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Existence of global attractors and convergence of solutions for the Cahn-Hilliard equation on manifolds with conical singularities

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ABSTRACT

We consider the Cahn-Hilliard equation on manifolds with conical singularities and prove existence of global attractors in higher order Mellin-Sobolev spaces with asymptotics. We also show convergence of solutions in the same spaces to an equilibrium point and provide asymptotic behavior of the equilibrium near the conical tips in terms of the local geometry.

1. Introduction

In this article, we show existence and regularity of global attractors as well as convergence results for the Cahn-Hilliard equation considered on *manifolds with conical singularities*. We model such a manifold as a $(n+1)$-dimensional compact manifold $\mathcal{B}$ with closed boundary $\partial\mathcal{B}$, $n \geq 1$, which is endowed with a degenerated Riemannian metric g that, in local coordinates $(x, y) \in [0, 1) \times \partial\mathcal{B}$ on a collar neighborhood of the boundary, has the following expression

$$g = dx^2 + x^2 h(x),$$

where $[0, 1) \ni x \mapsto h(x)$ is a smooth family of Riemannian metrics on $\partial\mathcal{B}$. We denote $\mathbb{B} = (\mathcal{B}, g)$ and $\partial\mathbb{B} = (\partial\mathcal{B}, h(0))$. The Laplacian on $\mathbb{B}$, in local coordinates $(x, y) \in [0, 1) \times \partial\mathcal{B}$ on the collar part, admits the following degenerate expression

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$$\Delta = \frac{1}{x^2}\Big((x\partial_x)^2 + (n-1+\frac{x\partial_x \det(h(x))}{2\det(h(x))})(x\partial_x) + \Delta_{h(x)}\Big),$$

where $\Delta_{h(x)}$ is the Laplacian on $(\partial\mathcal{B}, h(x))$. The operator Δ belongs to the class of *cone differential operators* or *Fuchs type operators*, see Section 3 for more details.

On $\mathbb{B}$ we consider the following problem

$$\begin{aligned} u'(t) + \Delta^2 u(t) &= \Delta(u^3(t) - u(t)), \quad t \in (0,T), \\ u(0) &= u_0, \end{aligned} \tag{1.1}$$

which is known as Cahn-Hilliard equation (CH for short); it is a diffusion interface equation that models phase separation of a binary mixture. On classical domains, (1.1) has been generalized and extensively studied in many directions and aspects, such as existence, regularity and convergence of solutions, existence of global attractors, etc. A sufficient number of related results can be found in [13].

However, on singular domains much less is known. Using the theory of cone differential operators, in [16] it was first shown short-time existence of CH on $\mathbb{B}$ for the case where $h(\cdot)$ is constant, by employing L^p-maximal regularity techniques. Those results were extended to arbitrary $\mathbb{B}$ and improved to higher regularity in [17]. Finally, global solutions and smoothing results were proved in [11]. Summarizing those results, let us assume that $\dim(\mathbb{B}) = n+1 \in \{2,3\}$, choose $s \geq 0$ and let the exponent γ be as follows

$$\frac{\dim(\mathbb{B})-4}{2} < \gamma < \min\Big\{-1+\sqrt{\Big(\frac{\dim(\mathbb{B})-2}{2}\Big)^2 - \lambda_1}, \frac{\dim(\mathbb{B})-4}{4}\Big\}, \tag{1.2}$$

where $0 = \lambda_0 > \lambda_1 > \cdots$ are the eigenvalues of the boundary Laplacian $\Delta_{h(0)}$ on $\partial\mathbb{B}$. Denote by $\mathcal{H}^{\eta,\rho}(\mathbb{B})$, $\eta, \rho \in \mathbb{R}$, the *Mellin-Sobolev space*, see Definition 2. Moreover, let $\mathbb{R}_\omega$ and $\mathbb{C}_\omega$ be the finite dimensional spaces of smooth functions on $\mathbb{B}$ that are locally constant close to the singularities, with values in $\mathbb{R}$ and $\mathbb{C}$ respectively, see Section 3 for details. Then, for any real-valued $u_0 \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega$, there exists a unique global solution in the following sense: for any $T > 0$ there exists a unique $u \in H^1(0,T;\mathcal{H}^{s,\gamma}(\mathbb{B})) \cap L^2(0,T;\mathcal{D}(\Delta_s^2))$ solving (1.1) on $[0,T] \times \mathbb{B}$. Furthermore, the solution u satisfies the regularity

$$u \in \bigcap_{s\geq 0} C^\infty((0,\infty);\mathcal{D}(\Delta_s^2)) \tag{1.3}$$

and

$$u \in C([0,\infty);\mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega) \hookrightarrow C([0,\infty);C(\mathbb{B})). \tag{1.4}$$

