Modern Physics Letters A
Vol. 30, No. 10 (2015) 1550047 (9 pages)

DOI: 10.1142/S0217732315500479

Role of division algebra in seven-dimensional gauge theory

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Received 14 November 2014
Accepted 5 January 2015
Published 6 March 2015

The algebra of octonions $\mathbb{O}$ forms the largest normed division algebra over the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$. The usual three-dimensional vector product is given by quaternions, while octonions produce seven-dimensional vector product. Thus, octonionic algebra is closely related to the seven-dimensional algebra, therefore one can extend generalization of rotations in three dimensions to seven dimensions using octonions. An explicit algebraic description of octonions has been given to describe rotational transformation in seven-dimensional space. We have also constructed a gauge theory based on non-associative algebra to discuss Yang–Mills theory and field equation in seven-dimensional space.

Keywords: Field equation; gauge theory; octonions.

PACS Nos.: 11.15.q, 03.50.De

1. Introduction

According to the Hurwitz theorem[1,2] the only composition algebras are real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, quaternions $\mathbb{H}$ and octonions $\mathbb{O}$ out of which octonion forms the largest division algebra (see e.g. Refs. 3 and 4). Complex numbers $\mathbb{C}$ can be described as pairs of real numbers $\mathbb{R}$. Quaternion $\mathbb{H}$ forms noncommutative algebra and can be defined as pair of complex numbers. The algebra of octonions $\mathbb{O}$ defined on the eight-dimensional real vector space $\mathbb{R}^8$ and is a non-associative extension of the algebra of quaternions $\mathbb{H}$. Octonion represents an alternative (the alternation of associator under the permutation of arguments) and composition (it satisfy the property $N(xy) = N(x)N(y)$ for the norm $N(y) = y\bar{y} = \bar{y}y$; $x, y \in \mathbb{O}$) algebra. These composition algebras[2] are responsible for many important mathematical structures of interest for physicists in various ways to describe properties of electromagnetism,[5] quantum mechanics,[6] field theory,[7] gauge theory,[8–10] gravitation,[11,12] and they have various applications in seven-dimensional space[13–15] as well. Various applications of octonion have been widely discussed by Baez.[16] It has also played an important role in the context of various physical problems of higher-dimensional space-like supersymmetry[17,18] and superstring theory.[19]

Here, we have described the matrix formulation of the vector product in seven-dimensional space with the help of octonion algebra. This formulation gives extension of non-Abelian Maxwell equations from three- to seven-dimensional space.

Our presentation is organized as follows. A brief description of octonions and its algebraic properties have been given in Sec. 2. The rotational transformation in seven-dimensional space has been constructed by using the algebraic property of octonions in Sec. 3. Such rotational transformations give SO(7) group generators as shown in Sec. 4. SO(7) Lagrangian and seven-dimensional Yang–Mills theory have been described in Sec. 5, which give gauge field and field equation in seven-dimensional space. Concluding remarks have been given in Sec. 6.

2. Octonions

The octonions, also known as Cayley–Graves numbers, are an algebraic structure defined on the eight-dimensional real vector space such that two octonions can be added, multiplied and divided, except that multiplication is neither commutative or associative. Its elements can be represented as

$$\begin{aligned} y &= e_0y_0 + e_1y_1 + e_2y_2 + e_3y_3 + e_4y_4 + e_5y_5 + e_6y_6 + e_7y_7 \\ &= e_0y_0 + \sum_{a=1}^{7} e_a y_a \,, \end{aligned} \tag{1}$$

where y_k $(k = 0, 1, \ldots, 7)$ are real numbers, e_a $(a = 1, 2, \ldots, 7)$ are imaginary octonion units and e_0 is the multiplicative unit element. The set of octets $(e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7)$ is known as the octonion basis and its elements satisfy the following multiplication rules:

$$\begin{aligned} e_0 &= 1\,, \quad \text{i.e. } e_0e_a = e_ae_0 = e_a\,, \\ e_ae_b &= -\delta_{ab}e_0 + f_{abc}e_c\,, \qquad (a, b, c = 1, 2, \ldots, 7)\,, \end{aligned} \tag{2}$$

where δ_{ab} is the usual Krönecker symbol and the structure constants f_{abc} are completely antisymmetric, take the value 1 for the following combinations:

$$f_{abc} = +1 \quad \forall\ (abc) = (123),\ (471),\ (257),\ (165),\ (624),\ (543),\ (736) \tag{3}$$

and vanish otherwise.

