Integrable quantum spin chains with free fermionic and parafermionic spectrum

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We present a general study of a large family of exactly integrable quantum chains with multispin interactions. The exact integrability follows from the algebraic properties of the energy density operators defining the quantum chains. The Hamiltonians are characterized by a parameter $p = 1, 2, \ldots$ related to the number of interacting spins in the multispin interaction. In the general case, the quantum spins are of infinite dimension. In special cases, characterized by the parameter $N = 2, 3, \ldots$, the quantum chains describe the dynamics of $Z(N)$ quantum spin chains with open boundary conditions. The simplest case $p = 1$ corresponds to the free fermionic quantum Ising chain ($N = 2$) or the $Z(N)$ free parafermionic quantum chain. The eigenenergies of the quantum chains are given in terms of the roots of special polynomials, and for general values of p the quantum chains are characterized by a free fermionic ($N = 2$) or free parafermionic ($N > 2$) eigenspectrum. The models have a special critical point when all coupling constants are equal. At this point, the ground-state energy is exactly calculated in the bulk limit, and our analytical and numerical analyses indicate that the models belong to universality classes of critical behavior with dynamical critical exponent $z = (p + 1)/N$ and specific-heat exponent $\alpha = \max\{0, 1 - (p + 1)/N\}$.

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I. INTRODUCTION

The study of free fermionic systems like the Ising model in a transverse field or the quantum XY model has proved to be an important step towards the understanding of quantum many-body interacting systems [1,2]. Since the entire spectrum of these models is obtained exactly in a finite geometry, they provide the ideal framework for the study of new mathematics and physics in condensed matter, statistical physics, and quantum information theory.

Extensions of the models with a free eigenspectrum have been also introduced. One example is the $Z(N)$ generalization of the Ising model in a transverse field introduced by Baxter [3,4] in which the eigenspectrum is described by free parafermions [5–11]. More recently, a new free fermionic $Z(2)$ model characterized by a three-spin interaction was discovered by Fendley [12]. The work [12] motivated us for the discovery of a large family of exactly integrable quantum spin chains. In general, the associated quantum spin chains act on an infinite-dimensional vector space, even in a finite lattice. However, in special cases the dimension is truncated and the models turn out describing $Z(N)$ quantum spin chains with multispin interactions and open boundary conditions [13].

The eigenspectrum of this new family of models has a free fermionic ($N = 2$) or free parafermionic ($N > 2$) nature. The solution of their free-particle eigenspectra cannot be obtained by the standard Jordan-Wigner transformation. As we show in this paper, the simplest way to solve their eigenspectra is by noticing that the general Hamiltonians are based on representations of a simple exchange algebra. The algebra defining the Hamiltonians has an integer parameter $p \geqslant 1$ and its representations turn out to be quantum chains with $(p + 1)$-multispin interactions.

The exchange algebra allows us to build a set of mutually commuting charges, including the Hamiltonian. The solution of the spectral problem, in the $Z(N)$-truncated cases, is obtained thanks to a product formula (or inversion relation for $N = 2$) satisfied by the generating function of the charges. This relation enables us to express the quasienergies of the free particles in terms of the roots of a family of polynomials. The polynomials, and consequently the eigenspectrum of the quantum chains, follow solely from the algebraic properties of the energy density operators defining the quantum chain. This implies that a given set of pseudoenergies describes the eigenspectrum of distinct quantum chains associated with different representations of the exchange algebra.

The commutativity of the charges built from the exchange algebra has been proved for $p = 1$ and arbitrary N in [5] and for $N = p = 2$ in [12]. In the communication [13] we announced the commutation for the truncated model with arbitrary values of p and N. In this paper we prove the announced commutation in the more general case where the Hilbert space associated to the quantum chain is not necessarily truncated.

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In the $Z(N)$-truncated models the product formula, or inversion relation, satisfied by the generating function is similar to the one satisfied by the transfer matrix of the τ_2 model [14–17], which is associated with the case where $p = 1$. For $p = N = 2$, the inversion relation has been proved in [12]. In [13] we conjectured the inversion relation for general p and general $Z(N)$-truncated cases. In this paper we prove this conjecture for $N = 2$ and 3, and general values of p. Our method is solely based on a recurrence relation satisfied by the generating function and can be extended for $N > 3$, case by case.

The polynomials determined by the product formula play a crucial role in the physical analysis of the Hamiltonians. The fact that the pseudoenergies of the free particles are related to the roots of the polynomials allows the evaluation of the eigenspectrum for quite large system sizes.

The quantum chains for general values of N and p have a special critical point. The mass gap and specific-heat calculations, at the critical point, allow us the calculation of the dynamical critical exponent z analytically and the specific-heat exponent α numerically.

We finally remark that the diagonalization of the Hamiltonians are performed independently of the representation of the exchange algebra. Nevertheless, once a representation is chosen, extra degeneracies can appear in the eigenspectrum.

This paper is organized as follows. In Sec. II we introduce the general family of integrable quantum Hamiltonians with the algebraic properties defining the energy density operators. We also present several representations of these models making connections with already known models. In Sec. III we prove that the Hamiltonians belong to a family of mutually commuting operators. Next, in Sec. IV we prove the inversion relation ($N = 2$) and the product formula ($N = 3$) for arbitrary p. The polynomial fixing the quasienergies is considered in Sec. V. In Sec. VI we study some of the physical properties of the multispin Hamiltonians. In Sec. VII some additional charges for specific representations are constructed. Our conclusions and further directions of investigation are given in Sec. VIII. Finally, in Appendices A and B, we present some technical details for the proof of Sec. V, and an example of application to a small quantum chain, respectively.

II. INTEGRABLE QUANTUM CHAINS

The integrable models we construct are defined in terms of M generators h_i ($i = 1, \ldots, M$):

$$\mathcal{H} = -\sum_{i=1}^{M} h_i. \tag{1}$$

The generators satisfy a simple exchange algebra characterized by an integer parameter $p = 1, 2, \ldots$ given by

$$\begin{aligned} h_i h_{i+m} &= \omega h_{i+m} h_i \quad \text{for} \quad 1 \leqslant m \leqslant p, \\ [h_i, h_j] &= 0 \quad \text{for} \quad |i-j| > p, \end{aligned} \tag{2}$$

where ω is a general complex c number. As we show in Sec. III, the relations (2) imply the exact integrability of (1) since we can construct a set of mutually commuting charges. This set is infinite in the bulk limit $M \to \infty$.

We define a word in the algebra by an arbitrary product of letters (generators) $h_1^{s_1} h_2^{s_2} \ldots h_M^{s_M}$, where $s_i = 0, 1, 2, \ldots$. It is important to stress that the relations (2) do not fix any power h_i^a of the generators and consequently the number of independent words of the algebra is infinite. In order to obtain a finite number of words (N^M) we can include aside from (2) a closure relation as, for example,

$$h_i^N = \lambda_i^N, \quad \omega = e^{2i\pi/N}, \tag{3}$$

where λ_i^N is a c number and $N = 2, 3, \ldots$. When (3) is taken into account together with relations (2), we are going to show that (1) has a free fermionic ($N = 2$) or free parafermionic ($N > 2$) eigenspectrum, i.e., the eigenenergies $E^{\{s_i\}}$ satisfy

$$-E^{\{s_i\}} = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \cdots + \omega^{s_{\overline{M}}}\epsilon_{\overline{M}}, \tag{4}$$

where we define

$$\overline{M} \equiv \text{int}\left(\frac{M+p}{p+1}\right) = \left\lfloor \frac{M+p}{p+1} \right\rfloor \tag{5}$$

and $s_i \in \{0, 1, \ldots, N-1\}$. The pseudoenergies ϵ_i ($i = 1, \ldots, \overline{M}$) are given by the roots of special polynomials, as we show in Sec. IV.

Let us remark that the algebra (2) and (3) for $p = 1$ has been shown to be important in the study of generalized Clifford algebra [18–25]. While the diagonalization of (1) in the truncated models is performed using only (2) and (3), representations of the algebra lead to interesting quantum spin chains with open boundary conditions (free boundary or with an impurity at the end of the chain).

A simple representation of (2) and (3) with $N = 2$ and $p = 1$ for odd values of M is given in terms of the spin-$\frac{1}{2}$ Pauli matrices $\sigma_i^{x,z}$:

$$\begin{aligned} h_{2i-1} &= \lambda_{2i-1}\sigma_i^x \quad \text{for} \quad i = 1, \ldots, \frac{M+1}{2}, \\ h_{2i} &= \lambda_{2i}\sigma_i^z\sigma_{i+1}^z \quad \text{for} \quad i = 1, \ldots, \frac{M-1}{2}, \end{aligned} \tag{6}$$

leading to the Hamiltonian

$$\mathcal{H}_I = -\sum_{i=1}^{L} \lambda_{2i-1}\sigma_i^x - \sum_{i=1}^{L-1} \lambda_{2i}\sigma_i^z\sigma_{i+1}^z, \tag{7}$$

also known as the free fermionic quantum Ising chain with $L = \frac{M+1}{2}$ sites and coupling constants $\{\lambda_i\}$.

Similarly, for $p = 1$, arbitrary N and odd M, a representation of (2) and (3) is given in terms of the $Z(N)$ generalizations of the $N \times N$ Pauli matrices satisfying

$$XZ = \omega ZX, \; X^N = Z^N = 1, \; Z^\dagger = Z^{N-1}, \tag{8}$$

and

$$\begin{aligned} h_{2i-1} &= \lambda_{2i-1}X_i \quad \text{for} \quad i = 1, \ldots, \frac{M+1}{2}, \\ h_{2i} &= \lambda_{2i}Z_i^\dagger Z_{i+1} \quad \text{for} \quad i = 1, \ldots, \frac{M-1}{2}. \end{aligned} \tag{9}$$

The Hamiltonian (1) in this case,

$$\mathcal{H}_B = -\sum_{i=1}^{L} \lambda_{2i-1}X_i - \sum_{i=1}^{L-1} \lambda_{2i}Z_i^\dagger Z_{i+1}, \tag{10}$$

reproduces the free parafermionic Baxter $Z(N)$ model with $L = \frac{M+1}{2}$ sites, introduced in [3,4].