The bi-Laplacian domain we choose is

$$\mathcal{D}(\Delta_s^2) = \{u \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega : \Delta u \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega\}, \tag{1.5}$$

where γ is always as (1.2). It satisfies

$$\mathcal{D}(\Delta_s^2) = \mathcal{D}(\Delta_{s,\min}^2) \oplus \mathbb{C}_\omega \oplus \mathcal{E}_{\Delta^2,\gamma}. \tag{1.6}$$

Here $\mathcal{E}_{\Delta^2,\gamma}$ is an s-independent finite dimensional space consisting of $C^\infty(\mathbb{B}^\circ)$-functions, that in local coordinates $(x,y) \in [0,1) \times \partial\mathcal{B}$, take the form $\omega(x)c(y)x^\rho \ln^k(x)$, $\rho \in \mathbb{C}$, $k \in \{0,1,2,3\}$, where $\mathbb{B}^\circ = \mathcal{B}\backslash\partial\mathcal{B}$ and $c \in C^\infty(\partial\mathbb{B})$. More precisely, there exists a discrete set of points Z_{Δ^2} in $\mathbb{C}$, determined only by the family of metrics $h(\cdot)$, such that the exponents ρ coincide with the set $Z_{\Delta^2} \cap \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\frac{n-7}{2} - \gamma, \frac{n-3}{2} - \gamma)\}$. The exponents k are also determined by $h(\cdot)$. In particular, when $h(\cdot) = h$ is constant, the set Z_{Δ^2} and the exponents k associated to each $\rho \in Z_{\Delta^2}$, are determined by n and the spectrum of Δ_h. The minimal domain $\mathcal{D}(\Delta^2_{s,\min})$ stands for the domain of the closure of $\Delta^2 : C_c^\infty(\mathbb{B}^\circ) \to \mathcal{H}^{s,\gamma}(\mathbb{B})$, and satisfies

$$\mathcal{H}^{s+4,\gamma+4}(\mathbb{B}) \hookrightarrow \mathcal{D}(\Delta^2_{s,\min}) \hookrightarrow \bigcap_{\varepsilon>0} \mathcal{H}^{s+4,\gamma+4-\varepsilon}(\mathbb{B}), \tag{1.7}$$

while

$$\mathcal{D}(\Delta^2_{s,\min}) = \mathcal{H}^{s+4,\gamma+4}(\mathbb{B}) \tag{1.8}$$

provided that

$$\{\gamma+1, \gamma+3\} \cap \bigcup_{\lambda_j \in \sigma(\Delta_{h(0)})} \Big\{ \pm \sqrt{\Big(\frac{\dim(\mathbb{B})-2}{2}\Big)^2 - \lambda_j} \Big\} = \emptyset. \tag{1.9}$$

Consequently, both spaces $\mathcal{D}(\Delta^2_s)$ and $\mathcal{E}_{\Delta^2,\gamma}$ are determined explicitly by $h(\cdot)$ and γ, see Corollary 18 for details.

These results allow us to define for any $s \geq 0$ a *semiflow* $T : [0,\infty) \times \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega \to \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega$ on real valued-functions by $T(t)u_0 := T(t,u_0) = u(t)$, see e.g. [26, Chapter 1, Section 1.1] for more details on semiflows, also known as *semigroups*. Let $X^s_{1,0}$ be the space of all real-valued functions $u \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega$ such that $\int_{\mathbb{B}} u d\mu_g = 0$, where $d\mu_g$ is the measure associated with the metric g. Then T can be restricted to $X^s_{1,0}$, see Section 4. Our main results are the following.

