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A NOTE ON 3n+1 GENERALIZED PROBLEM

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A Note on $3n+1$ Generalized Problem

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Abstract

Let us consider two natural numbers $m > d \geq 2$, a fixed set R of non-zero residue class representatives mod d, and we shall study the dynamics induced by Hasse function, defined by $H(n) = \frac{n}{d}$, if $n \equiv_d 0$, and $H(n) = \frac{mn-r}{d}$ if $r \in R$ and $n \equiv_d r$. A classical conjecture states that if $m < d^{\frac{d}{d-1}}$ then the orbit $\mathcal{O}(n)$ of n by H is bounded. We then prove that $\mathcal{O}(n)$ has Banach density zero which shows that if there is some unbounded orbit of H it must grow fast. We deduce this from the stronger fact that the set of *all distinct orbits* of H is small, in the sense that it has Banach density zero.

1 Introduction

We will study here some features of the generalizated Collatz problem, i.e., given two natural numbers d, m, with $m > d \geq 2$ and $\gcd(d;m) = 1$, let R_d be a complete system of non zero residues modulus d and $\varphi: \mathbf{N} \longrightarrow R_d$ the canonical projection of $\mathbf{N}$ in R_d. Then, we define the Hasse function, $H: \mathbf{N} \longrightarrow \mathbf{N}$ by[1]

$$H(x) = \begin{cases} \frac{x}{d} & \text{if } x \equiv_d 0 \\ \frac{mx-\varphi(mx)}{d} & \text{otherwise} \end{cases} \tag{1}$$

and we investigate the dynamics of the orbits of x by H.

We will consider here the case $m < d^{\frac{d}{d-1}}$. An old conjecture states that, in this situation, for all $x \in \mathbf{N}$ the orbit of x is bounded.

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[1] In this note we use $\mathbf{N}$ to denote the set of non-negative integers (including 0) and $\mathbf{N}^* = \mathbf{N} \setminus \{0\}$.

We remember that if $d = 2$ and $R_d = \{0, -1\}$ then we are in the classical Collatz problem, also called *Syracusse problem*, or $3n + 1$ *problem*. In this case we will call the function H by Collatz function and we will denote it T.

A very recent and good review of the state of the art in this problem can be found in the chapter 1 of the Wirsching's book ([Wir98]), we will present here only a brief discussion of some questions related with our work.

Two natural problems arise as we study this conjecture:

(i) *How "large" can the set of all "different" orbits of H be?*

(ii) *If the conjecture is false, how can an unbounded trajectory of H grow?*

In a classical paper of 1985, Lagarias ([Lag85]) shows that (for the $3n+1$ case) there exist $c_1 > 0$ and $\eta \in (0;1)$ such that

$$\#\left\{n \in \mathbb{N}: n \leq x,\ T^k(n) > n,\ \forall k \geq 1\right\} \leq c_1 x^{1-\eta}.$$

From this result, it is reasonable to claim that *if* there exists an unbounded trajectory for this case then it can't grow too slowly. In fact, in Corollary 1 of section 2, we show that this is true, in the sense of Banach density (for the definitions of density and Banach density of a subset of $\mathbb{N}$ see section 1).

In another work related with the second question Korec has proved, also for the Collatz function, in 1994 ([Kor94]) that the set

$$M_c = \{y \in \mathbb{N}:\ \exists n \in \mathbb{N} s.t. T^n(y) < y^c\}$$

has density one, for all $c > \log_4 3$.

For the Hasse function H, when $m < d^{\frac{d}{d-1}}$, the important result of Hepner ([Hep78]), which we will state in section 1, shows that the Korec's result is true in this situation for some $c_0 \in (0;1)$. However, unlike Korec's result, we don't have an estimative for c_0 in this case.

As to the first question, Korec and Znám in 1987 ([KZ87]) define a relation of equivalence in $\mathbb{N}$ by

$$a \sim_1 b \text{ iff there are integers } n \text{ and } m \text{ s.t. } T^n(a) = T^m(b)$$

and they showed that a complete set of representatives of $\mathbb{N}/\sim_1$ has density zero.

