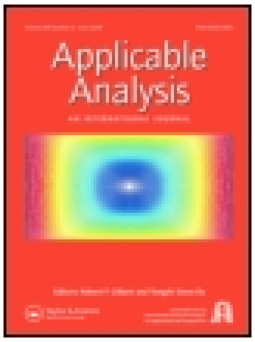

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A local/nonlocal diffusion model

Bruna C. dos Santos[a], Sergio M. Oliva[a] and Julio D. Rossi[b]

[a]IME-USP, Institute of Mathematics and Statistics, University of São Paulo, São Paulo, Brazil; [b]Department of Mathematics, FCEyN, University of Buenos Aires, Buenos Aires, Argentina

ABSTRACT
In this paper, we study some qualitative properties for solutions to an evolution problem that combines local and nonlocal diffusion operators acting in two different subdomains. The coupling takes place at the interface between these two domains in such a way that the resulting evolution problem is the gradient flow of an energy functional. We prove existence and uniqueness results, as well as that the model preserves the total mass of the initial condition. We also study the asymptotic behavior of the solutions. Besides, we show a suitable way to recover the heat equation at the whole domain from taking the limit at the nonlocal rescaled kernel. Finally, we propose a brief discussion about the extension of the problem to higher dimensions.

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1. Introduction and main results

Numerous records along the past decades provide examples of how the spreading and establishment of worldwide transport networks have contributed to the development of global pandemics. The Black Death, in the middle of the fourteenth century, is a classical example of one of the most aggressive and devastating epidemics in history. By killing more than 200 million people in Europe, the disease spread fastly across Europe through the silk routes [1]. Recently, another communicable disease takes place and surprise the whole world for its rapid spread and fatality becoming a global public health concern. First observed in Wuhan, a Hubei province, China, the virus spreads to many countries in a few months by air network, and once at a new place, the diffusion occurs slower [2].

Another example of fast diffusion appears in ecology. Studies have corroborated the hypothesis of successive invasion waves of mosquitoes of the species *Culex pipiens*, *Aedes aegypti*, and more recently, *Aedes albopictus*, facilitated by the worldwide shipping [3–5]. Due to their ability to survive and develop in artificial containers (such as tires and bamboo), mosquitoes have obtained a high success rate in invasions to new regions. Besides, the effect of climate change, in particular, the increase of the temperature has played a fundamental role in creating favorable conditions for the establishment and local propagation of these invasive species across their common geographical boundaries [3,6]. Further interesting behavior of propagation lines occurs on the wolves population [7]. The authors observed in this study that the wolves concentrate and move faster along seismic lines formed in areas of oil and gas exploration in the Western Canada Forest. Nonlocal patterns also play an important role in molecular interactions in dissimilar interfaces [8], continuum mechanics [9–11], peridynamics applied to elasticity and mechanics [12,13].

From a modeling perspective, empirical studies have shown that the spreading effect involves much more complex characteristics than the classical models have suggested, as the classical heat equation,

CONTACT Julio D. Rossi jrossi@dm.uba.ar

$\frac{\partial u}{\partial t}(x,t) = \Delta u(x,t)$, which is associated with a process (Brownian motion) and describes the random movement of a particle [14]. This type of modeling has largely ignored long-range dispersion.

On the other hand, an alternative to capture these features is the nonlocal diffusion equations. One popular choice is $\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))\,\mathrm{d}y$, where $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative kernel (these types of equations included the widely studied fractional Laplacians). Here the equation at a point x and time t depends on the values of the unknown u at all points in the set $x + \operatorname{supp} J$, which is what makes the diffusion nonlocal. These types of problems are associated to jump processes. Evolution equations of this form and variations of it have been recently widely used to model diffusion processes; see for instance [15–24]. For example in biology, if $u(x,t)$ is thought of as the density of a population at the point x at time t, and $J(x-y)$ is regarded as the probability distribution of jumping from location y to location x, then the rate at which individuals are arriving to position x from all other places is given by $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,\mathrm{d}y$, while the rate at which they are leaving location x to travel to all other sites is given by $-\int_{\mathbb{R}^N} J(y-x)u(x,t)\,\mathrm{d}y = -u(x,t)$. Therefore, in the absence of external or internal sources, the density u satisfies the nonlocal diffusion equation, see [19].

The two previous models (local or nonlocal) are well suited for homogeneous environments. When one deals with an inhomogeneous diffusion process one possibility is to add a diffusion coefficient and consider equations like $\frac{\partial u}{\partial t}(x,t) = \frac{\partial}{\partial x}(a(x,t)\frac{\partial u}{\partial x})(x,t)$ or $\frac{\partial u}{\partial t}(x,t) = \int_{\mathbb{R}^N} a(x,y)J(x-y)(u(y,t) - u(x,t))\,\mathrm{d}y$. However, by adding a diffusion coefficient one cannot deal with media that combine local and nonlocal diffusions in different regions. Therefore, to provide good models for inhomogeneous media we need to study couplings between local and nonlocal diffusion equations. These coupling strategies can include a transition region such that the local and nonlocal equations are superposed or a lower-dimensional interface separates the two regimes (we will describe these previous results below).

In [25], the effects of network transportation on enhancing biological invasion are studied. The proposed mathematical model consists of one equation with nonlocal diffusion in a one-dimensional domain coupled via boundary condition with a standard reaction–diffusion, in a two-dimensional domain. The results suggested that the fast diffusion enhances the spread in the domain in which the local diffusion takes place.

From a mathematical point of view, interesting properties arise from coupling local and nonlocal models. See for instance [26–31] and references therein. In [26], local and nonlocal problems were coupled through a prescribed region in which both types of equations overlap (the value of the solution in the nonlocal part of the domain is used as a Dirichlet boundary condition for the local part and vice versa). This type of coupling gives continuity of the solution in the overlapping region but does not preserve the total mass. In [26,28], numerical schemes using local and nonlocal equations were developed and used to improve the computational accuracy when approximating a purely nonlocal problem. In [30] (see also [29,31]), energy closely related to ours was studied, but the gradient flow of this energy (that it has all the nice properties listed above) gives an equation in the local region in which the coupling with the nonlocal part appears as an external source in the heat equation (that is complemented with zero flux boundary conditions in the whole boundary of the local region). In probabilistic terms, in the model described in [30], particles may jump across the interface between the two regions but cannot pass coming from the local side unless they jump.

Here, our aim is the study of coupling local and nonlocal diffusion equations and propose a model in which there is a sharp interface between the two regimes, and the coupling is done via the fluxes at the interface. In particular, we combine a local diffusion equation, the classical heat equation,

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t) \tag{1}$$

with a nonlocal diffusion equation with an integrable kernel

$$\frac{\partial u}{\partial t}(x,t) = \int J(x-y)(u(y,t) - u(x,t))\,\mathrm{d}y. \tag{2}$$

The kernel $J(z)$ is assumed to be nonnegative, continuous, symmetric, compactly supported with $supp(J) = [-R, R]$ and $\int J(z)\, dz = 1$ (these hypotheses on J will be assumed from now on).

The coupling of the problems (1) and (2) was thought in such a way that the following features (that are the usual ones when one deals with a diffusion problem) hold:

- The problem is well-posed in the sense that there are existence and uniqueness of solutions. Besides, a comparison principle holds.
- There is an energy functional such that the evolution problem can be obtained as the gradient flow associated with this energy.
- The total mass of the initial condition is preserved along with the evolution, naturally obtained by the Neumann boundary condition.
- Solutions converge exponentially fast to the mean value of the initial condition.

Effectively, we can think of our model in terms of a particle system. To simplify the exposition, we will restrict ourselves to a one-dimensional problem and comment on the extension to higher dimensions at the end of the paper. We split the domain $\Omega = (-1, 1)$ into two subdomains $(-1, 0)$ and $(0, 1)$ (to simplify we will restrict ourselves to this simple configuration). In $(-1, 0)$ particles move by Brownian motion (this gives the equation $\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)$, $x \in (-1, 0)$) with a reflexion at $x = -1$ (then $\frac{\partial u}{\partial x}(-1, 0) = 0$) and when the particle arrives to $x = 0$ it passes through to the other subdomain, $(0, 1)$ (this will give a flux boundary condition at $x = 0$). On the other hand, in $(0, 1)$ particles obey a pure jump process with jumping probability given by $J(x - y)$ (this gives an equation of the form (2) in $(0, 1)$, when a particle that is at $x \in (0, 1)$ wants to jump to a location $y \in (-1, 0)$ it enters the domain $(-1, 0)$ at the point $x = 0$ (particles are stuck there, giving the counterpart to the flux coming from $(-1, 0)$). This process has a density $w(x, t)$, which obeys an evolution equation associated with the gradient flow of a local/nonlocal energy that we describe in the next section. At this point, it is important to notice that we do not impose any continuity of the densities at the interface $x = 0$, but instead, we can ensure continuity of the densities inside the local and nonlocal subdomains $(-1, 0)$ and $(0, 1)$ by assuming continuity of the initial data.

1.1. A local/nonlocal diffusion model

As we mentioned, let us consider as the reference domain $\Omega = (-1, 1) \subset \mathbb{R}$ that is divided in two disjoint regions, the intervals $\Omega_l = (-1, 0)$ and $\Omega_{nl} = (0, 1)$, the local and nonlocal domains, respectively. We split a function $w \in L^2(-1, 1)$ as $w = u + v$, with $u = w\chi_{(-1,0)}$ and $v = w\chi_{(0,1)}$. For any

$$w = (u, v) \in \mathcal{B} := \left\{ w \in L^2(-1, 1) : u \in H^1(-1, 0), v \in L^2(0, 1) \right\}$$

we define the energy

$$\begin{aligned} E(u, v) := \frac{1}{2} \int_{-1}^{0} \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x + \frac{C_{J,1}}{4} \int_0^1 \int_0^1 J(x - y) \left(v(y) - v(x) \right)^2 \mathrm{d}y\, \mathrm{d}x \\ + \frac{C_{J,2}}{2} \int_0^1 \int_{-1}^0 J(x - y)\, (v(x) - u(0))^2\, \mathrm{d}y\, \mathrm{d}x, \end{aligned}$$

where $C_{J,1}$ and $C_{J,2}$ are fixed positive constants. Notice that, in this energy functional we have two terms

$$\frac{1}{2} \int_{-1}^{0} \left| \frac{\partial u}{\partial x} \right|^2 \mathrm{d}x \quad \text{and} \quad \frac{C_{J,1}}{4} \int_0^1 \int_0^1 J(x - y) \left(v(y) - v(x) \right)^2 \mathrm{d}y\, \mathrm{d}x$$

that are naturally associated with Equations (1) and (2), plus a coupling term

$$\frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)\,(v(x)-u(0))^2\,\mathrm{d}y\,\mathrm{d}x$$

that involves only the value of u at $x=0$.

We aim to write our model as the gradient flow associated with this energy, that is, (u,v) will be the solution of the abstract *ODE* problem

$$(u,v)'(t) = -\partial E\left[(u,v)(t)\right], \quad t\geq 0,$$

with $u(0)=u_0$, $v(0)=v_0$ and, $\partial E[(u,v)]$ denotes the subdifferential of E at the point (u,v). Let us compute the derivative of E at (u,v), in the direction of $\varphi\in C_0^\infty(-1,1)$,

$$\begin{aligned}\partial_\varphi E(u,v) &= \lim_{h\to 0}\frac{E(u+h\varphi,v+h\varphi)-E(u,v)}{h}\\ &= \int_0^{-1}\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x}\,\mathrm{d}x + \frac{C_{J,1}}{2}\int_0^1\int_0^1 J(x-y)(v(y)-v(x))(\varphi(y)-\varphi(x))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad + \frac{C_{J,2}}{2}\int_0^1\int_{-1}^0 J(x-y)(v(x)-u(0))(\varphi(x)-\varphi(0))\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Thus if u is smooth, we would have

$$\begin{aligned}\partial_\varphi E(u,v) &= \left\{\frac{\partial u}{\partial x}(0) - C_{J,2}\int_0^1\int_{-1}^0 J(x-y)(v(x)-u(0))\,\mathrm{d}y\,\mathrm{d}x\right\}\varphi(0) - \frac{\partial u}{\partial x}(-1)\varphi(-1)\\ &\quad -\int_{-1}^0\frac{\partial^2 u}{\partial x^2}\varphi\,\mathrm{d}x - C_{J,1}\int_0^1\left\{\int_0^1 J(x-y)(v(y)-v(x))\,\mathrm{d}y\right\}\varphi(x)\,\mathrm{d}x\\ &\quad + C_{J,2}\left\{\int_{-1}^0 J(x-y)(v(x)-u(0))\,\mathrm{d}y\right\}\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Since $\langle\partial E[u,v],\varphi\rangle = \partial_\varphi E(u,v)$, we can derive the local/nonlocal problem associated to this gradient flow. The evolution problem consists of two parts. A local part, composed of a heat equation with Neumann/Robin type boundary conditions,

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t),\\ \dfrac{\partial u}{\partial x}(-1,t)=0,\\ \dfrac{\partial u}{\partial x}(0,t) = C_{J,2}\int_{-1}^0\int_0^1 J(x-y)(v(y,t)-u(0,t))\,\mathrm{d}y\,\mathrm{d}x,\\ u(x,0)=u_0(x),\end{cases}\tag{3}$$

for $x\in(-1,0)$, $t>0$. Notice that we have a Robin-type boundary condition at $x=0$ that encodes the coupling with the nonlocal part of the problem.

