FRACTIONAL SUSCEPTIBILITY FUNCTIONS FOR THE QUADRATIC FAMILY: MISIUREWICZ–THURSTON PARAMETERS

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Abstract. For $f_t(x) = t - x^2$ the quadratic family, we define the fractional susceptibility function $\Psi^{\Omega}_{\phi,t_0}(\eta, z)$ of f_t, associated to a C^1 observable ϕ at a stochastic parameter t_0. We also define an approximate, "frozen," fractional susceptibility function $\Psi^{\mathrm{fr}}_{\phi,t_0}(\eta, z)$ such that $\lim_{\eta\to 1} \Psi^{\mathrm{fr}}_{\phi,t_0}(\eta, z)$ is the susceptibility function $\Psi_{\phi,t_0}(z)$ studied by Ruelle. If t_0 is Misiurewicz–Thurston, we show that $\Psi^{\mathrm{fr}}_{\phi,t_0}(1/2, z)$ has a pole at $z = 1$ for generic ϕ if $\mathcal{J}_{1/2}(t_0) \neq 0$, where $\mathcal{J}_\eta(t) = \sum_{k=0}^{\infty} \mathrm{sgn}(Df^k_t(c_1))|Df^k_t(c_1)|^{-\eta}$, with $c_1 = t$ the critical value of f_t. We introduce "Whitney" fractional integrals $I^{\eta,\Omega}$ and derivatives $M^{\eta,\Omega}$ on suitable sets Ω. We formulate conjectures on $\Psi^{\Omega}_{\phi,t_0}(\eta, z)$ and $\mathcal{J}_\eta(t)$, supported by our results on $M^{\eta,\Omega}$ and $\Psi^{\mathrm{fr}}_{\phi,t_0}(1/2, z)$, for the former, and numerical experiments, for the latter. In particular, we expect that $\Psi^{\Omega}_{\phi,t_0}(1/2, z)$ is singular at $z = 1$ for Collet–Eckmann t_0 and generic ϕ.

We view this work as a step towards the resolution of the paradox that $\Psi_{\phi,t_0}(z)$ is holomorphic at $z = 1$ for Misiurewicz–Thurston f_{t_0} [35, 17], despite lack of linear response [8].

Contents

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1. Introduction

For real[1] parameters $t \in (1,2)$, we consider the quadratic family

$$f_t(x) = t - x^2\,, \quad x \in [-2,2]\,.$$

The critical point is $c = c_{0,t} = 0$, the critical value is $c_{1,t} = t < |a_t|$ where $a_t := \frac{-1-\sqrt{1+4t}}{2} \in (-2,0)$ satisfies $f_t(a_t) = a_t = f_t(-a_t)$. More generally, we denote the postcritical points by $c_{k,t} = f_t^k(c)$ for $k \ge 0$.

We are interested in the set $\mathcal{S}$ of (so-called stochastic) parameters t for which f_t admits an absolutely continuous invariant probability measure $\mu_t = \rho_t dm$. The set $\mathcal{S}$ contains the Collet–Eckmann (CE) parameters t, i.e. those t such that there exist $\lambda_c > 1$ and $K_0 \ge 1$ with

$$|Df_t^k(c_{1,t})| \ge \lambda_c^k\,, \quad \forall k \ge K_0\,. \tag{1}$$

Linear response is the study of differentiability of the map $t \mapsto \mu_t$, on suitable subsets of $\mathcal{S}$, in a suitable topology in the image, viewing μ_t as a Radon measure or a distribution of higher order by introducing smooth observables ϕ. In the simpler setting of families $t \mapsto F_t$ of smooth expanding (or mixing smooth hyperbolic) maps with $\partial_t F_t = X_t \circ F_t$, the map

$$t \mapsto \mathcal{R}_\phi(t) := \int \phi \,\mathrm{d}\mu_t$$

[1]The map for $t = 2$ is the full parabola $2 - x^2$ on $[-2,2]$, which can only be perturbed by taking $t < 2$. For $t = 1$, we get a half-parabola on $[0,1]$.

is differentiable, and the derivative $\partial_s \mathcal{R}_\phi(s)$ at $s = t$ is, by [34], the value at $z = 1$ of the susceptibility function, which is the power series (see also [3, §1])

$$\Psi_\phi(z) := \Psi_{\phi,t}(z) = \sum_{k=0}^{\infty} z^k \int (\phi \circ F_t^k)' \, (X_t \rho_t) \, \mathrm{d}m \, .$$

Returning to the quadratic family f_t, it is well known since the work of Thunberg [40] (see [14, Theorem 1.30] for a more recent statement) that[2] $t \mapsto \mu_t$ is severely discontinuous if one does not restrict to Collet–Eckmann parameters with bounded constants. However, this map is continuous when restricted to a suitable (large) set of good parameters [42]. More recently, [8, Cor 1.6] showed that at almost every Collet–Eckmann parameter t, and for every 1/2 Hölder observable ϕ, the function $\mathcal{R}_\phi(s)$ is η-Hölder for all $\eta < 1/2$ at $s = t$, in the sense of Whitney, on a set $\Omega_{<1/2} = \Omega_{<1/2}(t)$ of Collet–Eckmann parameters having t as a density point.

One of the purposes of the present work is to reconcile two apparently contradictory results: In 2005, Ruelle [35] considered the full unimodal map f_t (and more generally, Chebyshev polynomials f_t of degree $D \geq 2$). He showed that the susceptibility function (note that $X_t := \partial_t f_t \circ f_t^{-1} \equiv 1$ for the quadratic family: Appendix B discusses the condition that X_t vanishes at endpoints)

$$\Psi_\phi(z) = \Psi_{\phi,t}(z) = \sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k)' \, \rho_t \, \mathrm{d}m \tag{2}$$

admits a meromorphic extension to $\mathbb{C}$. Ruelle also obtained the remarkable fact that the residue of the possible pole at $z = 1$ vanishes (for all observables $\phi \in C^1$). Soon thereafter, with Jiang [17], they generalised this result to the set MT of Misiurewicz–Thurston parameters, i.e., those t for which there exist $L \geq 1$ and $P \geq 1$ with $y = f_t^L(c)$ periodic of minimal period P, with $|Df_t^P(y)| > 1$. This raised the hope that $s \mapsto \mathcal{R}_\phi(s) := \int \phi(x) \rho_s(x) \, \mathrm{d}m$ could be differentiable (in the sense of Whitney, on an appropriate subset of $\mathcal{S}$) at $t \in \mathrm{MT}$, with $\partial_s \mathcal{R}_\phi(s)|_{s=t} = \Psi_\phi(1)$. In[3] 2015, however, with Benedicks and Schnellmann [8], one of us showed that for any mixing $t \in \mathrm{MT}$, there exist $\phi \in C^\infty$, and a set $\Omega_{1/2} = \Omega_{1/2}(t) \subset \mathcal{S}$ containing t as an accumulation point such that

$$0 < \liminf_{\substack{\delta \to 0 \\ t+\delta \in \Omega_{1/2}}} \frac{|\mathcal{R}_\phi(t+\delta) - \mathcal{R}_\phi(t)|}{\sqrt{|\delta|}} \leq \limsup_{\substack{\delta \to 0 \\ t+\delta \in \Omega_{1/2}}} \frac{|\mathcal{R}_\phi(t+\delta) - \mathcal{R}_\phi(t)|}{\sqrt{|\delta|}} < \infty \, . \tag{3}$$

A hard open question is whether t is a Lebesgue density point of $\Omega_{1/2}$: In the affirmative, (3) would not be compatible with Whitney-differentiability of $\mathcal{R}_\phi(t)$ at t in any natural sense, a strict paradox. Otherwise, the bounds (3), may be compatible with differentiability in the sense of Whitney, although this would still be counter-intuitive.

Aiming to shed[4] some light on this puzzling state of affairs, we introduce below, for $\Re\eta \in (0, 1)$ and an appropriate positive measure set $\Omega \subset \mathcal{S}$, a two-variable fractional susceptibility function $\Psi^{\Omega}_{\phi,t}(\eta, z)$ in §2.2. The idea is to replace ordinary

[2] As a Radon measure, say — using distributions of higher order does not help.

[3] In the decade between 2005 and 2015, the hope that $\mathcal{R}_\phi(s)$ could be differentiable in the sense of Whitney had already been diminished by the papers [10] and [36].

[4] Another goal is to give a probabilistic analysis (analogous to the central limit theorem of de Lima–Smania [22] in the piecewise expanding setting) of the breakdown of $C^{1/2}$ regularity of the acim in transversal families of smooth unimodal maps with a quadratic critical point.

derivatives by fractional derivatives (Marchaud derivatives are convenient, in particular because they vanish on constants). The main hurdle is that Ω has positive measure but does not contain any nontrivial interval: Despite the vast existing literature on fractional derivatives, we did not find any suitable notion of fractional derivatives on such sets (we propose a definition $M^{\eta,\Omega}$ in Section 7). In Conjectures A and A+, we formulate expected properties of $\Psi^{\Omega}_{\phi,t}(\eta,z)$. We also introduce in §2.2 an approximate, "frozen" fractional susceptibility function $\Psi^{\mathrm{fr}}_{\phi,t}(\eta,z)$, where the dynamics is frozen at a parameter t (so that Ω does not appear and ordinary Marchaud derivatives can be used), and we study its properties in Theorem C for $t \in \mathrm{MT}$. (We expect that the techniques of the proof can be extended to TSR parameters defined in (4), see Remarks 5.1 and D.2 and Footnote 16.)

We next briefly discuss the organisation of the paper and key points in the proof of our main rigorous result, Theorem C. Section 2 contains the definitions of the fractional susceptibility functions. (Another approximate fractional susceptibility function, the response function $\Psi^{\mathrm{rsp}}_{\phi,t}(\eta,z)$, is useful to prove Theorem C.) Sections 3 and 4 are devoted to preparatory material on fractional integrals and derivatives. We mention here that the case of piecewise expanding maps [7, 10, 9, 3] is easier, because the invariant density appearing there is a sum of a nice function with a countable sum of Heaviside functions. For the quadratic maps, the invariant density (50) involves a sum of quadratic spikes. The fact, used in [7, 10, 9], that the derivative of a Heaviside function is a Dirac mass is mirrored in the present work by Abel's remark that the one-sided half-integral of a quadratic spike is a Heaviside, so that its one-sided Marchaud half derivative is a Dirac mass (see Lemmas 3.1 and 4.4). However, *one-sided* derivatives do not seem appropriate to define reasonable fractional susceptibility functions. The *two-sided* half integrals, respectively derivatives, of quadratic spikes (Lemmas 3.2 and 4.4) involve an additional logarithmic, respectively[5] polar, term. The corresponding "iterated pole" is one of the features of Theorem C in Section 5 (see Lemma 5.6).

An unexpected ingredient of Theorem C is a new half-transversality condition $\mathcal{J}_{1/2}(t) \neq 0$ (see (10)). Conjecture B on sums $\mathcal{J}_\eta(t)$ in §6.2 is backed up by our numerical results in §6.1.

Finally, in §7.1 and §7.2, we introduce and study fractional Whitney–Riemann–Liouville integrals $I^{\eta,\Omega}$ and Whitney–Marchaud derivatives $M^{\eta,\Omega}$ (in particular a "Whitney version" of Abel's remark) which support our conjectures on $\Psi^{\Omega}_{\phi}(\eta,z)$ and $\Psi^{\mathrm{fr},\Omega}_{\phi}(\eta,z)$. More precisely, as a stepping stone between the frozen function $\Psi^{\mathrm{fr}}_{\phi}(\eta,z)$ and $\Psi^{\Omega}_{\phi}(\eta,z)$, we introduce yet another approximate "semifreddo" function $\Psi^{\Omega,\mathrm{sf}}_{\phi}(\eta,z)$ in §7.2. We expect that the approximate susceptibility functions $\Psi^{\mathrm{fr}}_{\phi}(\eta,z)$, $\Psi^{\mathrm{rsp}}_{\phi}(\eta,z)$, and $\Psi^{\Omega,\mathrm{sf}}_{\phi}(\eta,z)$ have the same qualitative behaviour as $\Psi^{\Omega}_{\phi}(\eta,z)$ (Remark 1.2). Proposition D, proved in §5.1, shows that the approximate functions $\Psi^{\mathrm{fr}}_{\phi}(\eta,z)$ and $\Psi^{\mathrm{rsp}}_{\phi}(\eta,z)$ tend to $\Psi_\phi(z)$ as $\eta \to 1$ as formal powers series in z (i.e., convergence of the coefficients of each individual z^k).

In the remainder of this Introduction, we flesh out the synopsis given above.

[5]It is natural that the half derivative of $\mathbf{1}_{x>c_k}(x-c_k)^{-1/2}$ involves $(x-c_k)^{-1}$, but we found no good reference for the computation.

1.1. **Conjecture A on the fractional susceptibility function $\Psi^\Omega_\phi(\eta, z)$.** We say that a parameter $t \in (1,2)$ is TSR if f_t is Collet–Eckmann and satisfies Tsujii's [41, (WR)] condition, i.e.,

$$\lim_{\eta\to 0^+} \liminf_{n\to\infty} \frac{1}{n} \sum_{\substack{1\le j\le n \\ |f_t^j(c)-c|<\eta}} \ln |f_t'(f_t^j(c))| = 0 \,. \tag{4}$$

TSR is implied by polynomial recurrence and implies Benedicks–Carleson exponential recurrence (see e.g. [11, Proposition 2.2 and references], also for a topological definition of TSR). Tsujii constructed in [41, Theorem 1 (I)] a positive measure subset $\Omega \subset \mathcal{S}$ of TSR parameters such that, setting $\Omega^c = \mathbb{R} \setminus \Omega$, and letting m denote Lebesgue measure,

$$\lim_{\delta\to 0} \frac{m([t-\delta, t+\delta]\cap \Omega^c)}{\delta^\beta} = 0 \,, \quad \forall t \in \Omega \,, \tag{5}$$

for all $\beta < 2$ (in particular each $t \in \Omega$ is a Lebesgue density point of Ω).

The transfer operator associated to f_t is defined on $L^1([-2,2], dm)$ by setting

$$\mathcal{L}_t\varphi(x) = \sum_{f_t(y)=x} \frac{\varphi(y)}{|Df_t(y)|} = \mathbf{1}_{x<t} \frac{\varphi(\sqrt{t-x}) + \varphi(-\sqrt{t-x})}{2\sqrt{t-x}} \,.$$

The dual of $\mathcal{L}_t$ fixes Lebesgue measure restricted to $I_t := [a_t, -a_t]$ so that $f_t(I_t) \subset (I_t)$.

For $t \in (1,2)$ a fixed TSR parameter, it is convenient to extend $s \mapsto f_s$ as a Lipschitz map to the whole line as follows: choosing $\epsilon = \epsilon(t) > 0$ such that $[t-\epsilon, t+\epsilon] \subset (1,2)$, and such that[6] $[c_{2,t}, c_{1,t}] \subset \mathrm{int}(\cap_{\tau\in[t-\epsilon,t+\epsilon]} I_\tau) =: I_{t,\epsilon}$, set

$$f_\tau = f_t \text{ if } |\tau - t| < \epsilon \,, f_\tau = f_{t-\epsilon} \text{ for all } \tau < t-\epsilon \,, \text{ and } f_\tau = f_{t+\epsilon} \text{ for all } \tau > t+\epsilon \,. \tag{6}$$

Then, for $\Omega \subset$ TSR having t as a Lebesgue density point, and ϕ a compactly supported C^1 function, the *fractional susceptibility function* $\Psi^\Omega_\phi(\eta, z) = \Psi^\Omega_{\phi,t,\epsilon}(\eta, z)$ for the quadratic family at t is the function of two complex variables η and z

$$\Psi^\Omega_\phi(\eta, z) = \frac{\eta}{2\Gamma(1-\eta)} \sum_{k=0}^\infty z^k \int \int_{\delta\in\mathbb{R}\cap(\Omega-t)} \phi(f^k_{t+\delta}(x)) \frac{(\mathcal{L}_{t+\delta} - \mathcal{L}_t)\rho_t(x)}{|\delta|^{1+\eta}} \mathrm{sgn}(\delta)\, \mathrm{d}\delta \mathrm{d}x \,,$$

(writing $\mathrm{d}x = \mathrm{d}m(x)$, $\mathrm{d}\delta = \mathrm{d}m(\delta)$), in the sense of formal power series in z, for fixed η with $\Re\eta \in (0,1)$. (Motivation and details are given in §2.2.)

For Ω satisfying (5) for some $\beta > 1$, we define a "Whitney–Marchaud" fractional derivative $M^{\eta,\Omega}$ in §7.2. For $\eta \in (0,1)$, Proposition F in §7.2 gives conditions on g and Ω ensuring that

$$\lim_{\zeta\uparrow\eta} \left(\frac{\Gamma(1-\zeta)}{\eta\cdot\Gamma(\eta-\zeta)} M^{\zeta,\Omega} g(t) \right) = \lim_{\delta\to 0, t+\delta\in\Omega} \frac{g(t+\delta)-g(t)}{\mathrm{sgn}(\delta)|\delta|^\eta} \,.$$

We can now state our main conjecture[7]:

[6] Recall that $\mathrm{supp}(\rho_t) = [c_{2,t}, c_{1,t}]$.

[7] The threshold for η below is $1/2$; for families with criticality d the expected threshold is $1/d$.

Conjecture A. *For almost every mixing*[8] $t \in \mathrm{TSR}$*, there exist* $\bar\lambda_t > 1$*,* $\epsilon > 0$*, and a set* $\Omega = \Omega(t) \subset \mathrm{TSR}$ *containing* t *and satisfying* (5) *for all* $\beta < 2$*, such that, for any compactly supported* C^1 *function* ϕ*, and any* $N \ge 1$*, the following holds:*

i. *For any* η *with* $0 < \Re\eta < 1/2$*, there exists a disc*[9] D_η *of radius* > 1 *such that* $\Psi^\Omega_{\phi,t}(\eta, z)$ *is holomorphic in* $\{(\eta, z) \mid 0 < \Re\eta < 1/2\,,\, z \in D_\eta\}$*.*

ii. *For any real* $0 < \eta < 1/2$*, we have the fractional response formula*

$$\Psi^\Omega_{\phi,t}(\eta, 1) = M^{\eta,\Omega}_{s=t} \int \phi(x)\rho_s(x)\,\mathrm{d}x\,. \tag{7}$$

iii. *The power series* $\Psi^\Omega_{\phi,t}(1/2, z)$ *defines a holomorphic function in the open unit disc. For a generic* C^N *function* $\tilde\phi$*: the unit circle is a natural boundary for this function; the limit as* $z \in (0, 1)$ *tends to* 1 *of* $\Psi^\Omega_{\tilde\phi,t}(1/2, z)$ *does not exist; the limit as* $z \in (0, 1)$ *tends to* 1 *of* $(z - 1)\Psi^\Omega_{\tilde\phi,t}(1/2, z)$*, if it exists, does not vanish.*

iv. *For any* η *with* $\Re\eta \in (1/2, 1)$ *there exists a disc* D_η *with radius in* $(1/\bar\lambda_t, 1)$ *such that the function* $\Psi^\Omega_\phi(\eta, z)$ *is holomorphic in* $\{(\eta, z) \mid 0 < \Re\eta < 1/2\,,\, z \in D_\eta\}$*. For any* η *with* $\Re\eta \in (1/2, 1)$ *and any generic* C^N *function* $\tilde\phi$ *we have that* $\Psi^\Omega_{\tilde\phi,t}(\eta, z)$ *has a singularity in the open unit disc.*

v. *We have* $\lim_{\eta\uparrow 1} \Psi^\Omega_{\phi,t}(\eta, z) = \Psi_{\phi,t}(z)$ *as formal power series in* z *(recall* (2)*).*

For families of piecewise expanding maps, a more precise version of [iii] for the ordinary susceptibility function $\Psi_{\phi,t}(z)$ (similar to Conjecture A+ below) was established [9, Theorem 1], using results in [7, 10]. (We expect that other results of [9], on the iterated logarithm law e.g., can be adapted to the quadratic family.)

Also in the piecewise expanding setting, the analogue of [i] and [ii] in Conjecture A, replacing 1/2 by 1, and taking Ω to be a neighbourhood of t, has been established in[10] [3].

We explain next how the conjectured properties of $\Psi^\Omega_\phi(\eta, z)$ reflect the behaviour described in [8] of the absolutely continuous invariant measure and may also contribute to resolve the paradox[11] arising from comparing the results of [35] and [17] with those of [8]. (The fractional susceptibility function being holomorphic in two variables also raises the hope to use tools such as Hartog's extension theorem.)

First, the η Hölder upper bounds of [8] on $\Omega_{<1/2}$ together with Proposition F and [ii] in Conjecture A would imply that, if $\Omega_{<1/2}$ satisfies (5) for some $\beta > 1$,

$$\lim_{\zeta\to\eta} \frac{\Psi^\Omega_{\phi,t}(\zeta, 1)}{\Gamma(\eta - \zeta)} = \frac{1}{\Gamma(1 - \eta)} \lim_{\delta\to 0, t+\delta\in\Omega} \frac{\mathcal{R}_\phi(t + \delta) - \mathcal{R}_\phi(t)}{\mathrm{sgn}(\delta)|\delta|^\eta} = 0\,,\ \forall \eta \in (0, 1/2)\,.$$

Next, if [ii] in Conjecture A could be established at any $\eta \in [1/2, 1)$ for which either side of (7) is well defined, then we would have for any t at which $\mathcal{R}_\phi(t)$ is

[8] Some results of [8] require polynomial recurrence. We expect that this is an artefact of the method used there, but maybe TSR must be strengthened to polynomial recurrence.

[9] All discs in the present work are centered at the origin.

[10] The weighted Marchaud derivatives in [3] could be useful to understand the logarithmic factors appearing in [8].

[11] The "averaging" response studied in [44, §3] and [45, (16)] does not resolve the paradox, see Appendix C.

Ω-Whitney 1/2 differentiable (Definition 7.4, Proposition F in §7.2)

$$\lim_{\zeta\uparrow 1/2} \frac{\Psi^{\Omega}_{\phi,t}(\zeta,1)}{\Gamma(1/2-\zeta)} = \frac{1}{\Gamma(1/2)} \lim_{\delta\to 0, t+\delta\in\Omega} \frac{\mathcal{R}_\phi(t+\delta)-\mathcal{R}_\phi(t)}{\mathrm{sgn}(\delta)|\delta|^{1/2}} . \tag{8}$$

(If $t \in \mathrm{MT}$, [8] furnishes upper and lower bounds on $(\mathcal{R}_\phi(t)-\mathcal{R}_\phi(t+\delta_n))/|\delta_n|^{1/2}$, for suitable sequences $\delta_n \to 0$, but the existence of the limit in the right-hand side of (8) is not known.)

If the last claim of [iii] in Conjecture A holds we also expect that, for Ω satisfying Tsujii's condition (5) for all $\beta < 2$, and generic ϕ,

$$\lim_{\zeta\uparrow 1/2} \frac{\Psi^{\Omega}_{\phi,t}(\zeta,1/2+\zeta)}{\Gamma(1/2-\zeta)} \neq 0 , \qquad \lim_{\eta\uparrow 1/2} \frac{\Psi^{\Omega}_{\phi,t}(\eta,1)}{\Gamma(1/2-\eta)} \neq 0 . \tag{9}$$

In view of Proposition F in §7.2, the above inequality would establish that $\mathcal{R}_\phi$ is not Ω-Whitney η-differentiable if $\eta > 1/2$ (Definition 7.4). In particular, the ordinary susceptibility function (2) at $z = 1$ could not be interpreted as a derivative. (The singularity of $\Psi_t(z)$ in the open unit interval could then be a "scar" of the singularity at $z = 1$ of $\Psi^{\Omega}_{\phi,t}(\eta,z)$ for some $\eta < 1$, presumably $\eta = 1/2$.) The inequalities (9) could be useful to determine whether t is a density point of the set $\Omega_{1/2}$ in (3).

Remark 1.1 (Tangential families). In view of the linear response result in [11], replacing the quadratic family by a "tangential" family $\tilde f_\tau$ of smooth unimodal maps all topologically conjugated to a TSR map $\tilde f_t$, we expect that, taking Ω a small enough neighbourhood of t, claims [i] and [ii] in Conjecture A, hold, replacing 1/2 by 1, and, in addition,

$$\lim_{\eta\uparrow 1} \Psi^{\Omega}_{\phi,\tilde f_t}(\eta,1) = \lim_{\tau\to t} \frac{\mathcal{R}_\phi(t)-\mathcal{R}_\phi(\tau)}{t-\tau} .$$

It would be interesting, but more challenging, to investigate whether "tangentiality" of a family $\tilde f_t$ at a single point t_0 implies some additional (Whitney) regularity of the response at t_0.

1.2. **Fractional transversality $\mathcal{J}_\eta$. Conjectures B and A+.** It is well known that all[12] Collet–Eckmann parameters t_1 are transversal (see [43, Theorem 3]) in the sense of Tsujii [41] (see also Appendix B), i.e.

$$\mathcal{J}(t) := \sum_{j=0}^{\infty} \frac{\partial_\tau f_\tau(c_{j,\tau})|_{\tau=t}}{Df^j_t(c_{1,t})} = \sum_{j=0}^{\infty} \frac{1}{Df^j_t(c_{1,t})} \neq 0 . \tag{10}$$

To state Conjecture A+ and the fractional transversality condition appearing in Theorem C (see §1.3), setting $\mathrm{sgn}(x) = \frac{x}{|x|}$ for $x \in \mathbb{R}_*$, and $\mathrm{sgn}(0) = 0$, we let

$$s_0 = 1 , \quad s_k := s_{k,t} = \mathrm{sgn}(Df^k_t(c_{1,t})) \in \{-1,+1\} , \quad k \geq 1 . \tag{11}$$

Then, we define, for $t > 1$ such that $f^k_t(c) \neq c$ for all $k \geq 1$, and $\eta > 0$,

$$\mathcal{J}_\eta(t) = \sum_{k=0}^{\infty} \frac{s_{k,t}}{|Df^k_t(c_{1,t})|^\eta} , \tag{12}$$

whenever the sum converges absolutely, and in this case we say that t satisfies the *η-summability condition.* Note that the parameter $t = 2$ (the full quadratic map)

[12] In fact all "summable" parameters, i.e. those for which $\mathcal{J}(t)$ is absolutely convergent, are transversal, see [20, Cor 1.b] and [4, Cor A.4].

satisfies $\mathcal{J}_{1/2}(2) = 0$. We expect that $t = 2$ is the only 1/2-summable parameter where the *fractional transversality condition* $\mathcal{J}_{1/2}(t) \neq 0$ fails: This is the main claim of **Conjecture B,** supported by numerics, in Section 6.