In the case where M is even, $N = 2$ and $p = 1$, an interesting representation of (1) is given by

$$h_{2i-1} = \lambda_{2i-1}\sigma_i^x \quad \text{for} \quad i = 1, \ldots, \frac{M-2}{2},$$
$$h_{2i} = \lambda_{2i}\sigma_i^z\sigma_{i+1}^z \quad \text{for} \quad i = 1, \ldots, \frac{M-2}{2},$$
$$h_{M-1} = \lambda_{M-1}\sigma_{M/2}^x, \quad h_M = \lambda_M\sigma_{M/2}^z, \tag{11}$$

and

$$\mathcal{H}_{\text{imp}} = -\sum_{i=1}^{L-1}\left(\lambda_{2i-1}\sigma_i^x + \lambda_{2i}\sigma_i^z\sigma_{i+1}^z\right) - \vec{S}_0\cdot\vec{\sigma}_L, \tag{12}$$

that represents a quantum Ising chain, with $L = M/2$ sites, interacting at one of its ends with a magnetic impurity with components $\vec{S}_0 = (S_0^x, S_0^z) = (\lambda_{M-1}, \lambda_M)$.

The simplest representation of (2) and (3) with $N = 2$ and $p = 2$ recovers the free fermionic three-spin interaction Hamiltonian

$$\mathcal{H}_F = -\sum_{i=1}^{L-2}\lambda_i\sigma_i^z\sigma_{i+1}^x\sigma_{i+2}^x \tag{13}$$

introduced in [12]. A general example of a free fermionic ($N = 2$) or free parafermionic ($N > 2$) with arbitrary values of $p = 1, 2, \ldots$ is given by [13]

$$\mathcal{H}_P = -\sum_{i=1}^{M} h_i = -\sum_{i=1}^{M}\lambda_i Z_i Z_{i+1}\ldots Z_{i+p-1}X_{i+p}. \tag{14}$$

A general interesting representation of (2) and (3), that we call *word representation*, is the one where the generators act on a vector space spanned by the basis $\{|s_1, \ldots, s_M>\}$ with a biunivocal correspondence with the N^M independent words formed by the normalized product of generators

$$|s_1, \ldots, s_M\rangle > \leftrightarrow (h_1/\lambda_1)^{s_1}\ldots(h_M/\lambda_M)^{s_M}, \tag{15}$$

where $s_i = 0, 1, \ldots, N-1$. The action of h_i $(i > p)$ in this basis gives

$$h_i|s_1, \ldots, s_M\rangle \leftrightarrow \Omega\lambda_i|s_1, \ldots, s_{i-1}, s_i^+, s_{i+1}, \ldots, s_M\rangle,$$
$$\text{with } \Omega = \omega^{\sum_{j=1}^{p} s_{i-j}} \text{ and } s_i^+ = s_i + 1, \bmod N.$$

From the properties of the $Z(N)$ matrices (Z_j, X_j) we can identify

$$h_i = \begin{cases} \lambda_i\left(\prod_{j=1}^{i-1} Z_j\right)X_i, & \text{if } 1 \leqslant i \leqslant p; \\ \lambda_i\left(\prod_{j=i-p}^{i-1} Z_j\right)X_i, & \text{if } p+1 \leqslant i \leqslant M, \end{cases} \tag{16}$$

in the basis $|\tilde{s}_1, \ldots, \tilde{s}_M\rangle$ where the $\{Z_i\}$ are diagonal, i.e.,

$$Z|\tilde{s}_i\rangle = e^{2i\pi\tilde{s}_i/N}|\tilde{s}_i\rangle\,, \quad X|\tilde{s}_i\rangle = |\tilde{s}_i + 1, \bmod N\rangle. \tag{17}$$

The Hamiltonian is then given, in this representation, by the $Z(N)$ quantum chain with multispin interactions:

$$\mathcal{H}_A = -\sum_{i=1}^{p}\lambda_i\left(\prod_{j=1}^{i-1} Z_j\right)X_i - \sum_{i=p+1}^{M}\lambda_i\left(\prod_{j=i-p}^{i-1} Z_j\right)X_i. \tag{18}$$

In the particular case $N = 2$ and $p = 1$ we have the simple nearest-neighbor interacting Hamiltonian

$$\mathcal{H}_A^{(p=1)} = -\lambda_1\sigma_1^x - \sum_{i=2}^{M}\lambda_i\sigma_{i-1}^z\sigma_i^x. \tag{19}$$

It is important to observe that the Hamiltonians (10), (14), and (18) are Hermitian in the fermionic cases ($N = 2$) and non-Hermitian in the parafermionic cases ($N > 2$).

III. EXACT INTEGRABILITY AND CONSERVED CHARGES

In this section we show that the general Hamiltonian (1) given in terms of the generators $\{h_i\}$ of the exchange algebra (2) is part of a set of commuting operators (charges), which becomes infinite in the bulk limit ($M \to \infty$), being exactly integrable.

The conserved charges follow directly from the algebraic rules (2). Indeed, the closure relation (3) is not needed in order to prove integrability. A given charge ℓ ($\ell = 0, 1, \ldots, \overline{M}$) is obtained by summing all the products of ℓ commuting generators $h_{j_1}h_{j_2}\ldots h_{j_\ell}$ with $j_1 < j_2 < \cdots < j_\ell$, i.e.,

$$H_M^{(0)} = \mathbb{1},$$
$$H_M^{(1)} = -\mathcal{H} = \sum_{j=1}^{M} h_j,$$
$$H_M^{(2)} = \sum_{j_1=1}^{M}\sum_{j_2=j_1+p+1}^{M} h_{j_1}h_{j_2},$$
$$\vdots$$
$$H_M^{(\overline{M})} = \sum_{j_1=1}^{M}\sum_{j_2=j_1+p+1}^{M}\cdots\sum_{j_{\overline{M}}=j_{\overline{M}-1}+p+1}^{M} h_{j_1}h_{j_2}\ldots h_{j_{\overline{M}}}. \tag{20}$$

Associated to these charges we define the generating function

$$G_M(u) = \sum_{\ell=0}^{\overline{M}}(-u)^\ell H_M^{(\ell)}, \tag{21}$$

where $u \in \mathbb{C}$ is a spectral parameter. We will prove that indeed

$$\left[H_M^{(\ell)}, H_M^{(\ell')}\right] = 0 \quad \forall\ \ell, \ell', \tag{22}$$

and consequently generating functions with distinct spectral parameters also commute, i.e.,

$$[G(u), G(u')] = 0. \tag{23}$$

The key property to prove (23) follows from the fact that both the charges (20) and the generating function (21) satisfy some recurrence relations that we now derive. Rewriting the last sum in $H_M^{(\ell)}$ in (20) as

$$\cdots\sum_{j_l=j_{\ell-1}+p+1}^{M} h_{j_1}h_{j_2}\ldots h_{j_\ell}$$
$$= \cdots\sum_{j_\ell=j_{\ell-1}+p+1}^{M-1} h_{j_1}h_{j_2}\ldots h_{j_\ell} + h_{j_1}\ldots h_{j_{\ell-1}}h_M, \tag{24}$$

we obtain the recurrence relation

$$H_M^{(\ell)} = H_{M-1}^{(\ell)} + h_M H_{M-(p+1)}^{(\ell-1)}, \tag{25}$$

with the initial conditions $H_M^{(0)} = \mathbb{1}$, $H_M^{(\ell)} = 0$ for $\ell < 0$ or $M \leqslant 0$. It follows from (25) and (21) that

$$G_M(u) = \sum_{\ell=0}^{\overline{M}} (-u)^\ell H_{M-1}^{(\ell)} - uh_M \sum_{\ell=0}^{\overline{M}-1} (-u)^\ell H_{M-(p+1)}^{(\ell)},$$

where we have used $H_{M-(p+1)}^{(-1)} = 0$.

Since, from (5) $\overline{M-1} = \overline{M-(p+1)}$, we identify from (21) the second summation in (26) as $G_{M-(p+1)}(u)$. Writing $M = j(p+1)+q$ with $j, q \in \mathbb{Z}$ and $0 \leqslant q \leqslant p$, we have that $\overline{M} = \overline{M-1}$ except for $q = 1$ where $\overline{M} = \overline{M-1}+1$. Since for $q = 1$, $\overline{M} > \overline{M-1}$ we have that $H_{M-1}^{(\overline{M})} = 0$, and consequently we identify, for all M, the first summation in (26) as $G_{M-1}(u)$. We have then the recurrence relation for the generating function

$$G_M(u) = G_{M-1}(u) - uh_M G_{M-(p+1)}(u), \tag{26}$$

with $G_M(u) = \mathbb{1}$ for $M \leqslant 0$.

The recurrences (25) and (26) are the basic identities we use in this paper. It is convenient, for further use, to iterate (25) and (26),

$$G_{M-(p+1)}(u) = G_{M-(p+1)-j}(u) - u\sum_{k=0}^{j-1} h_{M-(p+1)-k} G_{M-2(p+1)-k}(u), \tag{27}$$

$$H_{M-(p+1)}^{(\ell)} = H_{M-(p+1)-j}^{(\ell)} + \sum_{k=0}^{j-1} h_{M-(p+1)-k} H_{M-2(p+1)-k}^{(\ell-1)}, \tag{28}$$

for $j = 0, 1, 2, \ldots, M-(p+1)$, and also

$$G_{M-1}(u) = G_{M-(p+1)}(u) - u\sum_{k=1}^{p} h_{M-k} G_{M-(p+1)-k}(u), \tag{29}$$

$$H_{M-1}^{(\ell)} = H_{M-(p+1)}^{(\ell)} + \sum_{k=1}^{p} h_{M-k} H_{M-(p+1)-k}^{(\ell-1)}. \tag{30}$$

In order to demonstrate the involution (23) let us show initially that an arbitrary charge commutes with the generating function. For this sake it is convenient to define

$$\beta_{M,q}^{(\ell)} \equiv [H_M^{(\ell)}, G_M(u)]_q, \quad \ell = 1, \ldots, \overline{M} \tag{31}$$

where

$$[X, Y]_q = XY + qYX \tag{32}$$

is the q commutator. We want to show that $\beta_{M,-}^{(\ell)} = 0$, for any ℓ. In the remaining of this section, for simplicity, we will omit the explicit dependence of the generating function in the variable u.

