# ON THE USE OF ASYMPTOTIC HOMOGENIZATION TO PREDICT THE COUPLED COMPONENTS OF THE EFFECTIVE CONSTITUTIVE MATRIX OF LAMINATED COMPOSITES

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**Abstract.** *The objective of this work is to describe a numerical method, based on the rigorous mathematical foundation of Asymptotic Homogenization, used to assess the effective properties of laminates. Despite being a well-established method, the numerical assessment of the normal-shear coupled properties of the fourth-order elasticity matrix is rarely mentioned. At this work, a periodically laminated composite is described by a unit cell and the numerical relations of the Asymptotic Homogenization method are addressed in order to obtain all the 21 independent components of the elasticity matrix. Two main scenarios are considered: the first one comprises a stacking of orthotropic plies and the second one considers an isotropic interface in between the orthotropic plies. Several angles and stacking sequences are considered The results founded agree very well with the analytical results found in the literature.*

## 1. INTRODUCTION

The usage of composites has risen in the past decades and, consequently, the urge of reliable methods for the analysis of that kind of material is an important engineering task. An important aspect of designing composite structures lies on the determination of the effective mechanical properties of the material. Over the past year, several numerical homogenization methods have been used to accomplish that task, as seen in [1-4].

Here, one can highlight the Asymptotic Expansion Method (AHM) [5-11], which is a rigorous mathematical formulation used to assess the behavior of heterogeneous materials. On a linear elasticity context, the AHM considers a two-scale asymptotic approach to account for the small variations on the mechanical properties of the heterogeneous media hence obtaining the average or homogenized, properties of the media.

Despite the numerical homogenization methods for obtaining the effective properties of a heterogeneous medium are very common in literature, the assessment of the normal-shear coupled components of the elasticity is rarely observed.

Thus, this work focus on the discussion and application of the Asymptotic Homogenization Method to obtain all of the 21 independent components of the elasticity tensor of three-dimensional media, including the coupled components. A unit cell, under periodicity constraints, is considered in order to represent a heterogeneous media. In the present study, a periodically laminated composite is considered in two distinct scenarios, one with an isotropic interface between the plies and another with no interface. Initially, the main considerations regarding the method, as long as the mathematical formulation alongside the numerical solution, are presented. Next, several stacking sequences are analyzed considered the two aforementioned scenarios and the numerical results obtained with the AHM approach are compared to the analytical solution found in the literature. Finally, the conclusions and suggestions for further works are pointed out.

## 2. MATHEMATICAL BACKGROUND

The Asymptotic Homogenization Method (AHM) uses the concept of a Unit Cell to describe a heterogeneous media, in other words, there is a small portion of the domain that repeats itself in a pattern describing the whole domain. The method follows, basically, three assumptions [7]. The first one considers that, for the elasticity problem, the displacement field of the media can be written as an asymptotic expansion. The second one states that the analysis can be made by using two separate scales, one at the microscale (Unit Cell) and another one at the macroscale (Heterogeneous media) and that both are related by a small parameter $\epsilon \ll 1$. The third consideration is that the displacement field on opposite sides of the unit cell is the same, thus enforcing the periodicity constraints to the unit cell.

The mathematical development of the method is well established in the literature and, as shown by Hassani and Hinton (1998) [10], states that the homogenized fourth-order elasticity tensor of three-dimensional media can be found by

$$C^{H}_{ijkl}=\frac{1}{|\mathbf{Y}|}\int_{Y}\left(C_{ijkl}-C_{ijpq}\frac{\partial\chi^{kl}_{p}}{\partial y_{q}}\right)dY, \quad (1)$$

where $\mathbf{Y}$ is the dimension vector of the RVE, $\mathbf{y}$ is the coordinate vector and $\boldsymbol{\chi}^{kl}$ is the periodic solution of the equilibrium problem given by

$$\int_{Y}C_{ijpq}\frac{\partial\chi^{kl}_{p}}{\partial y_{q}}\frac{\partial v_{i}}{\partial y_{j}}dY=\int_{Y}C_{ijkl}\frac{\partial v_{i}}{\partial y_{j}}dY, \quad (2)$$

where v is a vector of virtual displacements. In order to obtain all of the 21 independent constants of the homogenized tensor, the equilibrium problem must be solved for six load cases kl. This method considers periodic media, thus not representing correctly a finite laminate. Equations (1) and (2) can be written, in a finite element form, respectively, as [11,12]

$$C^{H}_{ijkl}=\frac{1}{|Y|}\int_{Y}\left(C_{ijkl}-\mathbf{c}^{T}_{ij}\mathbf{B}\,\hat{\boldsymbol{\chi}}^{kl}\right)dY \quad (3)$$