Now, if f_t is Collet–Eckmann, setting $u_t = -\rho_t(0) \cdot \sqrt{\pi}/2 \neq 0$, we put

$$\mathcal{U}_{1/2}(z) = \mathcal{U}_{1/2,t}(z) := u_t \cdot \sum_{k=0}^{\infty} \frac{s_k}{z^k \sqrt{|Df_t^k(c_1)|}} \,. \tag{13}$$

The function $\mathcal{U}_{1/2}(z)$ is holomorphic outside of the disc of radius $1/\sqrt{\lambda_c}$, with $\mathcal{U}_{1/2,t}(1) \neq 0$ if and only if $\mathcal{J}_{1/2}(t) \neq 0$. We shall also need the power series

$$\mathcal{U}^+_{1/2}(z) = \mathcal{U}^+_{1/2,t}(z) := u_t \cdot \sum_{k=0}^{\infty} \frac{1}{z^k \sqrt{|Df_t^k(c_1)|}} \,. \tag{14}$$

The function $\mathcal{U}^+_{1/2}(z)$ is holomorphic outside of the disc of radius $1/\sqrt{\lambda_c}$, with

$$\mathcal{J}^+_{1/2}(t) := \sum_{k=0}^{\infty} \frac{1}{\sqrt{Df_t^k(c_{1,t})|}} = \frac{\mathcal{U}^+_{1/2,t}(1)}{u_t} \neq 0 \,.$$

Next, following [9] (where this function was denoted σ_ϕ) we set, for $\phi \in C^0$,

$$\Sigma_\phi(z) = \Sigma_{\phi,t}(z) := \sum_{\ell=1}^{\infty} \phi(c_{\ell,t}) z^{\ell-1} \,. \tag{15}$$

($\Sigma_\phi(z)$ is holomorphic in the open unit disc. If $t \in \mathrm{MT}$, then $\Sigma_\phi(z)$ is rational.)

Recall that if $\phi : \mathbb{R} \to \mathbb{C}$ is C^0, compactly supported, and C^1 at $y \in \mathbb{R}$, the Hilbert transform of ϕ at y is defined by the Cauchy principal value (see also §2.1)

$$(\mathcal{H}\phi)(y) := \frac{1}{\pi} p.v. \int \frac{\phi(x)}{y - x} \, \mathrm{d}x \,. \tag{16}$$

Then, for ϕ a C^1 function, we define a formal power series

$$\Sigma^{\mathcal{H}}_\phi(z) = \Sigma^{\mathcal{H}}_{\phi,t}(z) := \sum_{\ell=1}^{\infty} s_{\ell,t} \cdot \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_{\ell,t}) z^{\ell-1} \,. \tag{17}$$

Finally, for $r > 0$, $q > 1$, and a bounded sequence $\tilde\psi_t(\ell)$ of functions in the Sobolev space $H^r_q[-2,2] = \{\varphi \mid \mathbf{1}_{[-2,2]} \cdot \varphi \in H^r_q(\mathbb{R})\}$, we introduce the formal power series

$$\Sigma^{\tilde\psi_t}_t(z) = \sum_{\ell=1}^{\infty} s_{\ell,t} \cdot \tilde\psi_t(\ell) z^{\ell-1} \,. \tag{18}$$

We can now state the announced complements to [iii] in Conjecture A:

Conjecture A+. *For t, Ω, and ϕ as in Conjecture A, we have*

$$\Psi^\Omega_{\phi,t}(1/2, z) = \mathcal{U}_{1/2,t}(z)\Sigma_{\phi,t}(z) + \mathcal{W}^\Omega_{\phi,1/2,t}(z) + \mathcal{V}^\Omega_{\phi,1/2,t}(z) \,,$$

with $\mathcal{V}^\Omega_{\phi,1/2,t}(z)$ holomorphic in an open annulus $\mathcal{A}$ containing $\mathcal{S}^1$. Moreover, there exist $r > 0$, $q > 1$, and functions $\tilde\psi_t(\ell) \in H^r_q[-2,2]$, with $\int_{I_t} \tilde\psi_t(\ell) \, \mathrm{d}m = 0$, such that

$$\mathcal{W}^\Omega_{\phi,1/2,t}(z) = \mathcal{U}^+_{1/2,t}(z) \big[\Sigma^{\mathcal{H}}_{\phi,t}(z) + \sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k) \cdot \Sigma^{\tilde\psi_t}_t(z) \, \mathrm{d}m\big] \,.$$

Finally, $\Sigma_t^{\tilde\psi_t}(z)$*, and, for generic* $\tilde\phi \in C^N$ *(any* $N \geq 1$*) the functions* $\Sigma_{\tilde\phi,t}(z)$ *and* $\Sigma^{\mathcal{H}}_{\tilde\phi,t}(z)$ *are holomorphic in the open unit disc and have a natural boundary on* $\mathcal{S}^1$*.*

Remark 1.2 (Approximate Susceptibility Functions). We expect that claims [i], [iii], and [iv] (but not [ii]) of Conjecture A, as well as the claims of Conjecture A+, hold for the three approximate fractional susceptibility functions $\Psi^{\rm fr}_\phi(\eta,z)$, $\Psi^{\rm rsp}_\phi(\eta,z)$, and $\Psi^{\Omega,{\rm sf}}_\phi(\eta,z)$, keeping the same functions $\mathcal{U}_{1/2}$, $\mathcal{U}^+_{1/2}$, Σ_ϕ, $\Sigma^{\mathcal{H}}_\phi$, and $\Sigma^{\tilde\psi}$, and replacing $\mathcal{W}^{\Omega,reg}_{\phi,1/2}(z)$ and $\mathcal{V}^{\Omega}_{\phi,1/2}(z)$ by suitable $\mathcal{W}^{*}_{\phi,1/2}(z)$ and $\mathcal{V}^{*}_{\phi,1/2}(z)$, for $* =$ fr, rsp, and $(\Omega, {\rm sf})$, respectively. Claim [v] for the approximate fractional susceptibility functions $\Psi^{\rm fr}_\phi(\eta,z)$ and $\Psi^{\rm rsp}_\phi(\eta,z)$ is the content of Proposition D.

1.3. **Frozen and response susceptibilities: Theorem C and Proposition D.** We move to the rigorous results. To keep this "proof of concept" paper short, we will focus on the countable subset ${\rm MT} \subset \mathcal{S}$ of Misiurewicz–Thurston (MT) parameters. This toy model setting allows us to present new ideas with the least possible technicalities. In addition, the "paradox" discussed above occurs at MT parameters [8].

We shall mostly study here an approximate fractional susceptibility function, the *frozen fractional susceptibility function* (Definition (2.2))

$$\Psi^{\rm fr}_\phi(\eta,z) = \Psi^{\rm fr}_{\phi,t}(\eta,z) = \sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k)(x) M^\eta_s(\mathcal{L}_s \rho_t(x))|_{s=t}\, {\rm d}x\,,$$

where M^η_s is the two-sided Marchaud fractional[13] derivative of order η and ϕ is C^1 and supported in $[-2,2]$.

Sedro [39] has recently proved item [i] of Conjecture A for $\Psi^{\rm fr}_\phi(\eta,z)$ for Misiurewicz parameters.

Our main rigorous result, **Theorem C,** *stated in Section 5.2, furnishes the analogue of Conjecture A+ for* $\Psi^{\rm fr}(1/2,z)$*, considering parameters* $t \in {\rm MT}$. In the MT case, the functions Σ_ϕ, $\Sigma^{\mathcal{H}}_\phi$ and $\Sigma^{\tilde\psi}$ are rational and the singularities of $\Psi^{\rm fr}_\phi(1/2,z)$ on the unit circle are simple poles. (We also expect this to hold for $\Psi^{\Omega}_{\phi,t}(1/2,z)$ if $t \in {\rm MT}$.)

We also introduce (Definition 2.3) a *response fractional susceptibility function* by taking the Marchaud derivative with respect to x

$$\Psi^{\rm rsp}_\phi(\eta,z) = \Psi^{\rm rsp}_{\phi,t}(\eta,z) = \sum_{k=0}^{\infty} z^k \int_{I_t} M^\eta_x(\phi \circ f_t^k) \cdot \rho_t \, {\rm d}x\,.$$

The response function is related to the frozen susceptibility function (Proposition 2.5) and will be used to prove Theorem C. (See [3] for a fractional response function in the piecewise expanding setting.)

Although their value at 1 is not expected to coincide with $M^{\eta,\Omega}\mathcal{R}_\phi(t)$, we believe that $\Psi^{\rm fr}_\phi(\eta,z)$ and $\Psi^{\rm rsp}_\phi(\eta,z)$ share the qualitative properties of $\Psi^{\Omega}_\phi(\eta,z)$ (Remark 1.2). Finally, recalling (2), Proposition 2.5 and Lemma 2.4 imply (see §5.1):

Proposition D. *As formal power series,*

$$\lim_{\eta\uparrow 1} \Psi^{\rm fr}_\phi(\eta,z) = \lim_{\eta\uparrow 1} \Psi^{\rm rsp}_\phi(\eta,z) = \Psi_\phi(z)\,.$$

[13]We recall definitions in §4.1. A good introduction to fractional derivatives is the book [26]. See also the short introduction [31] and the treatise [37].

1.4. **Whitney fractional integrals and derivatives: Abel's remark and the semifreddo fractional susceptibility function $\Psi_\phi^{\Omega,\mathrm{sf}}(\eta, z)$.** In §7.1 we introduce Whitney fractional integrals $I^{\eta,\Omega}$ and prove Lemma E, the analogue of Abel's remark for $I^{1/2,\Omega}$ (and suitable sets Ω satisfying (5)). In §7.2, we introduce Whitney–Marchaud derivatives $M^{\eta,\Omega}$, and use them to define the *semifreddo fractional susceptibility function* $\Psi_\phi^{\Omega,\mathrm{sf}}(\eta, z)$, a stepping-stone to the fractional susceptibility function from its frozen version. Proposition F gives conditions ensuring $\lim_{\eta\uparrow 1} M^{\eta,\Omega} g(x) = g'_\Omega(x)$, where $g'_\Omega(x)$ is the Ω-Whitney derivative of g at $x \in \Omega$, from Definition 7.4. §7.2 also contains Proposition F on $\lim_{\eta\uparrow\zeta}\big[\frac{\Gamma(1-\eta)}{\Gamma(\zeta-\eta)}(M^{\eta,\Omega}g)(x)\big]$.

2. Defining fractional susceptibility functions

2.1. **Preliminaries. Hilbert transform. Gamma and Beta functions.** We next record classical facts for further use. First, the definition (16) of the Hilbert transform can be explicited as $(\mathcal{H}\phi)(y) = -\frac{1}{\pi}\lim_{\delta\downarrow 0}\int_\delta^\infty \frac{\phi(y+u)-\phi(y-u)}{u}\,\mathrm{d}u$. If ϕ is C^1 and compactly supported then $\mathcal{H}\phi$ coincides with the following distributional derivative

$$(\mathcal{H}\phi)(y) = \frac{d}{dy}\frac{1}{\pi}\int \phi(x)\log|y-x|\,\mathrm{d}x\,,$$

and the Cauchy principal value corresponds to integration by parts, since

$$\frac{d}{dy}\frac{1}{\pi}\int \phi(x)\log|y-x|\,\mathrm{d}x = \frac{d}{dy}\frac{1}{\pi}\int \phi(y-u)\log|u|\,\mathrm{d}u = \frac{1}{\pi}\int \phi'(x)\log|y-x|\,\mathrm{d}x.$$

Note that there exists $C<\infty$ such that for any compact interval J

$$|\mathcal{H}(\mathbf{1}_J\phi)(x)| \le C|J|\sup|\phi'|\,, \forall x \in \mathrm{int}(J)\,.$$

Euler's Gamma function is $\Gamma(\eta) = \int_0^\infty x^{\eta-1}e^{-x}\,\mathrm{d}x$ (recall that it has simple poles at $\eta = 0, -1, -2, \ldots$). The Beta function is defined for $\Re x > 0$ and $\Re y > 0$ by

$$B(x,y) = \int_0^1 u^{x-1}(1-u)^{y-1}\,\mathrm{d}u\,.$$

It satisfies $\Gamma(x)\Gamma(y) = B(x,y)\Gamma(x+y)$. Since $\Gamma(3/2) = \sqrt{\pi}/2$, $\Gamma(1) = \Gamma(2) = 1$, and $\Gamma(1/2) = \sqrt{\pi}$, we have $B(1/2,1/2) = \pi$ and $B(1/2,3/2) = \pi/2$. Recall also that $\sin(\pi/4) = \cos(\pi/4) = \sqrt{2}/2$.

2.2. **Susceptibility functions $\Psi_\phi^\Omega(\eta,z)$, $\Psi_\phi^{\mathrm{fr}}(\eta,z)$, $\Psi_\phi^{\mathrm{rsp}}(\eta,z)$. Proposition D.** We first motivate heuristically our definition of the fractional susceptibility function $\Psi_\phi^\Omega(\eta,z)$. The starting point is the right-hand side of (7) in [ii] from Conjecture A, i.e. the Marchaud derivative of $\mathcal{R}_\phi(t)$. Our first task is to rewrite

$$\mathcal{R}_\phi(s) - \mathcal{R}_\phi(t) = \int \phi\rho_s \mathrm{d}m - \int \phi\rho_t \mathrm{d}m$$

along the lines of [3]: If s belongs to a suitable subset of Ω of CE, then for every $r>0$, and $q>1$ there exists $\kappa<1$ such that for any bounded function ϕ supported in $[-2,2]$ and any $\psi \in H^r_q[-2,2]$ with $\int_{I_t}\psi\,\mathrm{d}m = 0$, there exists $C_{\phi,\psi}$ such that

$$|\int \phi\,\mathcal{L}_s^k(\psi)\,\mathrm{d}m| = |\int (\phi\circ f_s^k)\,\psi\,\mathrm{d}m| \le C_{\phi,\psi}\kappa^k\,,\quad \forall k\ge 1\,.$$

In particular, if ϕ is supported in I_t,

$$\int \phi\,(\mathrm{id}-\mathcal{L}_s)^{-1}(\psi)\,\mathrm{d}m=\sum_{k=0}^{\infty}\int \phi\mathcal{L}_s^k(\psi)\,\mathrm{d}m=\sum_{k=0}^{\infty}\int (\phi\circ f_s^k)\,\psi\,\mathrm{d}m\,. \tag{19}$$

If t also belongs to Ω, the fixed point property $\mathcal{L}_\tau\rho_\tau=\rho_\tau$ for $\tau=s,t$ implies

$$\int \phi\,(\mathrm{id}-\mathcal{L}_s)(\rho_s-\rho_t)\mathrm{d}m=\int \phi\,(\mathcal{L}_s-\mathcal{L}_t)\rho_t\,\mathrm{d}m\,.$$

Since $\int(\mathcal{L}_s-\mathcal{L}_t)\rho_t\,\mathrm{d}m=0$ (using that $f_t([c_{2,t},c_{1,t}])=[c_{2,t},c_{1,t}]\subset I_s$) if $|t-s|$ is small enough, we would like to multiply the factor of ϕ in both sides by $(\mathrm{id}-\mathcal{L}_s)^{-1}$ to recover $\mathcal{R}_\phi(s)-\mathcal{R}_\phi(t)$ and then attempt to implement the "recipe" in §4.1 for the Marchaud derivative. Writing $(\mathrm{id}-z\mathcal{L}_s)^{-1}=\sum_{k=0}^{\infty}z^k\mathcal{L}_s^k$, and using (19), this motivates our definition for the fractional susceptibility function:

Definition 2.1 (Ω-Whitney–Marchaud fractional susceptibility function)**.** For $t\in$ TSR and $\epsilon>0$ as in (6), let $\Omega\subset$ TSR have t as a Lebesgue density point. For $\Re\eta\in(0,1)$, the (Whitney–Marchaud) fractional susceptibility function $\Psi_\phi^\Omega(\eta,z)=\Psi_{\phi,t,\epsilon}^\Omega(\eta,z)$ (of the quadratic family, along Ω at t, for the observable $\phi\in C^1$) is the formal power series in z

$$\Psi_\phi^\Omega(\eta,z):=\frac{\eta}{2\Gamma(1-\eta)}\sum_{k=0}^{\infty}z^k\int\int_{\mathbb{R}\cap(\Omega-t)}\phi(f_{t+\delta}^k(x))\cdot \\ \cdot\frac{(\mathcal{L}_{t+\delta}-\mathcal{L}_t)\rho_t(x)}{|\delta|^{1+\eta}}\mathrm{sgn}(\delta)\,\mathrm{d}\delta\,\mathrm{d}x\,. \tag{20}$$

(The choice of ϵ implies that $x\mapsto(\mathcal{L}_{t+\delta}-\mathcal{L}_t)\rho_t(x)$ is supported in $I_{t,\epsilon}\subset I_t$.)

The coefficient of z^k in the power series (20) is a sum of improper integrals, for $\delta\in(-\infty,0)$ and $\delta\in(0,\infty)$. For each fixed $k\geq 1$, every δ such that $t+\delta\in\Omega$, and every $\psi_\delta\in L^1$ (and ϕ) supported in $I_{t,\epsilon}$, we have, since $I_{t,\epsilon}\subset I_t$,

$$z^k\int_{I_t}(\phi\circ f_{t+\delta}^k)(x)\cdot\psi_\delta(x)\,\mathrm{d}x=\int_{I_t}\phi(x)z^k(\mathcal{L}_{t+\delta}^k\psi_\delta)(x)\,\mathrm{d}x\,. \tag{21}$$

The presence of $(\mathrm{id}-z\mathcal{L}_{t+\delta})^{-1}$ in (21) is the reason we restrict the integral to good parameters $t+\delta\in\Omega$ (see also Appendix C).

In the present work, we mostly study the *frozen fractional susceptibility function:*

Definition 2.2 (Frozen susceptibility function)**.** Let t be a TSR parameter and choose $\epsilon>0$ as in (6). For $\eta\in(0,1)$ the frozen susceptibility function $\Psi_\phi^{\mathrm{fr}}(\eta,z)=\Psi_{\phi,t,\epsilon}^{\mathrm{fr}}(\eta,z)$ (of the quadratic family, at t for the observable $\phi\in C^1$) is the formal power series[14]

$$\Psi_\phi^{\mathrm{fr}}(\eta,z)=\sum_{k=0}^{\infty}z^k\int_{I_t}(\phi\circ f_t^k)(x)M_s^\eta(\mathcal{L}_s\rho_t(x))|_{s=t}\,\mathrm{d}x\,, \tag{22}$$

where M_t^η is the two-sided Marchaud fractional derivative of order η, in the parameter t, in the sense of distributions of order one (Definition 4.1). In other words, for fixed η, we have, as a formal power series in z,

$$\Psi_\phi^{\mathrm{fr}}(\eta,z)=\frac{\eta}{2\Gamma(1-\eta)}\sum_{k=0}^{\infty}z^k\int_{I_t}(\phi\circ f_t^k)(x)$$

[14]Recalling (6), the function $x\mapsto M_s^\eta(\mathcal{L}_s\rho_t(x))|_{s=t}$ is supported in $I_{t,\epsilon}\subset I_t$.

$$\cdot \lim_{\epsilon\to 0}\int_{|t|>\epsilon} \frac{\big((\mathcal{L}_{t+\delta}-\mathcal{L}_t)\rho_t\big)(x)}{|\delta|^{1+\eta}}\mathrm{sgn}(\delta)\,\mathrm{d}\delta\,\mathrm{d}x\,,$$

where the integral over $\mathrm{d}t$ is viewed as a distribution of order one.

Applying (21) to each term of (22), we find (formally)

$$\Psi^{\mathrm{fr}}_\phi(\eta,z)=\int_{I_t}\phi\,(\mathrm{id}-z\mathcal{L}_t)^{-1}(M^\eta_s(\mathcal{L}_s\rho_t(x))|_{s=t})\,\mathrm{d}x\,,$$

In Section 5, we shall prove Theorem C on the frozen susceptibility function for $\eta=1/2$ and Misiurewicz–Thurston parameters t.

Formulas for fractional response are not as neat as for linear response, since the usual Leibniz and chain rules are replaced by infinite expansions in the case of fractional derivatives. (See Eq. 2.209 in Section 2.7.3 of [32] for the chain rule. For the Leibniz formula, see §15 in [37].) However, we shall see in Proposition 2.5 that a simplification occurs for the frozen susceptibility function. This motivates the definition of a *response fractional susceptibility function:*

Definition 2.3 (Response susceptibility function)**.** For $\eta\in(0,1)$ and $\phi\in C^1$ is compactly supported, the response susceptibility function is defined by the following formal power series

$$\Psi^{\mathrm{rsp}}_\phi(\eta,z):=\sum_{k=0}^\infty z^k\int_{I_t}M^\eta_x(\phi\circ f^k_t)\cdot X_t\rho_t\,\mathrm{d}m=\sum_{k=0}^\infty z^k\int_{I_t}M^\eta_x(\phi\circ f^k_t)\cdot\rho_t\,\mathrm{d}m\,.$$

If $\eta\in(0,1/2)$, then

$$\Psi^{\mathrm{rsp}}_\phi(\eta,z)=-\sum_{k=0}^\infty z^k\int(\phi\circ f^k_t)\cdot M^\eta_x\big(\rho_t\big)\,\mathrm{d}x \tag{23}$$

follows from integration by parts for the Marchaud derivative[15] [37, (6.27)]. We will see in Lemma 5.2 that (23) in fact holds for all $\eta\in(0,1)$, up to taking the Marchaud derivative of ρ_t in the sense of distributions.

In the limit as $\eta\to 1$ the following easy lemma shows that the response susceptibility function converges to the Ruelle susceptibility function:

Lemma 2.4 (Ruelle susceptibility as a limit of response susceptibilities)**.** *Fix $t\in\mathcal{S}$ and a compactly supported $\phi\in C^1$, and let $\Psi_\phi(z)$ be Ruelle's susceptibility function* (2)*. Then, as formal power series in z,*

$$\lim_{\eta\uparrow 1}\Psi^{\mathrm{rsp}}_\phi(\eta,z)=\Psi_\phi(z)\,.$$

The proof of Lemma 2.4 does not use that $f_{t+\tau}(x)=f_t(x)+\tau$.

Proof of Lemma 2.4. Apply $\lim_{\eta\to1}M^\eta g=g'$ (e.g. [3]) to $g=\phi\circ f^k_t\in C^1$. □

Finally, using Lemma 5.2, we give the easy proof of the following remarkable result in §5.1 (the identity (25) greatly simplifies the proof of our main result on the frozen susceptibility function, Theorem C, for more general smooth unimodal maps it seems there is no way to bypass the study of $M^{1/2}_s(\mathcal{L}_s\rho_t)$):

[15] Use that ϕ and ρ_t are compactly supported while, on the one hand, we have $M^\eta_x(\phi\circ f^k_t)\in L^p_{loc}$ for all $p\ge1$, while $\phi\circ f^k_t\in L^s$ for all $s\ge1$, and, on the other hand, we have $M^\eta(\rho_t)\in L^r_{loc}$ for [39] any $1\le r<2(1+2\eta)^{-1}$, while $\rho_t\in L^{\tilde r}$ for all $1\le\tilde r<2$.

Proposition 2.5 (Relating the frozen and response susceptibility functions)**.** *For any*[16] *mixing* $t \in \mathrm{MT}$ *and* $\eta \in (0,1)$*, we have, as distributions of order one,*

$$M_s^\eta(\mathcal{L}_s \rho_t(x))|_{s=t} = -M_x^\eta \rho_t(x) + \frac{g_\eta(x)}{\Gamma(1-\eta)}\,, \tag{24}$$

where $g_\eta \in H_q^r$ *for some* $r > 0$ *and* $q > 1$*, with* $\sup_{\eta > \epsilon_1} \|g_\eta\|_{H_q^r} < \infty$ *for any fixed* $\epsilon_1 > 0$*, and* $\int_{\mathbb{R}} g_\eta(x) \mathrm{d}m = 0$*.*

In addition, there exists $\kappa < 1$ *and for any compactly supported* $\phi \in C^1$*, there exists* $\mathcal{V}^{\mathrm{rsp}}_{\phi,\eta}(z) = \sum_{j \geq 0} v_j z^j$ *holomorphic in the disc of radius* κ^{-1} *such that*

$$\Psi^{\mathrm{fr}}_\phi(\eta, z) - \mathcal{V}^{\mathrm{rsp}}_{\phi,\eta}(z) = \Psi^{\mathrm{rsp}}_\phi(\eta, z) \text{ as formal power series in } z. \tag{25}$$

Finally, we have, as formal power series, $\lim_{\eta \uparrow 1} \Psi^{\mathrm{fr}}_\phi(\eta, z) = \Psi_\phi(z)$*.*

Proposition 2.5 and Lemma 2.4 imply Proposition D: both the response and the frozen fractional susceptibility functions converge to the Ruelle susceptibility function as $\eta \to 1$. (However Ψ^{rsp}_ϕ and Ψ^{fr}_ϕ do not satisfy [ii] from Conjecture A.)

3. Half integrals of square root spikes

After recalling the definitions of Riemann–Liouville fractional integrals, we revisit Abel's computation of the one-sided half-integral of a square root spike and extend it to the two-sided half-integral. The corresponding statements, Lemma 3.1 and Lemma 3.2, will be used in Section 4 to compute Marchaud derivatives.