Inserting (25) and (26) in (31) with $q = -1$ we obtain

$$\beta_{M,-}^{(\ell)} = \beta_{M-1,-}^{(\ell)} - uh_M^2 \beta_{M-(p+1),-}^{(\ell-1)} + A + B, \tag{33}$$

where

$$A = \left[h_M H_{M-(p+1)}^{(\ell-1)}, G_{M-1}\right]_-, \tag{34}$$

$$B = -u\left[H_{M-1}^{(\ell)}, h_M G_{M-(p+1)}\right]_-. \tag{35}$$

Using (29) in A and (30) in B, we obtain

$$A = h_M \beta_{M-(p+1),-}^{(\ell-1)} - uh_M \sum_{j=1}^{p} \left[H_{M-(p+1)}^{(\ell-1)}, h_{M-j} G_{M-(p+1)-j}\right]_{-\omega} \tag{36}$$

and

$$B = -uh_M \beta_{M-(p+1),-}^{(\ell)} - \omega uh_M \sum_{j=1}^{p} \left[h_{M-j} H_{M-(p+1)-j}^{(\ell-1)}, G_{M-(p+1)}\right]_{-\omega^{-1}}. \tag{37}$$

Defining

$$X_{M,q}^{(\ell)}(j) = \left[H_{M-(p+1)}^{(\ell)}, h_{M-j} G_{M-(p+1)-j}\right]_q, \tag{38}$$

$$Y_{M,q}^{(\ell)}(j) = \left[h_{M-j} H_{M-(p+1)-j}^{(\ell)}, G_{M-(p+1)}\right]_q, \tag{39}$$

$$\gamma_{M,q}^{(\ell)}(j) = X_{M,-q}^{(\ell)}(j) + qY_{M,-q^{-1}}^{(\ell)}(j), \tag{40}$$

we can write

$$A + B = h_M \beta_{M-(p+1),-}^{(\ell-1)} - uh_M \beta_{M-(p+1),-}^{(\ell)} - uh_M \sum_{j=1}^{p} \gamma_{M,\omega}^{(\ell-1)}(j). \tag{41}$$

We now obtain a recurrence relation for $X_{M,q}^{(\ell)}$, $Y_{M,q}^{(\ell)}$, and $\gamma_{M,q}^{(\ell)}$. Inserting (28) in (38) and (27) in (39) we obtain, after straightforward manipulations,

$$X_{M,q}^{(\ell)}(j) = h_{M-j} \beta_{M-(p+1)-j,q}^{(\ell)} + \omega h_{M-j} \sum_{k=0}^{j-1} Y_{M-j,q\omega^{-1}}^{(\ell-1)}(p+1+k-j), \tag{42}$$

$$Y_{M,q}^{(\ell)}(j) = h_{M-j} \beta_{M-(p+1)-j,q}^{(\ell)} - uh_{M-j} \sum_{k=0}^{j-1} X_{M-j,q\omega}^{(\ell)}(p+1+k-j). \tag{43}$$

Using (43) in (42) and (42) in (43), with $q = \omega$, we obtain

$$\gamma_{M,\omega}^{(\ell)}(j) = (1+q) h_{M-j} \beta_{M-(p+1)-j,-}^{(\ell)} + \omega h_{M-j} \sum_{k=0}^{j-1} h_{M-(p+1)-k} \left(\beta_{M-2(p+1)-k,-}^{(\ell-1)} - u\beta_{M-2(p+1)-k,-}^{(\ell)} - u \sum_{k'=0}^{p+k-j} \gamma_{M-(p+1)-k,\omega}^{(\ell-1)}(k'-k+j) \right). \tag{44}$$

We then finally obtain, from (40) and (33),

$$\beta^{(\ell)}_{M,-} = \beta^{(\ell)}_{M-1,-} - uh_M\beta^{(\ell)}_{M-(p+1),-} - uh_M\sum_{j=1}^{p}\gamma^{(\ell-1)}_{M,\omega}(j) - h_M\beta^{(\ell-1)}_{M-(p+1),-}(uh_M-1), \tag{45}$$

with $\gamma^{(\ell)}_{M,\omega}(j)$ given by (44).

We have from (20) that $H^{(l)}_M = \beta^{(l)}_{M,-} = [H^{(l)}_M, G_M] = 0$ for $l \leqslant 0$, and $G^{(l)}_M = 1$ for $l \leqslant 0$, therefore, from (44) $\gamma^{(l)}_{M,q} = 0$ for $l \leqslant 0$.

Inserting these values in (45) we obtain, for $l = 1$,

$$\beta^{(1)}_{M,-} = \beta^{(1)}_{M-1,-} - uh_M\beta^{(1)}_{M-(p+1),-}. \tag{46}$$

Since for $M = 1$, $\beta^{(1)}_{l,-} = 0$ ($l \leqslant 0$), we obtain $\beta^{(1)}_{1,-} = 0$ and by iterating (46) we obtain $\beta^{(1)}_2 = \beta^{(1)}_3 = \cdots = \beta^{(1)}_M = 0$, for arbitrary M. From (44) $\gamma^{(1)}_{M,\omega}(j)$ only depends on $\beta^{(0)}_{M,-}$, $\beta^{(1)}_{M,-}$, and $\gamma^{(0)}_{M,\omega}$. This means that $\gamma^{(1)}_{M,\omega}(j) = 0$ for all M and from (45) with $l = 2$ we have

$$\beta^{(2)}_{M,-} = \beta^{(2)}_{M-1,-} - uh_M\beta^{(2)}_{M-(p+1),-}. \tag{47}$$

Since $\beta^{(l)}_{l,-} = 0$ for $l \leqslant 0$ we have $\beta^{(2)}_{1,-} = \beta^{(2)}_{2,-} = \cdots = \beta^{(2)}_{M,-} = 0$, for all M. Similarly as before $\gamma^{(2)}_{M,\omega}$ only depends on products involving $\beta^{(1)}_{M,-}$, $\beta^{(2)}_{M,-}$, and $\gamma^{(1)}_{M,\omega}$, therefore, $\gamma^{(2)}_{M,\omega} = 0$ and then, from (44),

$$\beta^{(3)}_{M,-} = \beta^{(3)}_{M-1,-} - uh_M\beta^{(3)}_{M-(p+1),-}. \tag{48}$$

Since $\beta^{(3)}_{\ell,-}$ for $\ell \leqslant 0$ we have $\beta^{(3)}_{M,-}$ for all M. This procedure iterates and we have our proof:

$$\beta^{(\ell)}_{M,-} = \left[H^{(\ell)}_M, G_M(u)\right] = 0 \quad \forall\, \ell = 1, \ldots, M. \tag{49}$$

Expanding this result in powers of u we obtain that all the distinct charges commute among themselves, i.e.,

$$\left[H^{(\ell)}_M, H^{(\ell')}_M\right] = 0 \quad \forall\, \ell, \ell'. \tag{50}$$

The relation (50) also implies that the generating functions with arbitrary values of the spectral parameter u commute, i.e.,

$$[G_M(u), G_M(v)] = 0. \tag{51}$$

The relations (50) or (51) imply that the Hamiltonian (1) with generators $\{h_i\}$ satisfying (2) is exactly integrable. The closure relation (3) is not a necessary condition for ensuring the exact integrability.

IV. INVERSION RELATION AND PRODUCT FORMULA FOR THE GENERATING FUNCTION

In this section, we show that when the generators $\{h_i\}$ defining the Hamiltonian (1) satisfy the closure relation (3) aside from the relations (2), the generating function (21) satisfies

$$G_M(u)G_M(\omega u)\ldots G_M(\omega^{N-1}u) = P^{(p)}_M(u^N)\mathbb{1}, \tag{52}$$

where $P^{(p)}_M(u^N)$ is a polynomial of degree $\overline{M}$ in u^N. Equation (52) implies (see Sec. V) that the Hamiltonians have a free fermion ($N = 2$) or a free parafermionic eigenspectrum ($N > 2$).

A. $N = 2$

Let us first consider the free fermion case ($N = 2$). This case has been considered for $p = 1$ in the context of the τ_2 model [6,17] and for $p = 2$ in [12] using a certain factorization of the generating function. Here we prove it for any value of $p \geqslant 1$ by showing that

$$\tau^{(2)}_M(u) \equiv G_M(u)G_M(-u) = P^{(p)}_M(u^2)\mathbb{1} \tag{53}$$

satisfies a recurrence relation.