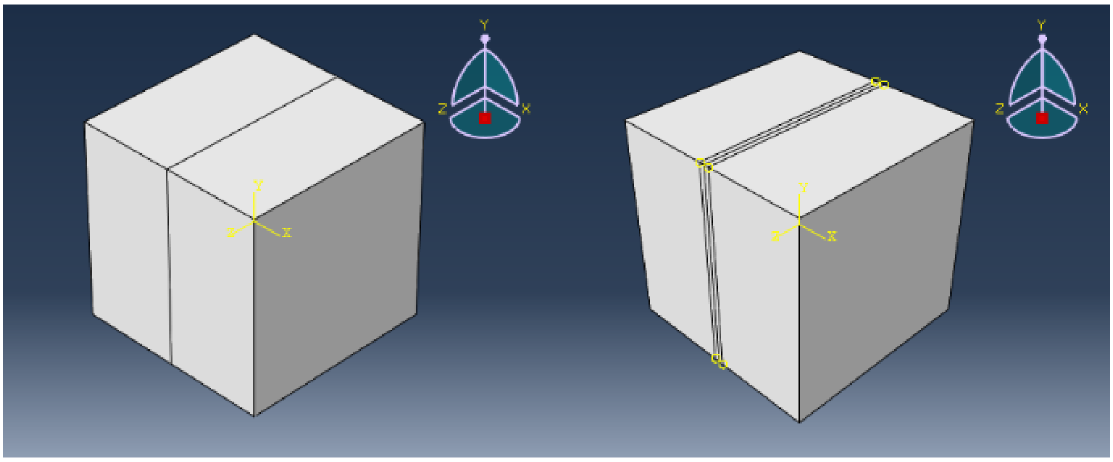
and

$$\int_{Y}\mathbf{B}^{T}\mathbf{C}\mathbf{B}dY\,\hat{\boldsymbol{\chi}}^{kl}=\int_{Y}\mathbf{B}^{T}\mathbf{c}_{kl}dY, \quad (4)$$

where **B** is the strain-displacement matrix, $\mathbf{c}_{kl}$ is the column of the elasticity matrix associated with the indexes kl and $\hat{\chi}^{kl}$ is the nodal solution of the equilibrium. As shown in [13], the right-hand side of Eq. (4) can be seen as an initial prescribed strain load case, thus, those force vectors enforce unitary strains in the given direction kl. This result is used in order to apply the load cases in the commercial finite element package Abaqus by the use of a UEXPAN subroutine, as shown by Yuan and Fish (2008) [14]. Additionally, the application of all boundary conditions and periodicity constraints, as well as the determination of the effective coefficients, Eq. (3), are made by a Python Script linked to Abaqus.

## 3. RESULTS AND DISCUSSION

Two scenarios are considered in this work. The first one consists of a periodically laminated composite in which two plies in given angles repeat themselves, stacked in the $x$ direction, to assemble the laminate. In the second scenario an additional isotropic layer, or an interface, is considered between the two plies. The unit cells used to describe both models are shown in Fig 1.



(a) (b)

Figure 1 – Unit cell models for: (a) Stacking sequence of $n$ plies; (b) stacking sequence of $n$ plies separated by an isotropic interface.

The analyses results show that the homogenized tensor, for a general case with stacking in $x$ direction, is given by

$$C^H_{ijkl} = \begin{pmatrix} C^H_{1111} & C^H_{1122} & C^H_{1133} & 0 & C^H_{1123} & 0 \\ & C^H_{2222} & C^H_{2233} & 0 & C^H_{2223} & 0 \\ & & C_{3333} & 0 & C^H_{3323} & 0 \\ & & & C^H_{1212} & 0 & C^H_{1213} \\ & & & & C^H_{2323} & 0 \\ \text{sym.} & & & & & C^H_{1313} \end{pmatrix} \quad (5)$$

in which the coupling between the components is expected.