Theorem 1. *Let $s \geq 0$, γ be as (1.2) and $\mathcal{D}(\Delta^2_s)$ be the bi-Laplacian domain described in (1.5)-(1.9).*
(i) (Global attractor) *The semiflow $T : [0,\infty) \times X^s_{1,0} \to X^s_{1,0}$ has an s-independent global attractor $\mathcal{A} \subset \cap_{r>0}\mathcal{D}(\Delta^2_r)$. Moreover, if B is a bounded set of $X^s_{1,0}$, then for any $r > 0$, $T(t)B$ is, for sufficiently large t, a bounded set of $\mathcal{D}(\Delta^2_r)$ and*

$$\lim_{t\to\infty} (\sup_{b\in B} \inf_{a\in\mathcal{A}} \|T(t)b - a\|_{\mathcal{D}(\Delta^2_r)}) = 0.$$

(ii) (Convergence to equilibrium) *If $u_0 \in X^0_{1,0}$, then there exists a $u_\infty \in \cap_{r>0}\mathcal{D}(\Delta^2_r)$ such that $\lim_{t\to\infty} T(t)u_0 = u_\infty$, where the convergence occurs in $\mathcal{D}(\Delta^2_r)$ for each $r \geq 0$.*

The definition of global attractor is recalled in Section 4. For proving part (i) of Theorem 1.1, we follow the strategy of Temam [26] to obtain estimates in a lower regularity space $H_0^{-1}(\mathbb{B})$, see Definition 9, and of [25] for obtaining higher regularity. For convergence to equilibrium, we first obtain the Lojasiewicz-Simon inequality due to [24], and proceed as [3], [9] and [19].

Concerning real-life applications of the above approach, recall first that the physical effects described by CH, as well as other evolution equations, occur in reality in many different types of domains and surfaces (manifolds), which are usually not smooth: many of them have edges, conical points, cusps, or even combinations of these and other types of singularities. In this context, conic manifolds are fundamental and a natural place to start. They describe simple point singularities, which, apart from their intrinsic interest, can be used to build more general ones [23], [22].

Moreover, whenever we are studying a smooth $(n+1)$-dimensional Riemannian manifold $\mathcal{M}$ endowed with a Riemannian metric f, an important question is: *how does the local geometry on* $\mathbb{M} = (\mathcal{M}, f)$ *affect the evolution?* An answer to this question arises as follows: fix a point o on $\mathcal{M}$ and denote by $d(o,z)$ the geodesic distance between o and $z \in \mathcal{M}\backslash\{o\}$, induced by the metric f. There exists an $r > 0$ such that $(x, y) \in (0, r) \times \mathbb{S}^n$ are local coordinates around o and moreover, the metric in these coordinates becomes $f = dx^2 + x^2 f_{\mathbb{S}^n}(x)$, where $\mathbb{S}^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$ is the unit sphere and $x \mapsto f_{\mathbb{S}^n}(x)$ is a smooth family of Riemannian metrics on $\mathbb{S}^n$. In case of $f_{\mathbb{S}^n}(\cdot)$ being smooth up to $x = 0$, we can regard $((\mathcal{M}\backslash\{0\}) \cup (\{0\} \times \mathbb{S}^n), f)$ as a conic manifold with one isolated conical singularity at o. On the other hand, since our problem involves the Laplacian, it becomes now degenerate. However, an application of our results shows that the asymptotic expansion of the solutions near o is provided by the expansion (1.6), where the boundary Laplacian $\Delta_{h(x)}$ now has to be replaced by the Laplacian $\Delta_{f_{\mathbb{S}^n}(x)}$ on $(\mathbb{S}^n, f_{\mathbb{S}^n}(x))$. Hence, in particular, through the structure of the spaces $\mathcal{E}_{\Delta^2,\gamma}$, we obtain an interplay between the spectrum of $\Delta_{f_{\mathbb{S}^n}(x)}$ and the evolution.

Though the strategies are mostly well established, the technical results that allow us to use them in the context of conical singularities are not, and, therefore, the strategies have to be adapted to this situation. For this reason new results on interpolation and embedding of Mellin-Sobolev spaces are developed in this article.

In Section 2, we define suitable function spaces to work on conic manifolds and study their embeddings. Section 3 is devoted to the domain description and the properties of the Laplacian and bi-Laplacian and to provide some facts about the complex interpolation of those spaces. Part (i) of Theorem 1 is proved in Section 4 and part (ii) in Section 5.

2. Function spaces

Fix a smooth non-negative function $\omega \in C^\infty(\mathbb{B})$ supported on the collar neighborhood $(x, y) \in [0, 1) \times \partial\mathcal{B}$ such that ω depends only on x and $\omega = 1$ near $\{0\} \times \partial\mathcal{B}$. Moreover denote by C_c^∞ the space of smooth compactly supported functions and by H^s, $s \in \mathbb{R}$, the usual Bessel potential spaces defined using the L^2-norm.