In order to simplify the notation, let us define the algebraic operation

$$\mathcal{L}(A(u); B(u)) = A(u)B(-u) - B(u)A(-u). \tag{54}$$

Using the fundamental relation (26) in (53) we obtain

$$\tau^{(2)}_M(u) = \tau^{(2)}_{M-1}(u) - u^2h^2_M\tau^{(2)}_{M-(p+1)}(u) + \Xi^{(2)}_M(u), \tag{55}$$

where

$$\Xi^{(2)}_M(u) = -uh_M\sum_{j=1}^{p}L_{M,j}(u), \tag{56}$$

with

$$L_{M,j}(u) \equiv \mathcal{L}(h_{M-(p+1)-j}G_{M-2(p+1)+j}(u); G_{M-(p+1)}(u)). \tag{57}$$

Using (27) for $G_{M-(p+1)}$ in (57) we obtain, for $j = 1, \ldots, p$,

$$L_{M,j}(u) = L_{M-1,j+1}(u) - uh_{M-(p+1)+j}L_{M-(p+1)-j,p+1-j}(u), \tag{58}$$

with the condition $L_{M,p+1}(u) = 0$ for all M. Since $G_M(u) = 1$ and $h_M = 0$ for $M \leqslant 0$, we have, from (57),

$$L_{M,j}(u) = 0 \quad \text{for} \quad M \leqslant p+1, \quad j = 1, \ldots, p. \tag{59}$$

The recurrence relation (58) then implies

$$L_{M,j}(u) = \Xi_M(u) = 0 \quad \forall\, M \tag{60}$$

and

$$\tau^{(2)}_M(u) = \tau^{(2)}_{M-1}(u) - u^2h^2_M\tau^{(2)}_{M-(p+1)}(u), \tag{61}$$

with $\tau^{(2)}_M(u) = 1$ for $M \leqslant 0$. This last expression gives $\tau^{(2)}_1(u) = 1 - u^2h^2_1$ and by iterating (61) we obtain that $\tau^{(2)}_M(u)$ is a polynomial in u^2, reproducing (53) for arbitrary values of p. As a consequence of the recurrence (61) the polynomials $P^{(p)}_M(u^2)$ also satisfy the recurrence

$$P^{(p)}_M(u^2) = P^{(p)}_{M-1}(u^2) - u^2\lambda^2_M P^{(p)}_{M-(p+1)}(u^2), \tag{62}$$

with the initial condition $P^{(p)}_M(u^2) = 1$ for $M \leqslant 0$.

It is important to notice from (62) that the polynomials $P^{(p)}_M(z) = P^{(p)}_M(\{\lambda^2_i\}, z)$ depend on the parameters $\{\lambda^2_i\}$ defining the closure relation (3) and the coupling constants of the Hamiltonian (1).

B. $N = 3$

We now consider the case $N = 3$. Extending the definition (53) for the case $N = 2$ we now define

$$\tau^{(3)}_M(u) \equiv G_M(u)G_M(\omega u)G_M(\omega^2 u) = P^{(p)}_M(u^3)\mathbb{1}, \tag{63}$$

where $\omega = e^{2i\pi/3}$ and, as we shall see, $P_M^{(p)}(u^3)$ is related to the same polynomial of degree $\overline{M}$ appearing in (53) for the $Z(2)$ case.

In Appendix A, we show the recurrence relation

$$\tau_M^{(3)}(u) = \tau_{M-1}^{(3)}(u) - u^3 h_M^3 \tau_{M-(p+1)}^{(3)}(u). \tag{64}$$

Since $\tau_1^{(3)}(u) = \mathbb{1}$, we obtain that $\tau_M^{(3)}(u)$ is given by the polynomial $P_M^{(p)}(u^3)$ as claimed in (63). Also from (3) $h_M^3 = \lambda_M^3$ the polynomial satisfies the recurrence relation

$$P_M^{(p)}(u^3) = P_{M-1}^{(p)}(u^3) - u^3 \lambda_M^3 P_{M-(p+1)}^{(p)}(u^3), \tag{65}$$

with $P_{M'}^{(p)} = 1$ for $M' \leqslant 0$. We can see that the polynomials $P_M^{(p)}(z) = P_M^{(p)}(\{\lambda_i^3\}, z)$ are the same ones that appeared in the case $N = 2$, where we replace the couplings $\{\lambda_i^2\}$ by $\{\lambda_i^3\}$.

C. $N > 3$

For $N > 3$, we can proceed in a similar way, and show that

$$\tau_M^{(N)}(u) = G_M(u) G_M(\omega u) \dots G_M(\omega^{N-1} u) = P_M^{(p)}(u^N)\mathbb{1} \tag{66}$$

with $\omega = e^{2i\pi/N}$, satisfies the recurrence relation

$$\tau_M^{(N)}(u) = \tau_{M-1}^{(N)}(u) - u^N h_M^N \tau_{M-(p+1)}^{(N)}(u), \tag{67}$$

with $\tau_M^{(N)}(u) = 1$ for $M \leqslant 0$. From (67) we obtain the recurrence relation for the polynomials $P_M^{(p)}(u^N) = P_M^{(p)}(\{\lambda_i^N\}, u^N)$:

$$P_M^{(p)}(u^N) = P_{M-1}^{(p)}(u^N) - u^N \lambda_M^N P_{M-(p+1)}^{(p)}(u^N), \tag{68}$$

with $P_{M'}^{(p)} = 1$ for $M' \leqslant 0$. A proof of (67), similar as we have done for the cases $N = 2$ and 3 (see Appendix A), for general N, is straightforward but lengthy. Anyway, we have checked (67) and hence (66) for several values of N and lattice sizes M.

Comparing the recurrence relations (67) and (68) we identify the coefficients $C_M^{(l,p)}$ in the expansion

$$P_M^{(p)}(z) = \sum_{\ell=0}^{\overline{M}} (-z)^\ell C_M^{(\ell,p)} \tag{69}$$

by replacing $h_j \leftrightarrow \lambda_j^N$ in (20), i.e.,

$$C_M^{(\ell,p)} = \sum_{j_1=1}^{M} \sum_{j_2=j_1+p+1}^{M} \cdots \sum_{j_\ell=j_{\ell-1}+p+1}^{M} \lambda_{j_1}^N \lambda_{j_2}^N \dots \lambda_{j_\ell}^N \tag{70}$$

for $\ell = 0, 1, \dots, \overline{M}$.

In the case where all $\lambda_i^N = 1$, $C_M^{(\ell)}$ is the number of distinct ways we can put ℓ particles with excluded volume of $(p+1)$ lattice units in a lattice with M sites, i.e.,

$$C_M^{(\ell,p)} = \binom{M - p(\ell-1)}{\ell} = \frac{[M - p(\ell-1)]!}{[M - p(\ell-1) - \ell]!\ell!}, \tag{71}$$

and $P_M^{(p)}(z)$ is the generalized hypergeometric polynomial known as ${}_{p+1}F_p$ [26]:

$$P_M^{(p)}(z) = {}_{p+1}F_p\left(\begin{matrix} -\frac{M+p}{p+1} & -\frac{M+p-1}{p+1} & -\frac{M+p-2}{p+1} & \cdots & -\frac{M}{p+1} \\ & -\frac{M+p}{p} & -\frac{M+p-1}{p} & \cdots & -\frac{M+1}{p} \end{matrix} ; \frac{(p+1)^{p+1}}{p^p} z \right) = \sum_{\ell=0}^{\overline{M}} (-1)^\ell \binom{M - p(1-\ell)}{\ell} z^\ell. \tag{72}$$

In this symmetric case the polynomial $P_M^{(p)}(-\tilde{z})$, as we can see from (69), is the grand canonical partition function of a polymer with monomers with size of $(p+1)$ lattice units and fugacity $\tilde{z}$ in a lattice of M sites. As we are going to see in the following sections the roots of $P_M^{(p)}(-\tilde{z})$ for any p and finite M are real and negative. This means that the grand canonical partition function is analytic. However, as $M \to \infty$ the largest root approaches 0 and therefore in the thermodynamic limit the polymer has a critical fugacity $\tilde{z} = \tilde{z}_c = 0$.

V. EIGENSPECTRUM OF THE FREE FERMIONIC AND FREE PARAFERMIONIC QUANTUM CHAINS

The eigenspectrum of any quantum chain (1) expressed in terms of the generators $\{h_i\}$ satisfying (2) and (3) are obtained from the zeros of the fundamental polynomials $P_M^{(p)}(z) \equiv P_M^{(p)}(\{\lambda_i\}, u^N)$ derived in the last section. This means that for a given parameter p, all the models with arbitrary N are ruled by the same polynomial.

It is interesting to mention that in the case where $p = 1$ and $\lambda_i = 1$ $(i = 1, \dots, M)$, that includes the critical quantum Ising chain (7) and the critical $Z(N)$ free parafermionic Baxter chain (10), these polynomials are related to the well-known Chebyshev polynomial of second type, i.e., $P_M^{(1)}(z) = z^{\frac{M+1}{2}} U_{M+1}(\frac{1}{2z^{\frac{1}{2}}})$. For $p = 1$ and arbitrary λ_i, the polynomials $P_M^{(1)}(z)$ are related to the FST polynomials (see [27,28]). In order to illustrate we present in Table I some polynomials at $\{\lambda_i = 1\}$ in the cases $p = 1, 2, 3$.

Since $[G_M(u), G_M(-u)] = 0$ and $G_M(0) = P_M^{(p)}(0) = 1$, by applying in the product formula (52) to a given eigenfunction of $G_M(u)$, with eigenvalue $\Lambda_M(u)$, we obtain

$$\Lambda(u) \dots \Lambda(\omega^{N-1} u) = P_M^{(p)}(u^N) = \prod_{i=1}^{\overline{M}} \left(1 - \frac{u^N}{z_i}\right), \tag{73}$$

where z_i are the roots of $P_M^{(p)}(z_i) = 0$. Solving (73) in terms of z_i we obtain

$$\Lambda_M(u) = \prod_{i=1}^{\overline{M}} \left(1 - u \frac{\omega^{s_i}}{z_i^{1/N}}\right) = \prod_{i=1}^{\overline{M}} (1 - u\, \omega^{s_i} \epsilon_i), \tag{74}$$

where $\epsilon_i = z_i^{-1/N}$ and $s_i \in \{0, 1, \dots, N-1\}$. In all the cases that we have considered in this paper, where the couplings are real, we have verified that the roots z_i are real positive and distinct, implying the existence of $N^{\overline{M}}$ distinct eigenvalues for the generating function $G_M(u)$. We can also expand (74) in

TABLE I. Example of polynomials $P_M^{(p)}(z)$, with coupling constants $\{\lambda_i = 1\}$, for $p = 1, 2$ and $p = 3$, and for $M = 1, \ldots, 11$.