Octonion multiplication can be described with the help of Fig. 1. Here, the cyclic symmetry for the multiplication of octonions can be obtained by ordering seven points clockwise on a circle with the numbering 1243657 (shown as $e_1, e_2, \ldots, e_7$). Different triangles like $e_1e_2e_3$ or $e_2e_4e_6$ represents the octonion multiplication and can be obtained by the successive rotations of each by an angle $2\pi/7$.

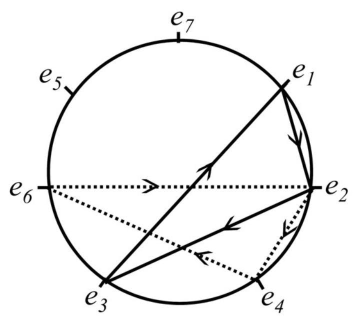

Fig. 1. Schematic representation of octonion multiplication.

The octonion basis elements satisfy the following additional relations:

$$
\begin{aligned}
[e_a, e_b] &= 2f_{abc}e_c\,, \\
\{e_a, e_b\} &= -2\delta_{ab}e_0\,, \\
e_a(e_b e_c) &\neq (e_a e_b)e_c\,,
\end{aligned} \tag{4}
$$

where the brackets [,] and { , } denote, as usual, the commutator and the anti-commutator, respectively. Octonion conjugate is defined as

$$
\begin{aligned}
\bar{y} &= e_0y_0 - e_1y_1 - e_2y_2 - e_3y_3 - e_4y_4 - e_5y_5 - e_6y_6 - e_7y_7 \\
&= e_0y_0 - \sum_{A=1}^{7} e_A y_A\,,
\end{aligned} \tag{5}
$$

where we have used the conjugates of basis elements as $\overline{e_0} = e_0$ and $\overline{e_A} = -e_A$. Hence, an octonion can be decomposed in terms of its scalar ($\mathrm{Sc}(y)$) and vector ($\mathrm{Vec}(y)$) parts as

$$
\begin{aligned}
\mathrm{Sc}(y) &= \frac{1}{2}(y + \bar{y})\,, \\
\mathrm{Vec}(y) &= \frac{1}{2}(y - \bar{y}) = \sum_{A=1}^{7} e_A y_A\,.
\end{aligned} \tag{6}
$$

Conjugates of product of two octonions and its own are described as

$$
\overline{(xy)} = \bar{y}\,\bar{x}\,; \qquad \overline{(\bar{x})} = x\,. \tag{7}
$$

The norm $N(y)$ and inverse y^{-1} (for a nonzero y) of an octonion are respectively defined as

$$
\begin{aligned}
N(y) &= y\bar{y} = \bar{y}y\,, \\
y^{-1} &= \frac{\bar{y}}{N(y)} \Rightarrow yy^{-1} = y^{-1}y = 1\,.
\end{aligned} \tag{8}
$$

The norm $N(y)$ of an octonion y is zero if $y = 0$, and is always positive otherwise. It also satisfies the following property of normed algebra

$$N(xy) = N(x)N(y) = N(y)N(x)\,. \tag{9}$$

Octonions are not associative in nature and thus do not form the group in their usual form. Non-associativity of octonion algebra $\mathbb{O}$ is provided by the associator $(x, y, z) = (xy)z - x(yz)\ \forall\, x, y, z \in \mathbb{O}$ defined for any three octonions. If the associator is totally antisymmetric for exchanges of any three variables, i.e. $[x, y, z] = -(z, y, x) = -(y, x, z) = -(x, z, y)$, then the algebra is called alternative. Hence, the octonion algebra is neither commutative nor associative but, is alternative.