The mechanical properties considered for this work are shown in Tab. 1, in which an orthotropic material is considered. In Tab. 1, L stands for the longitudinal direction (fiber direction) and T stands for the transversal direction (off-plane direction). The mechanical properties of the interface are considered as isotropic and are shown in Tab. 2.

Table 1 – Mechanical properties of the composite [15].

| $E_T$[GPa] | $E_L$[GPa] | $\nu_{LT}$ | $\nu_{TL}$ | $G_{LT}$[GPa] | $G_{TL}$[GPa] |
|---|---|---|---|---|---|
| 10.0 | 127.0 | 0.306 | 0.34 | 3.05 | 5.4 |

Table 2 – Mechanical properties of the isotropic interface [16].

| E[GPa] | $\nu$ |
|---|---|
| 5.033 | 0.4 |

The two first stacking sequences are given by $[0/45]_n$ and $[0/60]_n$, respectively. For these stacking sequences, both models, with and without an interface, are considered. The third and fourth stacking sequences are given, respectively, by $[45/\text{-}45]_n$ and $[60/\text{-}60]_n$, in which only the case with no interface is considered. For all cases, the trilinear isoparametric hexahedral element (C3D8) was chosen for the discretization of the domain. Also, a convergence analysis was performed, but no mesh dependency was noticed. All numerical results are compared with the analytical solution proposed by Brito-Santana et. al. (2018) [6].

The numerical results, as well as the comparison with the analytical solutions [6], for the stacking sequences of $[0/45]_n$ and $[0/60]_n$, with and without an interface, are shown, respectively in Tab. 3 and Tab. 4. All of the components not shown in the tables have zero as value.

Table 3 – Comparison between the numerical results (FEM) and the analytical solution (AHM) proposed by Brito-Santana et. al. (2018) [6]. Results for a stacking sequence of $[0/45]_n$ with and without an isotropic interface between the plies, respectively.

| | $[0/45]_n$ | | $[0/int/45]_n$ | |
|---|---|---|---|---|
| | **FEM** | **AHM** | **FEM** | **AHM** |
| $C_{1111}$[GPa] | 11.2272 | 11.2272 | 11.204231 | 11.204231 |
| $C_{1122}$[GPa] | 4.665885 | 4.665875 | 4.7969965 | 4.7969871 |
| $C_{1133}$[GPa] | 3.935435 | 3.935425 | 4.1044887 | 4.1044793 |
| $C_{1123}$[GPa] | 0.365226 | 0.365225 | 0.3462549 | 0.3462539 |
| $C_{2222}$[GPa] | 86.862419 | 86.862544 | 83.030546 | 83.030664 |
| $C_{2233}$[GPa] | 18.791236 | 18.791256 | 18.175068 | 18.175086 |
| $C_{2223}$[GPa] | 14.9004382 | 14.911131 | 15.36449 | 14.169626 |
| $C_{3333}$[GPa] | 27.265519 | 27.265544 | 26.394939 | 26.394961 |
| $C_{3323}$[GPa] | 14.8980619 | 14.887369 | 14.884001 | 14.148225 |
| $C_{1212}$[GPa] | 4.6189063 | 4.6189063 | 4.2758872 | 4.2758871 |
| $C_{1213}$[GPa] | 0.5638665 | 0.6396010 | 0.4744482 | 0.5377226 |
| $C_{1313}$[GPa] | 3.4911733 | 3.4911733 | 3.3269909 | 3.3269908 |
| $C_{2323}$[GPa] | 19.136369 | 19.136394 | 18.26884 | 18.268863 |

As one can see, the results for all homogenized components are very close to the analytical solution. The numerical results for the normal and shear components of the elasticity tensor, as well as the results for the normal coupling components, are exactly as the analytical ones. On the other hand, the normal-shear coupling components present a small difference in comparison to the analytical ones.