Although this result was proved by Korec and Znám for the $3n + 1$ context, it is not difficult to extend it for the general situation of Hasse function H, when $m < d^{\frac{d}{d-1}}$.

In our work we shall consider this general situation, i.e., the function H when $m < d^{\frac{d}{d-1}}$, and we will improve the result of Korec and Znám, precisely, we consider the stronger relation in $\mathbf{N}$

$$a \sim b \text{ iff there is an integer } k \text{ s.t. } H^k(a) = H^k(b)$$

and we will prove that a complete set of representatives of $\mathbf{N}/\sim$ has density zero. Moreover, we shall prove (Theorem 1) that such a set has Banach density zero.

A direct consequence of this statement is that *all the orbits of n by H* has Banach density zero (Corollary 1). This result gives a more precise answer to the question (ii) above as we give here a direct measure of the orbits of H.

This paper comprehends this introduction and 2 more sections. In section 1 we shall state the basic definitions and state some fundamental results that we will need later in the text. In section 2 we will develop the necessary tools to prove Theorem 1.

2 Basic Results

Consider, as in the introduction, integers m, d, with $m > d \geq 2$. Suppose that $\gcd(m;d) = 1$ and $m < d^{\frac{d}{d-1}}$. Let A be a complete system of non zero residues modulus d and $\varphi\colon \mathbf{N} \longrightarrow R_d$ the canonical projection of $\mathbf{N}$ in R_d.

We will study the dynamics induced in the set $\mathbf{N}^*$ of positive integers by Hepner's function $H\colon \mathbf{N}^* \longrightarrow \mathbf{N}^*$ defined by (1).

Since we are interested in studying *"how large are some subsets of* $\mathbf{N}$*"* (or, *"how small they are"*), we introduce here the concept of Banach density of a subset of $\mathbf{N}$. First, let us consider the more simple (and usual) concept of density.

Definition 1 *A subset $B \subset \mathbf{N}$ has density μ if*

$$\lim_{n\to+\infty} \frac{\#(B \cap \{1,\ldots,n\})}{n} = \mu.$$

When this limit exists it will be denoted by $\rho(B)$. Although this concept is very "natural", we will use in this article a more subtil concept, which gives a more uniform measure of the "size" of B.

Definition 2 *The Banach density of a subset $B \subset \mathbf{N}$ is*

$$\limsup_{n\to\infty} \left(\max_{a\in\mathbf{N}^*} \frac{\#(B \cap \{a,\ldots,a+n-1\})}{n} \right).$$

The Banach density of B will be indicated by $\rho_b(B)$.

Of course, the Banach density of B always exists and if, there exist $\rho(B)$ and $\rho_b(B)$ then $\rho_b(B) \leq \rho(B)$. Therefore, in order to show that B is "small" the information $\rho_b(B) = 0$ is more significative than $\rho(B) = 0$.

We will start now the study of the dynamics of H.

The following function $\ell\colon \mathbf{N} \times \mathbf{N}^* \longrightarrow \mathbf{N}$ will play an important role in this note

$$\ell(n;k) = \#\{0 \leq s \leq k-1 : U^s(n) \equiv 0(\text{mod d})\}. \tag{2}$$

Lemma 1 *If n, k and r are positive integers then*

$$H^k(n + rd^k) = H^k(n) + rm^{k-\ell(n;k)}.$$

Proof: We will proceed by induction in k.

The case $k = 0$ is obvious.