Definition 2 *(Mellin-Sobolev spaces).* Let $\gamma \in \mathbb{R}$ and consider the map

$$M_\gamma : C_c^\infty(\mathbb{R}_+ \times \mathbb{R}^n) \to C_c^\infty(\mathbb{R}^{n+1}) \quad \text{defined by} \quad u(x, y) \mapsto e^{(\gamma - \frac{n+1}{2})x} u(e^{-x}, y).$$

Let $\kappa_j : U_j \subseteq \partial\mathcal{B} \to \mathbb{R}^n$, $j \in \{1, ..., N\}$, $N \in \mathbb{N}\backslash\{0\}$, $\mathbb{N} := \{0, 1, 2, ...\}$, be a covering of $\partial\mathcal{B}$ by coordinate charts and let $\{\phi_j\}_{j\in\{1,...,N\}}$ be a subordinated partition of unity. For any $s, \gamma \in \mathbb{R}$, the Mellin Sobolev space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is defined to be the space of all distributions u on the interior $\mathbb{B}^\circ$ such that the norm

$$\|u\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = \sum_{j=1}^{N} \|M_\gamma(1 \otimes \kappa_j)_*(\omega\phi_j u)\|_{H^s(\mathbb{R}^{n+1})} + \|(1 - \omega)u\|_{H^s(2\mathbb{B})} \tag{2.1}$$

is defined and finite, where $2\mathbb{B}$ is the double of $\mathbb{B}$ and $*$ refers to the push-forward of distributions. Different choices of ω, covering and partition of unity give us the same spaces with equivalent norms. The space $\mathcal{H}^{s,\gamma}(\mathbb{B})$ is a Banach algebra, up to an equivalent norm, whenever $s > (n + 1)/2$ and $\gamma \geq (n + 1)/2$, see [18, Lemma 3.2].

If $s \in \mathbb{N}$, then $\mathcal{H}^{s,\gamma}(\mathbb{B})$ coincides with the space of all functions u in $H^s_{\mathrm{loc}}(\mathbb{B}^\circ)$ that satisfy

$$x^{\frac{n+1}{2}-\gamma}(x\partial_x)^k \partial_y^\alpha(\omega(x)u(x, y)) \in L^2([0, 1) \times \partial\mathcal{B}, \sqrt{\det(h(x))}\frac{dx}{x}dy), \quad k + |\alpha| \leq s. \tag{2.2}$$

In Section 3, we will associate the Mellin-Sobolev spaces with the Laplacian and bi-Laplacian.

Remark 3. Let $\mathsf{x} : \mathbb{B} \to [0, 1]$ be a smooth positive function on $\mathbb{B}^\circ$ that is equal to $\mathsf{x}(x, y) = x$ on the collar neighborhood $[0, 1) \times \partial\mathcal{B}$. Then $u \in \mathcal{H}^{0,\gamma}(\mathbb{B})$ iff $\mathsf{x}^{-\gamma}u \in L^2(\mathbb{B})$, where $L^2(\mathbb{B}) = \mathcal{H}^{0,0}(\mathbb{B})$. We define the spaces $L^p(\mathbb{B})$ using the measure $d\mu_g$ induced by the metric g. Note that $d\mu_g = \sqrt{\det(h(x))}x^n dxdy$ on the collar neighborhood. Moreover, for any $\alpha \in \mathbb{R}$, let $x^\alpha L^p(\mathbb{B}) := \{u : \int_{\mathbb{B}} |x^{-\alpha}u|^p d\mu_g < \infty\}$. Finally, recall that the inner product in $\mathcal{H}^{0,0}(\mathbb{B})$ induces an identification of the dual space of $\mathcal{H}^{s,\gamma}(\mathbb{B})$ with $\mathcal{H}^{-s,-\gamma}(\mathbb{B})$, see e.g. [11, Lemma 3.2 (ii)].

Besides the Mellin-Sobolev spaces, we define the following space.

Definition 4. Let $H^1(\mathbb{B})$ be the completion of $C_c^\infty(\mathbb{B}^\circ)$ with respect to the inner product

$$(u, v)_{H^1(\mathbb{B})} = \int_{\mathbb{B}} u\overline{v} d\mu_g + \int_{\mathbb{B}} \langle \nabla u, \overline{\nabla v} \rangle \, d\mu_g,$$

where ∇ and $\langle \cdot, \cdot \rangle$ are defined by the conical metric g.