M	$P_M^{(1)}(z)$	$P_M^{(2)}(z)$	$P_M^{(3)}(z)$
1	$1 - z$	$1 - z$	$1 - z$
2	$1 - 2z$	$1 - 2z$	$1 - 2z$
3	$1 - 3z + z^2$	$1 - 3z$	$1 - 3z$
4	$1 - 4z + 3z^2$	$1 - 4z + z^2$	$1 - 4z$
5	$1 - 5z + 6z^2 - z^3$	$1 - 5z + 3z^2$	$1 - 5z + z^2$
6	$1 - 6z + 10z^2 - 4z^3$	$1 - 6z + 6z^2$	$1 - 6z + 3z^2$
7	$1 - 7z + 15z^2 - 10z^3 + z^4$	$1 - 7z + 10z^2 - z^3$	$1 - 7z + 6z^2$
8	$1 - 8z + 21z^2 - 20z^3 + 5z^4$	$1 - 8z + 15z^2 - 4z^3$	$1 - 8z + 10z^2$
9	$1 - 9z + 28z^2 - 35z^3 + 15z^4 - z^5$	$1 - 9z + 21z^2 - 10z^3$	$1 - 9z + 15z^2 - z^3$
10	$1 - 10z + 36z^2 - 56z^3 + 35z^4 - 6z^5$	$1 - 10z + 28z^2 - 20z^3 + z^4$	$1 - 10z + 21z^2 - 4z^3$
11	$1 - 11z + 45z^2 - 84z^3 + 70z^4 - 21z^5 + z^6$	$1 - 11z + 36z^2 - 35z^3 + 5z^4$	$1 - 11z + 28z^2 - 10z^3$

powers of u [29]:

$$\Lambda_M^{\{s_i\}}(u) = \sum_{\ell=0}^{\overline{M}} (-1)^\ell e_\ell(\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\overline{M}}}\epsilon_{\overline{M}}) u^\ell, \qquad (75)$$

where

$$e_\ell(x_1, x_2, \ldots, x_n) = \sum_{1 \leqslant j_1 < j_2 < \cdots < j_\ell \leqslant n} x_{j_1} x_{j_2} \ldots x_{j_\ell} \qquad (76)$$

for $\ell = 0, 1, \ldots, n$ is the ℓth elementary symmetric polynomial in the variables $x_1, x_2, \ldots, x_n$.

Applying an eigenfunction of $G_M(u)$ with eigenvalue $\Lambda_M^{\{s_i\}}(u)$ in (21), since u is arbitrary, (75) implies that the eigenfunction is also an eigenfunction of the charges $\{H_M^{(\ell)}\}$ given in (20) with eigenvalues

$$q_{\{s_i\}}^{(\ell)} = e_l(\omega^{s_1}\epsilon_1, \ldots, \omega^{s_{\overline{M}}}\epsilon_{\overline{M}}), \quad \ell = 1, \ldots, \overline{M}. \qquad (77)$$

In particular, the Hamiltonian (1) has the free fermionic ($N = 2$) of free parafermionic ($N > 2$) spectrum, with eigenenergies

$$-E^{\{s_i\}} = q_{\{s_i\}}^{(1)} = \omega^{s_1}\epsilon_1 + \omega^{s_2}\epsilon_2 + \cdots + \omega^{s_L}\epsilon_{\overline{M}}. \qquad (78)$$

This means that not only the eigenspectrum of the general Hamiltonians (1), but also the ones of the extended Hamiltonians $\{H_M^{(\ell)}\}$ are given entirely in terms of the roots of the fundamental polynomials $P_M^{(p)}(\{\lambda_i\}, z)$.

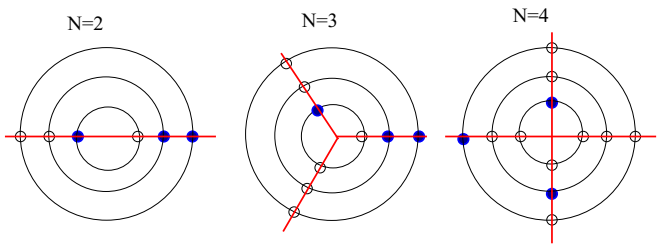

FIG. 1. Schematic representation of the eigenenergies of a quantum chain with $\overline{M} = 3$ and $N = 2, 3, 4$. There are three quasienergies that fix the radius of the $\overline{M}$ circles in the complex plane. There is a "circle exclusion principle" that imposes a single quasienergy in each circle (filled circles). The figures show for $N = 2$ and 3 the ground-state and first-excited-state energies, respectively.

In Fig. 1 we show schematically examples of eigenenergies for a $Z(N)$ quantum chain with $\overline{M} = 3$ and $N = 2, 3, 4$. In Appendix B we consider, as an example, the simple case of the $Z(N)$ Hamiltonian with $p = 3$ and with $M = 5$.

In the next section, by exploiting the solution for the polynomial roots of $P_M^{(p)}(z)$ we are going to derive the critical behavior of the Hamiltonian (1) at a special critical point.

VI. GROUND-STATE ENERGY AND CRITICAL EXPONENTS FOR THE QUANTUM CHAINS

For general values of the couplings $\{\lambda_i\}$ we should expect a quite rich phase diagram for the Hamiltonians (1) with $p \geqslant 2$. We restrict ourselves to the quantum chains at their symmetrical point where all the coupling constants $\lambda_i = 1$ ($i = 1, \ldots, M$) in (3) or any of its representations like (10), (14), and (18).

For $p = 1$, where for $N = 2$ the possible representations are the free fermionic Ising quantum chain (7) and (12), and for $N > 2$ the $Z(N)$ free parafermionc quantum chain (10), the models are critical with a dynamical critical exponent $z = 2/N$ and specific-heat exponent $\alpha = 1 - 2/N$ [9]. In [12] it has been showed that in the particular case $p = N = 2$ the symmetrical point is a multicritical point where $z = \frac{3}{2}$.

In [13] we have showed that for general values of p the polynomial roots $\{z_i\}$ of $P_M^{(p)}$, that give us the quasiparticle energies $\epsilon_i = z_i^{-1/N}$ in (78), for arbitrary M, and lattice sizes multiples of $(p + 1)$, can be parametrized by trigonometric functions

$$\epsilon_k = \frac{\sin^{\frac{p+1}{N}}(\mathfrak{p}_k)}{\sin^{\frac{1}{N}}\left(\frac{\mathfrak{p}_k}{p+1}\right)\sin^{\frac{p}{N}}\left(\frac{p\mathfrak{p}_k}{p+1}\right)}, \quad k = 1, \ldots, \overline{M} \qquad (79)$$

where $\mathfrak{p}_k$ is a quantum number. For $p = 1$, $\mathfrak{p}_k = k\pi/(\overline{M} + 1)$, for any M, and for $p > 1$ we have conjectured, and confirmed numerically, that as $M \to \infty$ the distribution density for the quantum numbers behaves as $\Delta\mathfrak{p}_k/\Delta k = \pi/\overline{M} = (p + 1)\pi/M$. Since the ground-state energy is obtained by taking in (78) the values $s_i = 0$ ($i = 1, \ldots, \overline{M}$), we obtain an exact expression for the ground-state energy per site:

$$e_\infty^{(p)} \equiv -\frac{E_0}{M} = -\frac{1}{M}\sum_{k=1}^{\overline{M}} \epsilon_k = -\frac{1}{(p+1)\pi}\int_0^\pi \epsilon(\mathfrak{p}) d\mathfrak{p}. \qquad (80)$$

For $N = 2$ and general values of p, (80) is given in terms of gamma functions:

$$e_\infty^{(p)} = -\frac{\Gamma\left(\frac{1}{2}+\frac{p}{2}\right)}{\Gamma\left(1+\frac{p}{2}\right)} \quad (N=2). \tag{81}$$

For general p and N [13], the integral (80) is expressed in terms of integral representation of the Lauricella hypergeometric series $F_D^{(p-1)}$ [30]. Moreover, in the cases $p = 1$, 2, and 3 (80) is given in terms of the gamma, ${}_2F_1$ and Appel functions F_1, i.e.,

$$e_\infty^{(1)} = -\frac{2^{\frac{2}{N}-1}\Gamma\left(\frac{1}{N}+\frac{1}{2}\right)}{\sqrt{\pi}\left(\frac{1}{N}+1\right)}, \tag{82}$$

$$e_\infty^{(2)} = -\frac{3^{\frac{3}{N}+\frac{1}{2}}\Gamma\left(\frac{3}{N}+1\right)}{2^{\frac{2}{N}+2}\sqrt{\pi}\Gamma\left(\frac{3}{N}+\frac{3}{2}\right)}\,{}_2F_1\left(\begin{matrix}\frac{1}{2} & \frac{1}{N}+\frac{1}{2}\\ & \frac{3}{N}+\frac{3}{2}\end{matrix};\frac{3}{4}\right), \tag{83}$$

$$e_\infty^{(3)} = -\frac{2^{\frac{8}{N}-\frac{3}{2}}\Gamma\left(\frac{4}{N}+1\right)}{3^{\frac{3}{N}}\sqrt{\pi}\Gamma\left(\frac{4}{N}+\frac{3}{2}\right)} \times F_1\left(\frac{1}{2};\frac{1}{2}-\frac{2}{N},\frac{3}{N};\frac{4}{N}+\frac{3}{2};\frac{1}{2},\frac{2}{3}\right). \tag{84}$$

In the case $p = 4$, we have

$$e_\infty^{(4)} = -\frac{5^{\frac{5}{N}}\sin\left(\frac{\pi}{5}\right)\Gamma\left(\frac{5}{N}+1\right)}{2^{\frac{8}{N}+1}\sqrt{\pi}\Gamma\left(\frac{5}{N}+\frac{3}{2}\right)} \times F_D^{(3)}\left(\frac{1}{2};\frac{1}{2}+\frac{2}{N},-\frac{5}{N},\frac{4}{N};\frac{5}{N}+\frac{3}{2};x_1,x_2,x_3\right), \tag{85}$$

where $F_D^{(3)}$ is the Lauricella function with three variables at $x_1 = \frac{1}{2+\frac{2}{\sqrt{5}}}$, $x_2 = \frac{2}{3+\sqrt{5}}$, and $x_3 = \frac{1}{1+\frac{1}{\sqrt{5}}}$. For a comparison of (81)–(85) with the numerical results obtained from the direct solution of the polynomial zeros $\{z_i\}$ we refer to [13].