Assume the result for $k-1$, then we have that

$$\begin{aligned} H^k(n+sd^k) &= H(H^{k-1}(n+dsd^{k-1})) = \\ &= H\left(H^{k-1}(n) + dsm^{k-1-\ell(n;k-1)}\right) \end{aligned} \tag{3}$$

Now we note that $H^{k-1}(n) \equiv H^{k-1}(n) + dsm^{k-1-\ell(n;k-1)}(\text{mod d})$, so we have:

(i) If $H^k(n) \equiv 0(\text{mod d})$ then $H^k(n) = \frac{H^{k-1}(n)}{d}$, $\ell(n;k) = \ell(n;k-1)+1$ and, by the definition of H,

$$H^k(n+sd^k) = \frac{H^{k-1}(n)}{d} + sm^{k-1-\ell(n;k-1)} = H^k(n) + sm^{k-\ell(n;k)};$$

(ii) If $H^k(n) \not\equiv 0(\text{mod d})$ then $\ell(n;k) = \ell(n;k-1)$ and a simple calculation shows that

$$H^k(n+sd^k) = H^k(n) + sm^{k-\ell(n;k)}. \qquad \blacksquare$$

As a direct consequence of this we have that

Lemma 2 *If $H^k(n) = H^k(r)$ and $\ell(n;k) = \ell(r;k)$ then for all s*

$$H^k(n+sd^k) = H^k(r+sd^k).$$

Now we state the important result of Hepner.

Proposition 1 (Hepner) *Let m, d, R_d and H as above, with $m < d^{\frac{d}{d-1}}$.*

There exist real numbers $\delta_1 = \delta_1(m,d)$ and $\delta_2 = \delta_(m,d)$ in the interval (0,1) such that, if $N(k) = \lfloor \log_d(k) \rfloor$ and

$$g(k) = \#\left\{n \le k \colon H^{N(k)}(n) \ge nk^{-\delta_1}\right\},$$

then $g(k)$ is $O(k^{\delta_2})$.

The reader can find the proof of this proposition in [Hep78].

We will use this result in several occasions in this paper, the first time to obtain

Proposition 2 *Let B be a subset of $\{1,\ldots,k\}$ such that $\#B > k^{1-\delta_1} + g(k)$ where δ_1 and g are given by Hepner's result. Then there are r_1 and r_2 in B, $r_1 \neq r_2$, such that $H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2)$.*

Proof: By Proposition 1, we have that there is a set $B_1 \subset B$ such that, $\#B_1 > k^{1-\delta_1}$ and

$$H^{\lfloor \log_d(k) \rfloor}(s) < sk^{-\delta_1} \le k^{1-\delta_1}, \ \forall s \in B_1.$$

Then, it follows from the pigeonhole principle that there are r_1 and r_2 in B_1, with $r_1 \neq r_2$ and

$$H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2). \qquad \blacksquare$$

Note that if A is a subset of $\mathbf{N}$ which doesn't have zero Banach density then there is a $k \in \mathbf{N}$ such that, for all $x \in \mathbf{N}^*$, $\#(A \cap \{x,\ldots,x+k-1\}) > k^{1-\delta_1} + g(k)$, because $g(k)$ is $O(k^{\delta_2})$ and δ_1 and δ_2 lay in (0;1).

We will use this observation in the next section.

3 Main Results

Lemma 3 (Fundamental Lemma) *Let A be a subset of $\mathbf{N}^*$ and numbers x and k in $\mathbf{N}^*$ such that*

$$\#(A \cap \{x,x+1,\ldots,x+k-1\} > 2(\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)) \qquad (4)$$

where δ_1 and $g(k)$ are given by Hepner's result, Proposition 1.

Then, there exist elements $r_1 \neq r_2$ in $A \cap \{x,x+1,\ldots,x+k-1\}$ such that

$$H^{\lfloor \log_d(k) \rfloor}(r_1) = H^{\lfloor \log_d(k) \rfloor}(r_2).$$

Proof: Let $z_1 \in \mathbf{N}^*$ such that $z_1 d^{\lfloor \log_d(k) \rfloor} < x \leq (z_1+1) d^{\lfloor \log_d(k) \rfloor}$.

It's easily verified that, if $y \in \{x, \ldots, x+k-1\}$, at least one of the numbers $y - z_1 \lfloor \log_d(k) \rfloor$ and $y - (z_1+1) \lfloor \log_d(k) \rfloor$ belongs to $\{1, \ldots, k\}$.