We complete the system with the nonlocal part,

$$\begin{cases}\dfrac{\partial v}{\partial t}(x,t) = C_{J,1}\int_0^1 J(x-y)\left(v(y,t)-v(x,t)\right)\mathrm{d}y - C_{J,2}\int_{-1}^0 J(x-y)\mathrm{d}y(v(x,t)-u(0,t)),\\ v(x,0)=v_0(x),\end{cases}\tag{4}$$

for $x\in(0,1)$, $t>0$. Here we have a nonlocal diffusion problem for v, where the coupling with the local part u appears as a source term in the equation, while the value of u appears only at the interface $x=0$.

The complete problem can be summarized as follows: we look for w defined by

$$w(x,t) = \begin{cases} u(x,t), & \text{if } x \in (-1,0), \\ v(x,t), & \text{if } x \in (0,1), \end{cases} \tag{5}$$

where (u, v) are the solutions to (3)–(4).

For this problem we have the following result:

Theorem 1.1: *Given $w_0 = (u_0, v_0) \in L^2(-1,1)$, there exists an unique mild solution*

$$w(\cdot,t) \in \mathcal{B} := \left\{ w \in L^2(-1,1) : u \in H^1(-1,0), v \in L^2(0,1) \right\}$$

to the local/nonlocal problem (5) with (u, v) satisfying (3)–(4) that is globally defined. If, $w_0 = (u_0, v_0)$, with $u_0 \in C([-1,0])$ and $v_0 \in C([0,1])$ then, the solution (u, v) is such that $u(\cdot,t) \in C([-1,0])$ and $v(\cdot,t) \in C([0,1])$ for every $t > 0$.

A comparison principle holds: if the initial data are ordered, $w_0 \geq z_0$, then the corresponding solutions are also ordered, they verify $w \geq z$ in $(-1,1) \times \mathbb{R}_+$.

Moreover, the total mass of the solution is preserved along the evolution, that is,

$$\int_{-1}^{1} w(x,t)\,\mathrm{d}x = \int_{-1}^{1} w_0(x)\,\mathrm{d}x = \int_{-1}^{0} u_0(x)\,\mathrm{d}x + \int_{0}^{1} v_0(x)\,\mathrm{d}x.$$

Remark 1.1: Notice that, for a continuous initial condition we obtain a solution (u, v) such that $u(\cdot,t) \in C([-1,0])$ and $v(\cdot,t) \in C([0,1])$ for every $t > 0$, but we are not imposing (nor obtaining) continuity across the interface, that is, we do not necessarily have $u(0,t) = v(0,t)$.

1.2. Asymptotic behavior

Once we proved the existence and uniqueness of a global solution, our next goal is to look for its asymptotic behavior as $t \to \infty$. We start by observing that the constants, $w(x,t) \equiv cte$, are stationary solutions of (3)–(4).

For the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},$$

with Neumann boundary conditions, it is well known that solutions have an exponential time decay to the mean value of the initial condition, that is,

$$\left\| u(\cdot,t) - \fint u_0 \right\|_{L^2} \leq C(u_0) e^{-\beta t}.$$

The same is valid (with a different β) for solutions to the nonlocal heat equation

$$\frac{\partial v}{\partial t}(x,t) = \int_{0}^{1} J(x-y)(v(y,t) - v(x,t))\,\mathrm{d}y,$$

with the additional assumption on the kernel, $M(J) := \int_{\mathbb{R}} J(z)|z|^2\,\mathrm{d}z < \infty$, see [32,33].

The problem (5) shares of the same behavior than each, local and nonlocal equation have, individually, that is, the solution of the coupled local/nonlocal problem converges exponentially to the mean value of the initial condition.

Theorem 1.2: *Given $w_0 \in L^2(-1,1)$, the solution to (5) with initial condition w_0 converges to its mean value as $t \to \infty$ with an exponential rate.*

$$\left\| w(\cdot,t) - \fint w_0 \right\|_{L^2(-1,1)} \leq Ce^{-2\beta_1 t}, \quad t > 0,$$

where $\beta_1 > 0$ depends only on J and Ω and, the constant C depends on the initial condition, w_0.

1.3. Rescaling the kernel

In the following, we will show that the solutions of the evolution problem (3)–(4), with the kernel J rescaled suitably, converges to the classical local problem (given by the heat equation) at the whole domain. The idea consists of to rescale the kernel J by a $\varepsilon > 0$ parameter

$$J^{\varepsilon}(x) := \frac{1}{\varepsilon^3} J\left(\frac{x}{\varepsilon}\right). \tag{6}$$

From now on, we choose (and fix) the constants $C_{J,1}$ and $C_{J,2}$ that appears before the nonlocal terms as

$$C_{J,1} := \frac{2}{M(J)} \text{ with } M(J) = \int_{\mathbb{R}} J(z)z^2\,dz \quad \text{and} \quad C_{J,2} := 1. \tag{7}$$

Summarizing, our goal is to show that, the solutions of the local heat equation with Neumann boundary conditions,

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), \quad x \in (-1,1), \quad t > 0, \\ \dfrac{\partial w}{\partial x}(-1,t) = \dfrac{\partial w}{\partial x}(1,t) = 0, \quad t > 0, \\ w(x,0) = w_0(x), \quad x \in (-1,1), \end{cases} \tag{8}$$

can be obtained as the limit as $\varepsilon \to 0$ of the solution w^{ε} to our local/nonlocal problem with J replaced by J^{ε}, given by (6). We will call $w^{\varepsilon} = (u^{\varepsilon}, v^{\varepsilon})$ the solution to (3)–(4) with the rescaled kernel and a fixed initial condition $w(x,0) = u_0(x)\chi_{(-1,0)}(x) + v_0(x)\chi_{(0,1)}(x)$.

We have the following result:

Theorem 1.3: *Let $w_0 \in L^2(-1,1)$. For each $\varepsilon > 0$, let w^{ε} be the solution to (3)–(4) with J replaced by J^{ε} ((6)) and, initial condition w_0. Then, it holds the following:*

$$\lim_{\varepsilon \to 0} \left(\max_{t \in [0,T]} \| w^{\varepsilon}(\cdot,t) - w(\cdot,t) \|_{L^2(-1,1)} \right) = 0,$$

where w is the solution to (8).

ORGANIZATION OF THE PAPER: The paper is organized as follows: in Section 2, we prove a key result concerning the control of the pure nonlocal energy by our local/nonlocal energy. In Section 3, we prove the existence and uniqueness of the problem, the total mass conservation property, and the asymptotic behavior of the solutions for large times. In Section 4, we deal with the rescaling of the kernel. Finally, in the final section (Section 5), we explain how to extend our results to higher dimensions.

2. Preliminaries

2.1. Control of the nonlocal energy

In this section, we prove the first important lemma that ensures the domination of the energy for the complete problem over the pure nonlocal energy.

Lemma 2.1: *Let*

$$(u,v) \in \mathcal{B} := \left\{u \in H^1(-1,0), v \in L^2(0,1)\right\}.$$

Then, there exists a constant $k := k(J,\Omega) > 0$ *such that*

$$\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x + \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)\left(v(y)-v(x)\right)^2 \,\mathrm{d}y\,\mathrm{d}x$$
$$+\frac{C_{J,2}}{2}\int_0^1\int_{-1}^{0} J(x-y)\left(v(x)-u(0)\right)^2 \,\mathrm{d}y\,\mathrm{d}x$$
$$\geq k\int_{-1}^{1}\int_{-1}^{1} J(x-y)\left(w(y)-w(x)\right)^2 \,\mathrm{d}y\,\mathrm{d}x. \tag{9}$$

Proof: Assume that the conclusion does not hold. This implies that, there exists a sequence $\{w_n\} \in L^2(-1,1)$, with $\{u_n\} \in H^1(-1,0)$ and $\{v_n\} \in L^2(0,1)$, such that it satisfies

$$\int_{-1}^{1}\int_{-1}^{1} J(x-y)(w_n(y)-w_n(x))^2 \,\mathrm{d}y\,\mathrm{d}x = 1, \tag{10}$$

and satisfying,

$$\int_{-1}^{1} w_n = \int_{-1}^{0} u_n + \int_0^1 v_n = 0, \tag{11}$$

and

$$1 \geq n\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u_n}{\partial x}\right|^2 + \frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)(v_n(y)-v_n(x))^2 \,\mathrm{d}y\,\mathrm{d}x\right.$$
$$\left.+\frac{C_{J,2}}{2}\int_{-1}^{0}\int_0^1 J(x-y)(u_n(0)-v_n(x))^2 \,\mathrm{d}x\,\mathrm{d}y\right), \tag{12}$$

for every $n \in \mathbb{N}$.

Taking the limit in n, in (12), we obtain

$$\lim_n\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u_n}{\partial x}\right|^2\right) = 0, \tag{13}$$

$$\lim_n\left(\frac{C_{J,1}}{4}\int_0^1\int_0^1 J(x-y)(v_n(y)-v_n(x))^2 \,\mathrm{d}y\,\mathrm{d}x\right) = 0, \tag{14}$$

and

$$\lim_n\left(\frac{C_{J,2}}{2}\int_{-1}^{0}\int_0^1 J(x-y)(u_n(0)-v_n(x))^2 \,\mathrm{d}x\,\mathrm{d}y\right) = 0. \tag{15}$$

From Equations (10) and (13), it implies a bound on the L^2-norm of w_n, so we can take a subsequence, also denoted $\{u_n\}$, which weakly converge for some limit in $H^1(-1,0)$. This limit is given by a constant A,

$$u_n \to A \quad \text{in } L^2(-1,0) \quad \text{and,}$$

$$u_n \to A \quad \text{uniformly in } (-1,0).$$

Note that, in particular, $u_n(0) \to A$.

Since the sequence $\{w_n\}$ is bounded in $L^2(-1,1)$ we have that $\{v_n\}$ is also bounded in $L^2(0,1)$ and then extracting a subsequence if necessary we can assume that their mean values converge, $\int_0^1 v_n(x)\,dx := B_n \to B$. Now, from [32] we know that there exists a constant c such that

$$\int_0^1 \int_0^1 J(x-y)(v_n(y)-v_n(x))^2\,dx\,dy \geq c \int_0^1 (v_n(x)-B_n)^2\,dx$$

Therefore, from Equation (14) we also can take a subsequence, also denoted as $\{v_n\}$, which strongly converges for some limit in $L^2(0,1)$ that is given by the constant B.

From (15), we obtain $A = B$. Moreover, from Equation (11) we get that $A + B = 0$, which leads to $A = B = 0$. On the other hand, we have

$$\int_{-1}^1 \int_{-1}^1 J(x-y)(w_n(y)-w_n(x))^2\,dy\,dx = 1,$$

which implies

$$\int_{-1}^1 \int_{-1}^1 J(x-y)(A-B)^2\,dy\,dx = 1,$$

that give us the contradiction. ■

The main advantage of this estimate is to observe that the constant obtained from (9) can be taken independent of ε, when we consider the rescaled kernel J^ε, as we will prove by Lemma 2.3.

2.2. A Poincaré type inequality

Let us consider w_ε as in the introduction, that is,

$$w_\varepsilon(x) = \begin{cases} u_\varepsilon(x), & \text{if} \quad x \in (-1,0) \\ v_\varepsilon(x), & \text{if} \quad x \in (0,1). \end{cases}$$

From [32] we have that

Lemma 2.2: *There exists a constant $C > 0$ (independent of ϵ) such that, for every $\{w_{\varepsilon_n}\} \in L^2(-1,1)$ it holds*

$$\int_{-1}^1 \left| w_{\varepsilon_n}(x) - \fint_{-1}^1 w_{\varepsilon_n}(x)\,dx \right|^2 dx \leq C \frac{1}{\varepsilon_n^3} \int_{-1}^1 \int_{-1}^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2\,dy\,dx. \quad (16)$$

As a consequence of (16) and the control of the nonlocal energy given by (9) we have the following Poincaré-type inequality [34].