3.1. Riesz potentials and Riemann–Liouville fractional integrals.

For any $\phi \in L^1$ and for $\eta \in (0,1)$, the Riesz potential fractional integral is defined for $\Re\eta > 0$, $\eta \neq 1, 3, 5, \dots$ by (see [37, (5.2)–(5.3), §12.1])

$$I^\eta \phi(t) = \frac{1}{2\Gamma(\eta)\cos(\eta\pi/2)} \int_{-\infty}^{\infty} \frac{\phi(\tau)}{|t-\tau|^{1-\eta}} \,\mathrm{d}\tau = \frac{I_+^\eta \phi(t) + I_-^\eta \phi(t)}{2\cos(\eta\pi/2)}\,, \tag{26}$$

where $I_\pm^\eta$ are the left- and right-sided Riemann–Liouville fractional integrals [37, (5.2)–(5.3)] (there is a typo in the second line of [37, (5.4)])

$$I_+^\eta \phi(t) = \frac{1}{\Gamma(\eta)} \int_{-\infty}^{t} \frac{\phi(\tau)}{(t-\tau)^{1-\eta}} \,\mathrm{d}\tau = \frac{1}{\Gamma(\eta)} \int_0^\infty \frac{\phi(t-y)}{y^{1-\eta}} \,\mathrm{d}y\,,$$
$$I_-^\eta \phi(t) = \frac{1}{\Gamma(\eta)} \int_{t}^{\infty} \frac{\phi(\tau)}{(\tau-t)^{1-\eta}} d\tau = \frac{1}{\Gamma(\eta)} \int_0^\infty \frac{\phi(t+y)}{y^{1-\eta}} \,\mathrm{d}y\,.$$

If $g_t(x)$ is a function of two variables x and t, we write $(I_t^\eta g_t)(x)$ to denote the fractional integral acting on the parameter t and evaluated at x and t, and similarly for the one-sided integrals $I^\eta_{-,t}$ and $I^\eta_{+,t}$.

Note for further use that, setting $Q\phi(t) = \phi(-t)$, $T_a\phi(t) = \phi(t+a)$, we have

$$I_\sigma^\eta \circ Q = Q \circ I_{-\sigma}^\eta\,, \quad I_\sigma^\eta \circ T_a = T_a \circ I_\sigma^\eta\,. \tag{27}$$

The case which will interest us most is $\eta = 1/2$, that is, "half Riesz potential integrals" or "half Riemann–Liouville integrals." In §3.2, we recall the proof of a key observation of Abel regarding the ordinary one-sided half-Riemann–Liouville integral of square root spikes, and we present its two-sided version, Lemma 3.2. (Lemma 3.2 will be a key ingredient to prove our main result in Section 5.)

[16]The proof shows that the proposition holds more generally, for example for mixing TSR parameters.

3.2. **Abel's remark: One-sided half integration of square-root spikes.** In this section, we recall a result of Abel on one-sided half integrals (Lemma 3.1) and extend it to two-sided half integrals (Lemma 3.2). The corresponding results will be used to prove Lemma 4.4 below about the half Marchaud derivative of a spike.

The following fact was probably first observed by Abel [1, 2] (see also [33]):

Lemma 3.1 (Abel's remark)**.** *Fix $k \ge 1$ and $\sigma \in \{-1, +1\}$. Consider the left and right square-root spikes (in x) at $c_k + t$*

$$\phi_{c_k,\sigma}(x,t) = (|x - c_k - t|)^{-1/2} \mathbf{1}_{\sigma x > \sigma(c_k+t)}\,, \quad x, t \in \mathbb{R}\,. \tag{28}$$

Then the one-sided Riemann–Liouville half integrals $I_\pm^{1/2}(\phi_{c_k,\mp})$ (with respect to t) are the following Heaviside jumps (in x) at $c_k + t$:

$$I_{-,t}^{1/2}(\phi_{c_k,+})(x,t) = \sqrt{\pi} \cdot \mathbf{1}_{x > c_k+t}(x)\,,\ I_{+,t}^{1/2}(\phi_{c_k,-})(x,t) = \sqrt{\pi} \cdot \mathbf{1}_{x < c_k+t}(x)\,.$$

Proof of Lemma 3.1. The half integral $I_-^{1/2,t}$ of $\phi_{c_k,+}(x,t)$ with respect to t is

$$\begin{aligned}(I_-^{1/2,t}\phi_{c_k,+})(x) &= \frac{1}{\Gamma(1/2)} \int_t^{+\infty} \frac{\phi_{c_k,+}(x,\tau)}{(\tau - t)^{1/2}}\,\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1/2)} \int_t^{+\infty} \frac{(x - c_k - \tau)^{-1/2}\mathbf{1}_{y > c_k+\tau}(x)}{(\tau - t)^{1/2}}\,\mathrm{d}\tau \\ &= \begin{cases} 0 & \text{if } c_k + t \ge x, \\ \frac{1}{\Gamma(1/2)} \int_t^{x-c_k} \frac{1}{((\tau-t)(x-c_k-\tau))^{1/2}}\,\mathrm{d}\tau & \text{if } c_k + t < x\,. \end{cases}\end{aligned}$$

If $c_k + t < x$, making the substitution $\tau = t + (x - c_k - t)u$, we get

$$\int_t^{x-c_k} \frac{1}{((\tau - t)(x - c_k - \tau))^{1/2}}\,\mathrm{d}\tau = \int_0^1 \frac{1}{(u(1-u))^{1/2}}\,\mathrm{d}u = B(1/2, 1/2)\,. \tag{29}$$

Recalling $B(1/2, 1/2) = \pi$ and $\Gamma(1/2) = \sqrt{\pi}$, we find

$$(I_{-,t}^{1/2}\phi_{c_k,+})(x,t) = \begin{cases} 0 & \text{if } c_k + t \ge x\,, \\ \sqrt{\pi} & \text{if } c_k + t < x\,. \end{cases}.$$

The other claim follows from (27) since

$$\phi_{c_k,-}(x,t) = \phi_{c_k,+}(x, 2(x - c_k) - t) = Q \circ T_{2(x-c_k)}(\phi_{c_k,+})(x,t)\,.$$

Indeed, we find

$$\begin{aligned}I_{+,t}^{1/2}\phi_{c_k,-}(x,t) &= I_{+,t}^{1/2} \circ Q \circ T_{2(x-c_k)}(\phi_{c_k,+})(x,t) \\ &= I_{-,t}^{1/2} \circ T_{2(x-c_k)}(\phi_{c_k,+})(x,-t) = I_{-,t}^{1/2}\phi_{c_k,+}(x, -t + 2(x - c_k))\,.\end{aligned}$$

Finally, $x > c_k - t + 2(x - c_k)$ if and only if $x < c_k + t$. □

Replacing the one-sided Riemann–Liouville fractional integral $I_\pm^\eta$ by the (two-sided) Riesz potential I^η from (26), Lemma 3.1 must be replaced by the following lemma, which includes an unbounded logarithm corresponding to the "other side."

Lemma 3.2 (Two-sided version of Abel's remark)**.** *For any real number $\mathcal{Z} > 1$, any integer $k \ge 1$ and any $x \in I$, the one-sided Riemann–Liouville half integrals of the $\mathcal{Z}$-truncated right and left square-root spikes*

$$\phi_{c_k,+,\mathcal{Z}}(x,t) = \frac{\mathbf{1}_{(c_k+t, c_k+t+\mathcal{Z})}(x)}{(x - c_k - t)^{1/2}}\,,\ \phi_{c_k,-,\mathcal{Z}}(x,t) = \frac{\mathbf{1}_{(c_k+t-\mathcal{Z}, c_k+t)}(x)}{(c_k + t - x)^{1/2}}\,, \tag{30}$$

satisfy, for $\sigma = \pm$ *and any* $|x - c_k - t| < \mathcal{Z}/2$,

$$I_\sigma^{1/2}(\phi_{c_k,\sigma,\mathcal{Z}})(x,t) = \frac{\sigma}{\sqrt{\pi}}\big(-\log|x-c_k-t| + \log\mathcal{Z} + G_{\mathcal{Z}}(\sigma(t-x+c_k))\big)\,,$$

where $G_{\mathcal{Z}}(y)$ *is analytic on* $|y| < \mathcal{Z}/2$, *with* $\lim_{\mathcal{Z}\to\infty}\sup_{|y|<\mathcal{Z}/2}|\partial_y G_{\mathcal{Z}}(y)| = 0$, *and*

$$\sup_{\mathcal{Z}>1}\sup_{|y|<\mathcal{Z}/2}\max\{|G_{\mathcal{Z}}(y)|, |\partial_y G_{\mathcal{Z}}(y)|, |\partial_y^2 G_{\mathcal{Z}}(y)|\} < \infty\,.$$

The elementary proof of the above crucial lemma (which will be used to prove Lemmas 4.4 and 4.5) is given in Appendix A.

Finally, the remark below will be used several times in the sequel:

Remark 3.3 (Phase and parameter half-integrals of a spike). Since $x > c_k + t - u$ if and only if $t < x + u - c_k$, we have for any $1 < \mathcal{Z} \le \infty$, recalling (30),

$$I_{-,t}^{1/2}(\phi_{c_k,+,\mathcal{Z}})(x,t) = I_{+,x}^{1/2}(\phi_{c_k,+,\mathcal{Z}})(x,t)\,,\ I_{+,t}^{1/2}(\phi_{c_k,-,\mathcal{Z}})(x,t) = I_{-,x}^{1/2}(\phi_{c_k,-,\mathcal{Z}})(x,t)\,,$$

and for any $1 < \mathcal{Z} < \infty$

$$I_{+,t}^{1/2}(\phi_{c_k,+,\mathcal{Z}})(x,t) = I_{-,x}^{1/2}(\phi_{c_k,+,\mathcal{Z}})(x,t)\,,\ I_{-,t}^{1/2}(\phi_{c_k,-,\mathcal{Z}})(x,t) = I_{+,x}^{1/2}(\phi_{c_k,-,\mathcal{Z}})(x,t)\,.$$

4. Marchaud derivatives applied to spikes and square roots

After recalling the definition of Marchaud derivatives M^η and extending them as distributions in §4.1, we show in §4.2 how $M^{1/2}$ acts on the singular components (spikes and square roots) of the invariant density ρ_t. The lemmas in this section will be crucial to prove Theorem C in Section 5.

4.1. One-sided and two-sided Marchaud derivatives $M_\pm^\eta$ and M^η.

Let $g : \mathbb{R} \to \mathbb{C}$ be bounded and γ-Hölder. We recall that the left-sided Marchaud fractional derivative (with lower limit $a = -\infty$) [37, pp. 110–111, Theorem 5.9, p. 225], where it is denoted by $\mathbf{D}_+^\eta$, see also [16, §2.2.2.3] is defined for $\eta \in (0,\gamma)$ and $x \in \mathbb{R}$, by

$$\begin{aligned}(M_+^\eta g)(x) &= \frac{\eta}{\Gamma(1-\eta)}\int_{-\infty}^x \frac{g(x)-g(y)}{(x-y)^{1+\eta}}\,\mathrm{d}y = \frac{\eta}{\Gamma(1-\eta)}\int_{-\infty}^0 \frac{g(x)-g(x+\tau)}{|\tau|^{1+\eta}}\,\mathrm{d}\tau \\ &= \frac{\eta}{\Gamma(1-\eta)}\int_0^\infty \frac{g(x)-g(x-\tau)}{\tau^{1+\eta}}\,\mathrm{d}\tau\,. \qquad (31)\end{aligned}$$

If g is bounded on $\mathbb{R}$ and differentiable[17] at x, the limit as $\eta \uparrow 1$ of $M_+^\eta(g)(x)$ is equal to the ordinary derivative $g'(x)$ (see e.g. [31, §3.2] or [3]).

The integral (31) is an improper integral. In the application of this paper, $g(t)$ will be bounded as $t \to \pm\infty$, so[18] the only delicate limit is $\tau \to 0$. Concretely, we will work with the expression (see [37, (5.59–5.60)])

$$(M_+^\eta g)(x) = \lim_{\epsilon\uparrow 0}(M_{+,\epsilon}^\eta g)(x) := \lim_{\epsilon\uparrow 0}\frac{\eta}{\Gamma(1-\eta)}\int_{-\infty}^{\epsilon}\frac{g(x)-g(x+\tau)}{|\tau|^{1+\eta}}\,\mathrm{d}\tau\,.$$

The right-sided Marchaud fractional derivative (with upper limit $b = +\infty$) is defined for $\eta \in (0,1)$ and $x \in \mathbb{R}$ by

$$M_-^\eta g(x) = \frac{\eta}{\Gamma(1-\eta)}\int_0^\infty \frac{g(x)-g(x+\tau)}{\tau^{1+\eta}}\,\mathrm{d}\tau$$

[17] If g is bounded and differentiable to the left at x, the limit as $\eta \uparrow 1$ of $M_+^\eta(g)(x)$ is equal to the left-sided derivative $g'_-(x)$, the notation is thus confusing.

[18] This is an advantage of Marchaud derivatives over Riemann–Liouville fractional derivatives.

$$= \lim_{\epsilon\downarrow 0}(M^\eta_{-,\epsilon} g)(x) = \lim_{\epsilon\downarrow 0} \frac{\eta}{\Gamma(1-\eta)} \int_\epsilon^\infty \frac{g(x)-g(x+\tau)}{\tau^{1+\eta}} \,\mathrm{d}\tau\,.$$

If g is bounded on $\mathbb{R}$ and differentiable at x (differentiable to the right is enough), then $\lim_{\eta\uparrow 1} M^\eta_-(g)(x) = -g'(x)$ (see e.g. [3]).

We define the two-sided Marchaud derivative by

$$M^\eta g(x) = \frac{M^\eta_+ g(x) - M^\eta_- g(x)}{2}\,.$$

Note that $M^\eta g(x) = \lim_{\epsilon\downarrow 0} M^\eta_\epsilon g(x)$ where

$$M^\eta_\epsilon g(x) = \frac{\eta}{2\Gamma(1-\eta)} \int_{|\tau|>\epsilon} \frac{g(x+\tau)-g(x)}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau)\,\mathrm{d}\tau\,. \tag{32}$$

Note for further use that, recalling $Qg(t) = g(-t)$, $T_a g(t) = g(t+a)$, we have

$$M^\eta_\sigma \circ Q = Q \circ M^\eta_{-\sigma}\,, \quad M^\eta_\sigma \circ T_a = T_a \circ M^\eta_\sigma\,,\ \sigma = \pm\,. \tag{33}$$

Therefore,

$$M^\eta \circ Q = -Q \circ M^\eta\,, \quad M^\eta \circ T_a = T_a \circ M^\eta\,. \tag{34}$$

We shall sometimes need to consider $M^\eta g$ (if g is not Hölder, for example) in the *sense of distributions (of order one):*

Definition 4.1 (Marchaud derivative in the sense of distributions of order one)**.** For $\eta \in (0,1)$ and a measurable function g such that the integral $G(y) = \int_{-\infty}^y g(u)\,\mathrm{d}u$ is well-defined and almost everywhere finite, with[19]

$$\lim_{\epsilon\to 0} M^\eta_\epsilon G(x) \in L^1_{loc}\,,$$

we define the two-sided Marchaud derivative of g in the sense of distributions of order one by setting, for any compactly supported C^1 function ψ,

$$\int (M^\eta g)(x)\psi(x)\,\mathrm{d}x := -\int \left[\lim_{\epsilon\to 0} M^\eta_\epsilon G(x)\right] \psi'(x)\,\mathrm{d}x\,. \tag{35}$$

The one-sided Marchaud derivatives M^η_- and M^η_+ in the sense of distributions are defined analogously (for M^η_-, it is convenient to set $G(y) = -\int_y^\infty g(u)\,\mathrm{d}u$.).

Note that (34) and (33) extend to the setting of Definition 4.1.

If $g_t(x)$ is a function of two variables x and t, then $(M^\eta_t g_t)(x)$ or $(M^\eta_s g_s)(x)|_{s=t}$ denote the Marchaud derivative acting on the parameter t and evaluated at x and t, and similarly for the one-sided derivatives $M^\eta_{-,t}$ and $M^\eta_{+,t}$.

Remark 4.2 (Marchaud in the sense of distributions)*.* If $g \in C^1$ is compactly supported then the definition (35) is in fact an identity which can be deduced from Fubini, Lebesgue dominated convergence, and integration by parts (for C^1 compactly supported ψ). Let us write the computation in the one-sided case:

$$\int (M^\eta_+ g)(x)\psi(x)\,\mathrm{d}x = \int \left[\lim_{\epsilon\uparrow 0} \frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^{\epsilon} \frac{g(x)-g(x+\tau)}{|\tau|^{1+\eta}}\,\mathrm{d}\tau\right] \psi(x)\,\mathrm{d}x$$
$$= \frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^0 \int \frac{g(x)-g(x+\tau)}{|\tau|^{1+\eta}}\,\psi(x)\,\mathrm{d}x\,\mathrm{d}\tau$$

[19]One could weaken this condition, up to exchanging the limit and the derivative in (35). We shall not need this more general notion.

$$= -\frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^{0} \int \frac{G(x) - G(x+\tau)}{|\tau|^{1+\eta}} \psi'(x) \, \mathrm{d}x \, \mathrm{d}\tau$$
$$= -\int \left[\lim_{\epsilon \uparrow 0} \frac{\eta}{\Gamma(1-\eta)} \int_{-\infty}^{\epsilon} \frac{G(x) - G(x+\tau)}{|\tau|^{1+\eta}} \mathrm{d}\tau \right] \psi'(x) \, \mathrm{d}x \, .$$

Remark 4.3 (Marchaud and Riemann–Liouville). If g is C^1 and and $|g'(\tau)| = O(|\tau|^{\eta-1-\epsilon})$ for some $\epsilon > 0$ as $\tau \to -\infty$ ([37, pp. 109–110]) then the left-sided Marchaud derivative of g coincides with the left-sided Riemann–Liouville derivative with lower limit $a = -\infty$ of g

$$(M_+^\eta g)(t) = \frac{d}{dt} I_+^{1-\eta}(g)(t) = \frac{1}{\Gamma(1-\eta)} \frac{d}{dt} \int_{-\infty}^{t} \frac{g(\tau)}{(t-\tau)^\eta} \mathrm{d}\tau \, .$$

Similarly, the right-sided Marchaud derivative of g coincides with the right-sided Riemann–Liouville derivative with upper limit $a = \infty$ of g

$$(M_-^\eta g)(t) = -\frac{d}{dt} I_-^{1-\eta}(g)(t) = \frac{-1}{\Gamma(1-\eta)} \frac{d}{dt} \int_{t}^{\infty} \frac{g(\tau)}{(\tau - t)^\eta} \mathrm{d}\tau \, .$$

The remark above will be used in the proof of Lemma 5.2. (Note that Lemma 4.4 is a generalisation of this remark, for g a one-sided spike and $\eta = 1/2$.)

4.2. **The half derivative $M^{1/2}$ of spikes, square roots, and C^1 functions.** The key fact we shall use is the following lemma about Marchaud derivatives (in the sense (35) of distributions) of spikes and truncated spikes, for $\mathcal{Z} > 1$,

$$\phi_{x_0,\sigma}(x) = \frac{\mathbf{1}_{\sigma x > \sigma x_0}}{\sqrt{|x - x_0|}} \, , \qquad \phi_{x_0,\sigma,\mathcal{Z}}(x) = \mathbf{1}_{0 < \sigma(x - x_0) < \mathcal{Z}} \cdot \phi_{x_0,\sigma}(x) \, .$$

Lemma 4.4 (Half Marchaud derivatives of a spike)**.** *For $x_0 \in \mathbb{R}$ and $\sigma = \pm$, the following holds: The one-sided half Marchaud derivatives satisfy, as distributions on continuous compactly supported functions,*

$$M_+^{1/2}(\phi_{x_0,+})(x) = \sqrt{\pi} \cdot \delta_{x_0} \, , \quad M_-^{1/2}(\phi_{x_0,-})(x) = \sqrt{\pi} \cdot \delta_{x_0} \, .$$

The two-sided half Marchaud derivative satisfies, as a distribution on C^1 compactly supported functions,

$$M^{1/2}(\phi_{x_0,\sigma})(x) = \frac{\sigma}{2\sqrt{\pi}} \cdot \left(\pi \delta_{x_0} - \frac{1}{x - x_0} \right) .$$

Finally, for any $\mathcal{Z} > 1$, the two-sided half Marchaud derivative satisfies, as a distribution on C^1 functions supported in $[x_0 - \mathcal{Z}/2, x_0 + \mathcal{Z}/2]$,

$$M^{1/2}(\phi_{x_0,\sigma,\mathcal{Z}})(x) = \frac{\sigma}{2\sqrt{\pi}} \cdot \left(\pi \delta_{x_0} - \frac{1}{x - x_0} + \Phi_{\mathcal{Z}}(\sigma(x_0 - x)) \right) ,$$

where $\Phi_{\mathcal{Z}}(y)$ is analytic on $|y| < \mathcal{Z}/2$, with $\lim_{\mathcal{Z} \to \infty} \sup_{|y| < \mathcal{Z}/2} |\Phi_{\mathcal{Z}}(y)| = 0$, and

$$\sup_{\mathcal{Z} > 1} \sup_{|y| < \mathcal{Z}/2} \max\{|\Phi_{\mathcal{Z}}(y)|, |\partial_y \Phi_{\mathcal{Z}}(y)|\} < \infty \, .$$

In view of the expansion (50) for the invariant density, we also need Marchaud derivatives of square roots and truncated square roots, defined for $\mathcal{Z} > 1$ by,

$$\bar{\phi}_{x_0,+,\mathcal{Z}}(x) = \mathbf{1}_{x_0 < x < x_0 + \mathcal{Z}} \cdot \sqrt{x - x_0} \, , \;\; \bar{\phi}_{x_0,-,\mathcal{Z}}(x) = \mathbf{1}_{x_0 - \mathcal{Z} < x < x_0} \cdot \sqrt{x_0 - x} \, .$$

Lemma 4.5 (Half Marchaud derivatives of a square root)**.** *Let $x_0 \in \mathbb{R}$. The one-sided Marchaud derivatives of square roots satisfy, for $\sigma = \pm$,*

$$M_\sigma^{1/2}\big(\mathbf{1}_{\sigma x > \sigma x_0}(\sqrt{|x-x_0|})\big) = \frac{\sqrt{\pi}}{2}\mathbf{1}_{\sigma x > \sigma x_0}(x)\,.$$

For $\mathcal{Z} > 1$, the two-sided Marchaud derivatives of truncated square roots satisfy

$$M^{1/2}(\bar\phi_{x_0,\sigma,\mathcal{Z}})(x) = \frac{\sigma}{2\sqrt{\pi}}\big(\pi\mathbf{1}_{\sigma x>\sigma x_0}(x) - \log|x-x_0| + \log\mathcal{Z}\big) + \bar\Phi_{\mathcal{Z}}(\sigma(x_0-x))\,,$$

where $\bar\Phi_{\mathcal{Z}}(y)$ is analytic on $|y| < \mathcal{Z}/2$, with $\lim_{\mathcal{Z}\to\infty}\sup_{|y|<\mathcal{Z}/2}|\bar\Phi_{\mathcal{Z}}(y)| = 0$, and

$$\sup_{\mathcal{Z}>1}\sup_{|y|<\mathcal{Z}/2}\max\{|\bar\Phi_{\mathcal{Z}}(y)|, |\partial_y\bar\Phi_{\mathcal{Z}}(y)|\} < \infty\,.$$

Lemma 4.6 (Action of Marchaud derivatives on C^1 functions)**.** *For any $\eta \in (0,1)$ and any C^1 function $g : \mathbb{R} \to \mathbb{R}$ with $\sup_{\mathbb{R}}|g'| < \infty$, the two-sided Marchaud derivative $M^\eta(g)$ is $(1-\eta)$-Hölder.*

Proof of Lemma 4.4. To show the claim on $M_+^{1/2}(\phi_{x_0,+})(x)$, we must show that, for any C^1 function ψ, compactly supported on a bounded interval J, we have

$$\int_J \psi(x) M_+^{1/2}\left(\frac{\mathbf{1}_{x>x_0}}{\sqrt{x-x_0}}\right)\mathrm{d}x = \sqrt{\pi}\psi(x_0)\,. \tag{36}$$

We shall use two facts. On the one hand, the distributional derivative of the Heaviside $\mathbf{1}_{x>y}$ is the Dirac mass at y, in particular, for any compactly supported C^1 function ψ, and any bounded (interval $[a,b]$ containing y, we have

$$\int_y^b \psi'(x)\,\mathrm{d}t = \int_a^b \mathbf{1}_{x>y}(x)\psi'(x)\,\mathrm{d}t = -\psi(y) + \psi(b)\,. \tag{37}$$

On the other hand, in view of Remark 3.3, Lemma 3.1 gives

$$I_{+,x}^{1/2}(\phi_{x_0,+})(x) = \frac{1}{\Gamma(1/2)}\int_{-\infty}^0 \frac{\phi_{x_0,+}(x+\tau)}{|\tau|^{1/2}}\,\mathrm{d}\tau = \sqrt{\pi}\mathbf{1}_{x>x_0}(x)\,. \tag{38}$$

We now move on to prove (36). We have, recalling (35) (in other words, integrating by parts with respect to x using Fubini, before taking the limit $\epsilon \to 0$),

$$\int_J \psi(x) M_{+,x}^{1/2}\phi_{x_0,+}(x)\,\mathrm{d}x = -\frac{1}{2\sqrt{\pi}}\int_J \psi'(x)\lim_{\epsilon\uparrow 0}\int_{-\infty}^{\epsilon}\frac{\tilde\phi_{x_0,+}(x) - \tilde\phi_{x_0,+}(x+\tau)}{|\tau|^{3/2}}\,\mathrm{d}\tau\,\mathrm{d}x\,,$$

where $\tilde\phi_{x_0,+}(x) = 0$ if $x_0 > x$ and, otherwise,

$$\tilde\phi_{x_0,+}(x) = \int_{-\infty}^x \phi_{x_0,+}(y)\,\mathrm{d}y = \int_{x_0}^x \frac{1}{\sqrt{y-x_0}}\,\mathrm{d}y = 2\sqrt{x-x_0}\,. \tag{39}$$

Next, for $x_0 - x < \epsilon < 0$, integrating by parts, we find,

$$\begin{aligned}
&\int_{-\infty}^{\epsilon}\frac{\tilde\phi_{x_0,+}(x) - \tilde\phi_{x_0,+}(x+\tau)}{2|\tau|^{3/2}}\,\mathrm{d}\tau \\
&\quad = 2\left[\int_{-\infty}^{x_0-x}\frac{\sqrt{x-x_0}}{2|\tau|^{3/2}}\,\mathrm{d}\tau + \int_{x_0-x}^{\epsilon}\frac{\sqrt{x-x_0}-\sqrt{x+\tau-x_0}}{2|\tau|^{3/2}}\,\mathrm{d}\tau\right] \\
&\quad = 2\left[\frac{\sqrt{x-x_0}}{\sqrt{|\tau|}}\Big|_{\tau=-\infty}^{\tau=x_0-x} + \int_{x_0-x}^{\epsilon}\frac{1}{2\sqrt{x+\tau-x_0}}\frac{1}{|\tau|^{1/2}}\,\mathrm{d}\tau\right.
\end{aligned}$$

$$+\frac{\sqrt{x-x_0}-\sqrt{x-x_0+\epsilon}}{|\epsilon|^{1/2}}-1\Bigg]$$

$$=2\Bigg[1+\int_{-\infty}^{\epsilon}\frac{\mathbf{1}_{x+\tau>x_0}(x)}{2\sqrt{x+\tau-x_0}}\frac{1}{|\tau|^{1/2}}\,\mathrm{d}\tau+\frac{\sqrt{x-x_0}-\sqrt{x-x_0+\epsilon}}{|\epsilon|^{1/2}}-1\Bigg]$$

$$=\int_{-\infty}^{\epsilon}\frac{\mathbf{1}_{x+\tau>x_0}(x)}{\sqrt{x+\tau-x_0}}\frac{1}{|\tau|^{1/2}}\,\mathrm{d}\tau+2\frac{\sqrt{x-x_0}-\sqrt{x-x_0+\epsilon}}{|\epsilon|^{1/2}}$$

$$=\int_{-\infty}^{\epsilon}\frac{\phi_{x_0,+}(x+\tau)}{|\tau|^{1/2}}\,\mathrm{d}\tau+2\frac{\sqrt{x-x_0}-\sqrt{x-x_0+\epsilon}}{|\epsilon|^{1/2}}\,. \tag{40}$$

Note that $\lim_{\epsilon\uparrow 0}\frac{\sqrt{x-x_0}-\sqrt{x-x_0+\epsilon}}{|\epsilon|^{1/2}}=0$ for any fixed $x>x_0$. Using first (38), and then (37) (recalling that ψ vanishes at the endpoints of J), we have,

$$-\frac{1}{\sqrt{\pi}}\int_J\psi'(x)\lim_{\epsilon\uparrow 0}\int_{-\infty}^{\epsilon}\frac{\phi_{x_0,+}(x+\tau)}{|\tau|^{1/2}}\,\mathrm{d}\tau\,\mathrm{d}x=-\int_J\psi'(x)I_+^{1/2}(\phi_{x_0,+})(x)\,\mathrm{d}x$$
$$=-\frac{\pi}{\sqrt{\pi}}\int_{J\cap[x_0,\infty)}\psi'(x)\,\mathrm{d}x=\sqrt{\pi}\cdot\psi(x_0)\,,$$

which concludes[20] the proof of (36) for $M_+^{1/2}(\phi_{x_0,+})$.