We investigate now certain properties of the space $H^1(\mathbb{B})$ and its connection with $\mathcal{H}^{s,\gamma}(\mathbb{B})$.

Remark 5. For the following computations, we note that
(1) The boundedness of $\int_{\mathbb{B}} |u|^2 \, d\mu_g$ is equivalent to

$$u \in L^2_{\text{loc}}(\mathbb{B}^\circ) \text{ and } (x, y) \mapsto x^{\frac{n+1}{2}} \omega(x)u(x, y) \in L^2([0, 1] \times \partial\mathcal{B}, \sqrt{\det(h(x))} \frac{dx}{x} dy).$$

(2) If $\int_{\mathbb{B}} |u|^2 \, d\mu_g < \infty$, then the boundedness of $\int_{\mathbb{B}} \langle \nabla u, \nabla \overline{u} \rangle \, d\mu_g$ is equivalent to

$$u \in H^1_{\text{loc}}(\mathbb{B}^\circ) \text{ and } (x, y) \mapsto x^{\frac{n-1}{2}} (x\partial_x)^k \partial_y^\alpha(\omega(x)u(x, y)) \in L^2([0, 1] \times \partial\mathcal{B}, \sqrt{\det(h(x))} \frac{dx}{x} dy), \; k + |\alpha| = 1.$$

The last statement can be easily proved once we recall that in local coordinates of $[0, 1) \times \partial\mathcal{B}$ we have

$$\langle \nabla u, \nabla \overline{v} \rangle = x^{-2}(x\partial_x u)(x\partial_x \overline{v}) + x^{-2} \sum_{i,j=1}^{n} h^{ij}(x, y)(\partial_{y_i} u)(\partial_{y_j} \overline{v}).$$

Along this paper, we use $\hookrightarrow$ and $\overset{c}{\hookrightarrow}$ to denote continuous and compact embedding, respectively. We recall that $\mathcal{H}^{s,\gamma}(\mathbb{B}) \hookrightarrow \mathcal{H}^{s',\gamma'}(\mathbb{B})$, when $s \geq s'$ and $\gamma \geq \gamma'$, and $\mathcal{H}^{s,\gamma}(\mathbb{B}) \overset{c}{\hookrightarrow} \mathcal{H}^{s',\gamma'}(\mathbb{B})$, when $s > s'$ and $\gamma > \gamma'$, see [23, Theorem 2.1.53].

Proposition 6. *For any $\beta < 1$, the following inclusions hold*

$$\mathcal{H}^{1,1}(\mathbb{B}) \oplus \mathbb{C}_\omega \hookrightarrow H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{1,\beta}(\mathbb{B}),$$

where $u \oplus v$ is identified with $u + v$ and the first inclusion is just $u \oplus v \mapsto u + v$. In particular, $H^1(\mathbb{B}) \overset{c}{\hookrightarrow} H^{0,0}(\mathbb{B})$.

Proof. We proceed in several steps. Let us denote by C positive constants that can change along the proof. For simplicity we ignore the term $\sqrt{\det(h(x))}$ in the proof, as it is uniformly bounded from above and below, and we abuse the notation $\int_{\partial\mathbb{B}}$ since the computations are made in local coordinates. We also note that it suffices to check the inclusion on the collar neighborhood and for functions $u \in C_c^\infty(\mathbb{B}^\circ)$.
Step 1: $\mathcal{H}^{1,1}(\mathbb{B}) \hookrightarrow H^1(\mathbb{B})$. We have

$$\int_0^1 \int_{\partial\mathbb{B}} |\omega(x)u(x, y)|^2 \, x^n dxdy \leq \int_0^1 \int_{\partial\mathbb{B}} \left| x^{\frac{n+1}{2}-1} \omega(x)u(x, y) \right|^2 \frac{dx}{x} dy.$$

Hence it is clear that

$$\int_{\mathbb{B}} |u|^2 \, d\mu_g \leq C \|u\|^2_{\mathcal{H}^{1,1}(\mathbb{B})} .$$