Lemma 2.3: *Let $w_\varepsilon \in \mathcal{B} := \{u_\varepsilon \in H^1(-1,0), v_\varepsilon \in L^2(0,1)\}$. Then there exists a constant $k := k(J,\Omega) > 0$, independent of ε, such that*

$$\frac{1}{2}\int_{-1}^0 \left|\frac{\partial w_\varepsilon}{\partial x}\right|^2 dx + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1 \int_0^1 \left(\frac{J(x-y)}{\varepsilon}\right)\left(w_\varepsilon(y)-w_\varepsilon(x)\right)^2\,dy\,dx$$
$$+ \frac{C_{J,2}}{2\varepsilon^3}\int_0^1 \int_{-1}^0 J\left(\frac{x-y}{\varepsilon}\right)(w_\varepsilon(x)-w_\varepsilon(0))^2\,dy\,dx$$

$$\geq k\frac{1}{\varepsilon^3}\int_{-1}^{1}\int_{-1}^{1}J\left(\frac{x-y}{\varepsilon}\right)\left(w_\varepsilon(y)-w_\varepsilon(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x. \tag{17}$$

Proof: Let us argue by contradiction. Suppose that (17) is false. Then, for every $n\in\mathbb{N}$, there exists a subsequence $\varepsilon_n\to 0$, and $\{w_{\varepsilon_n}\}\in L^2(-1,1)\cap H^1(-1,0)$, such that

$$\int_{-1}^{1}w_{\varepsilon_n}=\int_{-1}^{0}u_{\varepsilon_n}+\int_{0}^{1}v_{\varepsilon_n}=0, \tag{18}$$

$$\frac{1}{\varepsilon_n^3}\int_{-1}^{1}\int_{-1}^{1}J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2\,\mathrm{d}y\,\mathrm{d}x=1, \tag{19}$$

and

$$\frac{1}{n}\geq\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2+\frac{C_{J,1}}{4\varepsilon_n^3}\int_{0}^{1}\int_{0}^{1}J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2\,\mathrm{d}y\,\mathrm{d}x\right.$$
$$\left.+\frac{C_{J,2}}{2\varepsilon_n^3}\int_{-1}^{0}\int_{0}^{1}J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(0)-w_{\varepsilon_n}(x))^2\,\mathrm{d}x\,\mathrm{d}y\right),\quad\forall n\in\mathbb{N}. \tag{20}$$

Taking the limit in n in (20), we obtain

$$\lim_n\left(\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2\right)=0, \tag{21}$$

$$\lim_n\left(\frac{C_{J,1}}{4\varepsilon_n^3}\int_{0}^{1}\int_{0}^{1}J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2\,\mathrm{d}y\,\mathrm{d}x\right)=0, \tag{22}$$

and

$$\lim_n\left(\frac{C_{J,2}}{2\varepsilon_n^3}\int_{-1}^{0}\int_{0}^{1}J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(0)-w_{\varepsilon_n}(x))^2\,\mathrm{d}x\,\mathrm{d}y\right)=0. \tag{23}$$

From (21) we have that w_{ε_n} is bounded in $H^1(-1,0)$, so passing to a subsequence, also denoted $\{w_{\varepsilon_n}\}$, such that $\varepsilon_n\to 0$, we have

$$\begin{aligned}&w_{\varepsilon_n}\rightharpoonup w\quad\text{in }H^1(-1,0),\\&w_{\varepsilon_n}\to w\quad\text{in }L^2(-1,0)\quad\text{and}\\&w_{\varepsilon_n}\quad\text{converges uniformly in }(-1,0).\end{aligned}$$

Thanks to (21), and by Fatou's Lemma, we also know that

$$\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w}{\partial x}\right|^2\leq\liminf_{\varepsilon_n\to 0}\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w_{\varepsilon_n}}{\partial x}\right|^2=0.$$

Hence, the limit w is a constant, let us call $w=A_1\in H^1(-1,0)$.

Now, we shall see that, $\{w_{\varepsilon_n}\}$ is also bounded in $L^2(0,1)$, and by taking a subsequence $\{w_{\varepsilon_n}\}$, as $\varepsilon_n\to 0$, $\{w_{\varepsilon_n}\}$ weakly converges in $L^2(0,1)$ to some limit w (that will be a constant A_2).

Thanks to (16) and, by the assumption (19), we have

$$\int_{-1}^{1}\left|w_{\varepsilon_n}(x)-\fint_{-1}^{1} w_{\varepsilon_n}(x)\,\mathrm{d}x\right|^2 \mathrm{d}x \le C,$$

which implies

$$\int_{-1}^{1} |w_{\varepsilon_n}(x)\,\mathrm{d}x|^2\,\mathrm{d}x = \int_{-1}^{0} |u_{\varepsilon_n}(x)\,\mathrm{d}x|^2\,\mathrm{d}x + \int_{0}^{1} |v_{\varepsilon_n}(x)\,\mathrm{d}x|^2\,\mathrm{d}x \le C. \tag{24}$$

Besides, from (24) we have that $\{v_{\varepsilon_n}\}$ is bounded in $L^2(0,1)$ and then, there exists a subsequence, also denoted by $\{v_{\varepsilon_n}\}$, which weakly converges for some limit $w \in L^2(0,1)$.

Performing a variable change in (22), $x = y + \varepsilon_n z$, we obtain

$$\begin{aligned}&\frac{C_{J,1}}{4\varepsilon_n^3}\int_0^1\int_0^1 J\left(\frac{x-y}{\varepsilon_n}\right)(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(x))^2\,\mathrm{d}y\,\mathrm{d}x\\ &\quad=\frac{C_{J,1}}{4}\int_0^1\int_{\frac{-y}{\varepsilon_n}}^{\frac{1-y}{\varepsilon_n}} J(z)\frac{(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(y+\varepsilon_n z))^2}{\varepsilon_n^2}\mathrm{d}z\mathrm{d}y. \end{aligned}\tag{25}$$

As the limit in (22) is zero, it follows that

$$\frac{C_{J,1}}{4}\int_0^1\int_{\frac{-y}{\varepsilon_n}}^{\frac{1-y}{\varepsilon_n}} J(z)\frac{(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(y+\varepsilon_n z))^2}{\varepsilon_n^2}\,\mathrm{d}z\,\mathrm{d}y \le C. \tag{26}$$

So, as a consequence of (26) and the weak convergence of $\{v_{\varepsilon_n}\}$ in $L^2(0,1)$, by [[32], Theorem 6.11] we have that, the limit $w \in H^1(0,1)$ and, moreover

$$\left(\frac{C_{J,1}}{4}J(z)\right)^{1/2}\frac{(w_{\varepsilon_n}(y)-w_{\varepsilon_n}(y+\varepsilon_n z))}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,1}}{4}J(z)\right)^{1/2} z\cdot\frac{\partial w}{\partial x}(y)$$

weakly in $L^2(0,1)\times L^2(\mathbb{R})$. Therefore, taking the limit $\varepsilon_n \to 0$ in (25) we get

$$\frac{1}{2}\int_0^1\left|\frac{\partial w}{\partial x}\right|^2 = 0.$$

Hence, $w = A_2$ is just a constant.

Finally, from (23), taking $\varepsilon_n \to 0$ and by the Monotone Convergence Theorem, we obtain that $A_1 = A_2$. On the other hand, from Equation (18) we get that $A_1 + A_2 = 0$, which contradicts (19). ∎

3. The local/nonlocal problem

3.1. Existence and uniqueness

Now, our goal is to show the existence and uniqueness of solutions. The main idea to prove this result is, given a function u defined for $x \in [-1,0]$ we will use it as an initial input for Equation (4) in $[0,1]$. The solution v of this problem is then used to solve Equation (3) in $[-1,0]$, which yields a function z. This procedure in two steps can be regarded as an operator H given by $H(u) = z$. Now our task is to look for a fixed point of H via contraction in an adequate norm, meaning that, there must exist $u = H(u)$, solving the equation for $x \in [-1,0]$ with its corresponding v solving the equation for $x \in [0,1]$.

Fix $T > 0$ and consider the Banach spaces

$$X_T = \{u \in C([-1,0] \times [0,T])\} \quad \text{and} \quad Y_T = \{v \in C([0,1] \times [0,T])\},$$

with the respective norms

$$\| u \|_l = \max_{t\in[0,T]} \max_{x\in[-1,0]} |u| \quad \text{and} \quad \| v \|_{nl} = \max_{t\in[0,T]} \max_{x\in[0,1]} |v|.$$

Given $T > 0$, we define the operator $H_1 : X_T \to Y_T$ as $H_1(u) = v$, where v is the unique solution of

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_0^1 J(x-y)\left(v(y,t) - v(x,t)\right) \mathrm{d}y - C_{J,2} \int_{-1}^0 J(x-y)\, \mathrm{d}y v(x,t) \\ \quad +C_{J,2} u(t,0) \displaystyle\int_{-1}^0 J(x-y)\, \mathrm{d}y, \\ v(0,x) = v_0(x), \end{cases}$$

for $x \in (0,1)$ and $t \in (0,T)$.

In the next lemma, we will show that this problem has an unique solution (that means that H_1 is well defined). In addition, we show continuous dependence on u.

Lemma 3.1: *There are constants $C_{J,i}$, $i = 1,2$, depending only on J, such that for $T \in (0, \frac{1}{2C_{J,1}+C_{J,2}})$, given $u(x,t) \in C([-1,0] \times [0,T])$ and $v_0 \in C([0,1])$, there exists an unique $v(x,t) \in C([0,1] \times [0,T])$, solution to (4). Moreover, if v_1 and v_2 are the solutions corresponding to u_1 and u_2 then*

$$\| v_1 - v_2 \|_{nl} \leq \frac{C_{J,2}T}{1-(2C_{J,1}+C_{J,2})T} \| u_1 - u_2 \|_l . \tag{27}$$

Proof: To show the existence and uniqueness, we will use a fixed point argument. Let us define an operator $A_u(v) : Y_T \to Y_T$ as

$$\begin{aligned} A_u(v)(t,x) &:= v_0(x) + C_{J,1} \int_0^t \int_0^1 J(x-y)(v(y,s) - v(x,s))\, \mathrm{d}y\, \mathrm{d}s \\ &\quad - C_{J,2} \int_0^t \int_{-1}^0 J(x-y) v(x,s)\, \mathrm{d}y\, \mathrm{d}s + C_{J,2} \int_0^t \int_{-1}^0 J(x-y) u(0,s)\, \mathrm{d}y\, \mathrm{d}s. \end{aligned}$$

Taking the difference $A_u(v_1) - A_u(v_2)$ we get

$$\begin{aligned} \| A_u(v_1) - A_u(v_2) \|_{nl} &\leq C_{J,1} \max_{t\in[0,T]} \max_{x\in[0,1]} \int_0^t \int_0^1 J(x-y)|v_1(y,s) - v_2(y,s)|\, \mathrm{d}y\, \mathrm{d}s \\ &\quad + C_{J,1} \max_{t\in[0,T]} \max_{x\in[0,1]} \int_0^t \int_0^1 J(x-y)|v_2(x,s) - v_1(x,s)|\, \mathrm{d}y\, \mathrm{d}s \\ &\quad + C_{J,2} \max_{t\in[0,T]} \max_{x\in[0,1]} \int_0^t \int_{-1}^0 J(x-y)|v_2(x,s) - v_1(x,s)|\, \mathrm{d}y\, \mathrm{d}s. \end{aligned}$$

Since $J \geq 0$ and $\int_{\mathbb{R}} J = 1$, applying Fubini's theorem, we obtain

$$\| A_u(v_1) - A_u(v_2) \|_{nl} \leq (2C_{J,1} + C_{J,2})T \| v_1 - v_2 \|_{nl} .$$

Choosing $T < \frac{1}{2C_{J,1}+C_{J,2}}$, A_u is a strict contraction, and hence it has an unique fix point.

To check the dependence on the data, let v_1 and v_2 be defined as $v_1 = A_{u_1}(v_1)$ and $v_2 = A_{u_2}(v_2)$. Indeed, following the same idea as before we will get

$$\| v_1 - v_2 \|_{nl} \leq (2C_{J,1} + C_{J,2})T \| v_1 - v_2 \|_{nl} + C_{J,2}T \| u_1 - u_2 \|_l,$$

which yields (27) and it completes the proof. ■

Remark 3.1: In particular, any positive constants C_i, $i = 1, 2$ will ensure the statement (27). More specifically, we specify these constants, as in (7), to recover the classical heat equation at the whole domain from rescaling the nonlocal kernel.

Remark 3.2: We also have existence and uniqueness in L^2, that is, given $u(x,t) \in L^2([-1,0] \times [0,T])$ and $v_0 \in L^2([0,1])$, there exists an unique $v(t,x) \in L^2([0,1] \times [0,T])$, solution to (4). The proof is analogous and hence we omit the details.

In addition, we have a comparison principle, if we have two ordered functions $u \geq \tilde{u}$ and two initial conditions $v_0 \geq \tilde{v_0}$ then the corresponding solutions verify $v(x,t) \geq \tilde{v}(x,t)$.

Now, we need to look back to the local part. Given $v \in C([0,1] \times [0,T])$, we will show that there exists a unique solution $u \in C([-1,0] \times [0,T])$ to (3), with u_0 as initial condition. We define $H_2 : Y_T \to X_T$ as the solution operator $H_2(v) = u$ and once again we will prove continuity of this operator.

Lemma 3.2: *Fix $T > 0$. Given $v(x,t) \in C([0,1] \times [0,T])$ and $u_0 \in C([-1,0])$, there exists an unique $u(x,t) \in C([-1,0] \times [0,T])$, solution to (3). Moreover, if u_1 and u_2 are the solutions corresponding to v_1 and v_2 then*

$$\| u_1 - u_2 \|_l \leq C_2 \| v_1 - v_2 \|_{nl} . \tag{28}$$

Proof: It is well known, see [35], that given $v(t,x) \in C([0,1] \times [0,T])$ and $u_0 \in C([-1,0])$, the problem (3) has an unique solution $u(t,x) \in C([-1,0] \times [0,T])$. Therefore, the operator H_2 is well defined.

To show the bound (28), we will use a comparison argument.

