reference configuration.

The total potential energy is written as:

$$\Pi = \int_V \frac{1}{2}\sigma\varepsilon dV - \sum FX \tag{4}$$

In order to perform the integral indicated in equation (4) it is necessary to know the geometry of the studied body (the accepted geometric approximation) and its relation with the adopted strain measurement. Figure 2 gives the general geometry of a curve over a plane (frame bar).

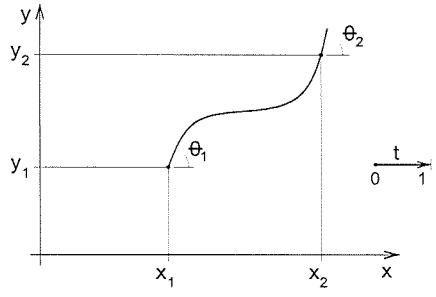

Figure 2. Curve in a 2D space.

This generic curve represents a configuration of the body. It can be parameterized as a function of a non-dimensional variable t (varying from 0 to 1). It should be noted that for two-dimensional problems one can state a linear approximation for position in x direction and a cubic approximation for y direction. When it is done an addition care should be taken when the curve stop to be a one valued function of x.

Adopting the above described approximation one writes:

$$x = X_1(1-t) + X_2 t \tag{5}$$

The value

$$l_x = (X_2 - X_1) \tag{6}$$

will be used in the formulation development. Relating t to y, following cubic approximation, one writes:

$$y = ct^3 + dt^2 + et + f \tag{7}$$

It is necessary to solve the generalized parameters c, d, e and f of equation (7) in order to write it as a function of the nodal parameters, i.e., positions $X_1, Y_1, X_2, Y_2, \Theta_1, \Theta_2$. It is done making $y(0) = Y_1$, $y(1) = Y_2$, $\partial y / \partial x(0) = \Theta_1$, $\partial y / \partial x(1) = \Theta_2$, resulting:

$$c = [\tan(\Theta_2) + \tan(\Theta_1)]l_x - 2l_y \tag{8a}$$

$$d = 3l_y - [\tan(\Theta_2) + 2\tan(\Theta_1)]l_x \tag{8b}$$

$$e = \tan(\Theta_1)l_x \tag{8c}$$

$$f = Y_1 \tag{8d}$$

It is interesting to note that the last two parameters will appear as arguments of tangent functions, i.e., $\tan(\Theta_1)$ and $\tan(\Theta_2)$.
The following strain evaluation is adopted:

$$\varepsilon = \frac{ds - ds_0}{ds_0} \tag{9a}$$

Logarithmic strain measurement

$$\varepsilon = \frac{ds - ds_0}{ds} \tag{9b}$$

where ds is the length of a fiber inside the domain (in this case it is parallel to the central line) in any position. ds_0 is its length in the reference configuration.

It is interesting to note that in majority of references this strain measurement is called linear strain. Any strain measurement can be used to solve geometrical non-linear problems if one takes its proper conjugate stress in order to calculate energy. For the second adopted strain its conjugate stress is the so called Cauchy (or true) stress.

At this point the novelty is to identify that the proposed strain determination can be achieved by relative length referred to the non-dimensional space represented here by variable 't', i.e.:

$$\varepsilon = \frac{ds - ds_0}{ds_0} = \frac{ds/dt - ds_0/dt}{ds_0/dt} \tag{10a}$$

or

$$\varepsilon = \frac{ds - ds_0}{ds} = \frac{ds/dt - ds_0/dt}{ds/dt} \tag{10b}$$

It is interesting to note that in majority of references this strain measurement is called linear strain. Any strain measurement can be used to solve geometrical non-linear problems if one takes its proper conjugate stress in order to calculate energy. For the adopted strain its conjugate stress is the so-called Cauchy (or true) stress.