The dynamical critical exponent is evaluated from the finite-size behavior of the mass gaps of the quantum Hamiltonians. The first-excited energy state is obtained in (78) by taking the set $s_1 = s_2 = \cdots = s_{\overline{M}-1} = 0$ and $s_{\overline{M}} = 1$. However, $\epsilon_{\overline{M}}$ is the smallest quasienergy, whose associated quantum number, for $M \to \infty$, behaves as $\mathfrak{p}_{\overline{M}} = \pi - a/M$, where a is a harmless constant. Therefore, the real part of the energy gap (complex for $N > 2$) has the leading behavior

$$\Delta_M^{(p)} = \mathrm{Re}(E_1 - E_0) = \mathrm{Re}(1-\omega)\epsilon(\mathfrak{p}_{\overline{M}}) \approx \left(\frac{a}{M}\right)^z, \tag{86}$$

where

$$z = (p+1)/N \tag{87}$$

is the dynamical critical exponent.

For the case $p = 1$ (87) recovers for $N = 2$ the known result for the conformally invariant quantum Ising chain ($z = 1$) and for the $Z(N)$, the Baxter free parafermionic model ($z = 2/N$), as calculated in [9]. The case $p = N = 2$ recovers the result $z = \frac{3}{2}$ derived in [12]. In summary, all the free interacting quantum chains (1) at their symmetric point $\{\lambda_i = 1\}$ are critical with the dynamical critical exponent given by (87). Since z is an increasing function of p, the correlation length of the critical chains, that goes as M^z, increases for a given lattice size M, as we increase the parameter p. This is physically expected since we increase the range of noncommuting operators in the Hamiltonian, and hence the quantum correlations.

In order to better characterize the critical universality classes at this critical point $\{\lambda_i = 1\}$ for the general free interacting models (1), let us perturb the couplings $\{\lambda_i\}$ around $\{\lambda_i = 1\}$. To simplify, let us restrict our analysis for the cases where M is a multiple of $(p+1)$, i.e., $M = (p+1)\overline{M}$. We consider perturbed Hamiltonians where all the couplings are kept at $\lambda_i = 1$, except for the couplings $\lambda_{(p+1)k} = \lambda$, with $k = 1, \ldots$. For example, for $p = 1$ the sequence of couplings are $(1, \lambda, 1, \lambda, \ldots, 1, \lambda)$ and for $p = 2$ we have $(1, 1, \lambda, 1, 1, \lambda, \ldots, 1, 1, \lambda)$.

In general, for anisotropic scaling, the dynamical critical exponent is given by $z = \nu_\perp/\nu_\parallel$, where $\nu_\perp$ and $\nu_\parallel$ are the correlation length exponents in the time and space directions, respectively. The specific heat at the critical point, from the finite-size scaling theory (FSS) of critical behavior [31], should have, as $M \to \infty$, the power-law behavior

$$C_M(\lambda = 1) \approx M^{\alpha/\nu_\parallel}, \tag{88}$$

specified by the critical exponent α. The specific heat is given by the second derivative of the ground-state energy [32]

$$C(\lambda, M) = -\frac{\lambda^2}{M}\frac{\partial^2 E_0(\lambda, M)}{\partial\lambda^2} = -\frac{\lambda^2}{M}\sum_{i=1}^{\overline{M}}\frac{\partial^2\epsilon_i}{\partial\lambda^2}. \tag{89}$$

Solving for the zeros $\{z_i\}$ of the polynomials $P_M^{(p)}(z)$ we obtain the quasienergies $\epsilon_i(\lambda, M)$.

Before considering the cases $p > 1$, let us consider the case $p = 1$. In this case we have verified *surprisingly* that the corresponding generalized Chebyshev polynomials have exact zeros z_i, producing quasienergies

$$\epsilon_j = z_j^{-1/N}, \quad z_j^{-1} = 1 + \lambda^N + 2\lambda^{N/2}\cos k_j, \tag{90}$$

where $k_j = 2\pi j/(M+2)$, for finite M and arbitrary λ. We recall that for $N = 2$ these are the quasienergies for the Hamiltonian (12) describing an Ising quantum chain with an impurity at one of its ends. In the case where M is odd, whose representation (7) is the standard Ising quantum chain, the roots $\{z_i\}$ are exactly known only at $\lambda = 1$ [9].

The specific heat is obtained from (89) and (91):

$$C(\lambda, M) = -\frac{\lambda^2}{M}\sum_{j=1}^{\overline{M}}\left\{(1-N)\frac{(\lambda^{N-1}+\lambda^{\frac{N}{2}-1}\cos k_j)^2}{(1+\lambda^N+2\lambda^{N/2}\cos k_j)^{2-1/N}} + \frac{(N-1)\lambda^{N/2}+(N/2-1)\lambda^{N/2-2}\cos k_j}{(1+\lambda^N+2\lambda^{N/2}\cos k_j)^{1-1/N}}\right\}. \tag{91}$$

At $\lambda = 1$, since $k_j = 2\pi j/(M+2)$, all terms in the above sums are of $o(1)$, except for the ones where $j = \overline{M} - k = M/2 - k$ with k of $o(1)$. These last terms will dominate the sum. Since for these terms $1 + \cos k_j \sim o(1/M^2)$, we get for $M \to \infty$

$$C(1, M) \sim \frac{1}{M}(M^2)^{1-1/N} \sim M^{1-2/N}, \tag{92}$$

giving us the critical exponent

$$\alpha = 1 - 2/N. \tag{93}$$

TABLE II. The specific-heat critical exponents α for the free fermionic ($N = 2$) and free parafermionic models ($N = 3$–9),and parameters $p = 2$–5. The exact predicted values (95) are shown in brackets. It is shown the extrapolated results of the estimator (94) using sequences of lattice sizes up to $M_{\max}$, shown in the last line of the table.

N	$p = 2$	$p = 3$	$p = 4$	$p = 5$
2	0.000 [0]	0.000 [0]	0.000 [0]	0.000 [0]
3	0.051 [0]	0.001 [0]	0.000 [0]	0.000 [0]
4	0.252 [0.25]	0.092 [0]	0.000 [0]	0.000 [0]
5	0.400 [0.4]	0.206 [0.2]	0.002 [0]	0.000 [0]
6	0.500 [0.5]	0.334 [0.333...]	0.002 [0]	0.000 [0]
7	0.571 [0.571...]	0.428 [0.428...]	0.287 [0.285...]	0.153 [0.142...]
8	0.625 [0.625]	0.500 [0.5]	0.375 [0.375]	0.252 [0.25]
9	0.666 [0.666...]	0.567 [0.555...]	0.444 [0.444...]	0.334 [0.333...]
$M_{\max}$	2100	2560	9450	10680

This exponent can also be derived for the case where M is odd, but the lack of an exact expression for the zeros renders the derivation lengthy.

In the general cases $p > 1$ we do not have an analytical solution for the roots as in (91) and we have to evaluate them numerically. We also have to calculate numerically the derivatives of the quasiparticle energies ϵ_i at $\lambda = 1$. By taking the specific-heat values at two distinct lattice sizes M_1 and M_2 we produce the finite-size estimator

$$\alpha_{M_1,M_2} = \frac{\ln\left(C(1,M_1)/C(1,M_2)\right)}{\ln(M_1/M_2)}, \tag{94}$$

that should tend towards α as $M_1, M_2 \to \infty$. In Table II we give the results obtained by extrapolating sequences of α_{M_1,M_2} for the models with $p = 2, \ldots, 5$ and $N = 2, \ldots, 9$. The results were obtained by using van den Broeck Schwartz [33] extrapolants up to lattice sizes $M_{\max}$, shown in the last line of the table. The zeros of $P_M^{(p)}(z)$ were numerically evaluated with 50 decimal digits by using multiple precision calculations.

Our results indicate the conjecture

$$\alpha = \max\{0, 1 - (p+1)/N\}, \tag{95}$$

i.e., the models will have a vanishing critical exponent for the specific heat if $N \leqslant (p+1)$. The conjectured values (95), that extend (93), are shown in brackets in Table II. We see a quite good agreement with (95).

In the cases where $p = 1$ and M odd, like the quantum Ising chain (7) and the free parafermionic models (10) it was numerically observed that the specific heat has a peak in a pseudocritical point $\tilde{\lambda}_M$, that approaches the true critical point $\lambda_c = 1$ as $|\tilde{\lambda}_M - \lambda_c| \sim M^{-\nu_\parallel}$, with the value $\nu_\parallel = 1$. However, when we consider the case $p = 1$ with M even, whose quasiparticle energies are given by (91) for $N = 2$, we verify that the pseudocritical point approaches the critical point as $|\tilde{\lambda}_M - \lambda_c| \sim M^{-1.94}$. For other models with $p > 1$, the exponents change as we consider different ℓ sequences of lattice sizes $M = j\overline{M} + \ell$ ($j = 0, 1, \ldots$, $\ell = 0, 1, \ldots$). In fact, the finite-size behavior $M^{-\nu_\parallel}$ is not a consequence of the FSS theory, and it is not generally expected for open chains [32]. We conjecture that for all the models with any p and N we have the simple scaling where $\nu_\parallel = 1$, as verified in the models with $p = 1$ and M odd.

VII. ADDITIONAL COMMUTING CHARGES AND COMPLETE SET OF COMMUTING OBSERVABLES

Distinct representations of the Hamiltonians (1), with M generators $\{h_i\}$ satisfying (2) and (3), with given values of p and N, have distinct dimensions. In Sec. III we have showed the existence of $\overline{M}$ commuting charges ($H_M^{(\ell)}$, $\ell = 1, \ldots, \overline{M}$), independently of the representation. The eigenvalues of these charges are given in terms of the $\overline{M}$ roots of the polynomials $P_M^{(p)}(z)$ and by the set of "$Z(N)$ signals" $\{s_1, \ldots, s_{\overline{M}}\}$ (78). These are all the possible values of the eigenvalues. In the generic case the dimensions of the representation of (1) is bigger than $N^{\overline{M}}$, implying degeneracy in energy as well in all the $\overline{M}$ commuting charges $\{H_M^{(\ell)}\}$.