The basic associator with three octonion basis units satisfies the following relation

$$[e_a, e_b, e_c] = 2f_{abcd}e_d\,, \tag{10}$$

where f_{abcd} is a fully antisymmetric four index object.

$$f_{abcd} = -1\ \forall\ (abcd) = (4567),\ (3751),\ (6172),\ (5214),\ (7423),\ (1346),\ (2635)\,.$$

Octonion structure constant satisfies the following properties,

$$\begin{aligned} f_{mij}f_{mkl} &= f_{ijkl} - \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\,, \\ f_{aij}f_{bij} &= 6\delta_{ab}\,. \end{aligned} \tag{11}$$

Octonions satisfy weak associative properties called alternativity and flexibility as follows:

- Alternative property

$$\begin{aligned} a(ab) &= a^2b\,, \\ ba^2 &= (ba)a\,. \end{aligned} \tag{12}$$

- Flexible property

$$a(ba) = (ab)a\,. \tag{13}$$

The sub-algebra generated by (e_0, e_7) is isomorphic to the algebra $\mathbb{C}$ of the complex numbers while the sub-algebra generated by (e_0, e_1, e_2, e_3) is isomorphic to the algebra $\mathbb{H}$ of quaternions.

3. Rotation in Seven-Dimensional Space

An infinitesimal rotation by an angle θ in the plane (i, j) is obtained by the following operator,

$$R_{ij} = 1 + \theta f_{ijk}e_k\,, \tag{14}$$

R_{ij} represents the rotations corresponding to various combinations of i and j and it gives the three same rotational transformations of vectors corresponding to indices

(i, j, k) of structure constants f_{ijk} of octonions. Since the sub-algebra of octonions composed by e_0 and any other three units of octonion with nonvanishing structure constants is isomorphic to the quaternionic algebra. It means that the product of octonions in this subspace is associative. Therefore, we can describe rotation operator R_{ij} acts on a vector x to form a rotated vector x' as,

$$x' = R_{ij} x R_{ij}^{-1} \,. \tag{15}$$

Rotation operator in plane $(1, 2)$, i.e. R_{12} becomes as

$$R_{12} = 1 + \theta e_3 \,. \tag{16}$$

Therefore, the rotated vector x' takes the form

$$\begin{aligned} x' = {} & e_1(x_1 - 2\theta x_2) + e_2(x_2 + 2\theta x_1) + e_3 x_3 + e_4(x_4 + 2\theta x_5) \\ & + e_5(x_5 - 2\theta x_4) + e_6(x_6 - 2\theta x_7) + e_7(x_7 + 2\theta x_6) \,. \end{aligned} \tag{17}$$

It shows that under R_{12} rotation, vectors transform as

$$\begin{aligned} x_1' &\Rightarrow x_1 - 2\theta x_2 \,, \\ x_2' &\Rightarrow x_2 + 2\theta x_1 \,, \\ x_3' &\Rightarrow x_3 \,, \\ x_4' &\Rightarrow x_4 + 2\theta x_5 \,, \\ x_5' &\Rightarrow x_5 - 2\theta x_4 \,, \\ x_6' &\Rightarrow x_6 - 2\theta x_7 \,, \\ x_7' &\Rightarrow x_7 + 2\theta x_6 \,. \end{aligned} \tag{18}$$

It shows that rotation R_{12} changes all the vectors except x_3. Rotation R_{12} yields three simultaneous rotations by an angle 2θ in three planes $(1, 2)$, $(5, 4)$ and $(6, 7)$ and leaves x_3 unchanged. The above transformations of x' under R_{12} can be written as

$$x'[R_{12}] = x - 2\theta f_{123} J_3 x \,. \tag{19}$$

In general, we can write

$$x'[R_{ij}] = x - 2\theta f_{ijk} J_k x \,, \tag{20}$$

where $i, j, k = 1, 2, \ldots, 7$ and J_k represents the real 7×7 matrix associated with octonions.