Before to start the proof, we will make some observations that can simplify our problem. First, note that due to the symmetry of the kernel and the fact that $\int_{-1}^{1} J(r)dr = 1$, it is reasonably to assume

$$C_2 = C_{J,2} \int_{-1}^{0} \int_{0}^{1} J(x-y)\, \mathrm{d}y\, \mathrm{d}x = \frac{C_{J,2}}{2}. \tag{29}$$

To obtain the estimate (28), let us consider $z = u_1 - u_2$, where both u_1 and u_2 satisfy (3) with the same initial condition $u_0(x)$ and two different functions v_1 and v_2, respectively. Then $z(x,t)$ is a solution to the following problem:

$$\begin{cases} \dfrac{\partial z}{\partial t}(x,t) = \dfrac{\partial^2 z}{\partial x^2}(x,t), \\ \dfrac{\partial z}{\partial x}(-1,t) = 0, \\ \dfrac{\partial z}{\partial x}(0,t) = c_{J,2} \displaystyle\int_{-1}^{0} \int_{0}^{1} J(x-y) \left[v_1(y,t) - v_2(y,t) - (u_1(0,t) - u_2(0,t)) \right] \mathrm{d}y\, \mathrm{d}x, \\ z(x,0) = 0. \end{cases}$$

Using (29), we can get the following estimate:

$$\left| C_{J,2} \int_{-1}^{0} \int_{0}^{1} J(x-y)(v_1(y,t) - v_2(y,t))\, \mathrm{d}y\, \mathrm{d}x \right|$$

$$
\begin{aligned}
&\le C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)|v_1(y,t)-v_2(y,t)|\,\mathrm{d}y\,\mathrm{d}x\\
&\le C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)\max_{t\in[0,T]}\max_{y\in[0,1]}\left|v_1(y,t)-v_2(y,t)\right|\mathrm{d}y\,\mathrm{d}x\\
&= C_2\parallel v_1-v_2\parallel_{nl}.
\end{aligned}
$$

Hence, we can define

$$
w(t,x)=\frac{z(t,x)}{C_2\parallel v_1-v_2\parallel_{nl}}, \tag{30}
$$

and w satisfies the following problem:

$$
\begin{cases}
\dfrac{\partial w}{\partial t}(x,t)=\dfrac{\partial^2 w}{\partial x^2}(x,t),\\
\dfrac{\partial w}{\partial x}(-1,t)=0,\\
\dfrac{\partial w}{\partial x}(0,t)\le -C_2w(0,t)+1 \quad \text{and} \quad \dfrac{\partial w}{\partial x}(0,t)\ge -C_2w(0,t)-1,\\
w(x,0)=0.
\end{cases} \tag{31}
$$

The idea now is to verify that the problem (31) possess a pair of sub- and supersolutions, $\underline{w}$, $\overline{w}$, respectively, which satisfy the comparison principle for then return to the estimate (30) and derive (28).

For this, we will recall that a function $\overline{w}(x,t)$ is called a supersolution for the problem (31) if it satisfies

$$
\begin{cases}
\dfrac{\partial \overline{w}}{\partial t}(x,t)\ge\dfrac{\partial^2 w}{\partial x^2}(x,t),\\
\dfrac{\partial \overline{w}}{\partial x}(-1,t)\le 0,\\
\dfrac{\partial \overline{w}}{\partial x}(0,t)\ge -C_2w(0,t)+1,\\
\overline{w}(x,0)=0.
\end{cases}
$$

Respectively, a function $\underline{w}(x,t)$ is called a subsolution if it satisfies the reverse inequalities.

Let us introduce an auxiliary function. Given $\xi<0$ and $0<a<1$ we can define

$$
g(\xi)=\frac{1}{a}f(\xi a), \quad \text{with } g'(0)=f'(0)=1. \tag{32}
$$

Here, the function f is chosen such that the following conditions hold:

Given $\xi_0>1$, f is increasing in $(-\xi_0,0]$, $C^2(-\xi_0,0)$, and $f\equiv 1$ in $(-\infty,-\xi_0]$.

Let us fix $T<\frac{a^2}{2\xi_0^2}$. For each $t\in[0,T]$ and $x\in[-1,0]$ we define

$$
\overline{w}(x,t)=(T+t)^{1/2}g\left(\frac{x}{(T+t)^{1/2}}\right),
$$

with g given by (32).

Claim: $\overline{w}$ is a supersolution for (31). Let us check the afirmation in the following items.

(i) We want to prove that

$$
\frac{\partial \overline{w}}{\partial t}\ge\frac{\partial^2 \overline{w}}{\partial x^2}.
$$

Differentiating $\overline{w}$ with respect to t and x, we would like to verify

$$\frac{1}{2}g\left(\frac{x}{(T+t)^{1/2}}\right) - \frac{1}{2}x(T+t)^{-1/2}g'\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right).$$

Observe that, since $x \in [-1,0]$ and $g'(\frac{x}{(T+t)^{1/2}}) > 0$, we only need to check that

$$\frac{1}{2}g\left(\frac{x}{(T+t)^{1/2}}\right) \geq g''\left(\frac{x}{(T+t)^{1/2}}\right). \tag{33}$$

To deal with this, let us call $\eta = \frac{x}{(T+t)^{1/2}}$. According to the definition of g, to prove (33) is equivalent to prove

$$\frac{1}{2a}f(a\eta) \geq af''(a\eta). \tag{34}$$

We know that, for each $\xi \leq 0$ and $0 < a < 1$,

$$\frac{f(\xi a)}{2} = \begin{cases} 1/2, & \text{if } \xi a < -\xi_0, \\ \frac{f(\xi a)}{2}, & \text{if } -\xi_0 \leq \xi a < 0. \end{cases}$$

Moreover, as $f \in C^2(-\xi_0, 0)$ and increasing in the same interval, we obtain

$$f''(\xi a) \leq \begin{cases} 0, & \text{if } \xi a < -\xi_0, \\ M, & \text{if } -\xi_0 \leq \xi a < 0, \end{cases}$$

where $M = \max_{-\xi_0 \leq \xi \leq 0} |f''(\xi)|$.

Hence, given M, we can choose $0 < a < 1$ in order to have the estimate $\frac{1}{2} \geq Ma^2$. With this in mind we are able to verify (34). Indeed,

(a) (a)If $-\xi_0 \leq \xi a < 0$, it follows that

$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq Ma^2 \geq f''(\xi a)a^2.$$

(b) (b)If $\xi a < -\xi_0$, we have that

$$\frac{f(\xi a)}{2} \geq \frac{1}{2} \geq 0 \geq f''(\xi a)a^2.$$

(ii) We want to verify that $\overline{w}$ satisfies

$$\frac{\partial \overline{w}}{\partial x}(-1,t) \leq 0.$$

At $x = -1$ we have

$$\frac{\partial \overline{w}}{\partial x}(-1,t) = g'\left(\frac{-1}{(T+t)^{1/2}}\right) = f'\left(\frac{-a}{(T+t)^{1/2}}\right). \tag{35}$$

We know that $f \equiv 1$ in $(-\infty, -\xi_0)$. Then, taking $T < \frac{a^2}{2\xi_0^2}$, we obtain that $\frac{-a}{(T+t)^{1/2}} < -\xi_0$ and therefore

$$f'\left(\frac{-a}{(T+t)^{1/2}}\right) = 0,$$

which it proves (35).

(iii) We want to check that

$$\frac{\partial \overline{w}}{\partial x}(0,t) \geq -C_2\overline{w}(0,t) + 1.$$

Differentiating $\overline{w}$ with respect to x, we aim to check if

$$g'(0) \geq -C_2(T+t)^{1/2}g(0) + 1.$$

Since we assume $g'(0) = f'(0) = 1$ and $g(0) \geq 0$, we get

$$1 \geq -C_2(T+t)^{1/2}g(0) + 1,$$

which it proves the item *iii*).

(iv) Finally, we aim to verify that $\overline{w}(x,0) \geq 0$.
Indeed, we have

$$\overline{w}(x,0) = (T)^{1/2}\underbrace{g\left(\frac{x}{T^{1/2}}\right)}_{>0} > 0.$$

With these four items we proved that $\overline{w}$ is a supersolution of (31).
The same analysis can be done to check that

$$\underline{w}(t,x) = -(T+t)^{1/2}g\left(\frac{x}{(T+t)^{1/2}}\right)$$

is a subsolution for the problem (31).
So, by the comparison principle, the solution $w(x,t)$ of the problem (31) verifies

$$\underline{w}(x,t) \leq w(x,t) \leq \overline{w}(x,t).$$

Hence, we get the estimate

$$|\overline{w}(x,t)| \leq \max_{x\in[-1,0]} \max_{t\in[0,T]} \left|(T+t)^{1/2}g\left(\frac{x}{(T+t)^{1/2}}\right)\right| = \frac{1}{a}(2T)^{1/2} < \frac{1}{\xi_o} < 1.$$

Therefore, going back to our original variable, z, we obtain the following:

$$\frac{|z(x,t)|}{C_2 \parallel v_1 - v_2 \parallel_{nl}} = \frac{|u_1 - u_2|}{C_2 \parallel v_1 - v_2 \parallel_{nl}} = w(x,t) \leq \overline{w}(x,t) \leq 1,$$

which implies that

$$\|u_1 - u_2\|_l \leq C_2 \parallel v_1 - v_2 \parallel_{nl},$$

and then the proof is complete. ∎

Remark 3.3: In this case, we also have existence and uniqueness in L^2, as in Remark 3.2. Given $v(x,t) \in L^2([0,1]\times[0,T])$ and $u_0 \in L^2([-1,0])$, there exists an unique $u(x,t) \in C^1([-1,0];L^2[0,T])$, solution to (3).

Again, we have a comparison principle, if we have two ordered functions $v \geq \tilde{v}$ and two initial conditions $u_0 \geq \tilde{u_0}$ then the corresponding solutions verify $u(x,t) \geq \tilde{u}(x,t)$.

Finally, combining the two lemmas, we get the following theorem.

Theorem 3.3: *Given $w_0 \in C([-1,1])$ (or given $w_0 \in L^2([-1,1])$), there exists a unique solution to problem (3)–(4), which has w_0 as initial condition.*

Proof: Let $T \in (0, \frac{1}{2C_{J,1}+C_{J,2}})$. We consider the operator $H : X_T \mapsto X_T$ given by

$$H(u) := H_2(H_1(u)) = H_2(v),$$

and we obtain, from our previous results,

$$\begin{aligned}\|H_2(H_1(u_1)) - H_2(H_1(u_2))\|_l &= \|H_2(v_1) - H_2(v_2)\|_l \leq C_2\|v_1 - v_2\|_{nl} \\ &\leq C_2 \frac{C_{J,2}T}{1-(2C_{J,1}+C_{J,2})T}\|u_1 - u_2\|_l,\end{aligned}$$

which proves that H is a strict contraction for T small enough. Therefore, there is a fixed point

$$u = H(u)$$

that gives us a unique solution $(u, v = H_1(u))$ in $(0, T)$. Since T can be chosen independently of the initial condition, the fixed point argument can be iterated to obtain a global solution for our problem. ■

3.2. Conservation of mass

As we expected, the model (5) preserves the total mass of the solution.

Theorem 3.4: *The solution w of the problem (5), with initial condition $w_0 \in C([-1, 1])$ satisfies*

$$\int_{-1}^{1} w(x, t)\,\mathrm{d}x = \int_{-1}^{1} w_0(x)\,\mathrm{d}x, \quad \textit{for every } t \geq 0.$$

Proof: Notice that

$$\int_{-1}^{1} w(x, t)\,\mathrm{d}x = \int_{-1}^{0} u(x, t)\,\mathrm{d}x + \int_{0}^{1} v(x, t)\,\mathrm{d}x,$$

and, the same is valid for w_0

$$\int_{-1}^{1} w_0 = \int_{-1}^{0} u_0 + \int_{0}^{1} v_0.$$

From the kernel's symmetry, and by Fubbini's Theorem, we obtain

$$\begin{aligned}&\frac{\partial}{\partial t}\left(\int_{-1}^{1} w(x, t)\,\mathrm{d}x\right) \\ &\quad = \int_{-1}^{0} \frac{\partial^2 u}{\partial x^2}(x, t)\,\mathrm{d}x + C_{J,1}\int_{0}^{1}\int_{0}^{1} J(x-y)(v(y, t) - v(x, t))\,\mathrm{d}y\,\mathrm{d}x \\ &\qquad - C_{J,2}\int_{0}^{1}\int_{-1}^{0} J(x-y)v(x, t)\,\mathrm{d}y\,\mathrm{d}x + C_{J,2}u(0, t)\int_{0}^{1}\int_{-1}^{0} J(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad = \frac{\partial u}{\partial x}(0, t) - \frac{\partial u}{\partial x}(-1, t) - C_{J,2}\int_{0}^{1}\int_{-1}^{0} J(x-y)v(x, t)\,\mathrm{d}y\,\mathrm{d}x + C_{J,2}u(0, t)\int_{0}^{1}\int_{-1}^{0} J(x-y)\,\mathrm{d}y\,\mathrm{d}x \\ &\quad = 0.\end{aligned}$$

This shows that the total mass is independent of t. ■

3.3. Comparison principle

Thanks to the linearity of the operator, if we have two solutions to the local/nonlocal problem (3)–(4) then the difference is also a solution. Besides, given a nonnegative initial data u_0, v_0, the solution is nonnegative for every positive time (this follows from our construction of the solutions as a fixed point and Remarks 3.2 and 3.3). Therefore, we have the following result:

Proposition 3.5: *Let $u_0 \geq \tilde{u}_0$ and $v_0 \geq \tilde{v}_0$ then the corresponding solutions to the local/nonlocal problem (3)–(4) verify*

$$u(x,t) \geq \tilde{u}(x,t), \quad v(x,t) \geq \tilde{v}(x,t),$$

for every $t \geq 0$.

To go one step further, let us define what we understand by sub and supersolutions.