Let us consider some free fermionic representations ($N = 2$) with $p = 1$. For M odd the representations of the Ising quantum chain (7) have dimension $2^{(M+1)/2}$ and since in this case $\overline{M} = (M+1)/2$ all the eigenvalues can be indexed by the roots of $P_M^{(p)}(z)$ and signals $\{s_i\}$. That is, all the $2^{(M+1)/2}$ eigenfunctions are distinctly characterized by the complete set of commuting observables (CSCO) $\{H_M^{(\ell)}; \ell = 1, \ldots, (M+1)/2\}$.

In the case where M is even, as in (12), the dimension of the Hilbert space is $2^{M/2}$ and since $\overline{M} = M/2$ the charges (20) form again a CSCO. This construction is an interesting way to see the fully exact integrability of the Ising quantum chains in a finite lattice.

In the case where the free fermionic models are in the representation (19) with dimension 2^M the situation is distinct since the number of conserved charges $\{H_M^{(\ell)}\}$ is $(M+1)/2$ or $M/2$ if M is odd or even, respectively. However, we can identify an extra set of $Z(2)$ gauge operators $\{g_M^{(i)}\}$:

$$\begin{aligned} &\sigma_1^x\sigma_2^z,\ \sigma_3^x\sigma_4^z, \ldots,\ \sigma_{M-1}^x\sigma_M^z,\ \sigma_M^x \quad (M \text{ even});\\ &\sigma_1^x\sigma_2^z,\ \sigma_3^x\sigma_4^z, \ldots,\ \sigma_{M-1}^x\sigma_M^z \quad (M \text{ odd}), \end{aligned} \tag{96}$$

that aside from commuting among themselves and with $\{H_M^{(\ell)}\}$ are independent. Since this extra set (96) has $(M-1)/2$ or $M/2$ charges, for M odd or even, respectively, we have a total of M commuting charges forming again a CSCO. Differently from the charges $\{H_M^{(\ell)}\}$ whose eigenvalues are obtained from the roots of the polynomials $P_M^{(p)}(z)$, the gauge charges $\{g_M^{(i)}\}$ have eigenvalues ± 1 since $(g_M^{(i)})^2 = 1$. The commutation

$[g_M^{(i)}, H_M^{(\ell)}] = 0$ implies that all the eigenvalues of the Hamiltonian (19), as well as all the charges $\{H_M^{(\ell)}\}$ will have, apart from accidental degeneracies, a degeneracy $2^{(M-1)/2}$ or 2^M for M odd or even, respectively.

The preceding discussion for $p = 1$ and $N = 2$ is easily generalized for the free parafermionic cases where $N > 2$. However, this is not the case for $p > 1$. In fact, let us consider the fermionic models ($N = 2$) with $p = 2$ and let us restrict ourselves to the representation (18):

$$\mathcal{H}_A = -\lambda_1\sigma_1^x - \lambda_2\sigma_1^z\sigma_2^x - \lambda_3\sigma_1^z\sigma_2^z\sigma_3^x - \lambda_4\sigma_2^z\sigma_3^z\sigma_4^x - \cdots - \lambda_M\sigma_{M-2}^z\sigma_{M-1}^z\sigma_M^x. \quad (97)$$

The number of commuting charges $\{H_M^{(\ell)}\}$, whose eigenvalues are given in terms of the roots of the polynomial $P_M^{(2)}(z)$ is $2^{\overline{M}}$, while the dimension of the representation is 2^M. We verified by direct diagonalizations that for small values of M all the eigenvalues of (97) have the same degeneracy $2^M/2^{\overline{M}}$. As in the case $p = 1$ we can also identify an extra set $\{g_M^{(i)}\}$ of $Z(2)$ gauge operators:

$$\sigma_1^x\sigma_2^z\sigma_3^z,\ \sigma_4^x\sigma_5^z\sigma_6^z,\ \ldots \quad (98)$$

forming the commuting set $\{g_M^{(i)}, H_M^{(\ell)}\}$. Since there are $= \lfloor M/3 \rfloor$ gauge operators and $(g_M^{(i)})^2 = 1$, we can explain $2^{\tilde{M}}$ of this degeneracy. Although we believe in their existence we have not found the extra charges that will complete the CSCO of (97).

The previous discussions can be easily extended for $N > 2$ and $p > 2$, and again we can explain part of the degeneracies of the Hamiltonian in their word representation. We remark that for $p = 2$ and $N = 2$ the observed degeneracies in the spectrum have been explained using an underlying extended supersymmetry of the model [12]. It would be interesting to see if this construction can be extended to other values of p and N.

VIII. CONCLUSIONS

We have demonstrated that the general quantum spin chains (1) with M density operators $\{h_i\}$ satisfying the p-exchange algebra (2) and (3) are exactly integrable. In the bulk limit $M \to \infty$ the Hamiltonians belong to an infinite set of conserved charges. In the generic case the number of independent words we can form in the algebra is infinite, even for finite M. However, by including the closure relation (3) in the algebra, the number of independent words, or the dimension of the Hilbert space associated to the Hamiltonian (1), is finite for finite M. We can have then several possible Hamiltonians described for the case $N = 2$ in terms of spin-$\frac{1}{2}$ Pauli matrices and for $N > 2$ by the generalized $Z(N)$ Pauli matrices (8). In these cases we show that for arbitrary values of the parameter p, all the models have a free fermionic ($N = 2$) or free parafermionic ($N > 2$) eigenspectrum. The eigenenergies are given in terms of the zeros of the polynomials $P_M^{(p)}(z)$. In the case $p = 1$ and $\{\lambda_i\} = 1$, where the corresponding Hamiltonians are the quantum Ising chain ($N = 2$) or the $Z(N)$ Baxter parafermionic chain ($N > 2$), at their critical point, $P_M^{(p)}(z)$ is related to the Chebyshev polynomial of second type. We presented several representations of the algebraic Hamiltonian (1), for several values of p. In particular, we have also constructed the Hamiltonian (1) on its word representation, a representation with a one-to-one equivalence among the independent words in the algebra and the basis vectors spanning the associated Hilbert space.

Although the models, for arbitrary N and p, have a quite rich phase diagram we only consider, in this paper, the isotropic point where all the couplings $\{\lambda_i = 1\}$ [see (3)]. We have showed that the models are critical at this point for any p and N. We calculated at this critical point the exact ground-state energy for general p and N, and it turns out to be expressed in terms of integral representations of Lauricella series. From our extensive numerical studies we have conjectured that at this isotropic point the dynamical critical exponent is given by $z = (p + 1)/N$ and the specific-heat exponent by $\alpha = \max\{0, 1 - (p + 1)/N\}$. This is interesting since most of the known critical chains are conformally invariant ($z = 1$), and therefore the multispin quantum chains provide an excellent laboratory to understand the universal behavior of the shared quantum information measures, like the von Neumann or Rényi entanglement entropies, in critical quantum chains without an underlying conformal symmetry. The polynomials $P_M^{(p)}(z)$ may also play an important role in the study of the entanglement entropy (see [34]).

An interesting direction of investigation is to consider extensions of the algebra (2) and (3) to include models with periodic boundary conditions. See, for example, the interesting recent paper [35]. In this case, for $p = 1$ and $N = 2$ the algebra (2) and (3) are related to the Temperley-Lieb and Onsager algebras. For periodic boundary conditions, the eigenspectrum seems to be not a simple free particle one, but probably a composition of free particle spectra. As observed in [10] the $Z(N)$ Baxter free parafermionic quantum chain, that corresponds to the case $p = 1$, has an anomalous behavior for the ground-state energy per site for $N > 2$, where the models are non-Hermitian. It should be also interesting to prove if this anomaly happens for general values of p.

Another interesting point for further investigation is the construction of the integrable quantum chains within the quantum inverse scattering method. For $p = 1$ and arbitrary N, the generating function (21) corresponds to the transfer matrix of the τ_2 model (see [7]). For $p = N = 2$, a solution of the Yang-Baxter equation with a 9×9 R-matrix has been given in [12]. It would be important to generalize this approach to arbitrary p and N. In addition, let us mention that the algebra (2) and (3) can be Baxterized. In this case, it would be nice to see if the generating function can be written directly in terms of $\check{R}$ matrices.

We conclude mentioning that although we have not constructed the general raising and lowering fermionic and parafermionic operators, related to the quantum chains (1), we believe that this construction should follow the one introduced by Fendley [12] for $p = N = 2$ exploiting the general product formula (52).

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APPENDIX A: DERIVATION OF THE RECURRENCE RELATION FOR $\tau_M^{(3)}(u)$

In this Appendix we derive the recurrence relation (64) of Sec. IV. To simplify the notation, let us denote

$$G_M^{(i)} \equiv G_M^{(i)}(u) = G_M(\omega^i u), \quad i = 0, 1, 2 \tag{A1}$$

so that

$$\tau^{(3)}(u) = G_M^{(0)} G_M^{(1)} G_M^{(2)}. \tag{A2}$$

Using the fundamental relation (26) in $G_M^{(i)}$ $(i = 0, 1, 2)$ we obtain

$$\tau_M^{(3)}(u) = \tau_{M-1}^{(3)}(u) - u^3 h_M^3 \tau_{M-(p+1)}^{(3)}(u) + \Xi_p^{(3)}(u), \tag{A3}$$

where

$$\Xi_p^{(3)}(u) = A_1(M)u + A_2(M)u^2 \tag{A4}$$

and

$$\begin{aligned} A_2(M) &= -\mathcal{L}(G_{M-1}; G_{M-(p+1)}h_M; G_{M-(p+1)}h_M)_{\omega^2}, \\ A_1(M) &= -\mathcal{L}(G_{M-(p+1)}h_M; G_{M-1}; G_{M-1})_{\omega}, \end{aligned} \tag{A5}$$

where we have introduced the $Z(3)$ cyclic commutator, defined as

$$\begin{aligned} &\mathcal{L}(A(u); B(u); C(u))_\Omega \\ &\quad = A(u)B(\omega u)C(\omega^2 u) + \Omega C(u)A(\omega u)B(\omega^2 u) \\ &\qquad + \Omega^2 B(u)C(\omega u)A(\omega^2 u). \end{aligned} \tag{A6}$$

To proceed, it is interesting to define the generalized operators

$$A_1^{(j)}(M) = -\mathcal{L}(G_{M-(p+1)}h_M; G_{M-j}; G_{M-j})_\omega \tag{A7}$$

and

$$\begin{aligned} A_2^{(j,k)}(M) = -\mathcal{L}(&G_{M-(p+1)}; \\ &\times h_{M-j}G_{M-(p+1)-j}; h_{M-k}G_{M-(p+1)-k})_{\omega^2}. \end{aligned} \tag{A8}$$

We see that $A_1 = A_1^{(1)}$.