For the claim[21] on $M_-^{1/2}(\phi_{x_0,-})$, we use (33) and

$$\phi_{x_0,+}(x)=\phi_{x_0,-}(2x_0-x)\,. \tag{41}$$

Next, we show the claim on the two-sided Marchaud derivative $M^{1/2}(\phi_{x_0,+})(x)$. We will apply Lemma 3.2. We first claim that $M_-^{1/2}(\phi_{x_0,+}-\phi_{x_0,+,\mathcal{Z}})(x)$ is C^1 (in fact, C^∞) on $x<x_0+\mathcal{Z}$. Indeed

$$2\Gamma(1/2)\cdot M_-^{1/2}(\phi_{x_0,+}-\phi_{x_0,+,\mathcal{Z}})(x)$$
$$=\int_0^\infty\frac{\mathbf{1}_{x>x_0+\mathcal{Z}}\cdot\phi_{x_0,+}(x)-\mathbf{1}_{x+\tau>x_0+\mathcal{Z}}\cdot\phi_{x_0,+}(x+\tau)}{\tau^{3/2}}\,\mathrm{d}\tau\,.$$

If $x<x_0+\mathcal{Z}$, we find

$$M_-^{1/2}(\phi_{x_0,+}-\phi_{x_0,+,\mathcal{Z}})(x)=-\frac{1}{2\Gamma(1/2)}\int_{x_0-x+\mathcal{Z}}^\infty\frac{\phi_{x_0,+}(x+\tau)}{\tau^{3/2}}\,\mathrm{d}\tau$$
$$=-\frac{1}{2\Gamma(1/2)}\int_{x_0-x+\mathcal{Z}}^\infty\frac{1}{\sqrt{x+\tau-x_0}}\frac{1}{\tau^{3/2}}\,\mathrm{d}\tau=:\widetilde{G}_{\mathcal{Z}}(x-x_0)\,. \tag{42}$$

Clearly, if $y<\mathcal{Z}/2$,

$$|\widetilde{G}_{\mathcal{Z}}(y)|=\left|\frac{1}{2\Gamma(1/2)}\int_{-y+\mathcal{Z}}^\infty\frac{1}{(y+\tau)^{1/2}\cdot\tau^{3/2}}\,\mathrm{d}\tau\right|$$
$$\le\left|\frac{1}{2\sqrt{\pi}}\frac{1}{(\mathcal{Z}/2)^{1/2}}\int_{-y+\mathcal{Z}}^\infty\tau^{-3/2}\,\mathrm{d}\tau\right|=\frac{1}{\sqrt{\pi}}\frac{1}{(\mathcal{Z}/2)^{1/2}}\frac{1}{\sqrt{\mathcal{Z}-y}}\le\frac{2}{\sqrt{\pi}}\frac{1}{\mathcal{Z}}\,. \tag{43}$$

We next focus on $M_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x)$. Just like in the proof of (36), taking a C^1 function ψ compactly supported in an interval J, we integrate by parts:

$$\int_J\psi(x)M_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x)\,\mathrm{d}x$$

[20]In particular we have shown that $M_+^{1/2}(\phi_{x_0,+})=dI_+^{1/2}(\phi_{x_0,+})$, as expected, see Remark 4.3.

[21]In particular, $M_-^{1/2}(\phi_{x_0,-})=-dI_-^{1/2}(\phi_{x_0,-})$, as expected, see Remark 4.3.

$$= -\frac{1}{2\Gamma(1/2)} \int_J \psi'(x) \lim_{\epsilon \downarrow 0} \int_\epsilon^\infty \frac{\tilde{\phi}_{x_0,\mathcal{Z}}(x) - \tilde{\phi}_{x_0,\mathcal{Z}}(x+\tau)}{\tau^{3/2}} \, \mathrm{d}\tau \, \mathrm{d}x \,,$$

where $\tilde{\phi}_{x_0,\mathcal{Z}}(x) = 0$ if $x < x_0$ and, otherwise,

$$\tilde{\phi}_{x_0,\mathcal{Z}}(x) = \int_{-\infty}^x \phi_{x_0,+,\mathcal{Z}}(y) \, \mathrm{d}y = \int_{-\infty}^x \frac{\mathbf{1}_{x_0+\mathcal{Z}>y>x_0}}{\sqrt{y-x_0}} \, \mathrm{d}y$$

$$= \int_{x_0}^{\min(x, x_0+\mathcal{Z})} \frac{1}{\sqrt{y-x_0}} \, \mathrm{d}y = \begin{cases} 2\sqrt{\mathcal{Z}} \text{ if } x > x_0 + \mathcal{Z} \,, \\ 2\sqrt{x-x_0} & \text{if } x < x_0 + \mathcal{Z} \,. \end{cases} \tag{44}$$

Next, integrating by parts again, we find for $x \in (x_0, x_0+\mathcal{Z})$, that, for any $0 < \epsilon < x_0 - x + \mathcal{Z}$,

$$- \int_\epsilon^\infty \frac{\tilde{\phi}_{x_0,\mathcal{Z}}(x) - \tilde{\phi}_{x_0,\mathcal{Z}}(x+\tau)}{2\tau^{3/2}} \, \mathrm{d}\tau$$

$$= -2 \int_\epsilon^{x_0-x+\mathcal{Z}} \frac{\sqrt{x-x_0} - \sqrt{x+\tau-x_0}}{2\tau^{3/2}} \, \mathrm{d}\tau - 2 \int_{x_0-x+\mathcal{Z}}^\infty \frac{\sqrt{x-x_0} - \sqrt{\mathcal{Z}}}{2\tau^{3/2}} \, \mathrm{d}\tau$$

$$= 2 \Big[\int_\epsilon^{x_0-x+\mathcal{Z}} \frac{1}{2\sqrt{x+\tau-x_0}} \frac{1}{\tau^{1/2}} \, \mathrm{d}\tau + \frac{\sqrt{x-x_0} - \sqrt{\mathcal{Z}}}{\sqrt{x_0-x+\mathcal{Z}}} - \frac{\sqrt{x-x_0} - \sqrt{x-x_0+\epsilon}}{\sqrt{\epsilon}} - \frac{\sqrt{x-x_0} - \sqrt{\mathcal{Z}}}{\sqrt{x_0-x+\mathcal{Z}}} \Big]$$

$$= \int_\epsilon^{x_0-x+\mathcal{Z}} \frac{\phi_{x_0,+}(x+\tau)}{\tau^{1/2}} \, \mathrm{d}\tau - 2 \frac{\sqrt{x-x_0} - \sqrt{x-x_0+\epsilon}}{\sqrt{\epsilon}} \,.$$

As $\epsilon \downarrow 0$, the right hand-side above tends to $\int_0^\infty \frac{\phi_{x_0,+,\mathcal{Z}}(x+\tau)}{\tau^{1/2}} \mathrm{d}\tau$ for any fixed $x < x_0$.

If $x < x_0$, we find for any $0 < \epsilon < x_0 - x$,

$$\int_\epsilon^\infty \frac{\tilde{\phi}_{x_0,\mathcal{Z}}(x) - \tilde{\phi}_{x_0,\mathcal{Z}}(x+\tau)}{2\tau^{3/2}} \, \mathrm{d}\tau$$

$$= 2 \int_{x_0-x}^{x_0-x+\mathcal{Z}} \frac{\sqrt{x+\tau-x_0}}{2\tau^{3/2}} \, \mathrm{d}\tau + 2 \int_{x_0-x+\mathcal{Z}}^\infty \frac{\sqrt{\mathcal{Z}}}{2\tau^{3/2}} \, \mathrm{d}\tau$$

$$= 2 \Big[\int_{x_0-x}^{x_0-x+\mathcal{Z}} \frac{1}{2\sqrt{x+\tau-x_0}} \frac{1}{\tau^{1/2}} \, \mathrm{d}\tau - \frac{\sqrt{\mathcal{Z}}}{\sqrt{x_0-x+\mathcal{Z}}} + \frac{\sqrt{\mathcal{Z}}}{\sqrt{x_0-x+\mathcal{Z}}} \Big]$$

$$= \int_0^{x_0-x+\mathcal{Z}} \frac{\phi_{x_0,+}(x+\tau)}{\tau^{1/2}} \, \mathrm{d}\tau = \int_0^\infty \frac{\phi_{x_0,+,\mathcal{Z}}(x+\tau)}{\tau^{1/2}} \, \mathrm{d}\tau \,. \tag{45}$$

So, recalling the definition of $I_-^{1/2}$, for any $x < x_0 + \mathcal{Z}$, we have

$$-\frac{1}{2\Gamma(1/2)} \lim_{\epsilon \downarrow 0} \int_\epsilon^\infty \frac{\tilde{\phi}_{x_0,\mathcal{Z}}(x) - \tilde{\phi}_{x_0,\mathcal{Z}}(x+\tau)}{\tau^{3/2}} \, \mathrm{d}\tau = I_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x) \,.$$

Summarising, and recalling (42), we have shown that if $J \cap [x_0 + \mathcal{Z}/2, \infty) = \emptyset$, then

$$\int_J \psi(x) M_+^{1/2}(\phi_{x_0,+})(x) \, \mathrm{d}x = - \int_J \psi'(x) I_+^{1/2}(\phi_{x_0,+})(x) \, \mathrm{d}x = \sqrt{\pi} \psi(x_0) \,, \tag{46}$$

and[22]

$$\int_J \psi(x) M_-^{1/2}(\phi_{x_0,+})(x) \, \mathrm{d}x = \int_J \psi(x) \widetilde{G}_{\mathcal{Z}}(x-x_0) \, \mathrm{d}x + \int_J \psi(x) M_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x) \, \mathrm{d}x$$

[22]In particular, $M_-^{1/2}(\phi_{x_0,+,\mathcal{Z}}) = -dI_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})$, as expected.

$$= \int_J \psi(x)\widetilde{G}_{\mathcal{Z}}(x-x_0)\,\mathrm{d}x + \int_J \psi'(x) I_-^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x)\,\mathrm{d}x\,.$$

Next, by Remark 3.3, Lemmas 3.1 and 3.2 give for $x_0 - \mathcal{Z}/2 < x < x_0 + \mathcal{Z}/2$ that

$$\begin{aligned} I_{-,x}^{1/2}(\phi_{x_0,+,\mathcal{Z}})(x) &= I_{+,t}^{1/2}(\phi_{x_0+t,+,\mathcal{Z}})(x)|_{t=0} \\ &= \frac{1}{\sqrt{\pi}}\big(-\log|x-x_0| + \log\mathcal{Z} + G_{\mathcal{Z}}(x_0-x)\big) \\ &= \frac{1}{\sqrt{\pi}}\big(-\log|x-x_0| + \log\mathcal{Z} + G_{\mathcal{Z}}(x_0-x)\big)\,, \end{aligned} \tag{47}$$

where $y \mapsto G_{\mathcal{Z}}(y)$ is analytic, with

$$\lim_{\mathcal{Z}\to\infty} \sup_{|y|<\mathcal{Z}/2} |\partial_y G_{\mathcal{Z}}(y)| = 0\,, \quad \sup_{\mathcal{Z}>1} \sup_{|y|<\mathcal{Z}/2} \max(|\partial_y G_{\mathcal{Z}}(y)|, |\partial_y^2 G_{\mathcal{Z}}(y)|) < \infty\,, \tag{48}$$

Recalling that ψ is C^1 and vanishes at the endpoints of J, we have shown that

$$\begin{aligned} 2\int_J \psi(x) M^{1/2}(\phi_{x_0,+})(x)\,\mathrm{d}x &= \int_J \psi(x)\big(M_+^{1/2}(\phi_{x_0,+}(x) - M_-^{1/2}(\phi_{x_0,+}(x))\big)\,\mathrm{d}x \\ &= \sqrt{\pi}\psi(x_0) - \int_J \psi(x)\widetilde{G}_{\mathcal{Z}}(x-x_0)\,\mathrm{d}x \\ &\qquad - \int_J \frac{\psi'(x)}{\sqrt{\pi}}\big[-\log|x-x_0| + \log\mathcal{Z} + G_{\mathcal{Z}}(x_0-x)\big]\,\mathrm{d}x \\ &= \sqrt{\pi}\psi(x_0) + \int_J \psi(x)\big[-\widetilde{G}_{\mathcal{Z}}(x-x_0) + \frac{\partial_x G_{\mathcal{Z}}(x_0-x)}{\sqrt{\pi}}\big]\,\mathrm{d}x \\ &\qquad - \frac{1}{\sqrt{\pi}}\int_J \frac{\psi(x)}{x-x_0}\,\mathrm{d}x\,, \end{aligned} \tag{49}$$

if $\mathcal{Z}$ is large enough (depending only on x_0 and J). Since the left-hand side above is independent of $\mathcal{Z}$, the function

$$\mathcal{G}(x-x_0) := -\widetilde{G}_{\mathcal{Z}}(x-x_0) + \partial_x G_{\mathcal{Z}}(x_0-x)/\sqrt{\pi}$$

does not depend on $\mathcal{Z}$. By (43) and the first claim of (48), we get $\mathcal{G}(y) = 0$. This establishes the claim on $M_-^{1/2}(\phi_{x_0,+})$ and the two-sided half derivative $M^{1/2}(\phi_{x_0,+})$. The claim on $M^{1/2}(\phi_{x_0,-})(x)$ then follows from (34) and (41).

Finally, the claims on truncated spikes $\phi_{x_0,\sigma,\mathcal{Z}}$ follow from (42) and (49) combined with the fact that if $\sigma(x-x_0) < \mathcal{Z}$ then $M_\sigma^{1/2}(\phi_{x_0,\sigma} - \phi_{x_0,\sigma,\mathcal{Z}})(x) = 0$. □

Proof of Lemma 4.5. In view of (38), to show the claim on $M_+^{1/2}(\mathbf{1}_{x>x_0}(\sqrt{x-x_0}))$, it is enough to check that for any continuous ψ vanishing at the endpoints of J,

$$\begin{aligned} \frac{1}{2\sqrt{\pi}}\int_J \psi(x) \lim_{\epsilon\uparrow 0}\int_{-\infty}^{\epsilon} &\frac{\mathbf{1}_{x>x_0}(\sqrt{x-x_0}) - \mathbf{1}_{x+\tau>x_0}(\sqrt{x+\tau-x_0})}{|\tau|^{3/2}}\,\mathrm{d}\tau\,\mathrm{d}x \\ &= \frac{1}{2}\int_J \psi(x) I_+^{1/2}(\phi_{x_0,+})(x)\,\mathrm{d}x\,. \end{aligned}$$

The above follows from (40) and (39). The claim on $M_-^{1/2}(\mathbf{1}_{x<x_0}(\sqrt{x_0-x}))$ then follows immediately from (33) and (41).

For the two-sided half derivative of truncated square roots, we note that

$$M_\sigma^{1/2}(\bar{\phi}_{x_0,\sigma} - \bar{\phi}_{x_0,\sigma,\mathcal{Z}})(x) = 0 \text{ if } \sigma(x-x_0) < \mathcal{Z}\,.$$

The claim on $M^{1/2}(\bar{\phi}_{x_0,\sigma,\mathcal{Z}})$ then follows from(45), (46), and (47). □

Proof of Lemma 4.6. Since $\eta \in (0,1)$, for any $h \in \mathbb{R}$, we have

$$
\begin{aligned}
&|M^\eta g(x+h) - M^\eta g(x)| \\
&\quad \le \frac{\eta}{2\Gamma(1-\eta)} \lim_{\epsilon \downarrow 0} \int_{|\tau|>\epsilon} \frac{|g(x+h+\tau) - g(x+h) - g(x+\tau) + g(x)|}{|\tau|^{1+\eta}} \, \mathrm{d}\tau \\
&\quad \le \frac{\eta}{\Gamma(1-\eta)} \sup |g'| \Big(\lim_{\epsilon \downarrow 0} \int_\epsilon^{|h|} \frac{\tau}{\tau^{1+\eta}} \, \mathrm{d}\tau + \int_{|h|}^\infty \frac{|h|}{\tau^{1+\eta}} \, \mathrm{d}\tau \Big) \\
&\quad = \frac{\eta}{\Gamma(1-\eta)} \sup |g'| \Big(\frac{|h|^{1-\eta}}{1-\eta} + \frac{|h|^{1-\eta}}{\eta} \Big) .
\end{aligned}
$$

□

5. Rigorous results on fractional susceptibility functions

Before stating Theorem C in §5.2 and proving it in §5.3, we recall an expansion for the invariant density ρ_t due to Ruelle in §5.1, and prove some of its consequences.

5.1. **Ruelle's formula for ρ_t. Fractional integration by parts. Exponential bounds. Proof of Proposition 2.5.** Let $f(x) = f_t(x) = t - x^2$ for $t \in \mathrm{MT}$, let $c_k = c_{k,t}$ and recall the sequence $s_k = s_{k,t}$ from (11). The starting point for the proof below of our main Theorem C is the expansion given by Ruelle [36, Theorem 9, Remark 16A] (in the slightly more general analytic Misiurewicz setting) for the invariant density ρ_t of f_t, supported in $[c_2, c_1]$:

$$
\begin{aligned}
\rho_t(x) = \psi_0(x) &+ \sum_{k=1}^\infty C_k^{(0)} \frac{\mathbf{1}_{w_0 < s_{k-1}(x-c_k) < 0}}{\sqrt{|x-c_k|}} \\
&+ \sum_{k=1}^\infty C_k^{(1)} \cdot \mathbf{1}_{w_1 < s_{k-1}(x-c_k) < 0} \cdot \sqrt{|x-c_k|} ,
\end{aligned} \tag{50}
$$

where[23] ψ_0 is a C^1 function, $w_1 < 0$, $w_0 < 0$, and where (for some $U_t \neq 0$)

$$
C_k^{(0)} = \frac{\rho_t(0)}{|Df_t^{k-1}(c_1)|^{1/2}} , \quad |C_k^{(1)}| \le \frac{U_t}{|Df_t^{k-1}(c_1)|^{3/2}} , \quad \forall k \ge 1 .
$$

Since $t \in \mathrm{MT}$, we have $c_{k+P} = c_k$ for $k \ge L$. Note also that, if $Df_t^P(c_L) > 0$, then the spikes and square roots along the postcritical orbit are all one-sided. If $Df_t^P(c_L) < 0$ then the spikes and square roots along the periodic part of the postcrititical orbit are all two-sided.

Remark 5.1. For more general TSR parameters, one could use [11, Prop 2.7] instead of (50). (To obtain an expansion involving spikes and square roots in the TSR setting, one could upgrade the results of [11], showing that if f_t is smooth enough then the smooth component of ρ_t belongs to W_1^r for large enough $r > 2$.)

In the remainder of this section, we show three consequences of (50).

First, we show that the integration by parts formula (23) holds for all $\eta \in (0,1)$ (this will be used to prove Proposition 2.5 which implies Proposition D):

[23] The cutoff is slightly different in Ruelle [36, Theorem 9, Remark 16A], who observes that "other choices can be useful."

Lemma 5.2 (Fractional integration by parts in the response susceptibility)**.** *Let $t \in \mathrm{MT}$. For any $\eta \in (0,1)$ and any compactly supported $\phi \in C^1$, we have, as formal power series,*

$$\sum_{k=0}^{\infty} z^k \int M_x^\eta(\phi \circ f_t^k) \cdot \rho_t \,\mathrm{d}x = -\sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k) \cdot M_x^\eta\big(\rho_t\big) \,\mathrm{d}x .$$

(By definition, the left-hand side above is just $\Psi_\phi^{\mathrm{rsp}}(\eta, z)$.)

Proof. Fix $\eta \in (0,1)$. It suffices to show that, for any compactly supported $\psi \in C^1$, we have

$$\int M_x^\eta(\psi)(x) \cdot \rho_t(x) \,\mathrm{d}x = -\int \psi(x) \cdot M_x^\eta(\rho_t)(x) \,\mathrm{d}x ,$$

where $M_x^\eta(\rho_t)$ is understood in the sense of distributions (of order one).

Since ψ is C^1 and compactly supported, we have

$$2M_x^\eta(\psi)(x) = \partial_x((I_+^{1-\eta} + I_-^{1-\eta})\psi)(x) = (I_+^{1-\eta} + I_-^{1-\eta})\psi'(x) . \tag{51}$$

(Use Remark 4.3 for the first equality and the definition of $I_\pm^{1-\eta}$ for the second.) Next, using the expansion (50) for $g(x) := \rho_t(x)$, we find that $G(y) := \int_{-\infty}^y g(u)\,\mathrm{d}u$ is the sum of a C^1 function with a (finite) sum of one- or two-sided truncated square roots along the postcritical orbit (see (44)). Thus (recalling (32))

$$\lim_{\epsilon \to 0} M_\epsilon^\eta G(x) = M^\eta G(x) \in L^1_{loc} .$$

The above claim is clear if $\eta < 1/2$. For $\eta = 1/2$, it follows from Lemma 4.5. Finally, for $\eta \in (1/2, 1)$, we may decompose $M_\pm^\eta = M_\pm^{\eta - 1/2} \circ M_\pm^{1/2}$, using the semigroup property ([19, Property 2.4], for[24] $m = 1$ and $\alpha = \eta - 1/2$, noting that $G \in L^1$ and $I_\pm^{3/2-\eta}(G) \in AC$ since $3/2 - \eta > 1/2$).

Now, on the one hand, by definition, we have

$$\begin{aligned}
2\int \psi(x)(M^\eta g)(x)\,\mathrm{d}x &= -2\int \psi'(x) \cdot M^\eta G(x)\,\mathrm{d}x \\
&= \int \psi'(x)\Big[\frac{\eta}{\Gamma(1-\eta)} \int \frac{G(x+\tau) - G(x)}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}x \\
&= \int \psi'(x)\Big[\frac{\eta}{\Gamma(1-\eta)} \int \frac{\int_x^{x+\tau} g(u)\,\mathrm{d}u}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau)\,\mathrm{d}\tau\Big]\,\mathrm{d}x , \qquad (52)
\end{aligned}$$

where (52) can be rewritten, integrating by parts in τ, as

$$\int \psi'(x) \frac{1}{\Gamma(1-\eta)} \int \frac{g(x+\tau)}{|\tau|^\eta}\,\mathrm{d}\tau\,\mathrm{d}x .$$

On the other hand, (51) followed by fractional integration by parts [37, (5.16)] gives

$$\begin{aligned}
2\int M_x^\eta(\psi)(x) \cdot g(x)\,\mathrm{d}x &= \int (I_+^{1-\eta} + I_-^{1-\eta})(\psi')(x) \cdot g(x)\,\mathrm{d}x \\
&= \int \psi'(x) \cdot (I_+^{1-\eta} + I_-^{1-\eta})(g)(x)\,\mathrm{d}x \\
&= \int \psi'(x) \cdot \frac{1}{\Gamma(1-\eta)} \int \frac{g(x+\tau)}{|\tau|^\eta}\mathrm{d}\tau\,\mathrm{d}x .
\end{aligned}$$

□

[24]The reference to Lemma 2.4 there should be replaced by Lemma 2.5.