4. Generators for SO(7) Group

In Eq. (19), J_3 is 7×7 matrix which represents rotation around the x_3 axis or in the $(1, 2)$, $(5, 4)$ and $(6, 7)$ planes, and it can be written as

$$J_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}. \tag{21}$$

We can write $J_a\,(a = 1, 2, \ldots, 7)$ in compact form as,

$$J_k = \sum_{i,j=1}^{7} f_{ijk} u_{ij} = \frac{1}{2} f_{ijk} U_{ij}\,, \tag{22}$$

where u_{ij} is a 7×7 matrix in which the ijth matrix element is unity and $U_{ij} = u_{ij} - u_{ji}$, rest other elements are zero. These 7×7 matrices satisfy the following commutation relations

$$J_k = \frac{1}{3} \sum_{i,j=1}^{7} f_{ijk} [J_i, J_j]\,. \tag{23}$$

Since $J_1, J_2, \ldots, J_7$ do not form any algebra so we will describe J_{ab} as the commutator of J_a and J_b, i.e.

$$J_{ab} = [J_a, J_b]\,. \tag{24}$$

There are 21 independents such as J_{ab}'s. Therefore, we consider these 21 generators as the generators of SO(7) group. Now using Eq. (22), we can write

$$\begin{aligned} J_{kr} = [J_k, J_r] &= \frac{1}{4} f_{pqr} f_{ijk} [U_{ij}, U_{pq}] \\ &= \frac{1}{4} f_{pqr} f_{ijk} [U_{iq}\delta^{jp} + U_{jp}\delta^{iq} - U_{ip}\delta^{jq} - U_{jq}\delta^{ip}] \\ &= \frac{1}{2} f_{pqr} [f_{ipk} U_{iq} + f_{qjk} U_{jp}]\,. \end{aligned} \tag{25}$$

Furthermore, we observe that J_m satisfies the following relationships

$$f_{qrki} U_{iq} + f_{rpjk} U_{jp} + 4U_{kr} = 4 f_{krm} J_m\,, \tag{26}$$

$$[U_{kr}, J_n] = -[f_{kn\alpha} U_{\alpha r} + f_{rn\beta} U_{k\beta}]\,. \tag{27}$$

Now, using Eq. (26) in Eq. (25), we obtain

$$J_{kr} = [-3U_{kr} + 2 f_{krm} J_m]\,. \tag{28}$$

Also with the help of Eqs. (28) and (27), the commutation relation between J_{ab} and J_{cd} can be obtained as

$$[J_{ab}, J_{cd}] = 2f_{abp}f_{cdq}f_{pqr}J_r - (3J_{bc}\delta_{ad} + 3J_{ad}\delta_{bc} - 3J_{bd}\delta_{ac} - 3J_{ac}\delta_{bd})\,, \tag{29}$$

where f_{abp}, f_{cdq}, f_{pqr} are the structure constants of octonions and $J_r = \frac{1}{3}f_{rxy}J_{xy}$. In what follows, by using Eq. (29), we define seven-dimensional gauge field, field tensor, Lagrangian and field equation.

5. Seven-Dimensional Field Equations

We start with covariant derivative

$$\begin{aligned} D_\mu &= \partial_\mu - igA_\mu^{ab}(J_{ab})\,, \\ D_\nu &= \partial_\nu - igA_\nu^{cd}(J_{cd})\,, \end{aligned} \tag{30}$$

where A_μ^{ab} is the gauge field. Commutation between D_μ and D_ν can be calculated as

$$[D_\mu, D_\nu] = -ig(\partial_\mu A_\nu^{cd})J_{cd} + ig(\partial_\nu A_\mu^{ab})J_{ab} - g^2[J_{ab}, J_{cd}]A_\mu^{ab}A_\nu^{cd}\,. \tag{31}$$

Field strength tensor $F_{\mu\nu}$ can be defined by the commutation of two covariant derivative as

$$[D_\mu, D_\nu] = -igF_{\mu\nu}\,. \tag{32}$$

We can write

$$\begin{aligned} -igF_{\mu\nu}^{xy}J_{xy} = -ig(\partial_\mu A_\nu^{cd})J_{cd} + ig(\partial_\nu A_\mu^{ab})J_{ab} - g^2A_\mu^{ab}A_\nu^{cd}\bigg(\frac{2}{3}f_{abp}f_{cdq}f_{pqr}f_{rxy}J_{xy} \\ - (3J_{bc}\delta_{ad} + 3J_{ad}\delta_{bc} - 3J_{bd}\delta_{ac} - 3J_{ac}\delta_{bd})\bigg). \end{aligned} \tag{33}$$