Therefore, it follows from (4) and the pigeonhole principle, that we can choose $z \in \{z_1, z_1+1\}$ such that, if

$$B = B(k;z) = \left\{ 1 \leq s \leq k \colon \exists\, q \in A,\ q - z d^{\lfloor \log_d(k) \rfloor} = s \right\}$$

then

$$\# B \geq (\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)).$$

Since $\ell(\cdot, \lfloor \log_d(k) \rfloor) \in \{0, \ldots, \lfloor \log_d(k) \rfloor\}$, we can apply once again the pigeonhole principle and find a subset B_1 of B with at least $k^{1-\delta_1} + g(k)$ elements such that if u and v are in B_1 then $\ell((u, \lfloor \log_d(k) \rfloor = \ell(v, \lfloor \log_d(k) \rfloor)$.

Now, apply Proposition 2 in order to obtain s_1 and s_2, $s_1 \neq s_2$, in B_1 such that $H^{\lfloor \log_d(k) \rfloor}(s_1) = H^{\lfloor \log_d(k) \rfloor}(s_2)$.

Then, since $\ell(s_1, \lfloor \log_d(k) \rfloor) = \ell(s_2, \lfloor \log_d(k) \rfloor)$, it follows from Lemma 2 that

$$H^{\lfloor \log_d(k) \rfloor}(s_1 + z d^{\lfloor \log_d(k) \rfloor}) = H^{\lfloor \log_d(k) \rfloor}(s_2 + z d^{\lfloor \log_d(k) \rfloor}).$$

By the definition of B it's obvious that $r_i = s_i + z d^{\lfloor \log_d(k) \rfloor} \in A$, for $i = 1, 2$ and this concludes the demonstration. ■

Now we are ready to state and prove our main result.

Consider in $\mathbf{N}^*$ the relation of equivalence

$$a \sim b \Leftrightarrow \exists k \in \mathbf{N} \colon H^k(a) = H^k(b) \tag{5}$$

Let $\mathcal{P}$ be a complete set of representatives of $\mathbf{N}^*/\sim$.

It seems natural to consider $\mathcal{P}$ as a set of all the different orbits of H. Now we will show that this set is "small".

Theorem 1 *The Banach density of $\mathcal{P}$ is zero.*

Proof: It is obvious that if u_1 and u_2 are distinct elements of $\mathcal{P}$ then $H^k(u_1) \neq H^k(u_2)$, for all $k \in \mathbf{N}$.

Then, by the Fundamental Lemma, for all a and k in $\mathbf{N}^*$, we have

$$\#(\mathcal{P} \cap \{a, \ldots, a+k-1\}) \leq 2(\lfloor \log_d(k) \rfloor + 1)(k^{1-\delta_1} + g(k)). \tag{6}$$

Since, by Proposition 1, $g(k)$ is $O(k^{\delta_2})$ and δ_1 and δ_2 belong to (0,1) the result follows when we take the limit $k \to +\infty$ in (6). ■

An important, now trivial, consequence of this is the next proposition.

Corollary 1 *The Banach density of the orbit $\mathcal{O}(n)$ of $n \in \mathbf{N}^*$ by H is zero.*

Proof: If $\mathcal{O}(n)$ is finite the result is obvious.

But, if $\mathcal{O}(n)$ is infinite then, for all u_1 and u_2 in $\mathcal{O}(n)$, with $u_1 \neq u_2$, and for all $k \in \mathbf{N}$, $H^k(u_1) \neq H^k(u_2)$ (otherwise, $\mathcal{O}(n)$ would be periodic).

Then we can choose a complete set of representatives $\mathcal{P}$ of $\mathbf{N}^*/\sim$ such that $\mathcal{O}(n) \subset \mathcal{P}$.

Since $\rho_b(\mathcal{P}) = 0$ the result follows. ■

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