Recurrence relation for $A_1^{(j)}(M)$. Using (26) in both the G_{M-j} of $A_1^{(j)}(M)$ we obtain

$$A_1^{(j)}(M) = \sum_{l=j}^{p} \left(\gamma_1^{(l)} + \gamma_2^{(l)}\right) - \sum_{l=j}^{p}\sum_{k=j}^{p} \gamma_3^{(l,k)}, \tag{A9}$$

where

$$\begin{aligned} \gamma_1^{(l)} &= u\mathcal{L}(G_{M-(p+1)}h_M; h_{M-j}G_{M-(p+1)-j}; G_{M-(p+1)})_\omega, \\ \gamma_2^{(l)} &= u\mathcal{L}(G_{M-(p+1)}h_M; G_{M-(p+1)}; h_{M-j}G_{M-(p+1)-j})_\omega, \end{aligned}$$

and

$$\begin{aligned} \gamma_3^{(l,k)} &= u^2\mathcal{L}(G_{M-(p+1)}h_M; \\ &\times h_{M-l}G_{M-(p+1)-l}; h_{M-k}G_{M-(p+1)-k})_\omega. \end{aligned} \tag{A10}$$

Expanding $\gamma_1^{(l)}$ and $\gamma_2^{(l)}$ we obtain

$$\gamma_1^{(l)} + \gamma_2^{(l)} = -uh_M\omega^2 A_1^{(p+1-l)}(M-l), \tag{A11}$$

where $l = 1, 2, \ldots, p$, and identify

$$\gamma_3^{(l,k)} = u^2 h_M A_2^{(l,k)}(M). \tag{A12}$$

Using (27) in (A8) we obtain

$$A_2^{(j,j)}(M) = -uh_{M-j}^2\omega \sum_{l=0}^{j-1} A_1^{(j-l)}[M-(p+1)-l].$$

Using this last expression with (A11) and (A12) in (A9) we get

$$A_1^{(j)}(M) = -uh_M\omega^2 \sum_{l=j}^{p} A_1^{(p+1-l)}(M-l) - u^3 h_M\omega \sum_{l=j}^{p} h_{M-l}^2 \sum_{l'=0}^{l-1} A_1^{(l-l')}[M-(p+1)-l'] - u^2 h_M \sum_{k=j}^{p}\sum_{l=j+1}^{k-1} \left(A_2^{(l,k)}(M) + A_2^{(k,l)}(M)\right). \tag{A13}$$

Also using (26) in (A8) we obtain, for $j \neq k$,

$$A_2^{(j,k)}(M) = \alpha_1^{(j,k)}(M) + \alpha_2^{(j,k)}(M),$$

where

$$\begin{aligned} \alpha_1^{(j,k)}(M) = \mathcal{L}(&G_{M-(p+1)-j}; h_{M-j}G_{M-(p+1)-j}; \\ &\times h_{M-k}G_{M-(p+1)-k})_{\omega^2}, \end{aligned} \tag{A14}$$

and for $(j < k)$

$$\alpha_2^{(j,k)}(M) = -uh_{M-j}\sum_{l=0}^{j-1} A_2^{(k-j,p+1+l-j)}(M-j),$$

$$\alpha_2^{(k,j)}(M) = -uh_{M-j}\sum_{l=0}^{j-1} A_2^{(p+1+l-j,k-j)}(M-j).$$

Combining $\alpha_1^{(j,k)}(M) + \alpha_1^{(k,j)}(M)$ $(j \neq k)$ we obtain, for $j < k$,

$$\alpha_1^{(j,k)}(M) + \alpha_1^{(k,j)}(M) = -\omega^2 h_{M-j} A_1^{(p+1+j-k)} \ (M-k).$$

Then, for $j < k$,

$$\begin{aligned} A_2^{(j,k)}(M) + A_2^{(k,j)}(M) = &-\omega^2 h_{M-j} A_1^{(p+1+j-k)}(M-k) \\ &-uh_{M-j}\sum_{l=0}^{j-1} \left(A_2^{(p+1+l-j,k-j)}(M-j)\right. \\ &\left. + A_2^{(k-j,p+1+l-j)}(M-j)\right). \end{aligned} \tag{A15}$$

Recurrence relation for $A_2(M)$. Using the relation (29) in $A_2(M)$, given in (A15), we obtain

$$A_2(M) = -uh_M^2 \sum_{j=1}^{p} \times \mathcal{L}(G_{M-(p+1)}; G_{M-(p+1)}; h_{M-j} G_{M-(p+1)-j})_\omega, \tag{A16}$$

giving us, from (A5),

$$A_2(M) = -uh_M^2 \omega \sum_{j=1}^{p} A_1^{(p+1-j)}(M-j). \tag{A17}$$

Equation (A14), with the recurrences (A15) and (A17), implies that $A_1(M) = A_1^{(1)}$, $A_1^{(l)}(M)$ $(l = 1, \ldots, M)$ and $A_2(M)$ depend only on the values of $A_1^{(k')}(M-j')$, $j' > 1, k' \leqslant p$, and on $A_2^{(l,l')}(M-j') + A_2^{(l',l)}(M-j)$, $j' > 1, 1 \leqslant l \neq l' \leqslant p$, i.e., it depends only on the values of $A_1^{(l)}(M')$, $A_2^{(j,k)}(M')$, evaluated for smaller lattices.

Since $h_M = 0$ $(M \leqslant 0)$, $G_M = \mathbb{1}$ $(M \leqslant 1)$ it is simple to verify from (A15) that $A_1^{(0)}(M) = A_2^{(j,k)}(M) = 0$ for $M \leqslant 1$. For $M = 2$ we see that $A_1^{(j)}(2) = 0$ $(j = 1, \ldots, p)$ and $A_2^{(j,k)}(2) + A_2^{(k,j)}(2) = 0$ $(k \neq j = 1, \ldots, p)$. From (A15) we also see that $A_2^{(j,k)}(3) + A_2^{(k,j)}(3) = 0$ $(k \neq j = 1, \ldots, p)$, and from (A14) we get $A_1^{(l)}$ $(l = 1, \ldots, p)$. Finally, iterating we obtain for (A5), (A15), and (A4)

$$A_1(M) = A_2(M) = \Xi_p^{(3)}(u) = 0 \tag{A18}$$

for any M and p. This implies that

$$\tau_M^{(3)}(u) = \tau_{M-1}^{(3)}(u) - u^3 h_M^3 \tau_{M-(p+1)}^{(3)}(u). \tag{A19}$$

APPENDIX B: EIGENSPECTRUM OF A QUANTUM CHAIN WITH $M = 5$ SITES AND PARAMETER $p = 3$

In this Appendix, for the sake of illustration, we give a simple example for the quantum chain (1) with $M = 5$ generators, parameter $p = 3$ and N arbitrary. In this case, $\overline{M} = \lfloor (5+3)/4 \rfloor = 2$.

The general Hamiltonian satisfying the algebra (2) and (3) is given by

$$\mathcal{H} = -(h_1 + h_2 + h_3 + h_4 + h_5). \tag{B1}$$

There are two conserved charges, i.e.,

$$H_5^{(1)} = -\mathcal{H}, \quad H_5^{(2)} = h_1 h_5.$$

One of the representations of the conserved charges is the word representation (16) where

$$\mathcal{H} = -\lambda_1 X_1 - \lambda_2 Z_1 X_2 - \lambda_3 Z_1 Z_2 X_3 - \lambda_4 Z_1 Z_2 Z_3 X_4 - \lambda_5 Z_2 Z_3 Z_4 X_5,$$
$$H_5^{(2)} = \lambda_1 \lambda_5 X_1 Z_2 Z_3 Z_4 X_5, \tag{B2}$$

where X_i, Z_i are the $Z(N)$ matrices (17) and the coupling constants $\{\lambda_i\}$ are defined by (3).

The fundamental polynomial is obtained by iterating (68) or by (69) (compare with Table I, for the case $\lambda_1 = \lambda_2 = 1$):

$$P_5^{(3)}(z) = 1 - (\lambda_1^N + \cdots + \lambda_5^N) z + \lambda_1^N \lambda_5^N z^2, \tag{B3}$$

whose roots z_1 and z_2 give us the quasienergies $\epsilon_1 = z_1^{-1/N}$ and $\epsilon_2 = z_2^{-1/N}$. The predicted N^2 eigenvalues of the Hamiltonian and second charge are obtained from (78) and (77):

$$E^{\{s_1,s_2\}} = -e^{i\frac{2\pi}{N}s_1}\epsilon_1 - e^{i\frac{2\pi}{N}s_2}\epsilon_2, \quad E_2^{\{s_1,s_2\}} = e^{i\frac{2\pi}{N}(s_1+s_2)}\epsilon_1\epsilon_2,$$

where $s_1, s_2 = 0, 1, \ldots, N-1$. In the word representation (B2) the Hamiltonian has N^5 eigenvalues. A direct diagonalization of (B2) shows us that all levels have the same degeneracy N^3. The ground-state energy (real for all N) is given by $E^{\{0,0\}} = -\epsilon_1 - \epsilon_2$, while the excited states have complex eigenvalues.

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