Comparing the coefficient of J^{xy} on both sides, we get the electromagnetic field tensor as

$$F_{\mu\nu}^{xy} = \partial_\mu A_\nu^{xy} - \partial_\nu A_\mu^{xy} + \frac{2g}{3}f_{abp}f_{cdq}f_{pqr}f_{rxy}A_\mu^{ab}A_\nu^{cd} + 12g(A_\mu^{ax}A_\nu^{dy}\delta_{ad})\,. \tag{34}$$

In general, we can write

$$F_{\mu\nu} = F_{\mu\nu}^{xy}J_{xy}\,. \tag{35}$$

With the help of electromagnetic field tensor (34), we can describe Lagrangian as

$$L = \frac{1}{2}F_{\mu\nu}F^{\mu\nu}\,. \tag{36}$$

Now, we will describe field equation in seven-dimensional space. Since Euler Lagrangian equation is given by

$$\frac{\partial L}{\partial A_\mu^{\alpha\beta}} - \partial_\nu\left[\frac{\partial L}{\partial(\partial_\nu A_\mu^{\alpha\beta})}\right] = 0\,. \tag{37}$$

By using Eq. (36), we can calculate

$$\frac{\partial L}{\partial A_\mu^{\alpha\beta}} = \frac{4g}{3} F_{\mu\nu}^{xy} f_{\alpha\beta m} f_{abn} f_{mn\lambda} f_{\lambda xy} A_\nu^{ab} - 12g[F_{\mu\nu}^{\alpha d} A_\nu^{cd} \delta_{\beta c} + F_{\mu\nu}^{\beta d} A_\nu^{\gamma d} \delta_{\alpha d}] \tag{38}$$

and also

$$\frac{\partial L}{\partial(\partial_\nu A_\mu^{\alpha\beta})} = 2F_{\nu\mu}^{\alpha\beta} . \tag{39}$$

Now with the help of Eqs. (38) and (39), Eq. (37) becomes,

$$\begin{aligned} \partial_\nu F_{\mu\nu}^{\alpha\beta} + \frac{2g}{3} F_{\mu\nu}^{xy} f_{\alpha\beta m} f_{abn} f_{mn\lambda} f_{\lambda xy} A_\nu^{ab} - 6g[F_{\mu\nu}^{\alpha d} A_\nu^{cd} \delta_{\beta c} + F_{\mu\nu}^{\beta d} A_\nu^{\gamma d} \delta_{\alpha d}] = 0 , \\ \partial_\nu F_{\mu\nu}^{\alpha\beta} + \frac{2g}{3} f_{\alpha\beta m} f_{mn\lambda} F_{\mu\nu}^{\lambda} A_\nu^{n} - 6g[F_{\mu\nu}^{\alpha d} A_\nu^{cd} \delta_{\beta c} + F_{\mu\nu}^{\beta d} A_\nu^{\gamma d} \delta_{\alpha d}] = 0 , \end{aligned} \tag{40}$$

which represents the field equation in seven-dimensional space. Therefore, we have described the importance of octonion algebra to obtain the non-Abelian field equations in seven-dimensional space.

6. Conclusion

We have described the algebraic properties of octonions to obtain the seven-dimensional rotational transformations. It gives the 7×7 matrix representation which is associated with octonions. We obtained generators of SO(7) group with the help of this matrix representation. Afterwards, we have described Lagrangian for SO(7) gauge group which provides Yang–Mills theory and non-Abelian Maxwell equations in seven-dimensional space. We have shown the generalization of Yang–Mills theory from SO(3) group to SO(7) group with the help of multiplication property of the structure constants of octonions. Thus, we have reproduced a non-associative octonionic gauge theory based on SO(7) Lie algebras.

Acknowledgments

P.K. acknowledges FAPESP (Brazil) — Process No. 2013/08090-9 for financial support.

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