Next, we use[25] the expansion (50) to get the following exponential bounds, useful to prove Theorem C:

Lemma 5.3 (Action of the transfer operator on Sobolev spaces H^r_q for $r>0$)**.** *Let* $t \in \mathrm{MT}$ *be a mixing parameter. Let* $r>0$, $q>1$. *There exist* $C<\infty$ *and* $\kappa<1$ *such that, for any*[26] $\psi \in H^r_q[-2,2]$ *and any bounded* φ *supported in* $[-2,2]$

$$|\int \varphi(f_t^j(x))\,\psi(x)\,\mathrm{d}x - \int \varphi(x)\,\mathrm{d}\mu_t \cdot \int_{I_t} \psi\,\mathrm{d}x| \le C\|\varphi\|_{L^\infty}\|\psi\|_{H^r_q}\kappa^j\,, \forall j \ge 0\,.$$

Lemma 5.3 applies to the Heaviside function $\psi = \mathbf{1}_{x>y}$.

Proof of Lemma 5.3. Since $q>1$ and we are in a one-dimensional setting, the Sobolev embeddings imply that, for any $\tilde r > 2$ (we may choose $\tilde r < 2+r$), there exists $\tilde C$ such that for any compactly supported $g \in H^{\tilde r}_q$

$$\|g\|_{C^1} \le \tilde C \|g\|_{H^{\tilde r}_q}\,.$$

Since $r>0$, using mollification, we can approach $\mathbf{1}_{[-2,2]}\psi$ by C^1 functions ψ_ϵ with

$$\|\psi_\epsilon\|_{C^1} \le \tilde C\|\psi_\epsilon\|_{H^{\tilde r}_q} \le \tilde C_0 \frac{\|\psi\|_{H^r_q}}{\epsilon^2}\,, \quad \|\psi-\psi_\epsilon\|_{L^q[-2,2]} \le \tilde C_1 \epsilon^r \|\mathbf{1}_{[-2,2]}\psi\|_{H^r_q}\,, \forall \epsilon>0\,.$$

Note that $(\mathbf{1}_{[c_2,c_1]}\psi_\epsilon)/\rho_t \in BV$, with BV norm bounded by $C_0\|\psi_\epsilon\|_{C^1[c_2,c_1]}$, because $\mathbf{1}_{[c_2,c_1]}/\rho_t \in BV$. (To check this, use that[27] $\inf_{[c_2,c_1]}\rho_t > 0$ and that ρ_t is the sum of a C^1 function together with finitely many square roots spikes and square roots, by (50), and consider separately each maximal interval bounded by postcritical points.)

Since it is easy to find $C_0<\infty$ and $\kappa<1$ (independent of φ, ψ) such that

$$\int_{\mathbb{R}\setminus I_t} |(\varphi\circ f_t^j)\,\psi|\,\mathrm{d}m \le C_0\kappa^j\|\varphi\|_{L^\infty}\|\psi\|_{L^q}\,,\ \forall j\ge 1\,,$$

and since we can write $\int_{c_1}^{a_t}(\varphi\circ f_t^j)\,\psi\,\mathrm{d}m = \int_{-a_t}^{c_2}(\varphi\circ f_t^{j-1})\,(\psi\circ f^{-1})|(f^{-1})'|\,\mathrm{d}m$, and

$$\int_{-a_t}^{c_2}(\varphi\circ f_t^j)\,\psi\,\mathrm{d}m = \int_{-a_t}^{c_2}\varphi(f_t^{j-[j/2]}(x))\frac{\psi(f^{-[j/2]}(x)}{(f^{[j/2]})'(f^{-[j/2]}(x))}\,\mathrm{d}x + \sum_{\ell=[j/2]}^{j-1}\int_{c_2}^{c_3}\varphi(f^\ell(x))\frac{\psi(f^{\ell-j}(x))}{|(f^{j-\ell})'(f^{\ell-j}(x))|}\mathrm{d}x\,,$$

(f^{-k} above denotes $(f^k|_{\cap_{j=1}^k f^{-j}[-a_t,c_2]})^{-1}$), which gives the limiting contribution $\int\varphi(x)\,\mathrm{d}\mu_t\cdot\int_{I_t\setminus[c_2,c_1]}\psi\,\mathrm{d}x$, the lemma follows from three facts. First,

$$\int_{[c_2,c_1]}(\varphi\circ f_t^j)\psi_\epsilon\,\mathrm{d}x = \int_{[c_2,c_1]}(\varphi\circ f_t^j)\frac{\psi_\epsilon}{\rho_t}\,\mathrm{d}\mu_t\,.$$

Second, there exist $\theta<1$, $C_1<\infty$ (independent of φ, ψ_ϵ, see [18, Theorem 1.1], by the principle of uniform boundedness, C_1 does not depend on ψ_ϵ) such that

$$|\int_{c_2}^{c_1}(\varphi\circ f_t^j)\frac{\psi_\epsilon}{\rho_t}\,\mathrm{d}\mu_t - \int\varphi\,\mathrm{d}\mu_t\int_{c_2}^{c_1}\psi_\epsilon\,\mathrm{d}m| \le C_1\|\varphi\|_{L^1}\left\|\frac{\mathbf{1}_{[c_2,c_1]}\psi_\epsilon}{\rho_t}\right\|_{BV}\theta^j\,,\ \forall j\ge 1\,.$$

[25]It would probably be possible to apply [47, Thm 2.II.b)] instead.

[26]If $\mathrm{supp}(\psi)\subset I_t$, the first term is $\int\varphi(x)\mathcal{L}_t^j(\psi(x))\,\mathrm{d}x$, thus the name of the lemma.

[27]See e.g. [46, Theorem 2c)], or, in the MT case [30].

Third,

$$\max\{|\int \varphi\, \mathrm{d}\mu_t \int_{I_t} (\psi_\epsilon - \psi)\, \mathrm{d}m|, |\int_{I_t} (\varphi \circ f_t^j)(\psi - \psi_\epsilon)\, \mathrm{d}m|\} \le \sup|\varphi| \|\psi - \psi_\epsilon\|_{L^q[-2,2]} \,.$$

To conclude, for each j choose $\epsilon = \theta^{j/(r+2)}$, so that $\frac{\theta^j}{\epsilon^2} = \epsilon^r = \theta^{jr/(r+2)} =: \kappa^j$. □

We can now provide the proof of Proposition 2.5:

Proof of Proposition 2.5. Setting $R_t(y) = \int_{-10}^{y} \rho_t(x)\, \mathrm{d}x$, it is easy to see (using e.g (50), or, in the TSR case, [11]) that $\mathbf{1}_{[-1,1]} \cdot M_x^\eta R_t(x)$ belongs to L^q for any $\eta \in (0,1)$ and any $1 \le q < 2$. So $\sup_{\eta \in (0,1)} \int_{-1}^{1} |M_x^\eta R_t(x)|\, \mathrm{d}x < \infty$. In particular, $M_x^\eta(\rho_t)(x)$ is well defined in the sense of distributions of order one[28] uniformly in $\eta \in (0,1)$.

Next, since $f_{t+\tau}(x) = f_t(x) + \tau$ for the quadratic family, we find, recalling (6),

$$(\mathcal{L}_{t+\tau}\rho_t)(x) = (\mathcal{L}_t\rho_t)(x - \tau) = \rho_t(x - \tau)\,, \forall x\,, \forall |\tau| < \epsilon_0\,,$$
$$(\mathcal{L}_{t+\tau}\rho_t)(x) = (\mathcal{L}_t\rho_t)(x \mp \epsilon_0) = \rho_t(x \mp \epsilon_0)\,, \forall x\,, \text{ if } \pm\tau > \epsilon_0\,.$$

This implies that for any $\eta \in (0,1)$ and any x, we have

$$\begin{aligned}
&\Gamma(1-\eta)\big(M_t^\eta(\mathcal{L}_s\rho_t(x))|_{s=t} + M_x^\eta\rho_t(x)\big) = \\
&\qquad = \frac{\eta}{2}\int_{|\tau|>\epsilon_0} \frac{\rho_t(x - \mathrm{sgn}(\tau)\epsilon_0) - \rho_t(x+\tau)}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau)\, \mathrm{d}\tau \\
&\qquad = \frac{1}{2}\frac{\rho_t(x-\epsilon_0) - \rho_t(x+\epsilon_0)}{\epsilon_0^\eta} - \frac{\eta}{2}\int_{|\tau|>\epsilon_0} \frac{\rho_t(x+\tau)}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau)\, \mathrm{d}\tau\,.
\end{aligned}$$

The above defines a function $g_\eta \in H_q^r$ for some $r > 0$ and $q > 1$ for any $\eta > 0$, uniformly in $\eta > \epsilon_1$, for any fixed $\epsilon_1 > 0$, and such that (24) holds. Clearly $\int_{\mathbb{R}} g_\eta(x)\mathrm{d}x = 0$. Since $M_x^\eta\rho_t(x)$ is a distribution of order one, $M_s^\eta(\mathcal{L}_s\rho_t(x))|_{s=t}$ (which is compactly supported) is also a distribution of order one.

We next establish the relation between the frozen and the response susceptibility functions: On the one hand, recalling (22) and using (24), we have

$$\Psi_\phi^{\mathrm{fr}}(\eta, z) = \sum_{k=0}^{\infty} z^k \int_{I_t} \phi(f_t^k(x))\Big(\frac{g_\eta(x)}{\Gamma(1-\eta)} - M_x^\eta\rho_t(x)\Big)\, \mathrm{d}x\,.$$

Lemma 5.3 holds for the term involving g_η. On the other hand, Lemma 5.2 gives

$$\Psi_\phi^{\mathrm{rsp}}(\eta, z) = -\sum_{k=0}^{\infty} z^k \int_{I_t} \phi(f_t^k(x)) M_x^\eta\rho_t(x)\, \mathrm{d}x\,.$$

The last claim of Proposition 2.5 follows from Lemma 2.4. □

5.2. **Theorem C on $\Psi_\phi^{\mathrm{fr}}(1/2, z)$ at MT parameters.** For a CE parameter t, recall $\lambda_c > 1$ from (1), $\mathcal{U}_{1/2}(z)$, $\mathcal{U}_{1/2}^+(z)$ from (13), (14), and $\Sigma_\phi(z)$, $\Sigma_\phi^{\mathcal{H}}(z)$, and $\Sigma_t^{\tilde\psi}(z)$ from (15), (17), and (18). Generalising (11), we put, $s_{k,1} = s_{k,1,t} = s_{k,t}$ and

$$s_{k,\ell} := s_{k,\ell,t} = \mathrm{sgn}(Df_t^k(c_\ell))\,, \quad k \ge 1\,, \ell \ge 1\,. \tag{53}$$

The following elementary lemma is proved at the end of §5.3 (see [9, Remark 1.2] for the case of piecewise expanding maps):

---

[28] This was already established in Lemma 5.2.

Lemma 5.4. *Let $t \in \mathrm{MT}$ with $f_t^P(c_L) = c_L$. Then $\mathcal{U}_{1/2,t}(z)$ is rational, with poles at the Pth roots of $\mathrm{sgn}(Df^P(c_L))|Df^P(c_L)|^{-1/2}$, and $\mathcal{U}_{1/2,t}(1) = u_t \mathcal{J}_{1/2}(t)$, and $\mathcal{U}^+_{1/2,t}(z)$ is rational, with poles at the Pth roots of $|Df^P(c_L)|^{-1/2}$, and $\mathcal{U}^+_{1/2,t}(1) = u_t \mathcal{J}^+_{1/2}(t)$.*

For $\phi \in C^1$, the function $\Sigma_{\phi,t}(z)$ is rational, with possible simple poles at the Pth roots of unity, while $\Sigma^{\mathcal{H}}_{\phi,t}(z)$ is rational, with possible simple poles at the Pth roots of $\mathrm{sgn}(Df^P(c_L))$. For any $r > 0$, $q > 1$ and any sequence $\tilde\psi(\ell) \in H^r_q[-2,2]$ such that $\tilde\psi(\ell) = \tilde\psi(\ell+p)$ for $\ell \ge L$, the function $z \mapsto \Sigma^{\tilde\psi}_t(z) \in H^r_q[-2,2]$ is rational, with possible simple poles at the Pth roots of $\mathrm{sgn}(Df^P(c_L))$.

Set $\mathcal{P}_t(z) = (z-1)\cdot\Sigma_{\phi,t}(z)$ and $\mathcal{P}^+_t(z) = (z-1)\cdot\Sigma^{\mathcal{H}}_{\phi,t}(z)$. Then we have $\mathcal{P}_t(1) = \frac{1}{P}\sum_{\ell=L}^{L+P-1}\phi(c_\ell)$.

If[29] *$\mathrm{sgn}(Df^P(c_L)) = +1$, then we have that $\mathcal{P}^+_t(1) = \frac{1}{P}\cdot\sum_{\ell=L}^{L+P-1} s_\ell\,\mathcal{H}(\mathbf{1}_{I_t}\phi)(c_\ell)$, and, setting $\mathcal{P}^{\tilde\psi}_t(z) = (z-1)\cdot\Sigma^{\tilde\psi}_t(z)$, that $\mathcal{P}^{\tilde\psi}_t(1) = \frac{1}{P}\sum_{\ell=L}^{L+P-1} s_\ell\tilde\psi_\ell$.*

Our main theorem is proved in §5.3 (it is reminiscent[30] of [10, 9, 36]):

Theorem C (Frozen susceptibility function $t \in \mathrm{MT}$)**.** *Let $t \in \mathrm{MT}$ be mixing with $f_t^P(c_L) = c_L$. There exist $\kappa < 1$ and a sequence $\tilde\psi_t(\ell) \in H^r_q[-2,2]$ with $\tilde\psi_t(\ell) = \tilde\psi(\ell+p)$ for $\ell \ge L$, and $\int_{I_t}\tilde\psi_\ell\,\mathrm{dm} = 0$ for all ℓ, such that the following holds: For any compactly supported ϕ in C^1,*

$$\Psi^{\mathrm{fr}}_{\phi,t}(1/2,z) = \mathcal{U}_{1/2,t}(z)\Sigma_{\phi,t}(z) + \mathcal{W}_{\phi,1/2,t}(z) + \mathcal{V}_{\phi,1/2,t}(z)\,,$$

where $\mathcal{V}_{\phi,1/2,t}(z)$ is holomorphic in the annulus $\{\lambda_c^{-1/2} < |z| < \kappa^{-1}\}$, while,

$$\mathcal{W}_{\phi,1/2,t}(z) = \mathcal{U}^+_{1/2,t}(z)\Sigma^{\mathcal{H}}_{\phi,t}(z) + \sum_{\ell=0}^{\infty}\int(\phi\circ f_t^\ell)\cdot\Sigma^{\tilde\psi_t}_t(z)\,\mathrm{d}m\,.$$

If $\mathrm{sgn}(Df^P(c_L)) = -1$, then $\Psi^{\mathrm{fr}}_\phi(1/2,z)$ has a simple pole at $z=1$, with residue

$$u_t\cdot\frac{\mathcal{J}_{1/2}(t)}{P}\sum_{k=L}^{L+P-1}\phi(c_k)\,. \tag{54}$$

If $\mathrm{sgn}(Df^P(c_L)) = 1$, then there[31] *exists $\tilde\psi^*_t \in (L^\infty[-2,2])^*$ with $\int_{I_t}\psi^*_t\,\mathrm{d}m = 0$, such that the residue of the simple pole at $z=1$ of $\Psi^{\mathrm{fr}}_\phi(1/2,z)/u_t$ is equal to*

$$\frac{\mathcal{J}_{1/2}(t)}{P}\cdot\sum_{k=L}^{L+P-1}\phi(c_k) + \frac{\mathcal{J}^+_{1/2}(t)}{P}\cdot\left(\sum_{\ell=L}^{L+P-1}s_\ell\cdot\mathcal{H}(\mathbf{1}_{I_t}\phi)(c_\ell) + \int\phi\cdot\tilde\psi^*_t\,\mathrm{d}m\right). \tag{55}$$

The vanishing of (55) is a codimension-one condition on ϕ. If $\mathcal{J}_{1,2}(t) \ne 0$ the vanishing of (54) is a codimension-one condition on ϕ. Thus, in view of Lemma 5.4, Theorem C establishes the analogue of Conjecture A[iii] and Conjecture A+ for the frozen susceptibility function at MT parameters.

[29] If $\mathrm{sgn}(Df^P(c_L)) = -1$, then, clearly, $\mathcal{P}^+_t(1) = \mathcal{P}^{\tilde\psi}_t(1) = 0$.

[30] With respect to [9] the term $\mathcal{W}_{\phi,1/2}(z)$ and the presence of the Hilbert transform are new.

[31] The notation $\int\phi\,\tilde\psi^*\,\mathrm{d}m$ represents the action of $\tilde\psi^* \in (L^\infty[-2,2])^*$ on $\phi \in L^\infty[-2,2]$. The formula defining $\tilde\psi^*_t$ is given in (62)–(63), it does not depend on ϕ.

Remark 5.5. The proof of Theorem C shows that the statements also hold for $\Psi_\phi^{\mathrm{rsp}}(1/2,z)$, up to replacing the function $\mathcal{V}_{\phi,1/2}(z)$ by $\mathcal{V}_{\phi,1/2}(z)-\mathcal{V}^{\mathrm{rsp}}_{\phi,1/2}(z)$ (using Proposition 2.5).

Besides Ruelle's expansion (50), the proof of Theorem C in §5.3 will be based on Proposition 2.5, Lemmas 4.4–4.6 and Lemma 5.3 above, and Lemma 5.6 below.

Lemma 5.6 (Action of the transfer operator on poles)**.** *For $\ell \ge 2$ with $c_{\ell-1} = c_{\ell-1,t} \neq 0$, set*

$$\tilde\chi_\ell(x) = \tilde\chi_{\ell,t}(x) = \frac{s_{1,\ell-1}\cdot \mathbf{1}_{x\ge c_1}}{x-c_\ell} + \frac{\mathbf{1}_{x<c_1}}{c_{\ell-1}\sqrt{c_1-x}} + \frac{\mathbf{1}_{x<c_1}}{c_{\ell-1}}\frac{\int_{c_\ell}^x \frac{1}{2\sqrt{c_1-u}}\mathrm{d}u}{x-c_\ell}\,.$$

Then, for any $k\ge 1$ such that $c_k \neq 0$, we have, setting $\chi_k=\chi_{k,t}=(x-c_{k,t})^{-1}$,

$$\mathcal{L}_t\chi_k = s_{1,k}\,\chi_{k+1} + \tilde\chi_{k+1}\,.$$

If $t\in \mathrm{MT}$, there exist $r>0$ and $q>1$ such that $\tilde\chi_\ell \in H^r_q[-2,2]$ for each $\ell\ge 2$.

Proof of Lemma 5.6. For any $x<t=c_1$ and $k\ge1$, using $c_k^2=c_1-c_{k+1}$ twice in the second equality of the third line, we find, inspired by the beginning of the proof of [43, Theorem 2],

$$\begin{aligned}
\mathcal{L}_t\chi_k(x) &= \frac{1}{2\sqrt{c_1-x}}\left(\frac{1}{\sqrt{c_1-x}-c_k} + \frac{1}{-\sqrt{c_1-x}-c_k}\right)\\
&= \frac{1}{c_k\sqrt{c_1-x}}\,\frac{-c_k^2}{(c_k^2-c_1+x)}\\
&= \frac{\sqrt{c_1-x}}{c_k}\,\frac{-c_k^2}{(c_1-x)(c_k^2-c_1+x)} = -\frac{\sqrt{c_1-x}}{c_k}\,\frac{c_1-c_{k+1}}{(c_1-x)(x-c_{k+1})}\\
&= -\frac{\sqrt{c_1-x}}{c_k}\left(\frac{1}{x-c_{k+1}}-\frac{1}{x-c_1}\right) = -\frac{\sqrt{c_1-x}}{c_k}\,(\chi_{k+1}(x)-\chi_1(x))\,.
\end{aligned}$$

Now, $-\sqrt{c_1-c_{k+1}}=c_k$ if $c_k<0$ that is, $s_{1,k}=1$, while $-\sqrt{c_1-c_{k+1}}=-c_k$ if $c_k>0$ that is, $s_{1,k}=-1$. Thus, using a Taylor series at c_{k+1}, we find

$$-\sqrt{c_1-x} = s_{1,k}\cdot c_k + \int_{c_{k+1}}^x \frac{1}{2\sqrt{c_1-u}}\mathrm{d}u\,,\ \ \forall x<c_1\,.$$

Therefore

$$\mathcal{L}_t\chi_k(x) = s_{1,k}\cdot\mathbf{1}_{x<c_1}\cdot\chi_{k+1}(x) + \frac{\mathbf{1}_{x<c_1}}{c_k\sqrt{c_1-x}} + \frac{\mathbf{1}_{x<c_1}}{c_k}\frac{\int_{c_{k+1}}^x\frac{1}{2\sqrt{c_1-u}}\mathrm{d}u}{x-c_{k+1}}\,.$$

In other words, setting

$$\tilde\chi_{k+1}(x) = \frac{s_{1,k}\cdot\mathbf{1}_{x\ge c_1}}{x-c_{k+1}} + \frac{\mathbf{1}_{x<c_1}}{c_k\sqrt{c_1-x}} + \frac{\mathbf{1}_{x<c_1}}{c_k}\frac{\int_{c_{k+1}}^x\frac{1}{2\sqrt{c_1-u}}\mathrm{d}u}{x-c_{k+1}}\,,$$

we have proved

$$\mathcal{L}_t\chi_k(x) = s_{1,k}\cdot\chi_{k+1}(x)+\tilde\chi_{k+1}(x)\,.$$

It is easy to find $r>0$ and $q>1$ such that all $\tilde\chi_\ell\in H^r_q[-2,2]$ if $t\in\mathrm{MT}$. □

Remark 5.7. We introduce notation useful for the proof of Theorem C. Let $\mathcal{Y}_t$ be the $L+P-1$-dimensional vector space generated by the functions

$$\chi_k=\chi_{k,t}=(x-c_{k,t})^{-1}\,,\ k=1,\dots L+P-1\,. \tag{56}$$

We write $\chi(\vec{Y}) = \sum_{k=1}^{L+P-1} Y_k \cdot \chi_k$ for $\vec{Y} = (Y_k) \in \mathbb{C}^{L+P-1}$. Then in view of Lemma 5.6 it is natural to introduce the finite $L+P-1 \times L+P-1$ matrix $\mathbb{S} = \mathbb{S}_t$ acting on $\mathcal{Y}_t$, with coefficients

$$\mathbb{S}_{k,j} = s_{1,j}\delta_{k,j+1} + s_{1,L+P-1}\delta_{k,L}\delta_{j,L+P-1}\,.$$

The eigenvalue zero of $\mathbb{S}_t$ has algebraic multiplicity $L-1$ but geometric multiplicity equal to one. Since $\mathrm{sgn}(Df_t^P(c_L)) = \prod_{k=L}^{L+P-1} s_{1,k} = s_{P,L} = \mathrm{sgn}(Df_t^P(c_L))$, the nonzero eigenvalues of $\mathbb{S}_t$ consist in the Pth roots of $\mathrm{sgn}(Df_t^P(c_L))$, they are simple.

5.3. **Proof of the main result (Theorem C).**

Proof of Theorem C. We already observed that $x \mapsto M_s^{1/2}(\mathcal{L}_s\rho_t(x))|_{s=t}$ is supported in $I_{t,\epsilon} \subset I_t$ (recall Footnote 14), while the support of μ_t is contained in I_t. Thus, for each compactly supported C^1 function $\tilde\phi$ such that $\tilde\phi(x) = \int \phi\, \mathrm{d}\mu_t =: \phi_*$ for all x in I_t, and each sequence of C^1 functions υ_k with $\upsilon_k(x) \equiv 1$ if $|x| \le k$ and $\upsilon_k(x) \equiv 0$ if $|x| \ge 2k$, Proposition 2.5 gives

$$\begin{aligned}\int \tilde\phi(x) M_t^{1/2}(\mathcal{L}_t\rho_t(x))\mathrm{d}x &= \phi_* \cdot \int_{I_t} M_t^{1/2}(\mathcal{L}_t\rho_t(x))\mathrm{d}x \\ &= \phi_* \cdot \lim_{k\to\infty} \int_{\mathbb{R}} \upsilon_k(x) M_t^{1/2}(\mathcal{L}_t\rho_t(x))\mathrm{d}x \\ &= \phi_* \cdot \lim_{k\to\infty} \Big(\int_{\mathbb{R}} \upsilon_k(x) M_x^{1/2}\rho_t(x)\mathrm{d}x + \int_{\mathbb{R}} \upsilon_k(x)\frac{g_{1/2}(x)}{\Gamma(1/2)}\mathrm{d}x\Big) = 0\,.\end{aligned}$$

(To show $\lim_{k\to\infty}\int_{\mathbb{R}} \upsilon_k(x) M_x^{1/2}\rho_t(x)\mathrm{d}x = 0$, recall Definition 4.1, note that $\upsilon_k'(x) = 0$ if $|x| > 2k$, and $G(y) := \int_{-\infty}^{y} \rho_t(x)\,\mathrm{d}x = 0$ if $y < c_2$ while $G(y) = 1$ if $y > c_1$, and use that the Marchaud derivative of any constant function vanishes.) Therefore,

$$\int \phi(f_t^j(x)) M_t^{1/2}(\mathcal{L}_t\rho_t(x))\mathrm{d}x = \int_{I_t} (\phi - \tilde\phi)(f_t^j(x)) M_t^{1/2}(\mathcal{L}_t\rho_t(x))\mathrm{d}x\,,\ \forall j \ge 0\,.$$

From now on, replacing ϕ by $\phi - \tilde\phi$ if necessary, we may thus assume that ϕ is compactly supported, C^1 and has zero average with respect to $\mathrm{d}\mu_t$. (This will allow us to exploit exponential decay of correlations from Lemma 5.3.)