Definition 3.6: The functions $\overline{u}$ and $\overline{v}$ are called supersolutions of the problem (3)–(4), in $[-1,1] \times [0,T]$ if, $\overline{u}, \overline{v}$ verify

$$\begin{cases} \dfrac{\partial \overline{u}}{\partial t}(t,x) \geq \dfrac{\partial^2 \overline{u}}{\partial x^2}(t,x), \quad x \in (-1,0), \quad t > 0, \\ \dfrac{\partial \overline{u}}{\partial x}(t,-1) \leq 0, \quad t > 0, \\ \dfrac{\partial \overline{u}}{\partial x}(t,0) \geq C_{J,2} \displaystyle\int_{-1}^{0}\int_{0}^{1} J(x-y)(\overline{v}(y,t) - \overline{u}(t,0))\,\mathrm{d}y\,\mathrm{d}x, \quad t > 0, \\ \dfrac{\partial \overline{v}}{\partial t}(t,x) \geq C_{J,1} \displaystyle\int_{0}^{1} J(x-y)(\overline{v}(y,t) - \overline{v}(x,t))\,\mathrm{d}y - C_{J,2} \int_{-1}^{0} J(x-y)(\overline{v}(x,t) - \overline{u}(t,0))\,\mathrm{d}y, \\ \quad x \in (0,1),\ t > 0, \\ \overline{u}(0,x) \geq u_0(x), \quad x \in (-1,0), \quad \overline{v}(0,x) \geq v_0(x), \quad x \in (0,1). \end{cases}$$

As usual, subsolutions, $\underline{u}, \underline{v}$, are defined analogously by reversing the inequalities.

Theorem 3.7: *Let $\overline{u}$ and $\overline{v}$ be supersolutions of the problem (3)–(4). If $(\overline{u}_0, \overline{v}_0) \geq 0$ then*

$$\overline{u}(x,t) \geq 0, \quad \text{and} \quad \overline{v}(x,t) \geq 0,$$

for every $t > 0$.

Moreover, given $(\overline{u}, \overline{v})$, a supersolution and, $(\underline{u}, \underline{v})$, a subsolution, with $\overline{u}_0 \geq \underline{u}_0$ and $\overline{v}_0 \geq \underline{v}_0$, then,

$$\overline{u}(x,t) \geq \underline{u}(x,t) \quad \text{and} \quad \overline{v}(x,t) \geq \underline{v}(x,t),$$

for every $t > 0$.

Proof: Let us define

$$\begin{cases} w = \overline{u} - \underline{u}, \\ z = \overline{v} - \underline{v}. \end{cases}$$

They are supersolutions of the problem, (3)–(4), respectively, with $w(x,0) \geq 0$ and $z(x,0) \geq 0$. In fact, we have that

(i) $\frac{\partial w}{\partial t} = \frac{\partial \overline{u}}{\partial t} - \frac{\partial \underline{u}}{\partial t} \geq \frac{\partial^2 \overline{u}}{\partial x^2} - \frac{\partial^2 \underline{u}}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}$;
(ii) $\frac{\partial w}{\partial x}(-1,t) = \frac{\partial \overline{u}}{\partial x}(-1,t) - \frac{\partial \underline{u}}{\partial x}(-1,t) \leq 0$;

(iii)

$$\begin{aligned}\frac{\partial w}{\partial x}(0,t) &= \frac{\partial \overline{u}}{\partial x}(0,t) - \frac{\partial \underline{u}}{\partial x}(0,t)\\ &\le C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)(\overline{v}(y,t)-\overline{u}(0,t))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad - C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)(\underline{v}(y,t)-\underline{u}(0,t))\,\mathrm{d}y\,\mathrm{d}x\\ &= C_{J,2}\int_{-1}^{0}\int_{0}^{1} J(x-y)(z(y,t)-w(0,t))\,\mathrm{d}y\,\mathrm{d}x;\end{aligned}$$

(iv)

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial \overline{v}}{\partial t} - \frac{\partial \underline{v}}{\partial t}\\ &\ge C_{J,1}\int_{0}^{1} J(x-y)(\overline{v}(y,t)-\overline{v}(x,t))dy - C_{J,2}\int_{-1}^{0} J(x-y)(\overline{v}(x,t)-\overline{u}(0,t))\,\mathrm{d}y\\ &\quad - \left(C_{J,1}\int_{0}^{1} J(x-y)(\underline{v}(y,t)-\underline{v}(x,t))dy - C_{J,2}\int_{-1}^{0} J(x-y)(\underline{v}(x,t)-\underline{u}(0,t))\,\mathrm{d}y\right)\\ &= C_{J,1}\int_{0}^{1} J(x-y)(z(y,t)-z(x,t))\,\mathrm{d}y - C_{J,2}\int_{-1}^{0} J(x-y)(z(x,t)-w(0,t))\,\mathrm{d}y.\end{aligned}$$

Once we have check that the pair (w,z) is a supersolution, we need to prove that $w \ge 0$ and $z \ge 0$, for all $t > 0$, which implies $\overline{u} \ge \underline{u}$ and $\overline{v} \ge \underline{v}$. To perform this task we need to show that its negative parts are identically zero, $w_- \equiv 0$ and $z_- \equiv 0$. Take $\varphi = w_- \ge 0$ and $\psi = z_- \ge 0$ as our test functions. Multiplying φ and ψ by w_t, z_t and integrating by parts, we obtain,

$$\begin{aligned}0 &\le \left(\int_{-1}^{0}\frac{\partial w}{\partial t}\varphi + \int_{0}^{1}\frac{\partial z}{\partial t}\psi\right)\\ &= -\int_{-1}^{0}\frac{\partial w}{\partial x}\frac{\partial \varphi}{\partial x}\,\mathrm{d}x - \frac{C_{J,1}}{2}\int_{0}^{1}\int_{0}^{1} J(x-y)(z(y)-z(x))(\psi(y)-\psi(x))\,\mathrm{d}y\,\mathrm{d}x\\ &\quad - C_{J,2}\int_{0}^{1}\int_{-1}^{0} J(x-y)(z(x)-w(0))(\psi(x)-\varphi(0))\,\mathrm{d}y\,\mathrm{d}x\\ &= -2E(w_-,z_-) \le 0.\end{aligned}$$

Thus $E(w_-,z_-) = 0$, which implies $w_- \equiv 0$ and $z_- \equiv 0$. This complete the proof. ■

3.4. Asymptotic decay

As we mentioned before, the coupled local/nonlocal diffusion problem shares the same property about asymptotic behavior than, the local and nonlocal problems, individually. In this section, we will derive the asymptotic behavior of the solution as $t \to \infty$. To perform this, we need to introduce an estimate that it was inspired by classical Poincare's inequality [32,34].

We start by analyzing the corresponding stationary problem. First, observe that for any constant k, $u = v = k$, is a solution to the problem (3)–(4). Besides, this constant solution is a minimizer of the energy (a simple inspection of the energy shows more, every minimizer is constant in the whole domain $(-1,1)$).

Let us take β_1 as

$$0 < \beta_1 = \inf_{u,v: \int_{-1}^{0} u + \int_0^1 v = 0} \frac{E(u,v)}{\int_{-1}^{0} (u(x))^2 \, \mathrm{d}x + \int_0^1 (v(x))^2 \, \mathrm{d}x}. \tag{36}$$

Before we prove the asymptotic decay of the solution, we need an extra result.

Lemma 3.8: *Let β_1 be given by (36), then*

$$\beta_1 > 0,$$

and moreover

$$E(u,v) \geq \beta_1 \left(\int_{-1}^{0} u(x)^2 \, \mathrm{d}x + \int_0^1 v(x)^2 \, \mathrm{d}x \right), \tag{37}$$

for every (u,v), solution of (3)–(4), such that it satisfies $\int_{-1}^{0} u + \int_0^1 v = 0$.

Proof: Let us argue by contradiction. Suppose that (37) is false. Then, there exists sequences $\{u_n\} \in H^1(-1,0)$ and $\{v_n\} \in L^2(0,1)$ such that

(i) $\int_{-1}^{0} u_n + \int_0^1 v_n = 0$,
(ii) $\int_{-1}^{0} (u_n)^2 + \int_0^1 (v_n)^2 = 1$ and,
(iii)

$$\frac{1}{2} \int_{-1}^{0} \left| \frac{\partial u_n}{\partial x} \right|^2 \mathrm{d}x + \frac{C_{J,1}}{4} \int_0^1 \int_0^1 J(x-y) \left(v_n(y) - v_n(x) \right)^2 \, \mathrm{d}y \, \mathrm{d}x + \frac{C_{J,2}}{2} \int_0^1 \int_{-1}^{0} J(x-y) \, \mathrm{d}y \, (v_n(x) - u_n(0))^2 \, \mathrm{d}x \leq \frac{1}{n}.$$

Consequently, taking the limit in n, in item (iii), we obtain

$$\lim_n \left(\frac{1}{2} \int_{-1}^{0} \left| \frac{\partial u_n}{\partial x} \right|^2 \right) = 0, \tag{38}$$

$$\lim_n \left(\frac{C_{J,1}}{4} \int_0^1 \int_0^1 J(x-y)(v_n(y) - v_n(x))^2 \, \mathrm{d}y \, \mathrm{d}x \right) = 0, \tag{39}$$

and

$$\lim_n \left(\frac{C_{J,2}}{2} \int_{-1}^{0} \int_0^1 J(x-y)(u_n(0) - v_n(x))^2 \, \mathrm{d}x \, \mathrm{d}y \right) = 0. \tag{40}$$

From (ii), we have that $\int_{-1}^{0} (u_n)^2 \leq 1$. Then, by (38), u_n is bounded in $H^1(-1,0)$, and hence there exists a subsequence $\{u_{nj}\} \in H^1(-1,0)$, which weakly converges for a limit $u \in H^1(-1,0)$. From the weak convergence in $H^1(-1,0)$, it follows the convergence of $\{u_{nj}\} \to u$ in $L^2(-1,0)$ and the uniform convergence in $[-1,0]$ to $u \in H^1(-1,0)$. Moreover, as $\frac{1}{2} \int_{-1}^{0} ((u_n)_x)^2 \to 0$ we have that the limit u is a constant, $u = k_1$. In particular, $u_{nj}(0) \to k_1$.

Also, from (ii) and by Cauchy–Schwarz inequality, we obtain that $\int_0^1 |v_n| \leq (\int_0^1 |v_n|^2)^{1/2} \leq 1$. Let $\{k_n\}$, be $\{k_n\} = \int_0^1 v_n$, with $|k_n| \leq 1$. We observe that, since $\{k_n\}$ is bounded in $L^2(0,1)$, we can extract a convergent subsequence k_{n_j}, which converges for some limit k_2, as $n_j \to \infty$.

Consider $\{z_{n_j}\} = \{v_n\} - \{k_{n_j}\}$, such that $\int_0^1 z_{n_j} = 0$. By [33], there exists a constant $c > 0$, such that

$$\int_0^1 \int_0^1 J(x-y)(z_{n_j}(y) - z_{n_j}(x))^2 \, dy \, dx \geq c \int_0^1 (z_{n_j}(x))^2 \, dx.$$

From (39) we get the following:

$$\int_0^1 \int_0^1 J(x-y)(z_{n_j}(y) - z_{n_j}(x))^2 \, dy \, dx = \int_0^1 \int_0^1 J(x-y)(v_n(y) - v_n(x))^2 \, dy \, dx \to 0,$$

which yields

$$0 \geq \lim_{n_j \to \infty} c \int_0^1 (z_{n_j}(x))^2 \, dx.$$

By the last inequality, we conclude that $z_{n_j} \to 0$ in $L^2(0,1)$, which leads to $(v_n - k_{n_j}) \to 0$ in $L^2(0,1)$, and thus we get that $v_n \to k_2$ in $L^2(0,1)$.

Taking the limit in (40), we obtain

$$\frac{C_{J,2}}{2} \int_{-1}^0 \int_0^1 J(x-y)(k_1 - k_2)^2 \, dx \, dy \to 0.$$

Hence $k_1 = k_2$. On the other hand, by *i*) we have $\int_{-1}^0 u_n + \int_0^1 v_n = 0$, so $k_1 = 0$ and $k_2 = 0$, then $k_1 = k_2 = 0$. But this is impossible since, by item *ii*) we have $\int_{-1}^0 (k_1)^2 + \int_0^1 (k_2)^2 = 1$. ∎

Remark 3.4: The value β_1 should be the first nontrivial eigenvalue for our problem (notice that $\beta = 0$ is an eigenvalue with $u = v = cte$ as eigenfunctions). However, due to the lack of compactness of the nonlocal part, it is not clear that the infimum defining β_1 is attained.

Now, we are ready to prove the exponential convergence of the solutions to the mean value of the initial datum as $t \to +\infty$.

Theorem 3.9: *Given $w_0 \in L^2(-1,1)$, the solution to (5), with initial condition w_0, converges to its mean value as $t \to \infty$, with an exponential rate,*

$$\left\| w(\cdot, t) - \fint w_0 \right\|_{L^2(-1,1)} \leq C(\|w_0\|_{L^2(-1,1)}) e^{-2\beta_1 t}, \quad t > 0,$$

where β_1 is given by (36) and $C(w_0) > 0$.