In this work it will be considered the reference configuration as a straight line. For this situation the reference approximation can be taken as:

$$x^0 = X_1^0 + l_x^0 t \tag{11}$$

$$y^0 = Y_1^0 + l_y^0 t \tag{12}$$

For the central line (passing trough the mass center of the bar) in the initial configuration one has:

$$\frac{ds^0}{dt} = \sqrt{\left(\frac{dx^0}{dt}\right)^2 + \left(\frac{dy^0}{dt}\right)^2} = \sqrt{(l_x^0)^2 + (l_y^0)^2} = l_0 \tag{13a}$$

or

$$ds^0 = l_0 dt \tag{14a}$$

In the same way for another general configuration one has:

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2} \tag{14b}$$

or

$$ds = \sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2}\, dt \tag{14c}$$

Applying strain measurement, for example equation (10a), for the central line one calculates:

$$\varepsilon^m = \frac{1}{l_0}\sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2} - 1 \tag{15}$$

Following the curvilinear coordinate 's' one can define an orthogonal co-ordinate to 's', called 'z', where one defines the strain dependence (Euler-Bernoulli hypothesis). Following this hypothesis one writes the strain as a function of the curvature difference plus the central line value.

$$\varepsilon = \varepsilon^m + \left(\frac{1}{r} - \frac{1}{r_0}\right) z \tag{16a}$$

As the initial curvature is zero (initial shape is a straight line) one has:

$$\varepsilon = \varepsilon^m + \frac{1}{r} z \tag{16b}$$

The exact curvature is given as a function of t only, as one can see in references [Wylie and Barret, 1995 and Leithould, 1977):

$$\frac{1}{r} = dx/dt * d^2y/dt^2 \Big/ \left(\sqrt{(dy/dt)^2 + (dx/dt)^2}\right)^3 \tag{17}$$

or, replacing the known expressions (approximation for positions):

$$\frac{1}{r} = l_x(6ct + 2d) \Big/ \left(\sqrt{l_x{}^2 + (3ct^2 + 2dt + e)^2}\right)^3 \tag{18}$$

Taking true stress, linear elasticity, and strain determination one writes the specific strain energy as:

$$u_e(espec)=\frac{E}{2}(\varepsilon^m+\frac{1}{r}z)^2=\frac{1}{2}\left(\left(\varepsilon^m\right)^2+2\varepsilon^m\frac{1}{r}z+\left(\frac{1}{r}z\right)^2\right) \tag{19}$$

In order to calculate the strain energy it is necessary to integrate the specific strain energy ($u_e(espec)$) over the volume of the analyzed body. For true stress and linear strain measurement it is done referring to the reference volume, as it is the zero energy situation and any change of energy should be referred to it. A definitive numerical proof for this affirmation is given in example three, where large strain is present.

To integrate the specific strain energy over the volume one starts by integrating it over the cross section as follows:

$$u_e=\frac{EA}{2}\left(\varepsilon^m\right)^2+\frac{EI}{2}\left(\frac{1}{r}\right)^2 \tag{20}$$

Now one integrates the strain energy per unit of length, equation (20), along the length of the bar, i.e..

$$U_e=\int_0^1\frac{EA}{2}\left(\varepsilon^m\right)^2+\frac{EI}{2}\left(\frac{1}{r}\right)^2 l_0dt=l_0\int_0^1 u_e dt \tag{21}$$

As the strain energy is known (written as a function of nodal parameters) it is necessary to differentiate the Total Potential Energy, expressions (1) or (4), regarding the nodal parameters, to obtain the equilibrium statement. In order to do so one should reorganize previous equations as follows:

$$\Pi=l_0\int_0^1 u_e dt\ -F_{x1}X_1-F_{y1}Y_1-M_1\Theta_1-F_{x2}X_2-F_{y2}Y_2-M_2\Theta_2 \tag{22}$$