Our starting point is then that $\Psi_\phi^{\mathrm{fr}}(1/2,z) = \Psi_\phi^{\mathrm{rsp}}(1/2,z) + \mathcal{V}_{\phi,1/2}^{\mathrm{rsp}}(z)$, with $\mathcal{V}_{\phi,1/2}^{\mathrm{rsp}}(z)$, holomorphic in the disc of radius $\kappa^{-1} > 1$, from Proposition 2.5. By Lemma 5.2,

$$\Psi_\phi^{\mathrm{rsp}}(1/2,z) = -\sum_{j=0}^{\infty} z^j \int_{I_t} \phi(f_t^j(x)) M_x^{1/2}(\rho_t)(x)\,\mathrm{d}x\,.$$

Therefore, using the expansion (50) for $\rho_t(x)$, and recalling $u_t = -\frac{\sqrt{\pi}}{2}\rho_t(0)$, Lemmas 4.4–4.6 imply that there exist $r > 0$, $q > 1$, and a function $\tilde g \in H_q^r[-2,2]$ such that $\Psi_\phi^{\mathrm{rsp}}(1/2,z)$ can be written as (using the[32] Hilbert transform (16))

$$\sum_{j=0}^{\infty} z^j \Big[\int \phi(f_t^j(x))\,\tilde g(x)\mathrm{d}x + \sum_{k\ge 1} \frac{u_t \cdot s_{k-1}}{\sqrt{|Df^{k-1}(c_1)|}}\big(\phi(c_{k+j}) + \mathcal{H}(\mathbf{1}_{I_t}\cdot(\phi\circ f_t^j))(c_k)\big)\Big]\,.$$

[32]The improper integral is well-defined and finite, since ϕ is C^1 and $[-a_t, c_1]$ contains a neighbourhood of each c_ℓ.

(Indeed, the Heaviside function and the logarithm from Lemma 4.5, the 1/2-Hölder contribution from Lemma 4.6, and — using the MT assumption — the functions $\mathbf{1}_{\mathrm{supp}(\phi\circ f_t^j)\setminus I_t}(x-c_k)^{-1}$ have uniformly bounded $H^r_q[-2,2]$ norms, for $r>0$, $q>1$.)

Next, recalling Proposition 2.5, set

$$\mathcal{V}^{reg}_{\phi,1/2}(z)=\mathcal{V}^{\mathrm{rsp}}_{\phi,1/2}(z)+\sum_{j=0}^{\infty}z^j\int\phi(f_t^j(x))\,\tilde g(x)\mathrm{d}x\,.$$

Since $\int\phi\,\mathrm{d}\mu_t=0$, Lemma 5.3 gives $\kappa<1$, independent of ϕ, such that the function $\mathcal{V}^{reg}_{\phi,1/2}(z)$ is holomorphic in the disc of radius $\kappa^{-1}>1$.

The rest of the proof is devoted to the study of the singular term of the susceptibility function, that is the formal power series

$$\Psi^{sing}_\phi(1/2,z):=\sum_{j=0}^{\infty}z^j\sum_{k\ge1}\frac{u_t\cdot s_{k-1}}{\sqrt{|Df^{k-1}(c_1)|}}\big(\phi(c_{k+j})+\mathcal{H}(\mathbf{1}_{I_t}\cdot(\phi\circ f_t^j))(c_k)\big)\,.$$

We first concentrate on the contribution of $\phi(c_{k+j})$, which can be rewritten as

$$\Psi^{sing,0}_\phi(1/2,z):=u_t\sum_{\ell=1}^{\infty}\phi(c_\ell)\sum_{j=0}^{\ell-1}z^j\frac{s_{\ell-j-1}}{\sqrt{|Df^{\ell-j-1}(c_1)|}}\,. \tag{57}$$

Following the arguments of [9, App. B, Remark 1.2] (see also [7, §5] and the proof of [10, Prop 4.6]), and recalling that our choices imply $X(c_\ell)\equiv1\equiv v(c_\ell)$ for all ℓ, we introduce for every $\ell\ge1$ the formal Laurent series (recalling (53))

$$\alpha_{1/2}(c_\ell,z)=-\sum_{k=1}^{\infty}z^{-k}\frac{s_{k,\ell}}{\sqrt{|Df^k(c_\ell)|}}\,.$$

Our assumptions imply that $\alpha_{1/2}(c_\ell,\cdot)$ is rational and that it is holomorphic in $|z|>1/\sqrt{\lambda_c}$. Recalling the definition (13) of $\mathcal{U}_{1/2}$, the coefficient of $\phi(c_\ell)$ in (57) is

$$\begin{aligned}&u_tz^{\ell-1}\sum_{j=0}^{\ell-1}z^{-(\ell-1-j)}\frac{s_{\ell-1-j}}{\sqrt{|Df^{\ell-1-j}(c_1)|}}\\ &\quad=z^{\ell-1}\Bigg(\mathcal{U}_{1/2}(z)-u_t\frac{s_{\ell-1}}{z^{\ell-1}\sqrt{Df^{\ell-1}(c_1)}}\sum_{k=1}^{\infty}\frac{s_{k,\ell}}{z^k\sqrt{|Df^k(c_\ell)|}}\Bigg)\,.\end{aligned} \tag{58}$$

Thus, we find

$$\Psi^{sing,0}_\phi(1/2,z)=\mathcal{U}_{1/2}(z)\sum_{\ell=1}^{\infty}\phi(c_\ell)z^{\ell-1}-u_t\sum_{\ell=1}^{\infty}\frac{\phi(c_\ell)s_{\ell-1}\alpha_{1/2}(c_\ell,z)}{\sqrt{|Df^{\ell-1}(c_1)|}}\,.$$

Next, our MT assumption implies that the function

$$\mathcal{V}^{sing,0}_{\phi,1/2}(z):=-u_t\sum_{\ell=1}^{\infty}\frac{\phi(c_\ell)s_{\ell-1}\alpha_{1/2}(c_\ell,z)}{\sqrt{|Df^{\ell-1}(c_1)|}}$$

is rational, and that it is holomorphic in the domain $\{|z|>1/\sqrt{\lambda_c}\}$.

It remains to consider the contribution of $\mathcal{H}(\mathbf{1}_{I_t}\cdot(\phi\circ f_t^j))(c_k)$, that is,

$$\Psi^{sing,1}_\phi(1/2,z):=-\frac{u_t}{\pi}\sum_{j=0}^{\infty}z^j\sum_{k\ge1}\frac{s_{k-1}}{\sqrt{|Df^{k-1}(c_1)|}}\int_{I_t}(\phi\circ f_t^j)\,\chi_k\,\mathrm{d}m\,,$$

with χ_k from (56).

Using the notation introduced in Remark (5.7), and introducing $\mathcal{M}_t : \mathcal{Y}_t \to H^r_q$ by

$$\mathcal{M}_t(\vec{Y}) = Y_{L+P-1}\tilde{\chi}_L + \sum_{k=1}^{L+P-2} Y_k \tilde{\chi}_{k+1} ,$$

Lemma 5.6 allows us to write, setting $\vec{Y}_k = (\delta_{j,k})_{j=1,\dots L+P-1} \in \{0,1\}^{L+P-1}$,

$$\begin{aligned}\sum_{j=0}^{\infty} z^j \mathcal{L}_t^j(\chi_k) &= \sum_{j=0}^{\infty} z^j \mathbb{S}_t^j(\vec{Y}_k) + \sum_{\ell=0}^{\infty} z^\ell \mathcal{L}_t^\ell \mathcal{M}_t \sum_{n=0}^{\infty} z^n \mathbb{S}_t^n(\vec{Y}_k) \\ &=: \sum_{j=0}^{\infty} z^j A_j(\vec{Y}_k) + \sum_{j=0}^{\infty} z^j B_j(\vec{Y}_k) . \end{aligned} \tag{59}$$

Since $\int_{I_t} (\phi \circ f_t^j)\, \chi_k \, \mathrm{d}m = \int_{I_t} \phi\, \mathcal{L}_t^j(\chi_k)\, \mathrm{d}m$, using A_j and B_j from (59), we write $\Psi_\phi^{sing,1}(1/2, z)$ as

$$-\frac{u_t}{\pi} \sum_{j=0}^{\infty} z^j \sum_{k\ge 1} \frac{s_{k-1}}{\sqrt{|Df^{k-1}(c_1)|}} \int_{I_t} \phi(x) \Big[A_j(\vec{Y}_k) + B_j(\vec{Y}_k) \Big] \mathrm{d}m . \tag{60}$$

We start with the terms for the A_j. Applying Lemma 5.6 (j times), we find,

$$-\frac{1}{\pi} \int_{I_t} \phi(x) A_j(\vec{Y}_k)\, \mathrm{d}m = -\frac{1}{\pi} s_{j,k} \int_{I_t} \phi(x) \chi_{k+j}(x)\, \mathrm{d}x .$$

Therefore, since $s_{k-1} \cdot s_{j,k} = s_{k+j}$, proceeding as for (58), but with the signs removed, the contribution of the A_j terms in (60) give

$$\begin{aligned} -\frac{u_t}{\pi} \sum_{j=0}^{\infty} z^j \sum_{k\ge 1} \frac{1}{\sqrt{|Df^{k-1}(c_1)|}} \int_{I_t} \phi(x) \cdot s_{k+j} \chi_{k+j}(x) \mathrm{d}x \\ = \mathcal{U}^+_{1/2}(z) \Sigma^{\mathcal{H}}_\phi(z) + \mathcal{V}^{sing,1}_{\phi,1/2}(z) , \end{aligned}$$

with $\mathcal{V}^{sing,1}_{\phi,1/2}(z)$ rational and holomorphic outside of the disc of radius $\sqrt{\lambda_c}^{-1}$.

Next, we analyse the contribution of the B_j in (60). Using $s_{k-1} \cdot s_{n,k} = s_{k+n}$, proceeding as for (58) with the signs removed, using $\int \phi\, \mathrm{d}\mu_t = 0$, and setting $\vec{\Upsilon}_t := -(u_t/\pi) \cdot \sum_{k \ge 1} \frac{s_{k-1}}{\sqrt{|Df^{k-1}(c_1)|}} \vec{Y}_k$, we find[33]

$$\begin{aligned} -\frac{u_t}{\pi} \sum_{j=0}^{\infty} \sum_{k=1}^{\infty} \frac{s_{k-1}\, z^j}{\sqrt{|Df^{k-1}(c_1)|}} \int_{I_t} \phi \cdot B_j(\vec{Y}_k)\, \mathrm{d}m \\ = \sum_{\ell=0}^{\infty} z^\ell \int_{I_t} \phi \cdot \mathcal{L}_t^\ell \mathcal{M}_t \sum_{n=0}^{\infty} z^n \mathbb{S}_t^n(\vec{\Upsilon}_t)\, \mathrm{d}m \\ = \sum_{\ell=0}^{\infty} z^\ell \int_{I_t} (\phi \circ f_t^\ell)\, \mathcal{M}_t \sum_{n=0}^{\infty} z^n \mathbb{S}_t^n(\vec{\Upsilon}_t)\, \mathrm{d}m \\ = -\frac{u_t}{\pi} \cdot \sum_{\ell=0}^{\infty} z^\ell \int_{I_t} (\phi \circ f_t^\ell) \cdot \sum_{n=0}^{\infty} z^n \sum_{k=1}^{\infty} \frac{s_{k-1} s_{n,k}}{\sqrt{|Df^{k-1}(c_1)|}} \mathcal{M}_t(\chi_{k+n})\, \mathrm{d}m \end{aligned} \tag{61}$$

[33] We identify $\chi_k = \chi(\vec{Y}_k)$ with $\vec{Y}_k$ in (61).

$$= -\frac{1}{\pi} \cdot \sum_{\ell=0}^{\infty} z^\ell \int_{I_t} (\phi \circ f_t^\ell) \big(\mathcal{U}^+_{1/2}(z) \Sigma_t^{\tilde\psi_t}(z) + \mathcal{V}^{sing,2}_{1/2}(z)\big)\, \mathrm{d}m\,,$$

with

$$\tilde\psi_t(\ell) = \tilde\chi_{\ell+1} - \rho_t \int_{I_t} \tilde\chi_{\ell+1} \mathrm{d}m\,, \tag{62}$$

and where $z \mapsto \mathcal{V}^{sing,2}_{1/2}(z) \in H^r_q[-2,2]$ is rational, and it is holomorphic outside of the disc of radius $\sqrt{\lambda_c}^{-1}$. Hence, using again $\int \phi \mathrm{d}\mu_t = 0$ and Lemma 5.3,

$$\mathcal{V}_{\phi,1/2}(z) := \mathcal{V}^{reg}_{\phi,1/2}(z) + \mathcal{V}^{sing,0}_{\phi,1/2}(z) + \mathcal{V}^{sing,1}_{\phi,1/2}(z) - \frac{1}{\pi} \cdot \sum_{\ell=0}^{\infty} z^\ell \int_{I_t} (\phi \circ f_t^\ell)\, \mathcal{V}^{sing,2}_{1/2}(z)\, \mathrm{d}m$$

is holomorphic in the annulus $\{\lambda_c^{-1/2} < |z| < \kappa^{-1}\}$.

Finally, the formulas (54) and (55) for the residues follow from Lemma 5.4. In particular, since $\int \phi \mathrm{d}\mu_t = 0$, using Lemma 5.3, we may take

$$\int \phi\, \tilde\psi_t^*\, \mathrm{d}m := -\frac{1}{\pi} \sum_{j=0}^{\infty} z^j \int_{I_t} (\phi \circ f_t^j) \sum_{\ell=L}^{L+P-1} s_\ell\, \tilde\psi_t(\ell)\, \mathrm{d}m\,. \tag{63}$$

□

Proof of Lemma 5.4. If $t \in \mathrm{MT}$, then $\mathcal{U}_{1/2}(z)$ is the rational function

$$\mathcal{U}_{1/2}(z) = \frac{u_t}{z^{L-1}} \Big(\sum_{\ell=0}^{L-1} \frac{s_\ell z^{L-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} + \sum_{\ell=L}^{L+P-1} \frac{s_\ell z^{L-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} \sum_{k=0}^{\infty} \frac{s_{kp,\ell}}{z^{kp}\sqrt{|Df_t^{kp}(c_L)|}} \Big)$$

$$= \frac{u_t}{z^{L-1}} \Big(\sum_{\ell=0}^{L-1} \frac{s_\ell z^{L-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} + \sum_{\ell=L}^{L+P-1} \frac{s_\ell z^{L+P-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} \frac{1}{z^P - \frac{s_{P,L}}{\sqrt{|Df_t^P(c_L)|}}} \Big)\,.$$

Similarly, $\mathcal{U}^+_{1/2}(z)$ is the rational function

$$\mathcal{U}^+_{1/2}(z) = \frac{u_t}{z^{L-1}} \Big(\sum_{\ell=0}^{L-1} \frac{z^{L-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} + \sum_{\ell=L}^{L+P-1} \frac{z^{L+P-1-\ell}}{\sqrt{|Df_t^\ell(c_1)|}} \frac{1}{z^P - \frac{1}{\sqrt{|Df_t^P(c_L)|}}} \Big)\,.$$

We show that $\Sigma_\phi(z)$ and $\Sigma^{\mathcal{H}}_\phi(z)$ are rational, with possible poles at the Pth roots of unity for $\Sigma_\phi(z)$, and at the Pth roots of $\mathrm{sgn}(Df^P(c_L)$ for $\Sigma^{\mathcal{H}}_\phi(z)$. Indeed,

$$\Sigma_\phi(z) = \sum_{\ell=1}^{\infty} \phi(c_\ell) z^{\ell-1} = \sum_{\ell=1}^{L-1} \phi(c_\ell) z^{\ell-1} + \frac{z^{L-1}}{1-z^P} \sum_{\ell=0}^{P-1} \phi(c_{L+\ell}) z^\ell\,.$$

The residue of $\Sigma_\phi(z)$ at 1 is thus $\frac{1}{P}\sum_{\ell=L}^{L+P-1} \phi(c_\ell)$. If $\mathrm{sgn}(Df^P(c_L)) = +1$ then

$$\Sigma^{\mathcal{H}}_\phi(z) = \sum_{\ell=1}^{L-1} s_\ell\, \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_\ell) z^{\ell-1} + \frac{z^{L-1}}{1-z^P} \sum_{\ell=0}^{P-1} s_{L+\ell}\, \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_{L+\ell}) z^\ell\,,$$

in which case the residue at 1 is $\frac{1}{P}\sum_{\ell=L}^{L+P-1} s_\ell \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_\ell)$. If $\mathrm{sgn}(Df^P(c_L)) = -1$

$$\Sigma^{\mathcal{H}}_\phi(z) = \sum_{\ell=1}^{L-1} s_\ell\, \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_\ell) z^{\ell-1} + \frac{z^{L-1}}{1+z^P} \sum_{\ell=0}^{P-1} s_{L+\ell}\, \mathcal{H}(\mathbf{1}_{I_t}\phi)(c_{L+\ell}) z^\ell\,.$$

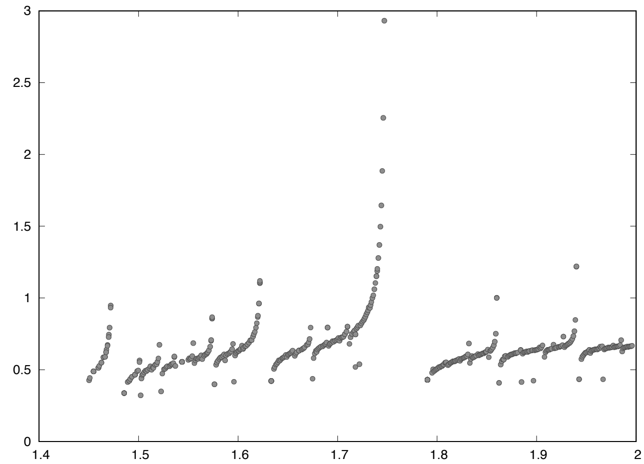

FIGURE 1. $\mathcal{J}_1(t)$ for Misiurewicz–Thurston (MT) parameters t.

The same argument gives that $\Sigma_t^{\tilde{\psi}}(z)$ is rational, with possible poles at the Pth roots of of $\text{sgn}(Df^P(c_L))$, and, when $\text{sgn}(Df^P(c_L)) = 1$, its residue at $z = 1$ is equal to $\frac{1}{P}\sum_{\ell=L}^{L+P-1} s_\ell\, \tilde{\psi}_\ell$. □

6. One-half transversality: Numerics and Conjecture B

Sums $\mathcal{J}_\eta(t)$ of the form (12) with $\eta = 1/2$ play an important role in our study of the fractional response in the quadratic family at a Misiurewicz–Thurston (MT) parameter t. In particular the "one half transversality condition" condition $\mathcal{J}_{1/2}(t) \neq 0$ is essential in Theorem C.

The sums (12) already appeared in the literature: For $\eta = 1$, we recover the Tsujii transversality condition $\mathcal{J}_1(t) \neq 0$ (see Tsujii [41]), which is satisfied for *every* MT parameter in the quadratic family (and in a far larger class of parameters, see Footnote 12). Figure 1 illustrates the graph of $\mathcal{J}_1(t)$ over hundreds of MT parameters. (We explain in §6.1, how these MT parameters were obtained.)

For $\eta = 1/2$, the set of 1/2-summable parameters is important in the study of unimodal maps. Nowicki and van Strien [29] proved (in particular) that quadratic maps that satisfy the 1/2-summability condition have an absolutely continuous invariant probability measure. It turns out that, in the complement of the hyperbolic parameters, *almost every parameter* satisfies the 1/2-summability condition (see Lyubich [23], and also Martens and Nowicki [24, §4]).

However, the condition $\mathcal{J}_{1/2}(t) \neq 0$ does not seem to have appeared in the literature. The reader may wonder when this condition holds. We do not have a definitive answer for this. As observed after the statement of Theorem C, it is easy to see that $\mathcal{J}_{1/2}(2) = 0$. This first came as a surprise to us, but it is in fact natural, as we explain next.

We already noticed that the piecewise expanding and piecewise analytic map F_t conjugated to f_t via the change of variable $\Lambda = \Lambda_t$ given by its invariant densities (see (73)) $(D_tF_t)^k(\Lambda(c_{1,t})_\pm) = s_k\sqrt{|(Df_t^k)(c_{1,t})|}$ we see that $\mathcal{J}_{1/2}(t) \neq 0$ is just the ordinary transversality assumption [10] of F_t for the vector field $v \equiv 1$ on

$[F_t(\Lambda(c_{1,t})), \Lambda(c_{1,t})]$. Noting that F_2 is just the full tent map with slopes ± 2, the fact that $\mathcal{J}_{1/2}(t) = 0$ for the full quadratic map mirrors the fact that the family of tent maps $\tilde{F}_t$ is tangential[34] so that F_2 is tangential for the vector field $v \equiv 1$.

Note also that the measure of maximal entropy of f_2 coincides with the absolutely continuous measure. See [28, §8, §9] for classical necessary conditions for this property to hold. More recently, based on [12, Theorem 2], Dobbs and Mihalache observed [13, Fact 5.2] that the measure of maximal entropy of an S-unimodal map f with positive entropy is absolutely continuous if and only if f is[35] pre-Chebyshev. The ([13, Proposition 5.1] only pre-Chebyshev quadratic map is $f_2 : x \to 2 - x^2$.

The parameter $t = 2$ corresponds to the simplest combinatorics $0 \mapsto c_1 \mapsto c_2 \mapsto c_2$. One can check that $\mathcal{J}_{1/2}(t) \neq 0$ holds for the parameter t corresponding to the next simplest Misiurewicz–Thurston combinatorics (beware that it is not mixing) $0 \mapsto c_1 \mapsto c_2 \mapsto c_3 \mapsto c_3$. Indeed, this parameter is $t = 1.54368\ldots$, and we have $f_t(c_3) = c_3$ with $\lambda_1 = f_t'(c_1) = -3.0874\ldots$, $\lambda_2 = Df_t(c_2) = -Df_t(c_3) = 1.6786\ldots$, so that a geometric series gives

$$\mathcal{J}^{1/2}(t) = 1 - \frac{1}{\sqrt{|\lambda_1|}} - \frac{1}{\sqrt{|\lambda_1 \lambda_2|}} \frac{1}{1 + 1/\sqrt{\lambda_2}} = 0.182959\ldots .$$

6.1. **Numerics.** We have performed numerical experiments to investigate $\mathcal{J}_{1/2}(t)$ for hundreds of Misiurewicz–Thurston parameters: We calculate 858 MT parameters t, with high accuracy, and we compute the corresponding sums $\mathcal{J}_{1/2}(t)$.

The algorithm consists into finding approximate values for Misiurewicz–Thurston parameters in the real line, and then use the Milnor–Thurston transformation to obtain such parameters with higher precision. Indeed, given a real Misiurewicz–Thurston parameter $t = c_1$ such that

$$f_{c_1}^{k+j}(0) = f_{c_1}^{k}(0)$$

for some $k \geq 1$ and $j \geq 1$, choose a point $x = (x_1, x_2, \ldots, x_{k+j-1}) \in \mathbb{R}^{k+j-1}$ such that $x_i \cdot f_{c_1}^i(0) > 0$ for every $1 \leq i < j + k$. Next, define

$$T(x_1, x_2, \ldots, x_{k+j-1}) = (y_1, y_2, \ldots, y_{k+j-1})$$

where $f_{x_1}(y_i) = x_{i+1}$ for $i < k + j - 2$, while $f_{x_1}(y_{k+j-1}) = x_k$, and $y_i \cdot f_{c_1}^i(0) > 0$ for every $0 < i < j + k$. Then Milnor and Thurston [27, Proof of Lemma 13.4] proved that $T^\ell(x)$ converges to $(c_1, f_{c_1}(c_1), \ldots, f_{c_1}^{k+j-1}(c_1))$ exponentially fast.

We explain next why some of the Misiurewicz–Thurston parameters found by this algorithm are not renormalizable, and hence mixing: The critical point of a Misiurewicz–Thurston map f is not periodic, and there is $L > 0$ such that $f^L(0)$ is periodic. Taking L minimal with this property, let $P \geq 1$ be such that $f^P(f^L(0)) = f^L(0)$. Suppose that P is a prime number, $P \neq 2$, the multiplier $Df^P(f^L(0))$ is positive, and f is renormalizable. Then the period of the first (and only) renormalization of f is P,

$$F = f^P : [|f^L(0)|, -|f^L(0)|] \to [-|f^L(0)|, |f^L(0)|]$$

[34] The topological entropy is the logarithm of the slope and thus constant, so there are no bifurcations. It is illuminating to construct explicitly the corresponding topological conjugacy.

[35] A unimodal map f is called pre-Chebyshev if f is exactly m times renormalisable, for some $m \geq 0$, each renormalisation being of period two, and, in addition, if J is the restrictive interval for the mth renormalisation, $f^{2^m}|_J : J \to J$ is smoothly conjugate on J to $x \mapsto 1 - 2|x|$ on $(-1, 1)$.

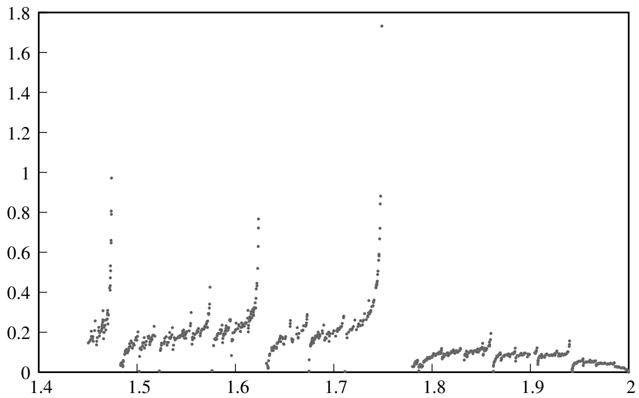

FIGURE 2. $\mathcal{J}_{1/2}(t)$ for MT parameters t.

is the first (and only) renormalization of f, and, additionally, $F(f^L(0)) = f^L(0)$, while $F(0)$ is not a fixed point of F, and $F^2(0) = f^L(0)$. In particular $f^{2P}(f^L(0)) = f^L(0)$. This implies $2P \geq L > P$. Our numerical experiment give Misiurewicz–Thurston maps f for which P is a prime number, $P \neq 2$, the multiplier $Df^P(f^L(0))$ is positive, but $2P \geq L > P$ does not hold, so that f is not renormalizable. Moreover the numerical experiment gives $J_{1/2}(f) \neq 0$.