Proof: As we know, $u = v = k$, k constant, is a solution of (3)–(4). In particular $h(x,t) = u(x,t) - k$ and $z(x,t) = v(x,t) - k$ are also a solution. If $k = \int_{-1}^0 u_0 + \int_0^1 v_0$, then h and z satisfy

$$\int_{-1}^0 h(x,t) \, dx + \int_0^1 z(x,t) \, dx = 0.$$

Let

$$f(t) = \frac{1}{2} \int_{-1}^0 h(x,t)^2 \, dx + \frac{1}{2} \int_0^1 z(x,t)^2 \, dx.$$

Differentiating f with respect to t, we obtain

$$f'(t) = \int_{-1}^0 h \frac{\partial h}{\partial t} \, dx + \int_0^1 z \frac{\partial z}{\partial t} \, dx$$

$$= \int_{-1}^{0} h\frac{\partial^2 h}{\partial x^2}\,\mathrm{d}x + C_{J,1}\int_0^1 z(x,t)\int_0^1 J(x-y)(z(y,t)-z(x,t))\,\mathrm{d}y\,\mathrm{d}x$$

$$- C_{J,2}\int_0^1 z(x,t)\int_{-1}^{0} J(x-y)z(x,t)\,\mathrm{d}y\,\mathrm{d}x + C_{J,2}\int_0^1 z(x,t)h(0,t)\int_{-1}^{0} J(x-y)\,\mathrm{d}y\,\mathrm{d}x$$

$$= h(0,t)\frac{\partial h}{\partial x}(0,t) - h(-1,t)\frac{\partial h}{\partial x}(-1,t) - \int_{-1}^{0}\left|\frac{\partial h}{\partial x}\right|^2\mathrm{d}x$$

$$+ C_{J,1}\int_0^1\int_0^1 J(x-y)(z(y,t)-z(x,t))z(x,t)\,\mathrm{d}y\,\mathrm{d}x$$

$$- C_{J,2}\int_0^1\int_{-1}^{0} J(x-y)(z(x,t)-h(0,t)dyz(x,t)\,\mathrm{d}x.$$

Applying Fubini's Theorem, and using the symmetry of the kernel, we obtain

$$f'(t) = -2E(h,z)(t).$$

Finally, by Lemma 3.8, we obtain the following:

$$2E(h,z) \geq 2\beta_1 \int_{-1}^{0} h^2 + \int_0^1 z^2 \geq 2\beta_1 f(t),$$

which implies

$$f'(t) \leq -2\beta_1 f(t).$$

Hence,

$$f(t) \leq e^{-2\beta_1 t} f(0),$$

where

$$f(0) = \frac{1}{2}\left(\int_{-1}^{0} h_0^2\,\mathrm{d}x + \int_0^1 z_0^2\,\mathrm{d}x\right) = C(\| w_0 \|_{L^2(-1,1)}).$$

From this follows that

$$\int_{-1}^{0} |u(t,x)-k|^2\,\mathrm{d}x + \int_0^1 |v(t,x)-k|^2\,\mathrm{d}x \leq C(\| w_0 \|_{L^2(-1,1)})e^{-2\beta_1 t} \to 0,$$

as $t \to \infty$. In particular, we have that $u \to k$ in $L^2(-1,0)$ and $v \to k$ in $L^2(0,1)$. ■

4. Rescaling the kernel. Convergence to the local problem

We derive a strong convergence in $L^2(-1,1)$, uniformly or bounded times, of the solutions of the rescaled problem (with J as in (6)) to the solution of the local problem (8) (the heat equation in the whole domain with homogeneous Newman boundary conditions) using the Brezis–Pazy Theorem through Mosco's convergence result. To perform this task, we need to provide another existence and uniqueness result for the problem (3)–(4), based on semigroup theory for m-accretive operators.

4.1. Existence and uniqueness of a mild solution

On the concept of solution. We will introduce now the concept of solution for the complete problem (5). We rely on serigroup theory and introduce the operator

$$B_J u(x) = \begin{cases} -\dfrac{\partial^2 u}{\partial x^2}(x) \quad \text{for } x \in (-1,0), \\ -C_{J,1} \displaystyle\int_0^1 J(x-y)(v(y)-v(x))\,\mathrm{d}y + C_{J,2} \int_{-1}^0 J(x-y)(v(y)-u(0))\,\mathrm{d}y \\ \text{for } x \in (0,1). \end{cases}$$

Let

$$D(B_J) := \Big\{(u,v) : u \in H^2(-1,0), v \in L^2(0,1) \text{ with } \frac{\partial u}{\partial x}(-1) = 0 \text{ and } \\ \frac{\partial u}{\partial x}(0) = -C_{J,2} \int_{-1}^0 J(x-y)(v(y)-u(0))\,\mathrm{d}y\Big\}$$

be the domain of the operator, and

$$B_J : D(B_J) \subset L^2(-1,1) \mapsto L^2(-1,1).$$

Now, according to [32], we can define a mild solution in $L^2(-1,1)$, of the abstract Cauchy problem by

$$\begin{cases} u'(t) = B_J(u(t)), & t > 0 \\ u(0) = u_0. \end{cases}$$

Moreover, given an initial condition in the domain of the operator, there exists a unique strong solution for this problem, provided by the semigroup related to B_J operator, see [32,36] for more details.

Following the ideas presented in [32], we will prove that, the operator B_J is completely accretive in $L^2(-1,1)$ and satisfies the range condition, $L^2(-1,1) \subset R(I+B_J)$. Once the B_J operator satisfies these two conditions, we can conclude that B_J is m-completely accretive in $L^2(-1,1)$. The range condition implies that for any $f \in L^2(-1,1)$ there exists $u \in D(B_J)$ such that, $u + B_J(u) = f$, and the resolvent, $(I+B_J)^{-1}$, is a contraction in $L^2(-1,1)$. With this in mind, by the Crandall–Ligget's Theorem we will obtain the existence and uniqueness of a mild solution for the coupled local/nonlocal evolution problem.

Theorem 4.1: *Given and initial condition $w_0 \in L^2(-1,1)$, there exists a mild solution w of the problem (5) that is a contraction in the L^2-norm.*

Proof: According to [32], it is enough to show that the operator B_J is completely accretive in $L^2(-1,1)$ and satisfies the range condition, $L^2(-1,1) \subset R(I+B_J)$. Consider the set

$$P_0 = \left\{q \in C^\infty(-1,1) : 0 \le q \le 1, \quad \text{supp}(q')\text{is compact and } 0 \notin \text{supp}(q)\right\}.$$

To prove the operator B_J is completely accretive, is equivalent to show that, given $w_1, w_2 \in D(B_J)$, and $q(w_1 - w_2)$, as a test function, we have that

$$\int_{-1}^1 (B_J(w_1(x)) - B_J(w_2(x)))q(w_1(x) - w_2(x))\,\mathrm{d}x \ge 0. \tag{41}$$

Using the weak form of the operator, we get

$$\int_{-1}^1 (B_J(w_1(x)) - B_J(w_2(x)))q(w_1(x) - w_2(x))\,\mathrm{d}x$$

$$
\begin{aligned}
&= \int_{-1}^{0} \frac{\partial(w_1 - w_2)}{\partial x} \frac{\partial[q(w_1 - w_2(x))]}{\partial x}\, dx \\
&+ \frac{C_{J,1}}{2} \int_0^1 \int_0^1 J(x-y)[(w_1 - w_2)(y) - (w_1 - w_2)(x)] \\
&\times [q(w_1(y) - w_2(y)) - q(w_1(x) - w_2(x))]\, dy\, dx \\
&+ C_{J,2} \int_0^1 \int_{-1}^{0} J(x-y)[(w_1 - w_2)(x) - (w_1 - w_2)(0)] \\
&\times [q(w_1(x) - w_2(x)) - q(w_1(0) - w_2(0))]\, dy\, dx.
\end{aligned}
$$

Since $J \geq 0$, using the Mean Value Theorem, we obtain that the inequality (41) is holds.

To derive that, B_J is completely accretive in $L^2(-1,1)$ we need to show that it satisfies the range condition

$$L^2(-1,1) \subset R(I + B_J).$$

Given $f \in L^2(-1,1)$, we consider the variational problem

$$I[u] = \min_{u \in L^2(-1,1)} \left\{ \frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \int_{-1}^{1} fu \right\}. \tag{42}$$

The existence of a unique minimizer u, of the variational problem (42), is proved using the direct method in the calculus of variations. This operator is continuous, monotone, and coercive in $L^2(-1,1)$. Indeed, using Young's inequality, we obtain

$$
\begin{aligned}
\frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \int_{-1}^{1} fu &\geq \frac{1}{2} \int_{-1}^{1} u^2 + E(u) - \left(\int_{-1}^{1} f^2 \right)^{1/2} \left(\int_{-1}^{1} u^2 \right)^{1/2} \\
&\geq \frac{3}{8} \int_{-1}^{1} u^2 + E(u) - C,
\end{aligned} \tag{43}
$$

and then

$$\lim_{\|u\|_{L^2(-1,1)} \to \infty} \frac{I(u)}{\| u \|_{L^2(-1,1)}} \geq \lim_{\|u\|_{L^2(-1,1)} \to \infty} \frac{\left(\frac{3}{8} \| u \|_{L^2(-1,1)} + E(u) - C \right)}{\| u \|_{L^2(-1,1)}} = +\infty.$$

Then, from [35], there exists a minimizing sequence $\{u_n\}$ in $H^1(-1,0) \cap L^2(-1,1)$, with $n \in \mathbb{N}$, such that

$$\frac{1}{2} \int_{-1}^{1} u_n^2 + E(u_n) - \int_{-1}^{1} fu_n \leq C, \quad \forall\, n \in \mathbb{N}.$$

Therefore $\| u_n \|_{L^2(-1,1)} \leq M$ and $\| u_n \|_{H^1(-1,0)} \leq M$, for all $n \in \mathbb{N}$. Hence, by the compact embedding theorem [[35], Rellich–Kondrachov Compactness Theorem], we can assume, taking a subsequence if necessary, that $u_n \rightharpoonup u$ in $L^2(-1,1)$, $u_n \to u$ in $L^2(-1,0)$, and by the reflexivity of $H^1(-1,0)$, we get that $u \in H^1(-1,0)$.

According to [35], as the functional $I(u)$ is bounded and convex, it follows that $I(u)$ is weakly lower semicontinuous,

$$I(u) \leq \liminf_{n \to \infty} I(u_n). \tag{44}$$

Thanks to (44) and (43), we can conclude that u is actually a minimizer of the variational problem (42). The uniqueness follows by the strict convexity of the functional. ∎

Remark 4.1: One can also show existence and uniqueness using Hille–Yosida Theorem. In fact, one can show that B_J is closed, its domain $D(B_J)$ is dense in $L^2(-1,1)$ and it holds that for every $\lambda > 0$,

$$\|(\lambda - B_J)^{-1}\|_{L^2(-1,1)} \leq \frac{1}{\lambda}.$$

Since we prove the existence and uniqueness of a mild solution to the local/nonlocal problem, we are ready to show that we can recover the local heat equation at the whole domain, (8), from a suitable rescaling of the kernel J. The convergence result proved here will be given at the Mosco sense. For more details, see [32].

Before we prove the main result of this section, we need to define the energy functional associated with the rescaled problem

$$E^{\varepsilon}(w^{\varepsilon}) := \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u^{\varepsilon}}{\partial x}\right|^2 + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^{\varepsilon}(x-y)\left(v^{\varepsilon}(y) - v^{\varepsilon}(x)\right)^2 \,\mathrm{d}y\,\mathrm{d}x$$
$$+ \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)\left(u^{\varepsilon}(0) - v^{\varepsilon}(x)\right)^2 \,\mathrm{d}x\,\mathrm{d}y,$$

if $w^{\varepsilon} \in D(E^{\varepsilon}) := H^1(-1,0) \times L^2(0,1)$, and $E^{\varepsilon}(w) := \infty$, otherwise. Analogously, we define the limit energy functional as

$$E(w) := \frac{1}{2}\int_{-1}^{1}\left|\frac{\partial w}{\partial x}\right|^2 \mathrm{d}x,$$

if $w \in D(E) := H^1(-1,1)$, and $E^{\varepsilon}(w) := \infty$, otherwise.

Given $w_0 \in L^2(-1,1)$, for each $\varepsilon > 0$, let w^{ε} be the solution to the evolution problem associated with the energy E^{ε}, and w be the solution associated to the functional E, considering the same initial condition.

Theorem 4.2: *Under the above assumptions, the solutions to the rescaled problem, w^{ε}, converge to w, the solution of (8). For any finite $T > 0$ we have*

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]} \| w^{\varepsilon}(\cdot,t) - w(\cdot,t) \|_{L^2(-1,1)}\right) = 0.$$

Proof: To prove this result, we will make use of the Brezis–Pazy Theorem (Theorem A.37, see [32]), for a sequence of m-accretive operators $B_{J^{\varepsilon}} \in L^2(-1,1)$ defined in the beginning of the section. To apply this result, we would like to show the convergence of the resolvents, that is

$$\lim_{\varepsilon\to 0}\left(I + B_{J^{\varepsilon}}\right)^{-1}\phi = (I + A)^{-1}\phi, \tag{45}$$

where $A(w) := -w_{xx}$ is the classic operator for the heat equation, and for every $\phi \in L^2(-1,1)$. If we can prove (45), then by the Brezis–Pazy Theorem, we get the convergence of the solutions w^{ε} to w in $L^2(-1,1)$ uniformly in $[0,T]$. To prove the convergence of resolvents, we will use a convergence result given by Mosco, checking the following statements:

(1) For every $w \in D(E)$, there exists a sequence $\{w^{\varepsilon}\} \in D(E^{\varepsilon})$, such that $w^{\varepsilon} \to w$ in $L^2(-1,1)$ and

$$E(w) \geq \limsup_{\varepsilon\to 0} E^{\varepsilon}(w^{\varepsilon}).$$

(2) If, $w^{\varepsilon} \to w$ weakly in $L^2(-1,1)$ and

$$E(w) \leq \liminf_{\varepsilon\to 0} E^{\varepsilon}(w^{\varepsilon}).$$

Let us start the proof by the assertion 2). We can suppose that the inferior limit is finite, otherwise, there is nothing to prove. Hence, we can assume that $E^{\varepsilon}(w^{\varepsilon}) \le C$. With this in mind, and because all the terms involved in the energy are positive, we have

(i) $\frac{1}{2}\int_{-1}^{0} |\frac{\partial u^{\varepsilon}}{\partial x}|^2 \,\mathrm{d}x \le C$;
(ii) $\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^{\varepsilon}(x-y)(v^{\varepsilon}(y)-v^{\varepsilon}(x))^2 \,\mathrm{d}y\,\mathrm{d}x \le C$;
(iii) $\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(0)-v^{\varepsilon}(x))^2 \,\mathrm{d}x\,\mathrm{d}y \le C$.