As there are no singularities in the strain energy integral one can write:

$$\frac{\partial\Pi}{\partial X_1}=l_0\int_0^1\frac{\partial u_e}{\partial X_1}dt\ -F_{x1}=0 \tag{23a}$$

$$\frac{\partial\Pi}{\partial Y_1}=l_0\int_0^1\frac{\partial u_e}{\partial Y_1}dt\ -F_{y1}=0 \tag{23b}$$

$$\frac{\partial\Pi}{\partial\Theta_1}=l_0\int_0^1\frac{\partial u_e}{\partial\Theta_1}dt\ -M_1=0 \tag{23c}$$

$$\frac{\partial\Pi}{\partial X_2}=l_0\int_0^1\frac{\partial u_e}{\partial X2}dt\ -F_{x2}=0 \tag{23d}$$

$$\frac{\partial\Pi}{\partial Y_2}=l_0\int_0^1\frac{\partial u_e}{\partial Y_2}dt\ -F_{y2}=0 \tag{23f}$$

$$\frac{\partial\Pi}{\partial\Theta_2}=l_0\int_0^1\frac{\partial u_e}{\partial\Theta_2}dt\ -M_2=0 \tag{23g}$$

The strategy is to develop the derivatives inside the integrals and after that integrate it numerically along the non-dimensional space by Gaussian quadrature. As it can be noted the numerical integral result is not linear regarding the nodal parameters. Therefore, one writes the above system of equations in the following compact notation:

$$\frac{\partial\Pi}{\partial v_i}=g_i(v_j)=f_i(v_j)-F_i=0 \tag{24}$$

where v_i is a generalized parameter and its indices are related to the nodal variables by $(1,2,3,4,5,6)=(X_1,Y_1,\Theta_1,X_2,Y_2,\Theta_2)$

In a vector representation one has:

$$g(v,F)=0 \tag{25}$$

or

$$f(v)-F=0 \tag{26}$$

Note that in this work the applied forces have been assumed not dependent regarding space. Space dependent forces are easily implemented if desired. The vector function g(v) is non-linear regarding nodal parameters (v and F), but equation (25) represents the minimum potential energy situation and therefore the equilibrium of the analyzed body. To solve equation (25) one can use the Newton-Rapson procedure, i.e.:

$$g(p)=0=g(p_0)+\nabla g(p_0)\Delta p \tag{27}$$

or

$$\Delta p=-[\nabla g(p_0)]^{-1}g(p_0) \tag{28}$$

Where p is any position and p_0 is the initial position.

At this point all usual words of non-linear analysis could be introduced, but the reader is invited to understand the procedure as a simple non-linear system solver. One can calculate the Hessian (of strain energy) matrix $\nabla g(p_0)$ from expressions (24) and (25), as:

$$\nabla g(p_0)=g_{i,k}(p_0)=f_{i,k}(p_0)-F_{i,\ell} \tag{29}$$

where i=1-6; k=1-6 for parametric positions and ℓ=7-12 for forces. It is easy to achieve the following representation:

$$\nabla g(p_0)=l_0\int_0^1 u_{e,ik}dt\Big|_{p_0}-\delta_{i\ell} \tag{29a}$$

For equation (28)+one needs to calculate $g(p_0)$, i.e.:

$$g(p_0)=l_0\int_0^1 u_{e,i}dt\Big|_{p_0}-F_i \tag{30}$$

The iterative (Newton-Rapson) process is summarized as: Assumes initially p_0 as the initial configuration (non-deformed). Calculates $g(p_0)$ following equation (30). For this p_0 calculates the Hessian matrix by unity of length, $u_{e,ik}\big|_{p_0}$. Integrates this value as indicated in (29a) and results the gradient of g at p_0. Solves the system of equations (28) and determines Δp. Updates position $p_0 = p_0+\Delta p$. Goes back to step 1 until Δp is sufficiently small.

To achieve good results, the total loading (or prescribed position) is divided in cumulative steps resulting in an incremental and iterative procedure.