The results for the stacking sequences of $[45/\text{-}45]_n$ and $[60/\text{-}60]_n$ alongside the comparison with the analytical solution are shown in Tab. 5, in which the components with value zero are not shown. It can be noticed that, for these stacking sequences, no coupling between components was noticed. That behavior is expected since an angle-ply composite with a stacking of n plies is here considered. Also, one can notice that the numerical matches the analytical results.

Table 4 – Comparison between the numerical results (FEM) and the analytical solution (AHM) proposed by Brito-Santana et. al. (2018) [6]. Results for a stacking sequence of $[0/60]_n$ with and without an isotropic interface between the plies, respectively.

| | $[0/60]_n$ | | $[0/int/60]_n$ | |
|---|---|---|---|---|
| | **FEM** | **AHM** | **FEM** | **AHM** |
| $C_{1111}$ [GPa] | 11.2272 | 11.2272 | 11.204231 | 11.204231 |
| $C_{1122}$ [GPa] | 4.48327 | 4.4832625 | 4.6238672 | 4.6238601 |
| $C_{1133}$ [GPa] | 4.118045 | 4.1180375 | 4.2776133 | 4.2776062 |
| $C_{1123}$ [GPa] | 0.316295 | 0.316294 | 0.2998655 | 0.2998646 |
| $C_{2222}$ [GPa] | 75.385418 | 75.385512 | 72.123195 | 72.123285 |
| $C_{2233}$ [GPa] | 15.369037 | 15.369038 | 14.923539 | 14.92354 |
| $C_{2223}$ [GPa] | 7.5100339 | 6.965385 | 7.1382954 | 6.6208787 |
| $C_{3333}$ [GPa] | 45.587018 | 45.587012 | 43.805442 | 43.805433 |
| $C_{3323}$ [GPa] | 19.176216 | 18.840873 | 18.221677 | 17.9031 |
| $C_{1212}$ [GPa] | 4.2514607 | 4.2514607 | 15.006611 | 3.9747577 |
| $C_{1213}$ [GPa] | 0.4786945 | 0.552791 | 0.4043827 | 0.4669028 |
| $C_{1313}$ [GPa] | 3.698712 | 3.698712 | 3.507817 | 3.5078169 |
| $C_{2323}$ [GPa] | 15.702289 | 15.702296 | 15.006611 | 15.006616 |

Table 5 – Comparison between the numerical results (FEM) and the analytical solution (AHM) proposed by Brito-Santana et. al. (2018) [6]. Results for a stacking sequence of $[45/-45]_n$ with and $[60/-60]_n$, respectively.

| | $[45/-45]_n$ | | $[60/-60]_n$ | |
|---|---|---|---|---|
| | **FEM** | **AHM** | **FEM** | **AHM** |
| $C_{1111}$ [GPa] | 11.2272 | 11.2272 | 11.2272 | 11.2272 |
| $C_{1122}$ [GPa] | 4.30066002 | 4.30065 | 3.93543 | 3.935425 |
| $C_{1133}$ [GPa] | 4.30066002 | 4.30065 | 4.66588 | 4.665875 |
| $C_{1123}$ [GPa] | 0 | 0 | 0 | 0 |
| $C_{2222}$ [GPa] | 43.3276002 | 43.32765 | 20.4033 | 20.403287 |
| $C_{2233}$ [GPa] | 32.5276001 | 32.52765 | 25.6535 | 25.653512 |
| $C_{2223}$ [GPa] | 0 | 0 | 0 | 0 |
| $C_{3333}$ [GPa] | 43.3276002 | 43.32765 | 80.0003 | 80.000288 |
| $C_{3323}$ [GPa] | 0 | 0 | 0 | 0 |
| $C_{1212}$ [GPa] | 3.8982249 | 3.8982249 | 3.4223376 | 3.4223377 |
| $C_{1213}$ [GPa] | 0 | 0 | 0 | 0 |
| $C_{1313}$ [GPa] | 3.8982249 | 3.8982249 | 4.527835 | 4.5278351 |
| $C_{2323}$ [GPa] | 32.8489763 | 32.849026 | 25.986757 | 25.98677 |

## 4. CONCLUSIONS

This work addresses the numerical determination of the elasticity tensor of a composite laminate. The numerical solution for the Asymptotic Homogenization Method is implemented in the finite element package software Abaqus and all the 21 components of the elasticity matrix are obtained. The results show that is possible to obtain the normal-shear coupled components of the elasticity tensor via the numerical solution of the AHM, which is rarely observed in the literature. Also, the results show a good concordance with analytical solutions.