From (i), it follows that, there exists a subsequence, also denoted by $\{u^{\varepsilon}\}$, such that

$$u^{\varepsilon} \rightharpoonup u \quad \in H^1(-1,0),$$

which implies

$$u^{\varepsilon} \to u \quad \text{in } L^2(-1,0), \quad \text{and} \quad u^{\varepsilon} \to u \quad \text{uniformlyin}(-1,0).$$

Consider the following:

$$\begin{aligned}
&\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(y)-v^{\varepsilon}(x))^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad \le \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(y)-u^{\varepsilon}(0))^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\qquad + \underbrace{\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(0)-v^{\varepsilon}(x))^2 \,\mathrm{d}x\,\mathrm{d}y.}_{\le C}
\end{aligned}$$

Let us show that

$$\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(y)-u^{\varepsilon}(0))^2 \,\mathrm{d}x\,\mathrm{d}y$$

is bounded. Performing a variable change and, observing that the $\mathrm{supp}(J) = B(0,R)$, we get

$$\begin{aligned}
&\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^{\varepsilon}(x-y)(u^{\varepsilon}(y)-u^{\varepsilon}(0))^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{C_{J,2}}{2\varepsilon^2}\int_{-1}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)\,\mathrm{d}z(u^{\varepsilon}(y)-u^{\varepsilon}(0))^2 \,\mathrm{d}x\,\mathrm{d}y \\
&\quad = \frac{C_{J,2}}{2\varepsilon^2}\int_{-R\varepsilon}^{0} f_{\varepsilon}(y)\frac{u^{\varepsilon}(y)-u^{\varepsilon}(0))^2}{\varepsilon}\frac{\mathrm{d}y}{\varepsilon}.
\end{aligned}$$

Changing variables again, using Holder's inequality, and the arithmetic-geometric inequality, it follows that

$$\begin{aligned}
&\frac{C_{J,2}}{2}\int_{-R}^{0} f_{\varepsilon}(\varepsilon w)\frac{1}{\varepsilon}\left((u^{\varepsilon}(0)-u^{\varepsilon}(\varepsilon w)\right)^2 \mathrm{d}w \\
&\quad \le \frac{C_{J,2}}{2}\int_{-R}^{0} f_{\varepsilon}(\varepsilon w)\left[(-w^2)^{1/2}\left(\int_{\varepsilon w}^{0}(u_x^{\varepsilon}(s))^2\,\mathrm{d}s\right)^{1/2}\right]\mathrm{d}w \\
&\le \frac{C_{J,2}}{2}\int_{-R}^{0} f_{\varepsilon}(\varepsilon w)\left[\frac{1}{2}(-w^2)+\frac{1}{2}\int_{\varepsilon w}^{0}(u_x^{\varepsilon}(s))^2\,\mathrm{d}s\right]\mathrm{d}w
\end{aligned}$$

$$\leq \frac{C_{J,2}}{8}\int_{-R}^{0}(-w^2)\,\mathrm{d}w + \frac{C_2}{8}\int_{-R}^{0}\left[\int_{\varepsilon w}^{0}(u_x^{\varepsilon}(s))^2\,\mathrm{d}s\right]\mathrm{d}w$$

$$= \tilde{C} + \frac{C_{J,2}}{8}\int_{-R}^{0}\underbrace{\left[\int_{-1}^{0}(u_x^{\varepsilon}(s))^2\,\mathrm{d}s\right]}_{\leq C}\mathrm{d}w. \tag{46}$$

Therefore, we conclude that

$$\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1}J^{\varepsilon}(x-y)(u^{\varepsilon}(y)-v^{\varepsilon}(x))^2\,\mathrm{d}x\,\mathrm{d}y$$

is bounded. By (46), we can write a new bounded energy functional,

$$\overline{E}(w^{\varepsilon}) := \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u^{\varepsilon}}{\partial x}\right|^2\mathrm{d}x + \frac{C_{J,1}}{4\varepsilon^3}\int_{0}^{1}\int_{0}^{1}J^{\varepsilon}(x-y)\left(v^{\varepsilon}(y)-v^{\varepsilon}(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x$$

$$+ \frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1}J^{\varepsilon}(x-y)\left(u^{\varepsilon}(y)-v^{\varepsilon}(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y \leq C.$$

By Lemma 2.1, there exists $k>0$ (independent of ε) such that

$$C \geq \overline{E}(w^{\varepsilon}) \geq k\frac{1}{\varepsilon^3}\int_{-1}^{1}\int_{-1}^{1}J^{\varepsilon}(x-y)\left(w^{\varepsilon}(y)-w^{\varepsilon}(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x. \tag{47}$$

Using (47), it follows from [32] that there exists a subsequence, also denoted $\{w^{\varepsilon}\}$, which converges in $L^2(-1,1)$ to a limit $w \in H^1(-1,1)$.

Moreover, taking the inferior limit at the first term of the energy $\overline{E}(w^{\varepsilon})$,

$$\liminf_{\varepsilon\to 0}\frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u^{\varepsilon}}{\partial x}\right|^2\mathrm{d}x \geq \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|^2\mathrm{d}x. \tag{48}$$

Now, using the fact that

$$\frac{C_{J,1}}{2\varepsilon^3}\int_{0}^{1}\int_{0}^{1}J^{\varepsilon}(x-y)(v^{\varepsilon}(y)-v^{\varepsilon}(x))^2\,\mathrm{d}x\,\mathrm{d}y$$

is bounded, by Theorem 6.11, in [32], there exists a subsequence, also denoted by $\{v^{\varepsilon}\}$, such that

$$v^{\varepsilon} \to v \quad \text{in } L^2(0,1),$$

the limit $v \in H^1(0,1)$ and, moreover

$$\left(\frac{C_{J,1}}{4}J(z)\right)^{1/2}\frac{\overline{v}^{\varepsilon}(x+\varepsilon z)-v^{\varepsilon}(x)}{\varepsilon} \rightharpoonup \left(\frac{C_{J,1}}{4}J(z)\right)^{1/2} z\cdot\frac{\partial v}{\partial x}, \tag{49}$$

weakly in $L^2(0,1)\times L^2(\mathbb{R})$. Then, taking the limit in Equation (49) and, after a variables change, we have that

$$\liminf_{\varepsilon\to 0}\frac{C_{J,1}}{4\varepsilon^3}\int_{0}^{1}\int_{0}^{1}J^{\varepsilon}(x-y)\left(v^{\varepsilon}(y)-v^{\varepsilon}(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x \geq \frac{1}{2}\int_{0}^{1}\left|\frac{\partial v}{\partial x}\right|^2\mathrm{d}x.$$

Moreover, we have

$$\liminf_{\varepsilon\to 0}\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_{0}^{1}J^{\varepsilon}(x-y)\left(u^{\varepsilon}(y)-v^{\varepsilon}(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y \geq 0. \tag{50}$$

Therefore, from (48)–(50) we conclude that

$$\liminf_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon) \geq \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial u}{\partial x}\right|^2 \mathrm{d}x + \frac{1}{2}\int_{0}^{1}\left|\frac{\partial v}{\partial x}\right|^2 \mathrm{d}x = E(w).$$

Now let us prove (1). Given $w \in H^1(-1,1)$ we choose as the approximating sequence $w_n^\varepsilon \equiv w^\varepsilon$. We have

$$E^\varepsilon(w^\varepsilon) := \frac{1}{2}\int_{-1}^{0}\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2 \mathrm{d}x + \frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x$$
$$+\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y$$

and we want to show that

$$\limsup_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon) \leq E(w). \tag{51}$$

The inequality (51) is holds if, we verify the following:

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x\right) = \frac{1}{2}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}x$$

and

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y\right) = 0. \tag{52}$$

Let us first show (52). Performing a change of variables and using Holder's inequality, Equation (52) can be written as

$$\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^{0}\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y$$
$$= \frac{C_{J,2}}{2\varepsilon^2}\int_{-1}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)\left(w^\varepsilon(y+\varepsilon z)-w^\varepsilon(0)\right)^2\,\mathrm{d}z\,\mathrm{d}y$$
$$= \frac{C_{J,2}}{2\varepsilon^2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{\frac{1-y}{\varepsilon}} J(z)\left(w^\varepsilon(y+\varepsilon z)-w^\varepsilon(0)\right)^2\,\mathrm{d}z\,\mathrm{d}y$$
$$= \frac{C_{J,2}}{2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{R} J(z)\left[\int_0^{y+\varepsilon z}\frac{\frac{\partial w^\varepsilon}{\partial x}(s)}{\varepsilon}\,\mathrm{d}s\right]^2\mathrm{d}z\,\mathrm{d}y$$
$$\leq \frac{C_{J,2}}{2}\int_{-R\varepsilon}^{0}\int_{\frac{-y}{\varepsilon}}^{R} J(z)\left[\int_0^{y+\varepsilon z}\left|\frac{\partial w^\varepsilon}{\partial x}(s)\right|^2\mathrm{d}s\right]\mathrm{d}z\frac{\mathrm{d}y}{\varepsilon}.$$

Changing variables again and since $\int_{-R}^{R} J(z)\,\mathrm{d}z = 1$, we obtain

$$\frac{C_{J,2}}{2}\int_{-R}^{0}\int_{-t}^{R} J(z)\left[\int_0^{\varepsilon(t+z)}\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}s\right]\mathrm{d}z\,\mathrm{d}t$$
$$\leq \frac{C_{J,2}}{2}\int_{-R}^{0}\int_{-R}^{R} J(z)\,\mathrm{d}z\left[\int_0^{2R\varepsilon}\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}s\right]\mathrm{d}t$$

$$\leq \frac{C_{J,2}}{2}\int_{-R}^{0}\left[\int_{0}^{2R\varepsilon}\left|\frac{\partial w^{\varepsilon}}{\partial x}\right|^{2}\mathrm{d}s\right]\mathrm{d}t.$$

Now, we observe that, as $\frac{\partial w}{\partial x}\in L^{2}(-1,1)$ then $|\frac{\partial w}{\partial x}|^{2}\in L^{1}(-1,1)$. Then, we have

$$\int_{0}^{2R\varepsilon}\left|\frac{\partial w^{\varepsilon}}{\partial x}\right|^{2}\mathrm{d}z\to 0$$

as $\varepsilon\to 0$, which yields (52).

Now, it remains to derive the following

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^{3}}\int_{0}^{1}\int_{0}^{1}J^{\varepsilon}(x-y)\left(w^{\varepsilon}(y)-w^{\varepsilon}(x)\right)^{2}\mathrm{d}y\,\mathrm{d}x\right)=\frac{1}{2}\int_{0}^{1}\left|\frac{\partial w^{\varepsilon}}{\partial x}\right|^{2}\mathrm{d}x.$$

Changing variables, and using Taylor's expansion, it follows that

$$\begin{aligned}
&\left|\frac{C_{J,1}}{4\varepsilon^{3}}\int_{0}^{1}\int_{0}^{1}J^{\varepsilon}(x-y)\left(w^{\varepsilon}(y)-w^{\varepsilon}(x)\right)^{2}\mathrm{d}y\,\mathrm{d}x\right|\\
&=\left|\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left(w^{\varepsilon}(x+\varepsilon z)-w^{\varepsilon}(x)\right)^{2}\mathrm{d}z\,\mathrm{d}x\right|\\
&\leq\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|w^{\varepsilon}(x+\varepsilon z)-w^{\varepsilon}(x)\right|^{2}\mathrm{d}z\,\mathrm{d}x\\
&=\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\varepsilon z+\frac{1}{2}\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\varepsilon^{2}z^{2}\right|^{2}\mathrm{d}z\,\mathrm{d}x\\
&\leq\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left(\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\varepsilon z\right|+\frac{1}{2}\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\varepsilon^{2}z^{2}\right|\right)^{2}\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

Now, using Minkowski's inequality

$$\begin{aligned}
&\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left(\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\varepsilon z\right|+\frac{1}{2}\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\varepsilon^{2}z^{2}\right|\right)^{2}\mathrm{d}z\,\mathrm{d}x\\
&\leq\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\varepsilon z\right|^{2}\mathrm{d}z\,\mathrm{d}x+\frac{C_{J,1}}{4\varepsilon^{2}}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|\frac{1}{2}\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\varepsilon^{2}z^{2}\right|\mathrm{d}z\,\mathrm{d}x\\
&\leq\frac{C_{J,1}}{4}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\right|^{2}|z|^{2}\mathrm{d}z\,\mathrm{d}x+\varepsilon^{2}\frac{C_{J,1}}{16}\int_{0}^{1}\int_{\frac{-x}{\varepsilon}}^{\frac{1-x}{\varepsilon}}J(z)\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\right|^{2}|z|^{4}\mathrm{d}z\,\mathrm{d}x\\
&\leq\frac{C_{J,1}}{4}\int_{0}^{1}\int_{-R}^{R}J(z)\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\right|^{2}|z|^{2}\mathrm{d}z\,\mathrm{d}x+\varepsilon^{2}\frac{C_{J,1}}{16}\int_{0}^{1}\int_{-R}^{R}J(z)\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\right|^{2}|z|^{4}\mathrm{d}z\,\mathrm{d}x\\
&\leq\frac{C_{J,1}}{4}\int_{0}^{1}\int_{\mathbb{R}}J(z)|z|^{2}\mathrm{d}z\left|\frac{\partial w^{\varepsilon}}{\partial x}(x)\right|^{2}\mathrm{d}x+\varepsilon^{2}\frac{C_{J,1}}{16}\int_{0}^{1}\int_{\mathbb{R}}J(z)\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\right|^{2}|z|^{4}\mathrm{d}z\,\mathrm{d}x.
\end{aligned}$$