3. Numerical Examples:

3.1. Clamped beam:

This example consists on an initially horizontal clamped beam that is subjected to a crescent rotation, as well as a crescent bending moment, at its free end. The adopted parameters are: L=500cm, A=20cm^2, E=2x10^{6}N/cm^2 and I=2000cm^4. Position control is dictated by an applied rotation of θ=6.2831853 divided into 100 steps. Load control is dictated by an applied bending moment of M=50,265,482.48Nm divided into 100 steps. It has been 100 finite elements to run this example. In spite of 100 steps, to run this problem only eight are enough. The adopted tolerance is 10^{-5} in relative position. In figure 4 the horizontal and vertical positions of the free node, achieved by the program and the analytical solution, are compared.

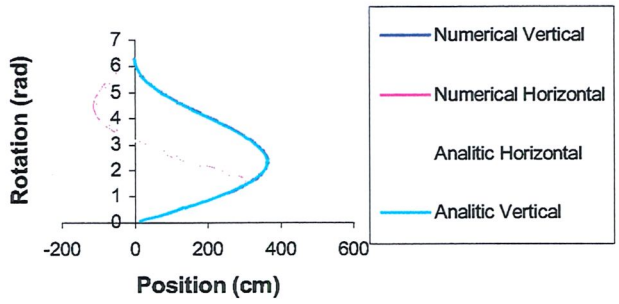

Figure 4: Horizontal and vertical positions for analitical and numerical cases (rotation control).

In figure 5a the deformed shapes during 'loading' are depicted. The same problem is solved with 10 finite elements and 6 load increments. In figure 5b the achieved results are compared with the analytical one.

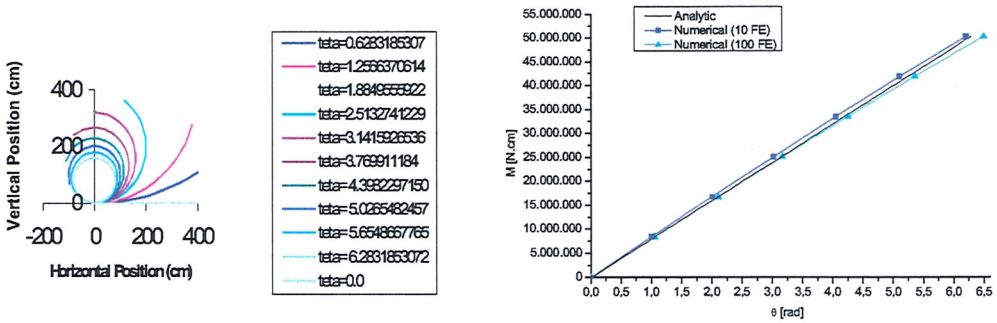

Figure 5: a)Deformed shapes for different prescribed rotation (teta); Comparisom for 10 and 100 FEM

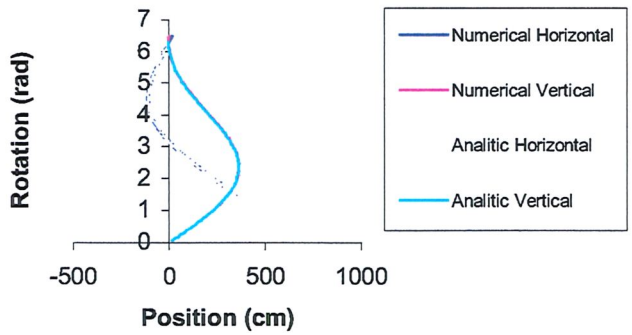

Figure 8 -- Horizontal and vertical positions for analytical and numerical cases (load control 100 elements)

It should be noted that for 100 finite elements and 100 load steps 7 iterations is the maximum obtained in the non-linear process, but for 10 finite elements and 6 steps this number grows to 40 iterations at the final process. The explanation for such behavior is the adoption of the initial position as the starting process at each load step, for this example one can chose a better initial position because the analytical solution is known but it is not a fair procedure. Another point that must be clarified is that the formulation is called exact because it does not used the so called first, second or third order expansions in order to built its mathematical expression, but the actual positions of the body. However, it involves the polynomial approximation for the overall body positions, and so it is an approximate technique. The adopted approximation is not able to reproduce exactly the circumference, so it presents better results as one increases discretization.