The resulting graph for $\mathcal{J}_{1/2}(t)$ can be seen in Figure 2. (To be compared with Figure 1 for the graph of the Tsujii transversality condition $\mathcal{J}_1(t)$.) The value of $\mathcal{J}_{1/2}(t)$ seems to be always strictly positive except at $t = 2$, where it vanishes. However, $\mathcal{J}_{1/2}(t)$ appears to be close to zero (see Figure 3 for a close-up) at a few values of t. The "almost vanishing of $\mathcal{J}_{1/2}(t)$" phenomenon seems to occur when the real Misiurewicz–Thurston parameter t is such that f_t is renormalisable with deepest (i.e., last) renormalisation has[36] topological entropy $\log 2$ (that is, there is a periodic point $-x_* \in \mathbb{R}$ with period $n \geq 2$ such that the intervals $f_t^k[-x_*, x_*]$, for $k = 0, \dots, n-1$, are pairwise disjoint, except possibly at their boundaries, with $f_t^n[-x_*, x_*] \subset [-x_*, x_*]$, and the unimodal map $g\colon [-1,1] \to [-1,1]$ defined by $g(x) = x_*^{-1} f_t^n(x_* x)$ satisfies $g(-1) = g(1) = -1$ and $g(0) = x_*$). Moreover it seems that this deepest renormalisation is close to a quadratic polynomial on the interval of renormalisation. (This last property happens when the so called "complex bounds" are large enough. This occurs for instance in the first renormalisation of parameters very close to $t = 2$, see e.g. Douady and Hubbard [15, Proof of Thm 5].)

Note that if the deepest renormalisation is conjugated to the Ulam–von Neumann map, then [8] does not give any lower bound for the regularity of the SRB measure.

6.2. **Conjecture B on one-half-transversality for the quadratic family.** Our numerical experiments support and motivate the following conjecture.

Conjecture B. *For the quadratic family f_t, we have:*

i. *For every real MT parameter $t \neq 2$, we have $\mathcal{J}_{1/2}(t) > 0$.*
ii. *In fact,* $\inf\{\mathcal{J}_{1/2}(t) \mid t \text{ a real MT parameter}, t \neq 2\} = 0$.

[36] In particular, this deepest renormalisation is topologically conjugated to f_2.

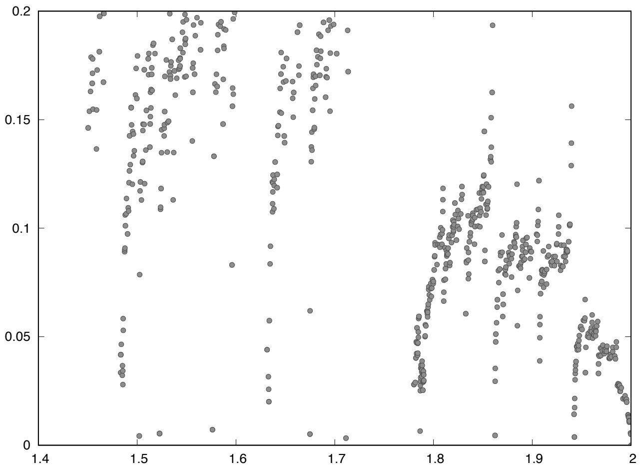

FIGURE 3. Close-up of $\mathcal{J}_{1/2}(t)$ for MT parameters t.

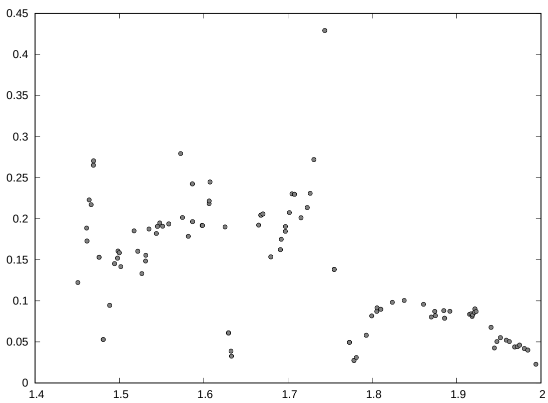

FIGURE 4. $\mathcal{J}^{\rm per}_{1/2}(t)$ for periodic parameters t.

iii. *More generally, if $t \neq 2$ satisfies $1/2$-summability, then $\mathcal{J}_{1/2}(t) > 0$.*
iv. *The parameter $\eta = 1/2$ is a critical exponent in the following sense: If $\eta > 1/2$ then $\mathcal{J}_\eta(t) > 0$ for every real MT parameter t. If $\eta \in (0, 1/2)$ there are infinitely many real parameters t such that $\mathcal{J}_\eta(t) < 0$.*

G. Levin suggested that we perform experiments also for parameters such that $f_t^P(c) = c$ for some $P \geq 1$. For such t, we set

$$\mathcal{J}^{\rm per}_\eta(t) = \sum_{k=0}^{P-1} \frac{\mathrm{sgn}((Df_t^k(c_1)))}{|Df_t^k(c_1)|^\eta}.$$

In view of the resulting data, which is presented in Figure 4, we expect that claim [i] of Conjecture B also holds for all real periodic parameters.

Finally, note that if $t > 2$ then $c_1 > 2$ and, for all $k \geq 1$, we have $f_t^{k+1}(c_1) > f_t^k(c_1) > 2$, so that $|Df_t^{k+1}(c_1)| > |Df_t^k(c_1)| > 4$, while $\mathrm{sgn}((Df_t^{k+1}(c_1)) =$

$-\mathrm{sgn}((Df_t^k(c_1)))$. Thus, the sum (12) converges absolutely for any $\eta > 0$ and

$$\mathcal{J}_\eta(t) > 1 - \frac{1}{4^\eta} > 0\,,\ \forall t > 2\,,\ \forall \eta > 0\,.$$

7. Whitney fractional integrals $I^{\eta,\Omega}$ and derivatives $M^{\eta,\Omega}$

7.1. **Abel's remark for Whitney fractional integrals $I^{1/2,\Omega}$ (Lemma E).** For $\Omega \subset (1,2)$ satisfying (5) and $t \in \Omega$, it is natural to consider the one-sided Ω-*(Whitney–)Riemann–Liouville fractional integrals of* $\phi \in L^1$ on Ω defined by

$$(I_+^{\eta,\Omega}\phi)(t) = \frac{1}{\Gamma(\eta)} \int_{\Omega\cap(-\infty,t]} \frac{\phi(\tau)}{(t-\tau)^{1-\eta}}\,\mathrm{d}\tau\,,$$
$$(I_-^{\eta,\Omega}\phi)(t) = \frac{1}{\Gamma(\eta)} \int_{\Omega\cap[t,\infty)} \frac{\phi(\tau)}{(\tau-t)^{1-\eta}}\,\mathrm{d}\tau\,,$$

and the two-sided corresponding object defined by

$$I^{\eta,\Omega}\phi(t) = \frac{1}{2\Gamma(\eta)\cos(\eta\pi/2)} \int_\Omega \frac{\phi(\tau)}{|t-\tau|^{1-\eta}}\,\mathrm{d}\tau\,.$$

Recalling the spikes $\phi_{c_k,\pm}$ from (28), we give an analogue of Lemma 3.1.

Lemma E (Abel's remark on Ω)**.** *Let $\Omega \subset (1,2)$ be a compact positive Lebesgue measure set satisfying Tsujii's property* (5) *for all $\beta < 2$. For any $k \ge 1$, the one-sided Ω-Riemann–Liouville half integrals of the square root spikes satisfy*

$$I_{-,t}^{1/2,\Omega}(\phi_{c_k,+})(x,t) = A^\Omega_{c_k,+}(x,t) + \sqrt{\pi}\cdot \mathbf{1}_{x>c_k+t}(x)\,,$$
$$I_{+,t}^{1/2,\Omega}(\phi_{c_k,-})(x,t) = A^\Omega_{c_k,-}(x,t) + \sqrt{\pi}\cdot \mathbf{1}_{x<c_k+t}(x)\,,$$

where $A^\Omega_{c_k,\pm}(x,t) \le 0$ are defined by $A^\Omega_{c_k,\sigma}(x,t) = 0$ if $\sigma x \le \sigma(c_k+t)$, and

$$A^\Omega_{c_k,\sigma}(x,t) = -\frac{1}{\Gamma(1/2)} \int_0^1 \frac{\mathbf{1}_{\Omega^c}(t+\sigma(x-(c_k+t))u)}{u^{1/2}(1-u)^{1/2}}\,\mathrm{d}u \text{ if } \sigma x > \sigma(c_k+t)\,.$$

In addition, $x \mapsto A^\Omega_{c_k,\sigma}(x,t)$ is η Hölder, for $\sigma = \pm$ and for all $\eta < 1/2$.

We refrain from stating the Ω version of the two-sided Lemma 3.2. (Since (69) cannot be used, the proof must be more "hands on." See also Remark 7.1.)

Proof of Lemma E. We handle $\phi_{c_k,+}(x,t)$. The case of $\phi_{c_k,-}(x,t)$ is symmetric. The Whitney half integral $I_{-,t}^{1/2,\Omega}$ of the spike $\phi_{c_k,+}(x,t)$ with respect to t is

$$\begin{aligned}(I_{-,t}^{1/2,\Omega}\phi_{c_k,+})(x,t) &= \frac{1}{\Gamma(1/2)} \int_{\tau\in[t,\infty]\cap\Omega} \frac{\phi_{c_k,+}(x,\tau)}{(\tau-t)^{1/2}}\,\mathrm{d}\tau \\ &= \frac{1}{\Gamma(1/2)} \int_{\tau\in[t,\infty]\cap\Omega} \frac{\mathbf{1}_{y>c_k+\tau}(x)}{(x-c_k-\tau)^{1/2}(\tau-t)^{1/2}}\,\mathrm{d}\tau \\ &= \begin{cases} 0 & \text{if } c_k+t \ge x, \\ \frac{1}{\Gamma(1/2)}\int_t^{x-c_k} \left(\frac{\mathbf{1}_\Omega(\tau)}{(\tau-t)(x-c_k-\tau)}\right)^{1/2}\mathrm{d}\tau & \text{if } c_k+t < x\,. \end{cases}\end{aligned}$$

If $c_k + t < x$, making the substitution $\tau = t + (x - c_k - t)u$, we get

$$(I_{-,t}^{1/2,\Omega}\phi_{c_k,+})(x,t) = \frac{1}{\Gamma(1/2)} \int_0^1 \frac{\mathbf{1}_\Omega(t+(x-c_k-t)u)}{u^{1/2}(1-u)^{1/2}}\,\mathrm{d}u\,.$$

Recalling (29), the function $A^{\Omega}_{c_k,-}(x,t) := (I^{1/2,t}_{\Omega,-}\phi_{c_k,+})(x,t) - \sqrt{\pi}\mathbf{1}_{x>c_k+t}$ vanishes for $c_k + t \geq x$, while for $x > c_k + t$ we have,

$$-\Gamma(1/2)\cdot A^{\Omega}_{c_k,-}(x,t) = \int_0^1 \frac{\mathbf{1}_{\Omega^c}(t+(x-c_k-t)u)}{u^{1/2}(1-u)^{1/2}}\,\mathrm{d}u\,.$$

Next, we show that $A^{\Omega}_{c_k,-}(x,t)$ is η Hölder for all $\eta < 1/2$ at $c_k + t$. Fixing $q \in (2, 1/\eta)$ and $\tilde{q} < 2$ such that $1/q + 1/\tilde{q} = 1$, first note that $\|u^{-1/2}(1-u)^{-1/2}\|_{L^{\tilde{q}}([0,1])} < \infty$. Then, setting $\epsilon = x - c_k - t > 0$, we have

$$-\Gamma(1/2)\cdot A^{\Omega}_{c_k,-}(x,t) \leq \|\mathbf{1}_{\Omega^c}(t+\epsilon u)\|_{L_q([0,1])}\cdot\|u^{-1/2}(1-u)^{-1/2}\|_{L^{\tilde{q}}([0,1])}\,,$$

by the Holder inequality. Next, using (5), we find for any $\beta > 2$,

$$\|\mathbf{1}_{\Omega^c}(t+\epsilon u)\|_{L_q([0,1])} = \left(\int_0^{\epsilon}\frac{\mathbf{1}_{\Omega^c}(t+v)}{\epsilon}\,\mathrm{d}v\right)^{1/q} \leq C_\beta\cdot\epsilon^{(\beta-1)/q}\,.$$

By taking $\beta < 2$ close enough to 2 we may ensure $(\beta-1)/q \geq \eta$. Recalling that $x = \epsilon + c_k + t$, this proves that $A^{\Omega}_{c_k,-}(x,t)$ is η Hölder at $c_k + t$.

Finally, we prove that for any $\eta < 1/2$ there exists $C_\eta < \infty$ such that

$$|A^{\Omega}_{c_k,-}(x_2,t) - A^{\Omega}_{c_k,-}(x_1,t)| \leq C_\eta|x_2 - x_1|^\eta\,,\quad \forall x_i > c_k + t\,,\ i = 1,2\,.$$

For this, assuming without loss of generality that $x_1 < x_2$, we have

$$\begin{aligned}
&\Gamma(1/2)|A^{\Omega}_{c_k,-}(x_2,t) - A^{\Omega}_{c_k,-}(x_1,t)| \\
&\quad= \int_t^{x_2-c_k}\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_2-c_k-\tau}\sqrt{\tau-t}}\,\mathrm{d}\tau - \int_t^{x_1-c_k}\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_1-c_k-\tau}\sqrt{\tau-t}}\,\mathrm{d}\tau \\
&\quad= \int_{x_1-c_k}^{x_2-c_k}\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_2-c_k-\tau}\sqrt{\tau-t}}\,\mathrm{d}\tau \qquad (64)\\
&\qquad+ \int_t^{x_1-c_k}\left(\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_2-c_k-\tau}\sqrt{\tau-t}} - \frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_1-c_k-\tau}\sqrt{\tau-t}}\right)\mathrm{d}\tau\,. \qquad (65)
\end{aligned}$$

Using the change of variable $\tau = (x_1 - c_k) + (x_2 - x_1)u$, with $\mathrm{d}\tau = (x_2 - x_1)\mathrm{d}u$, for the integral in (64), and the Hölder inequality for $1/\tilde{q} + 1/q = 1$ with $1 < \tilde{q} < 2$ and $2 < q < 1/\eta$, we find

$$\begin{aligned}
&\int_{x_1-c_k}^{x_2-c_k}\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{x_2-c_k-\tau}\sqrt{\tau-t}}\,\mathrm{d}\tau \\
&\quad\leq \left(\int_{x_1-c_k}^{x_2-c_k}(\mathbf{1}_\Omega(\tau))^q\,\mathrm{d}\tau\right)^{1/q}\left(\int_{x_1-c_k}^{x_2-c_k}\left(\frac{1}{\sqrt{x_2-c_k-\tau}\sqrt{\tau-t}}\right)^{\tilde{q}}\mathrm{d}u\right)^{1/\tilde{q}} \\
&\quad\leq |x_1-x_2|^{1/q}\left(\int_0^1\left(\frac{1}{\sqrt{u+(x_1-c_k-t)/(x_2-x_1)}\sqrt{1-u}}\right)^{\tilde{q}}\mathrm{d}u\right)^{1/\tilde{q}} \\
&\quad\leq |x_1-x_2|^{1/q}\left(\int_0^1\left(\frac{1}{\sqrt{u}\sqrt{1-u}}\,\mathrm{d}u\right)^{\tilde{q}}\right)^{1/\tilde{q}} \leq C_\eta|x_1-x_2|^\eta\,.
\end{aligned}$$

It remains to estimate (65). We rewrite the integral as

$$\int_t^{x_1-c_k}\frac{\mathbf{1}_\Omega(\tau)}{\sqrt{\tau-t}}\frac{1}{\sqrt{x_1-c_k-t}}\left(\frac{\sqrt{x_1-c_k-\tau}}{\sqrt{x_2-c_k-\tau}}-1\right)\mathrm{d}\tau\,.$$

Now, for any $1 < \tilde q < 2$, we have

$$\frac{1}{\sqrt{\tau - t}\sqrt{x_1 - c_k - t}} \in L^{\tilde q}([t, x_1 - c_k])\,,\ \text{uniformly in } x_1 > c_k + t\,.$$

So, by the Hölder inequality, setting $w = x_1 - c_k$ and $\delta = x_2 - x_1$, it suffices to take $2 < q < 1/\eta$, and $\tilde q < 2$ with $1/\tilde q + 1/q = 1$, and estimate

$$\begin{aligned}\int_t^w \left(\sqrt{1 - \frac{\delta}{w + \delta - \tau}} - 1\right)^q \mathrm{d}\tau &\le \int_t^w \left(\frac{\delta}{w + \delta - \tau}\right)^{q/2} \mathrm{d}\tau \\ &\le \delta^{q/2} \int_t^w \left(\frac{1}{w + \delta - \tau}\right)^{q/2} \mathrm{d}\tau \\ &= \delta^{q/2} \cdot \frac{1}{1 - q/2} (w + \delta - \tau)^{1 - q/2}\Big|_{\tau = t}^{w}\,.\end{aligned}$$

Finally, if $2 < q < 1/\eta$, we have $\left(\delta^{q/2}\delta^{1-q/2}\right)^{1/q} \le \delta^{1/q} \le |x_1 - x_2|^\eta$. □

7.2. **The semifreddo function. Proposition F.** Let g be a γ-Hölder function defined on a closed subset $\Omega \subset \mathbb{R}$ of positive Lebesgue measure. Then, for any $\eta < \gamma$, by analogy with the notion of the derivative in the sense of Whitney, we define[37] the left-sided *Whitney–Marchaud derivative of* g on Ω to be

$$\begin{aligned}(M_+^{\eta,\Omega} g)(x) &= \frac{\eta}{\Gamma(1-\eta)} \int_{\Omega \cap (-\infty, x]} \frac{g(x) - g(y)}{(x - y)^{1+\eta}} \,\mathrm{d}y \\ &= \frac{\eta}{\Gamma(1-\eta)} \int_{\Omega - x \cap (-\infty, 0]} \frac{g(x) - g(x + \tau)}{|\tau|^{1+\eta}} \,\mathrm{d}\tau\,.\end{aligned}$$

We then define

$$(M_-^{\eta,\Omega} g)(x) = \frac{\eta}{\Gamma(1-\eta)} \int_{\Omega - x \cap [0, \infty)} \frac{g(x) - g(x + \tau)}{\tau^{1+\eta}} \,\mathrm{d}\tau\,.$$

and

$$(M^{\eta,\Omega} g)(x) = \frac{\eta}{2\Gamma(1-\eta)} \int_{\Omega - x} \frac{g(x + \tau) - g(x)}{|\tau|^{1+\eta}} \mathrm{sgn}(\tau) \,\mathrm{d}\tau\,.$$

Remark 7.1 (Boundary of Ω). Beware that integration by parts with respect to the variable t is problematic for $M^{\eta,\Omega}$ since $\partial\Omega$ is wild in our application. (In particular, the analogue of Proposition 2.5 is not obvious.)

Remark 7.2. In view of Conjecture A, it is desirable to prove versions of Lemma 4.4 (as well as Lemmas 4.5 and 4.6) for $M^{1/2,\Omega}$, if Ω is compact and satisfies Tsujii's condition (5) for suitable β. Although it seems possible to bypass the (problematic) integration by parts in τ in the proof of Lemma 4.4 by using instead an infinite Taylor series for $g(\tau)$, we refrain from including this analysis here.

We define yet another susceptibility function:

[37] This definition is meaningful if x is a point of Ω with nonzero Lebesgue density, see also Proposition F. In our setting, we may use the stronger condition (5).

Definition 7.3 (Semifreddo fractional susceptibility function)**.** The *semifredddo fractional susceptibility function* at $t \in$ TSR and along Ω is the following formal power series

$$\Psi_\phi^{\Omega,\mathrm{sf}}(\eta, z) := \sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k)(x)(M_s^{\eta,\Omega}\mathcal{L}_s\rho_t(x)|_{s=t})\,\mathrm{d}x\,.$$

This function lies "between" $\Psi_\phi^{\mathrm{fr}}(\eta, z)$ and $\Psi_\phi^{\Omega}(\eta, z)$ since $\Psi_\phi^{\Omega,\mathrm{sf}}(\eta, z)$ is

$$\frac{\eta}{2\Gamma(1-\eta)} \sum_{k=0}^{\infty} z^k \int (\phi \circ f_t^k)(x) \int_{\mathbb{R}\cap(\Omega - t)} \frac{\big((\mathcal{L}_{t+\delta} - \mathcal{L}_t)\rho_t\big)(x)}{|\delta|^{1+\eta}} \mathrm{sgn}(\delta)\,\mathrm{d}\delta\,\mathrm{d}x\,.$$

The results in this section together with Theorem C motivate the statement on $\Psi^{\Omega,\mathrm{sf}}(\eta, z)$ in Remark 1.2. (In addition, we expect that Proposition F should allow to prove that $\lim_{\eta\uparrow 1} \Psi_\phi^{\Omega,\mathrm{fr}}(\eta, z) = \Psi_\phi(z)$, as formal power series.)

The following notion of differentiability seems to be relevant in our context:

Definition 7.4 (Ω-Whitney differentiability)**.** Let g be a function defined on a closed subset $\Omega \subset \mathbb{R}$. We say that g is Ω-Whitney differentiable at $t \in \Omega$ if there exists $g'_\Omega(t) \in \mathbb{C}$ such that

$$\lim_{\delta \to 0, t+\delta \in \Omega} \frac{g(t+\delta) - g(t)}{\delta} = g'_\Omega(t)\,.$$

For $\zeta \in (0, 1)$, we say that g is Ω-Whitney ζ-differentiable at $t \in \Omega$ if there exists $g^\zeta_\Omega(t) \in \mathbb{C}$, such that

$$\lim_{\delta \to 0, t+\delta \in \Omega} \frac{g(t+\delta) - g(t)}{\mathrm{sgn}(\delta)|\delta|^\zeta} = g^\zeta_\Omega(t)\,.$$

The following proposition shows that $M^{\eta,\Omega}$ is naturally related to Ω-Whitney differentiability for large enough sets Ω:

Proposition F ($M^{\eta,\Omega}$ and Ω-Whitney differentiability)**.** *Let $\Omega \subset \mathbb{R}$ satisfy Tsujii's condition* (5) *for some $\beta > 1$. Then for any bounded function $g : \mathbb{R} \to \mathbb{C}$ which is Ω-Whitney differentiable at $t \in \Omega$, we have $\lim_{\eta\uparrow 1}(M^{\eta,\Omega} g)(t) = g'_\Omega(t)$.*

Moreover, for any $\zeta \in (0, 1)$ and any bounded $g : \mathbb{R} \to \mathbb{R}$ which is Ω-Whitney ζ-differentiable at $t \in \Omega$, we have,

$$\lim_{\eta\uparrow\zeta} \left(\frac{\Gamma(1-\eta)}{\zeta \cdot \Gamma(\zeta - \eta)} (M^{\eta,\Omega} g)(t) \right) = g^\zeta_\Omega(t)\,.$$

Proof of Proposition F. Let us assume to fix ideas that $t = 0$ and $g'_\Omega(0) \geq 0$.

We first prove that $\lim_{\eta\uparrow 1} M_+^{\eta,\Omega} g(0) = g'_\Omega(0)$ by showing that, for any $\epsilon > 0$, we have $(1-\epsilon)(g'(0) - \epsilon) \leq \liminf_{\eta\uparrow 1} M_+^{\eta,\Omega} g(0) \leq \limsup_{\eta\uparrow 1} M_+^{\eta,\Omega} g(0) \leq g'_\Omega(0) + \epsilon$.

First, since $g'_\Omega(0) = \lim_{h\to 0, h\in\Omega} \frac{g(h)-g(0)}{h}$, we can find $\delta > 0$ such that

$$-\epsilon \leq \frac{g(h) - g(0)}{h} - g'_\Omega(0) \leq \epsilon\,, \quad \forall h \in \Omega\,,\ |h| \leq \delta\,.$$

Then, since $0 < \eta < 1$, we can write

$$M_+^{\eta,\Omega} g(0) = \frac{\eta}{\Gamma(1-\eta)} \left(\int_{\Omega\cap[0,\delta]} \frac{g(0) - g(-t)}{t} \frac{1}{t^\eta}\,\mathrm{d}t + \int_{\Omega\cap[\delta,\infty)} \frac{g(0) - g(-t)}{t^{1+\eta}}\,\mathrm{d}t \right)$$

$$\leq \frac{\eta}{\Gamma(1-\eta)}\left((g'_\Omega(0)+\epsilon)\int_0^\delta \frac{1}{t^\eta}\,\mathrm{d}t + \int_\delta^\infty \frac{2\sup|g|}{t^{1+\eta}}\,\mathrm{d}t\right)$$
$$\leq \frac{\eta}{(1-\eta)\Gamma(1-\eta)}(g'_\Omega(0)+\epsilon)\delta^{1-\eta} + \frac{2}{\Gamma(1-\eta)}\sup|g|\,\delta^{-\eta}\,.$$

Using that

$$(1-\eta)\Gamma(1-\eta) = \Gamma(2-\eta) \text{ with } \Gamma(1)=1\,, \quad \text{while } \lim_{\eta\uparrow 1}\Gamma(1-\eta)=\infty \text{ and } \lim_{\eta\uparrow 1}\delta^{1-\eta}=1\,, \tag{66}$$

we find $\limsup_{\eta\uparrow 1} M_+^{\eta,\Omega} g(0) \leq g'_\Omega(0)+\epsilon$.