Since $\int_{\mathbb{R}}J(z)|z|^{2}\,\mathrm{d}z=M(J)$, $\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}$ is bounded, and $\int_{\mathbb{R}}J(z)|z|^{4}\,\mathrm{d}z$ is finite, we can conclude that

$$\limsup_{\varepsilon\to 0}\left(\varepsilon^{2}\frac{C_{J,1}}{16}\int_{0}^{1}\int_{\mathbb{R}}J(z)|z|^{4}\mathrm{d}z\left|\frac{\partial^{2}w^{\varepsilon}}{\partial x^{2}}(\xi)\right|^{2}\mathrm{d}x\right)=0,$$

and

$$\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}.M(J)}{4}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}(x)\right|^2|z|^2\,\mathrm{d}x\right)=\frac{1}{2}\int_0^1\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}x.$$

Finally, we have

$$\begin{aligned}
&\limsup_{\varepsilon\to 0} E^\varepsilon(w^\varepsilon)\\
&=\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{-1}^0\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}x+\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x\right.\\
&\quad\left.+\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^0\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y\right)\\
&=\limsup_{\varepsilon\to 0}\left(\frac{1}{2}\int_{-1}^0\left|\frac{\partial w^\varepsilon}{\partial x}\right|^2\mathrm{d}x\right)\\
&\quad+\limsup_{\varepsilon\to 0}\left(\frac{C_{J,1}}{4\varepsilon^3}\int_0^1\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(y)-w^\varepsilon(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x\right)\\
&\quad+\limsup_{\varepsilon\to 0}\left(\frac{C_{J,2}}{2\varepsilon^3}\int_{-1}^0\int_0^1 J^\varepsilon(x-y)\left(w^\varepsilon(0)-w^\varepsilon(x)\right)^2\,\mathrm{d}x\,\mathrm{d}y\right)\\
&\le\frac{1}{2}\int_{-1}^0\left|\frac{\partial w}{\partial x}\right|^2\mathrm{d}x+\frac{1}{2}\int_0^1\left|\frac{\partial w}{\partial x}\right|^2\mathrm{d}x\\
&=E(w),
\end{aligned}$$

as we wanted to show. ■

Remark 4.2: Our convergence result can be also read as: take, as before, $w^\varepsilon=(u^\varepsilon,v^\varepsilon)$. Then, for any finite $T>0$ we have

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]}\parallel u^\varepsilon(\cdot,t)-u(\cdot,t)\parallel_{L^2(-1,1)}\right)=0$$

and

$$\lim_{\varepsilon\to 0}\left(\max_{t\in[0,T]}\parallel v^\varepsilon(\cdot,t)-v(\cdot,t)\parallel_{L^2(-1,1)}\right)=0.$$

The limit pair (u,v) is the unique solution to two heat equations

$$\begin{cases}\dfrac{\partial u}{\partial t}(x,t)=\dfrac{\partial^2 u}{\partial x^2}(x,t), & x\in(-1,0),t\in(0,T),\\ \dfrac{\partial u}{\partial x}(-1,t)=0, & \end{cases}$$

$$\begin{cases}\dfrac{\partial v}{\partial t}(x,t)=\dfrac{\partial^2 v}{\partial x^2}(x,t), & x\in(0,1),t\in(0,T),\\ \dfrac{\partial v}{\partial x}(1,t)=0, & \end{cases}$$

with the coupling

$$u(0,t) = v(0,t), \quad \frac{\partial u}{\partial x}(0,t) = \frac{\partial v}{\partial x}(0,t)$$

and initial conditions

$$u(x,0) = u_0(x), \quad v(x,0) = v_0(x).$$

Notice that the coupling gives continuity, and continuity of the derivative of the function

$$w(x,t) = \begin{cases} u(x,t), & \text{if } x \in (-1,0) \\ v(x,t), & \text{if } x \in (0,1) \end{cases}$$

that therefore, turns out to be a solution to

$$\begin{cases} \dfrac{\partial w}{\partial t}(x,t) = \dfrac{\partial^2 w}{\partial x^2}(x,t), & x \in (-1,1),\ t > 0, \\ \dfrac{\partial w}{\partial x}(-1,t) = \dfrac{\partial w}{\partial x}(1,t) = 0, & t > 0, \\ w(x,0) = w_0(x), & x \in (-1,1). \end{cases}$$

5. Extension to higher dimensions

In this final section, we will briefly describe how our results can be extended to higher dimensions. Take Ω, as a bounded smooth domain in $\mathbb{R}^N$ and split it into two subdomains Ω_l and Ω_{nl}, $\Omega = \Omega_l \cup \Omega_{nl}$. Let us call Σ, the interface between Ω_l and Ω_{nl} inside Ω, that is,

$$\Sigma = \overline{\Omega_l} \cap \overline{\Omega_{nl}} \cap \Omega.$$

We will assume that Ω_l has a Lipschitz boundary (in order to solve a heat equation with Newman boundary conditions, we need some regularity of the boundary).

As before, we split $w \in L^2(\Omega)$ as $w = u + v$, with $u = w\chi_{\Omega_l}$ and $v = w\chi_{\Omega_{nl}}$. Fix a nonnegative continuous kernel $G : \Omega_{nl} \times \Sigma \mapsto \mathbb{R}$. For any

$$w = (u,v) \in \mathcal{B} := \left\{ w \in L^2(\Omega) : u|_{\Omega_l} \in H^1(\Omega_l), v \in L^2(\Omega_{nl}) \right\}$$

we define the energy

$$E(u,v) := \frac{1}{2} \int_{\Omega_l} |\nabla u|^2 \, dx + \frac{C_{J,1}}{4} \int_{\Omega_{nl}} \int_{\Omega_{nl}} J(x-y) \left(v(y) - v(x) \right)^2 \, dy \, dx$$
$$+ \frac{C_{J,2}}{2} \int_{\Omega_{nl}} \int_{\Sigma} G(x,z) \left(v(x) - u(z) \right)^2 \, d\sigma(z) \, dx.$$

Remark that in this energy we have

$$\int_{\Omega_{nl}} \int_{\Sigma} G(x,z) \left(v(x) - u(z) \right)^2 \, d\sigma(z) \, dx \tag{53}$$

as coupling term. This integral can be obtained from an integral of the form

$$\iint_A J(x-y) \left(v(x) - u(z) \right)^2 \, dy \, dx$$

assuming the following geometric condition on the interface Σ; for every $x \in \Omega_l$ and every $y \in \Omega_{nl}$ with $x - y \in \mathrm{supp}(J)$ there exists a unique $z \in \Sigma$ that belongs to the segment that joins x with y

(hence $z = z(x, y)$). To provide examples, notice that this geometric condition holds if Σ is almost flat. This assumption is useful since, from a probabilistic viewpoint, when a particle wants to jump from $y \in \Omega_{nl}$ to $x \in \Omega_l$ we want that it gets stuck at the interface (and then we want that there exist a unique point on Σ that belongs to the segment $[x, y]$, otherwise, some selection principle has to be assumed and, the selected point on the interface will not depend continuously on x and y, in general). This assumption is used to make the change of variables

$$z = ax + (1 - a)y$$

in

$$\iint_A J(x - y)\,(v(x) - u(z))^2 \,\mathrm{d}y\,\mathrm{d}x$$

with $A = \{(x, y) : x \in \Omega_{nl}, y \in \Omega_l, \text{ with } z \in \Sigma, z = ax + (1 - a)y\}$ to obtain the coupling term in our energy (53). The kernel G is nonnegative and comes from the change of variables that involves a jacobian $D(x, z)$.

With this energy, $E(u, v)$, the associated evolution problems reads as

$$\begin{cases} \dfrac{\partial u}{\partial t}(x,t) = \Delta u(x,t), \\ \dfrac{\partial u}{\partial \eta}(z,t) = 0, \quad z \in \partial\Omega_l \cap \partial\Omega, \\ \dfrac{\partial u}{\partial \eta}(z,t) = C_{J,2} \displaystyle\int_{\Omega_{nl}} G(x,z)(v(y,t) - u(z,t))\,\mathrm{d}x, \qquad z \in \Sigma, \\ u(x,0) = u_0(x). \end{cases} \tag{54}$$

for $x \in \Omega_l$, $t > 0$, and

$$\begin{cases} \dfrac{\partial v}{\partial t}(x,t) = C_{J,1} \displaystyle\int_{\Omega_{nl}} J(x-y)\big(v(y,t) - v(x,t)\big)\,\mathrm{d}y - C_{J,2} \int_{\Sigma} G(x,z)(v(x,t) - u(z,t))\,\mathrm{d}\sigma(z), \\ v(x,0) = v_0(x), \end{cases} \tag{55}$$

for $x \in \Omega_{nl}$, $t > 0$.

For this problem (54)–(55), we can also prove existence and uniqueness following the same steps that we made for the one-dimensional case. In fact, the strategy of building a solution as a fixed point of the composition of the maps that solves the problem for u (given v) and for v (fixing u) also works here. Remark that we obtain a solution $u(x, t)$ that is in $H^1(\Omega_l)$ for $t > 0$ and hence $u(z, t)$ is defined on Σ in the sense of traces (and belongs to $L^2(\Sigma)$ for $t > 0$). The more abstract approach using semigroup theory also works here. Consider the operator

$$B_J(u,v) = \begin{cases} -\Delta u \quad \text{for } x \in \Omega_l, \\ -C_{J,1} \displaystyle\int_{\Omega_{nl}} J(x-y)(v(y) - v(x))\,\mathrm{d}y + C_{J,2} \int_{\Sigma} G(x,z)(v(x) - u(z))\,\mathrm{d}\sigma(z) \\ \quad \text{for } x \in \Omega_{nl}, \end{cases}$$

with domain

$$D(B_J) := \Big\{(u,v) : u \in H^2(\Omega_l), v \in L^2(\Omega), \text{ with } \frac{\partial u}{\partial \eta}(z) = 0 \text{ on } \partial\Omega \cap \partial\Omega_l$$
$$\text{and } \frac{\partial u}{\partial \eta}(z) = -C_{J,2} \int_{\Omega_{nl}} G(x,z)(v(x) - u(z))\,\mathrm{d}x \text{ on } \Sigma, \Big\}$$

and proceed as we did previously.

The total mass is preserved. In fact, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\left(\int_{\Omega} w(x,t)\,\mathrm{d}x\right) &= \int_{\Omega_l} \Delta u(x,t)\,\mathrm{d}x + C_{J,1}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)(v(y,t)-v(x,t))\,\mathrm{d}y\,\mathrm{d}x \\
&\quad - C_{J,2}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)(v(x,t)-u(z,t)\,\mathrm{d}\sigma(z)\,\mathrm{d}x \\
&= \int_{\partial\Omega_l} \frac{\partial u}{\partial \eta}(x,t)\,\mathrm{d}x - C_{J,2}\int_{\Omega_{nl}}\int_{\Sigma} G(x,z)(v(x,t)-u(z,t)\,\mathrm{d}\sigma(z)\,\mathrm{d}x \\
&= 0.
\end{aligned}
$$

The key control of the nonlocal energy,

$$
\begin{aligned}
&\frac{1}{2}\int_{\Omega_l} |\nabla u|^2\,\mathrm{d}x + \frac{C_{J,1}}{4}\int_{\Omega_{nl}}\int_{\Omega_{nl}} J(x-y)\left(v(y)-v(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x && (1)\\
&\quad + \frac{C_{J,2}}{2}\int_{\Omega_{nl}}\int_{\Sigma} J(x,z)\,(v(x)-u(z))^2\,\mathrm{d}\sigma(z)\,\mathrm{d}x \\
&\geq k\int_{\Omega}\int_{\Omega} J(x-y)\left(w(y)-w(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x. && (56)
\end{aligned}
$$

can be proved, as before, arguing by contradiction.

With the key inequality (56), we can show that solutions converge to the mean value of the initial condition, as $t \to \infty$ with an exponential rate.

$$
\left\| w(\cdot,t) - \fint w_0 \right\|_{L^2(\Omega)} \leq Ce^{-\beta_1 t}, \quad t > 0.
$$

In fact, we have that

$$
0 < \beta_1 = \inf_{w:\int_\Omega w=0} \frac{E(w)}{\int_\Omega (w(x))^2\,\mathrm{d}x}
$$

is strictly positive. This fact can be proved by contradiction as we did before, but it also follows from (56) and the results in [32] since we have

$$
\beta_1 = \inf_{w:\int_\Omega w=0} \frac{E(w)}{\int_\Omega (w(x))^2\,\mathrm{d}x} \geq \inf_{w:\int_\Omega w=0} \frac{k\int_\Omega\int_\Omega J(x-y)\left(w(y)-w(x)\right)^2\,\mathrm{d}y\,\mathrm{d}x}{\int_\Omega (w(x))^2\,\mathrm{d}x} > 0.
$$

The approximation of the heat equation with Neumann boundary conditions under rescales of the kernel is left open. We believe that the result holds with extra assumptions on the coupling kernel G.

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