3.2. Clamped column:

A clamped bar with length L=2m subjected to the action of a crescent compressive load from 0 to 37100kN (load step = 100kN) as depicted in figure 9 is analyzed. The physical properties are: $I=2{,}425\times10^{-5}\ m^4$, $A=0.0175m^2$ and $E=210\times10^{9}Pa$. Following this characteristics the Euler critical load is $P_{cr}=3141{,}5kN$. In figure 10 the lateral position

curves for top initial eccentricities of: L/1000, L/100 and L/10, considered following a parabolic initial shape (cubic) compatible with the boundary conditions are depicted. In figure 10 the critical reference value is used to show the precision of the results. In figure 11 the deformed shapes for the eccentricity L/100 is showed for various load values. This figure also shows exactly the interval of load where small loads increment enforce large change of position. Load control has been adopted.

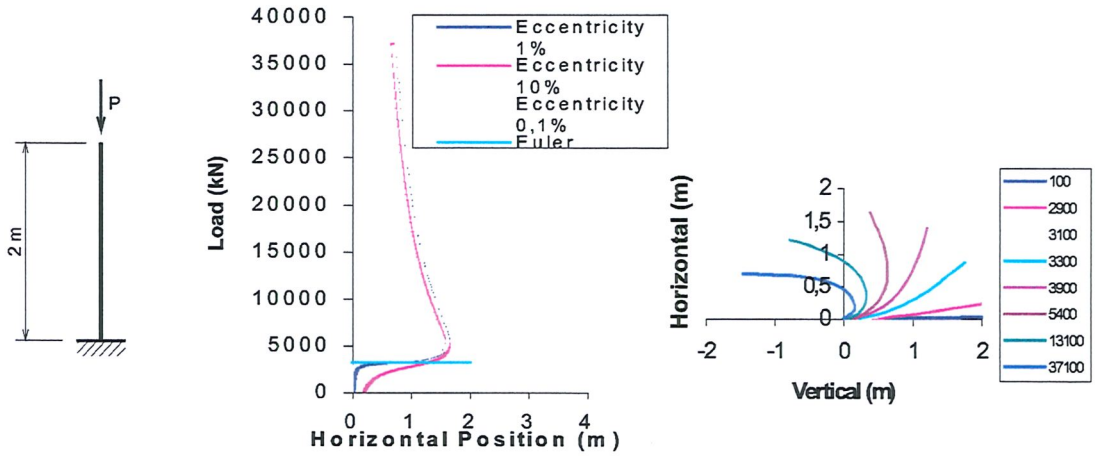

Figure – 9: Analyzed Column, lateral displacement and deformed shapes

Figure 10 shows the capability of the proposed formulation to predict the pre and post-critical behavior for columns. There were adopted 10 finite elements to run this analysis and a tolerance in position of 10^{-6}.

3.3. Pined fixed diamond frame

In reference (Jenkins et all, 1966) this example is solved analytically. The authors used elliptic integrals and numerical evaluation of the resulting integrals are developed for this example (Mattiasson, 1981). The following properties are adopted to run the problem: L=1, E=1, I=1 and A=1000. In figure 19 one can see the diamond frames and the measured "displacement". This variable is referred here only to make comparisons with literature (Mattiasson, 1981), it is calculated by difference between positions. Symmetry is adopted.

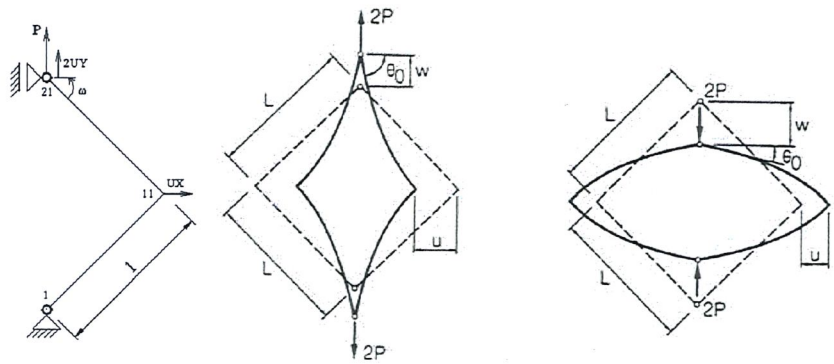

Figure 19 – Diamond frame, boundary conditons for tension and compression situations.