For future works, the approach can be extended in order to considered finite laminates. That can be achieved by relieving the periodicity constraints on the off-plane direction, in other words, a theoretical infinite laminate no longer needs to be used thus enabling the method to be used in a wider range of composites.

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## REFERENCES

[1] F. Otero, S. Oller, X. Martinez, O. Salomón. Numerical homogenization for composite materials analysis. Comparison with other micro mechanical formulations. *Composite Structures*, v.122, p.405-416. (2015).

[2] H. Brito-Santana, R. de Medeiros, R. Rodriguez-Ramos, V. Tita. Different interface models for calculating the effective properties in piezoelectric composite materials with imperfect fiber–matrix adhesion. *Composite Structures*, v.151, p.70-80. (2016).

[3] R. Rodríguez-Ramos, R. de Medeiros, R. Guinovart-Díaz, J. Bravo-Castillero, J. A. Otero, V. Tita. Different approaches for calculating the effective elastic properties in composite materials under imperfect contact adherence. *Composite Structures*, v.99, p.264-275. (2013).

[4] F. Montero-Chacón, S. Zaghi, R. Rossi, E. García-Pérez, I. Heras-Pérez, X. Martínez, S. Oller, M. Doblaré. Multiscale thermo-mechanical analysis of multi-layered coatings in solar thermal applications. *Finite Elements in Analysis and Design*, v.127, p.31-43. (2017).

[5] H. Brito-Santana, R. de Medeiros, A.J.M. Ferreira, R. Rodríguez-Ramos, V. Tita. Effective elastic properties of layered composites considering non-uniform imperfect adhesion. *Applied Mathematical Modelling*, v.59, p.183-204. (2018).

[6] H. Brito-Santana, B.G. Christoff, A.J.M. Ferreira, F. Lebon, R. Rodríguez-Ramos, V. Tita. Delamination influence on elastic properties of laminated composites. *Acta Mechanica*, v.230, p.821-837. (2019).

[7] S.J. Hollister, N. Kikuchi. Homogenization theory and digital imaging: a basis for studying the mechanics and design principles of bone tissue. *Biotechnology and bioengineering*, v.43, p.586-596. (1994).

[8] E. Sanchez-Palencia. Homogenization in mechanics. A survey of solved and open problems. *Rend. Sem. Mat. Univ. Politec. Torino*, v.44, p.1-45. (1986).

[9] J. Guedes, N. Kikuchi. Preprocessing and postprocessing for materials based on the homogenization method with adaptive finite element methods. *Computer methods in applied mechanics and engineering*, v.83, p.143-198. (1990).

[10] B. Hassani, E. Hinton. A review of homogenization and topology optimization I—homogenization theory for media with periodic structure. *Computers & Structures*, v.69, p.707-717. (1998).

[11] Z. Fang, W. Sun, J.T. Tzeng. Asymptotic homogenization and numerical implementation to predict the effective mechanical properties for electromagnetic composite conductor. *Journal of Composite Materials*, v.38, p.1371-1385. (2004).

[12] B. Hassani, E. Hinton. A review of homogenization and topology optimization II—analytical and numerical solution of homogenization equations. *Computers & structures*, v.69, p.719-738. (1998).

[13] Christoff, B. G. *Analysis and Optimization of three dimensional microstructures*. Master Thesis. Santa Catarina State University. (2016).

[14] Z. Yuan, J. Fish. Toward realization of computational homogenization in practice. *International Journal for Numerical Methods in Engineering*, v.73, p.361-380. (2008).

[15] V. Tita, J. De Carvalho, D. Vandepitte. Failure analysis of low velocity impact on thin composite laminates: Experimental and numerical approaches. *Composite Structures*, v.83, p.413-428. (2008).

[16] J.T. Verbis, S.E. Kattis, S.V. Tsinopoulos, D. Polyzos. Wave dispersion and attenuation in fiber composites. *Computational mechanics*, v.27, p.244-252. (2001).

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