The proof that $\liminf_{\eta\uparrow 1} M_+^\eta g(0) \geq (g'_\Omega(0)-\epsilon)(1-\epsilon)$ is a bit trickier and will use (5) for $\beta>1$. By the above computation for the limsup, taking η close enough to 1 for fixed δ it suffices to show that

$$\liminf_{\eta\uparrow 1}\frac{\eta}{\Gamma(1-\eta)}\int_{\Omega\cap[0,\delta]}\frac{g(0)-g(-t)}{t^{1+\eta}}\,\mathrm{d}t \geq g'_\Omega(0)-\epsilon\,. \tag{67}$$

Since

$$\frac{\eta}{\Gamma(1-\eta)}\int_{\Omega\cap[0,\delta]}\frac{g(0)-g(-t)}{t}\frac{1}{t^\eta}\,\mathrm{d}t \geq \frac{\eta}{\Gamma(1-\eta)}\int_{\Omega\cap[0,\delta]}(g'_\Omega(0)-\epsilon)\frac{1}{t^\eta}\,\mathrm{d}t\,,$$

using again $(1-\eta)\Gamma(1-\eta)=\Gamma(2-\eta)$, the bound (67) reduces to

$$\liminf_{\eta\uparrow 1}\int_{\Omega\cap[0,\delta]}\frac{1-\eta}{t^\eta}\,\mathrm{d}t \geq 1\,.$$

We already know that $\lim_{\eta\uparrow 1}\int_{[0,\delta]}\frac{1-\eta}{t^\eta}\,\mathrm{d}t = \lim_{\eta\uparrow 1}\delta^{1-\eta}=1$, so it suffices to show

$$\lim_{\eta\uparrow 1}\int_{\Omega^c\cap[0,\delta]}\frac{1-\eta}{t^\eta}\,\mathrm{d}t = 0\,.$$

This will follow from the fact that

$$\lim_{\eta\uparrow 1}\int_{\Omega^c\cap[0,\delta]}\frac{1}{t^\eta}\,\mathrm{d}t < \infty\,.$$

The above bound, i.e. uniform Lebesgue integrability of $F_\eta(\tau)=\mathbf{1}_{\Omega^c\cap[0,\delta]}(\tau)\cdot\tau^{-\eta}$ as $\eta\uparrow 1$, will follow from (5) for $\beta>1$ at $t=0$. Indeed, for any fixed $1<\beta<2$, there exists $C_\beta<\infty$ such that for all $t>\delta^{-\eta}$ we have

$$m\{\tau\in\Omega^c\cap[0,\delta] \mid \tau^{-\eta}>t\} = m\{\tau\in\Omega^c\cap[0,\delta] \mid \tau < t^{-1/\eta}\} \leq C_\beta t^{-\beta/\eta}\,.$$

Observe next that

$$\limsup_{\eta\uparrow 1}\int_{\delta^{-\eta}}^\infty C_\beta t^{-\beta/\eta}\,\mathrm{d}t < \infty \text{ if } \beta>1\,, \text{ and } \limsup_{\eta\uparrow 1}\delta\int_0^{\delta^{-\eta}}\,\mathrm{d}t = 1\,.$$

To conclude, just apply the characterisation of Lebesgue integrability of a nonnegative measurable function F given by $\int_0^\infty m(\tau \mid F(\tau)>t)\,\mathrm{d}t<\infty$, see e.g. [21, §1.5, p. 14, (2)], and use $t\mapsto g(-t)$ to get $\lim_{\eta\uparrow 1} M_-^{\eta,\Omega}g(0) = -g'_\Omega(0)$.

The second claim of Proposition F follows from the same replacing (66) by $(\zeta-\eta)\Gamma(\zeta-\eta)=\Gamma(1+\zeta-\eta)$, with $\Gamma(1)=1$, while $\lim_{\eta\uparrow\zeta}\Gamma(\zeta-\eta)=\infty$ and $\lim_{\eta\uparrow\zeta}\delta^{\zeta-\eta}=1$. □

Appendix A. Proof of Lemma 3.2. (Abel's remark, two-sided)

Recall that if $|x| < 1$ then

$$\sqrt{1+x} = 1 + \sum_{n=1}^{\infty} b_n x^n \text{ with } b_n = \frac{(1/2)(-1/2)\cdots(1/2-n+1)}{n!}\,. \tag{68}$$

(In particular $b_1 = 1/2$ and $b_2 = -1/8$.) We shall also use the fact [25, 195.01] that for any real numbers $a > \tau$ and $b > \tau$

$$\partial_\tau \log\bigl(|\sqrt{a-\tau} - \sqrt{b-\tau}|\bigr) = \frac{1}{2}\frac{1}{\sqrt{a-\tau}\sqrt{b-\tau}}\,. \tag{69}$$

Proof of Lemma 3.2. Set $x_0 = x - c_k$. To prove the claim on $I^{1/2}_{+,t}(\phi_{c_k,+,\mathcal{Z}})(x,t)$, it is enough to show that, for any $c_k + t - \mathcal{Z} < x < c_k + t + \mathcal{Z}$,

$$\sqrt{\pi}\cdot I^{1/2}_{+,t}(\phi_{c_k,+,\mathcal{Z}})(x,t) = -\log|x_0 - t| + \log\mathcal{Z} + G_{\mathcal{Z}}(t - x_0)\,, \tag{70}$$

where, for $y \in (-\mathcal{Z}/2, \mathcal{Z}/2)$,

$$G_{\mathcal{Z}}(y) = -2\log H_{\mathcal{Z}}(y)\,, \quad H_{\mathcal{Z}}(y) = \frac{\sqrt{1+\frac{y}{\mathcal{Z}}} - 1}{y/\mathcal{Z}} > 0\,.$$

Indeed, using (68), we have (the power series below are absolutely convergent)

$$H_{\mathcal{Z}}(y) = 1/2 + \sum_{j=1}^{\infty} b_{j+1}\left(\frac{y}{\mathcal{Z}}\right)^j\,,\ H_{\mathcal{Z}}(0) = 1/2\,,$$

$$\partial_y G_{\mathcal{Z}}(y) = -\frac{2}{H_{\mathcal{Z}}(y)}\cdot\sum_{j=1}^{\infty} j\cdot b_{j+1}\frac{y^{j-1}}{\mathcal{Z}^j}\,,$$

$$\partial_y^2 G_{\mathcal{Z}}(y) = \frac{2}{(H_{\mathcal{Z}}(y))^2}\cdot\sum_{j=1}^{\infty} j\cdot b_{j+1}\frac{y^{j-1}}{\mathcal{Z}^j} - \frac{2}{H_{\mathcal{Z}}(y)}\cdot\sum_{j=2}^{\infty} j(j-1)b_{j+1}\frac{y^{j-2}}{\mathcal{Z}^j}\,.$$

In particular, $\lim_{\mathcal{Z}\to\infty}\sup_{y\in(-\mathcal{Z}/2,\mathcal{Z}/2)}|\partial_y G_{\mathcal{Z}}(y)| = 0$, and

$$\sup_{\mathcal{Z}}\ \sup_{y\in(-\mathcal{Z}/2,\mathcal{Z}/2)} \max\{|G_{\mathcal{Z}}(y)|, |\partial_y G_{\mathcal{Z}}(y)|, |\partial_y^2 G_{\mathcal{Z}}(y)|\} < \infty\,.$$

We proceed to show (70). From the definition (26) of $I^{1/2}_{+,t}$, we get

$$\begin{aligned}\sqrt{\pi}I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x,t) &= \int_{-\infty}^{t}\frac{\phi_{c_k,+,\mathcal{Z}}(x,\tau)}{\sqrt{t-\tau}}\,\mathrm{d}\tau\\ &= \int_{x-c_k-\mathcal{Z}}^{\min(t,x-c_k)}\frac{1}{\sqrt{x-c_k-\tau}}\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau\,.\end{aligned}$$

Recalling that $x_0 = x - c_k$, if $x < c_k + t$, then we find

$$\sqrt{\pi}I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x,t) = \int_{x_0-\mathcal{Z}}^{x_0}\frac{1}{\sqrt{x_0-\tau}}\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau\,.$$

(This term did not appear in Lemma 3.1.) Using (69), we find

$$\begin{aligned}\int_{x_0-\mathcal{Z}}^{x_0}\frac{1}{\sqrt{x_0-\tau}\sqrt{t-\tau}}\,\mathrm{d}\tau &= 2\log(|\sqrt{x_0-\tau} - \sqrt{t-\tau}|)\Big|_{x_0-\mathcal{Z}}^{x_0}\\ &= \log(t-x_0) - 2\log(\sqrt{t-x_0+\mathcal{Z}} - \sqrt{\mathcal{Z}})\,.\end{aligned}$$

If in addition $c_k + t - \mathcal{Z} < x$, then, by (68) we find

$$-2\log(\sqrt{t-x_0+\mathcal{Z}} - \sqrt{\mathcal{Z}}) = -2\log\Big(\sqrt{\mathcal{Z}}\Big(\sqrt{1+\frac{t-x_0}{\mathcal{Z}}} - 1\Big)\Big)$$

$$= -2\log\Big(\frac{t-x_0}{\sqrt{\mathcal{Z}}}\sum_{n=1}^{\infty} b_n \left(\frac{t-x_0}{\mathcal{Z}}\right)^{n-1}\Big)$$

$$= +\log\mathcal{Z} - 2\log(t-x_0) - 2\log\Big(1/2 + \sum_{j=1}^{\infty} b_{j+1}\left(\frac{t-x_0}{\mathcal{Z}}\right)^{j}\Big), \tag{71}$$

and we have shown that if $c_k + t - \mathcal{Z} < x < c_k + t$ then

$$\sqrt{\pi} I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x,t) = -\log(t-x_0) + \log\mathcal{Z} - 2\log\Big(\frac{1}{2} + \sum_{j=1}^{\infty} b_{j+1}\left(\frac{t-x_0}{\mathcal{Z}}\right)^{j}\Big). \tag{72}$$

We now consider the case $c_k + t < x < c_k + t + \mathcal{Z}$. Then

$$\sqrt{\pi} I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x,t) = \int_{x_0-\mathcal{Z}}^{t} \frac{1}{\sqrt{x_0-\tau}}\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau\,.$$

Using again (69), and we find

$$\int_{x_0-\mathcal{Z}}^{t} \frac{1}{\sqrt{x_0-\tau}}\frac{1}{\sqrt{t-\tau}}\,\mathrm{d}\tau = \log(x_0-t) - 2\log(\sqrt{\mathcal{Z}} - \sqrt{t-x_0+\mathcal{Z}})\,.$$

Similarly as for (71), we find, using (68),

$$-2\log(\sqrt{\mathcal{Z}} - \sqrt{t-x_0+\mathcal{Z}}) = -2\log\Big(\sqrt{\mathcal{Z}}\Big(1 - \sqrt{1+\frac{t-x_0}{\mathcal{Z}}}\Big)\Big)$$

$$= -2\log\Big(\frac{x_0-t}{\sqrt{\mathcal{Z}}}\sum_{n=1}^{\infty} b_n \left(\frac{t-x_0}{\mathcal{Z}}\right)^{n-1}\Big)$$

$$= +\log\mathcal{Z} - 2\log(x_0-t) - 2\log\Big(\frac{1}{2} - \sum_{j=1}^{\infty} b_{j+1}\left(\frac{t-x_0}{\mathcal{Z}}\right)^{j}\Big).$$

We have thus shown that if $c_k + t - \mathcal{Z} < x < c_k + t + \mathcal{Z}$ then

$$\sqrt{\pi}\cdot I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x,t) = -\log|x_0-t| + \log\mathcal{Z} - 2\log\Big(\frac{1}{2} + \sum_{j=1}^{\infty} b_{j+1}\left(\frac{t-x_0}{\mathcal{Z}}\right)^{j}\Big).$$

With (72), the above identity shows the claim on the right-handed spike ($\sigma = +$).

For the left-handed spike, we have, recalling $x_0 = x - c_k$ and (27),

$$\phi_{c_k,-,\mathcal{Z}}(x,t) = \phi_{c_k,+,\mathcal{Z}}(x, 2x_0 - t) = Q \circ T_{2x_0}(\phi_{c_k,+,\mathcal{Z}})(x,t)\,.$$

Thus, we find

$$I^{1/2}_{-,t}\phi_{c_k,-,\mathcal{Z}}(x,t) = I^{1/2}_{-,t}\circ Q \circ T_{2x_0}(\phi_{c_k,+,\mathcal{Z}})(x,t)$$
$$= I^{1/2}_{+,t}\circ T_{2x_0}(\phi_{c_k,+,\mathcal{Z}})(x,-t) = I^{1/2}_{+,t}\phi_{c_k,+,\mathcal{Z}}(x, -t+2x_0)\,.$$

Finally, note that $-(x - c_k - t) = x - c_k - (2x_0 - t)$. □

Appendix B. Vanishing of X_t at the image of endpoints

It is sometimes convenient to assume that X_t vanishes at the endpoints ± 1. This can be achieved in several ways, as we explain now. For $t \in (1,2)$, setting

$$\tilde{f}_t(y) = \frac{f_t(|a_t|y)}{|a_t|} = \frac{t}{|a_t|} - |a_t|y^2 = a_t y^2 - \frac{t}{a_t}\,,$$

gives a family of maps $\tilde{f}_t$ preserving $[-1,1]$, with $c_{0,t} = 0$, and such that

$$\tilde{f}_t(-1) = -1 = \tilde{f}_t(1)\,,\ \forall\, t \in (-1,2)\,,$$

so that $\partial_t \tilde{f}_t$ vanishes at the endpoints -1 and 1. This is a transversal family of quadratic maps in the sense of Tsujii [41], or [6, 5]. The formula for $\partial \tilde{f}_t$ being unwieldy, it is convenient to work with the family $\tilde{h}_t : [0,1] \to [0,1]$ defined by $\tilde{h}_t(x) = tx(1-x)$, $t \in (1,4]$. The critical point of each $\tilde{h}_t$ is $1/2$, and $X_t(x) = \partial_t \tilde{h}_t \circ \tilde{h}_t^{-1} = tx$ (there is a typo in [8, eq. (2)] where it is stated incorrectly that $X_t^{\tilde{h}_t}(x) \equiv t$). Then $\tilde{h}_t(0) = \tilde{h}_t(1) = 0$ for all t, so that $\partial_t \tilde{h}_t$ vanishes at the endpoints 0 and 1. A variant of $\tilde{h}_t$ is $h_t(x) = t(1-x^2) - 1$ for $t \in (1,2]$ on $[-1,1]$ (there, $c = 0$ and $h_t(-1) = h_t(1) = -1$). However, the formulas for f_t are easier to manipulate than those of $\tilde{f}_t$, h_t, or $\tilde{h}_t$, compensating for the non vanishing of the vector field at the endpoints of a common invariant interval. In addition[38], for any fixed $t_0 \in (1,2)$ and all t close enough to t_0, we may extend f_t on $[-2,2]$ to a C^4 map, also called f_t, with negative Schwarzian derivative, such that Df_t is positive on $[-2, t-t^2]$ and negative on $[t,2]$, with $f_t(-2) = f_t(2) = -2$ and $f_t - f_{t_0} = O(|t-t_0|)$. (The extended family f_t is not needed in the present paper, but we expect it should be useful to prove equality [ii] in Conjecture A in future works.) Finally, using that $c_{2,t} = t - t^2 > -|a_t|$, one can easily show that for any $t_0 \in (1,2)$ there are $\epsilon > 0$ and an interval $I'_{t_0} \subset (-2,2)$ such that $f_t^k(I'_{t_0}) \subset I'_{t_0}$ for all $t \in (t_0 - \epsilon, t_0 + \epsilon)$. In other words, f_t for $t \in (t_0 - \epsilon, t_0 + \epsilon)$ is a transversal family of unimodal maps on I'_{t_0} in the sense of Tsujii [41] since, recalling (10), if $\mathcal{J}(t_1)$ is absolutely convergent then $\mathcal{J}(t_1) \neq 0$.

Appendix C. Averaging

For regular parameters t (also called "hyperbolic"), the physical measure $\mu_t^{sink} = P^{-1} \sum_{j=1}^{P} \delta_{x_{t,j}}$ is atomic, supported on an attracting periodic orbit $f^P(x_{1,t}) = x_{1,t}$ with $P = P(t)$, and can be obtained as

$$\lim_{k\to\infty} \sum_{j=0}^{P-1} \int \mathcal{L}_t^{k+j}(\psi)\phi \,\mathrm{d}m = \lim_{k\to\infty} \sum_{j=0}^{P-1} \int \psi(\phi \circ f_t^{k+j}) \,\mathrm{d}m = \int \psi \,\mathrm{d}m \cdot \frac{1}{P} \sum_{j=1}^{P} \phi(x_{j,t})$$

The convergence is however not uniform in any interval of regular parameters so one cannot a priori sum over k even if $\int_{I_t} \psi \,\mathrm{d}m = 0$.

Since almost every parameter is either regular or stochastic [23] it is natural to consider, for a C^1 observable, say, a Collet–Eckmann parameter t, and $\epsilon > 0$, the double Lebesgue integral

$$\mathcal{A}_\epsilon(t) := \int_{[-\epsilon,\epsilon]} \int \phi(x) \mathrm{d}\mu_{t+\delta}(x) \,\mathrm{d}\delta\,,$$

[38]See [41, Lemma 2.1] for an analogous remark.

where $\mu_{t+\delta} = \mu_{t+\delta}^{sink}$ for regular parameters, and $\mu_{t+\delta} = \rho_{t+\delta} dm$ is the SRB measure for stochastic parameters. Then it is not hard to see, using Lebesgue differentiation, that the (ordinary) t-derivative $\mathcal{A}'_\epsilon(t)$ exists for almost every $\epsilon > 0$ and coincides with $\int \phi(x) \mathrm{d}\mu_{t+\epsilon}(x)$ (see also [44, §3] and [45, (16)]). This does not resolve the paradox described in the introduction, since the derivative depends on ϵ and does not coincide with $\Psi_\phi(1)$ in general. (Note that the "weakening of the linear response problem" in the introduction of [8] — existence and value of the limit as $\epsilon \to 0$ of the derivative $\mathcal{A}'_\epsilon$ — does not explain the paradox either.)

Appendix D. Complements to the proof of Theorem C

We record here interesting facts which are not needed for our proofs.

Remark D.1 (Spectrum on a pole-extended Banach space). For $t \in \mathrm{MT}$, let $\Lambda = \Lambda_t : [-a_t, a_t] \to [0,1]$ be the absolutely continuous bijection defined by $\Lambda_t(x) = \int_{-a_t}^x \rho_t(u)\mathrm{d}u$. Then (see [30, 38]) the map $F_t : [0,1] \to [0,1]$ defined by $F_t(\Lambda_t(x)) = \Lambda_t(f_t(x))$ is Markov (for the partition J_ℓ defined by the endpoints $\Lambda_t(c_k)$, $k = 0, \dots L+P-1$), and F_t is C^1 on the interior of each interval of monotonicity J_ℓ, with $\inf |F_t'| > 1$. At the endpoints, we have[39]

$$(D_t F_t)^k (\Lambda_t(c_{1,t})_\pm) = s_k \sqrt{|(Df_t^k)(c_{1,t})|} \tag{73}$$

(taking right or left-sided limits in the left-hand side according to the dynamical orbit). In fact, $G_t := 1/F_t'$ extends to a C^1 map on the closure of each J_ℓ, with $\sup G_t'' < \infty$. On the Banach space $\mathcal{B}_{\Lambda_t}$ of bounded functions ϕ on $[0,1]$ such that each $\phi|_{\mathrm{int} J_\ell}$ is C^1 and admits a C^1 extension to the closure of J_ℓ, the transfer operator $\mathcal{L}_t^\Lambda \phi(y) = \sum_{F_t(z)=y} \phi(z)/|F_t'(z)|$ thus has spectral radius equal to one, with a simple eigenvalue at 1, for the eigenvector $\rho_t^\Lambda(y) := \rho_t(\Lambda_t^{-1}(y))$, and the rest of the spectrum is contained in a disc of radius κ strictly smaller than 1. Then $\mathcal{B}_t = \{\phi \circ \Lambda_t \,,\, \phi \in \mathcal{B}_{\Lambda_t}\}$ is a Banach space for the norm induced by $\mathcal{B}_{t,\Lambda}$ and the operator $\mathcal{L}_t$ on $\mathcal{B}_t$ inherits the spectral properties of $\mathcal{L}_t^\Lambda$ on $\mathcal{B}_{\Lambda_t}$. Any element of $\mathcal{B}_t$ belongs to $L^1(dm)$, with $\int_{I_t} |\varphi| \,\mathrm{d}m \le \|\varphi\|_{\mathcal{B}_t}$. Recall the notations $\mathcal{Y}_t$, χ_k, $\chi(\vec{Y})$, and $\mathcal{M}_t$ from Remark 5.7. Then we claim that we may extend $\mathcal{L}_t : \mathcal{B}_t \to \mathcal{B}_t$ to a bounded operator $\mathbb{L}_t$ on the Banach space $\mathbb{B}_t := \mathcal{B}_t \oplus \mathcal{Y}_t$, whose nonzero spectrum is the union of the Pth roots of $\mathrm{sgn}(Df_t^P(c_L))$ with the nonzero spectrum of $\mathcal{L}_t$ on $\mathcal{B}_t$. Morever the following holds if $\mathrm{sgn}(Df^p(c_L)) = +1$: First, setting $\mathcal{M}_t^0(\vec{Y}) = \mathcal{M}_t(\vec{Y}) - \rho_t \int_{I_t} \mathcal{M}_t(\vec{Y})\mathrm{d}m$, and letting $\vec{S}_t$ be the fixed point of $\mathbb{S}_t$,

$$\psi_t^* := (\mathrm{id} - \mathcal{L}_t)^{-1}(\mathcal{M}_t^0(\vec{S}_t)) \in \mathcal{B}_t \,,$$

and the (rank-two) spectral projector Π_1 for the eigenvalue 1 of $\mathbb{L}_t$ satisfies

$$\Pi_1(\varphi, \chi(\vec{Y})) = \int \varphi \mathrm{d}m \cdot \rho_t + \frac{\langle \vec{S}_t^*, \vec{Y} \rangle}{\langle \vec{S}_t^*, \vec{S}_t \rangle} \cdot (\psi_t^* + \chi(\vec{S}_t)) \,.$$

Second, if $\int_{I_t} \mathcal{M}_t(\vec{S}_t)\,\mathrm{d}m = 0$ then $\psi_t^* + \chi(\vec{S}_t) \in \mathbb{B}_t$ is a fixed point[40] of $\mathbb{L}_t$, while if $\int_{I_t} \mathcal{M}_t(\vec{S}_t)\,\mathrm{d}m \neq 0$ then there exists a non zero $\nu_t^N \in \mathbb{B}_t^*$ such that the (rank-one) nilpotent operator for the eigenvalue 1 of $\mathbb{L}_t$ satisfies $N_1^2 = 0$ and $\Pi_{\mathcal{B}_t} \circ N_1 = \nu_t^N \cdot \rho_t$.

[39] This is an exercise left to the reader.

[40] In this case, the eigenvalue 1 has geometric multiplicity two, i.e. there is no Jordan block.

We justify the claims above: First, there exists $\kappa < 1$ such that, on $\mathcal{B}_t$,

$$\mathcal{L}_t^j(\varphi) = \int \varphi \, \mathrm{d}y \cdot \rho_t + \mathcal{Q}_t^j(\varphi)\,,\ j \ge 1\,,$$

where for any $\epsilon > 0$ there exists C such that $\|\mathcal{Q}_t^j\|_{\mathcal{B}_t} \le C(\kappa+\epsilon)^j$ for all $j \ge 1$. Note that $\mathcal{M}_t(\vec{Y}) = \Pi_{\mathcal{B}_t}\big(\mathbb{L}_t(\chi(\vec{Y}))\big)$. Identifying $\chi(\vec{Y})$ and $\vec{Y}$, we have

$$\mathbb{L}_t(\varphi, \chi(\vec{Y})) = \begin{pmatrix} 1 & 0 & \rho_t \cdot \int \mathcal{M}_t \, \mathrm{d}m \\ 0 & \mathcal{Q}_t & \mathcal{M}_t^0 \\ 0 & 0 & \mathbb{S} \end{pmatrix} \begin{pmatrix} \rho_t \cdot \int \varphi \, \mathrm{d}m \\ \varphi - \rho_t \cdot \int \varphi \, \mathrm{d}m \\ \vec{Y} \end{pmatrix}.$$

In particular if $1/z$ does not belong to the spectrum of $\mathcal{L}_t$ or $\mathbb{S}_t$, then

$$\big(\mathrm{id} - z\mathbb{L}_t\big)^{-1} = \begin{pmatrix} \frac{1}{1-z} & 0 & -\frac{\rho_t \int \mathcal{M}_t (\mathrm{id}-z\mathbb{S})^{-1} \, \mathrm{d}m}{1-z} \\ 0 & (\mathrm{id} - z\mathcal{Q}_t)^{-1} & -(\mathrm{id} - z\mathcal{Q}_t)^{-1} \mathcal{M}_t^0 (\mathrm{id} - z\mathbb{S})^{-1} \\ 0 & 0 & (\mathrm{id} - z\mathbb{S})^{-1} \end{pmatrix}.$$

If $\int_{I_t} \mathcal{M}_t(\vec{S}_t) \, \mathrm{d}m = 0$ (with $\vec{S}_t$ the fixed point of $\mathbb{S}_t$) then a direct computation gives that $\mathbb{L}_t$ inherits a (second) fixed point $\psi_t^* + \chi(\vec{S}_t) \in \mathbb{B}_t$ from the fixed point $\vec{S}_t$ of $\mathbb{S}_t$. If $\int \mathcal{M}_t(\vec{S}_t) \, \mathrm{d}m \ne 0$ then the eigenvalue 1 of $\mathbb{L}_t$ has algebraic multiplicity two but geometric multiplicity one, and the associated nilpotent N_1 satisfies our claim. In both cases, the claim on Π_1 follows.

Remark D.2. In the Collet–Eckmann case with an infinite postcritical orbit, the finite matrix $\mathbb{S}_t$ appearing in the proof of Theorem C will be replaced by a shift to the right, also denoted $\mathbb{S}_t$, weighted by $s_{1,k} = \pm 1$, acting on a space of infinite sequences (for example $\ell^\infty(\mathbb{Z}_+)$). Then $\mathbb{S}_t$ does not have any eigenvalues, and its spectrum is contained in the closed unit disc. Also, $\mathbb{M}_t(z) := (\mathrm{id} - z\mathbb{S}_t)^{-1}$ is the infinite matrix with $(\mathbb{M}_t(z))_{j,j} = 1$, $(\mathbb{M}_t(z))_{\ell,j} = (-1)^{1+\ell-j} z^{\ell-j} \prod_{k=\ell}^{j-1} s_{1,k} = (-1)^{1+\ell-j} z^{\ell-j} s_{j-1,\ell}$ for $j < \ell$, and $(\mathbb{M}_t(z))_{\ell,j} = 0$ for other ℓ, j.

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