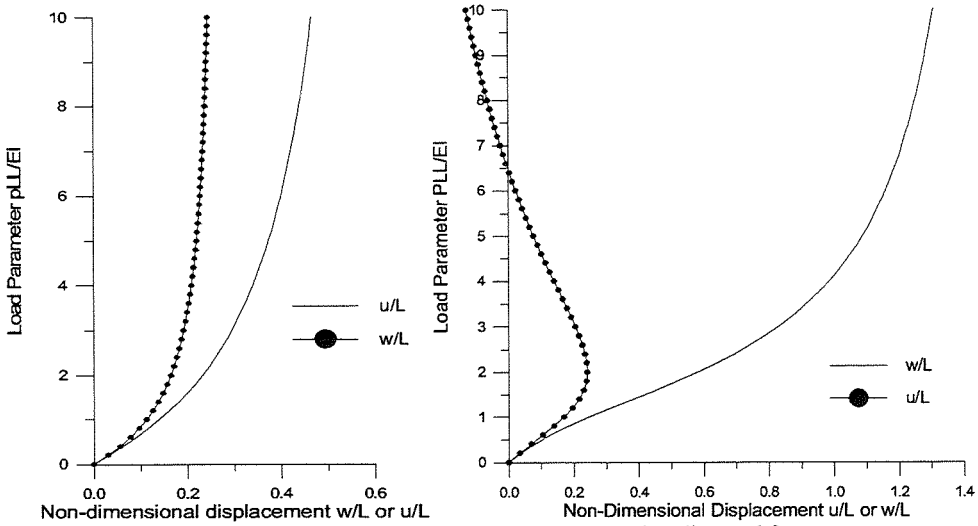

Figure 20: Displacement versus load for the tension and compression diamond frame.

It has been used 20 finite elements to run the problem. In figures 20 and 21 the measured variables are numerically depicted, the reference value is not plotted because the maximum obtained error (in the last step for both tension and compression cases) are $\varepsilon_{tension} = 1\%$ and $\varepsilon_{compression} = 0.3\%$ when compared with the analytical solution

(Mattiasson, 1981; Jenkins et all, 1966). The adopted tolerance is $Tol = 10^{-5}$ in position. As a curiosity let's see the final deformed shape (position) at figures 22 and 23.

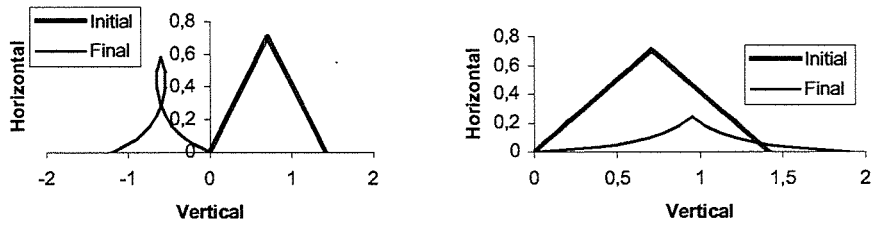

Figure 22 – Initial and final shapes for traction and compressed diamond frame

4. Conclusions:

In this work a consistent and simple formulation is proposed to solve geometrically non-linear plane frame problems. It is based on position description, not displacement, and calculates strains from relative lengths and curvatures as an alternative to deformation functions. The developed formulation presents a high degree of convergence and accuracy. The number of iterations falls as the number of degrees of freedoms rises. The formulation is general and suitable for practical applications.

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