Moreover

$$\int_0^1 \int_{\partial\mathbb{B}} \frac{1}{x^2} \left(|x\partial_x(\omega u)|^2 + \sum_{i,j=1}^{n} h^{ij}(x,y)\partial_{y_i}(\omega u)\partial_{y_j}(\omega \overline{u}) \right) x^n dxdy$$

$$\leq C \sum_{k+|\alpha|=1} \int_0^1 \int_{\partial\mathbb{B}} \left| x^{\frac{n-1}{2}} (x\partial_x)^k \partial_y^\alpha (\omega(x)u(x,y)) \right|^2 \frac{dx}{x} dy,$$

which implies that

$$\int_{\mathbb{B}} \left\langle \nabla u, \overline{\nabla u} \right\rangle d\mu_g \leq C \|u\|^2_{\mathcal{H}^{1,1}(\mathbb{B})} .$$

We conclude that

$$\|u\|_{H^1(\mathbb{B})} \leq C \|u\|_{\mathcal{H}^{1,1}(\mathbb{B})} .$$

Step 2: For each $\varepsilon > 0$, we have $\mathsf{x}^\varepsilon \in H^1(\mathbb{B})$. Let $0 < r < 1$ and $\chi_r : \mathbb{B}^\circ \to [0,1]$ be such that $\chi_r(x,y) = 1 - \omega(x/r)$, for $(x,y) \in [0,1) \times \partial\mathcal{B}$ and χ_r be equal to 1 outside the collar neighborhood. It is enough to prove that $\lim_{r\to 0} \chi_r \mathsf{x}^\varepsilon = \mathsf{x}^\varepsilon$ in $H^1(\mathbb{B})$, as $\chi_r \mathsf{x}^\varepsilon \in C_c^\infty(\mathbb{B}^\circ)$. For this, we must prove that

$$\text{(i)} \quad \lim_{r\to 0} \int_0^1 \int_{\partial\mathbb{B}} |\omega(\chi_r x^\varepsilon - x^\varepsilon)|^2 x^n dxdy = 0,$$

$$\text{(ii)} \quad \lim_{r\to 0} \int_0^1 \int_{\partial\mathbb{B}} \left| \partial_{y_j}(\omega\chi_r x^\varepsilon) - \partial_{y_j}(\omega x^\varepsilon) \right|^2 x^{n-2} dxdy = 0,$$

$$\text{(iii)} \quad \lim_{r\to 0} \int_0^1 \int_{\partial\mathbb{B}} |x\partial_x(\omega\chi_r x^\varepsilon) - x\partial_x(\omega x^\varepsilon)|^2 x^{n-2} dxdy = 0.$$

Note that (i) follows directly from the dominated convergence theorem and (ii) is identically zero. For (iii), we have that the integral is smaller or equal to two times

$$\int_0^1 \int_{\partial\mathbb{B}} |\chi_r x\partial_x(\omega x^\varepsilon) - x\partial_x(\omega x^\varepsilon)|^2 x^{n-2} dxdy + \int_0^1 \int_{\partial\mathbb{B}} |\omega(x)x^\varepsilon \partial_x \chi_r|^2 x^n dxdy.$$

Only the last term is important, as we can handle the first one directly with dominated convergence theorem. Note that $|\omega(x)x^\varepsilon\partial_x\chi_r|^2 x^n = |\omega(x)x/r(\partial_x\omega)(x/r)|^2 x^{2\varepsilon+n-2}$ and that, for $n \geq 1$, the integrand is smaller than the integrable function $\|x\partial_x\omega\|^2_{L^\infty([0,\infty))} x^{2\varepsilon+n-2}$. Moreover

$$\lim_{r\to 0} |\omega(x)x/r(\partial_x\omega)(x/r)|^2 x^{2\varepsilon+n-2} = 0.$$

The result now follows again by the dominated convergence theorem.
Step 3: $\mathbb{C}_\omega \hookrightarrow H^1(\mathbb{B})$. It suffices to show that the constant function equal to one belongs to $H^1(\mathbb{B})$ by showing that $\lim_{\varepsilon\to 0} \mathsf{x}^\varepsilon = 1$ in $H^1(\mathbb{B})$. To this end, it is enough to show that

$$\text{(i)} \quad \lim_{\varepsilon\to 0} \int_0^1 \int_{\partial\mathbb{B}} |\omega(1-x^\varepsilon)|^2 \, x^n dx dy = 0,$$

$$\text{(ii)} \quad \int_0^1 \int_{\partial\mathbb{B}} \left|\partial_{y_j}(\omega) - \partial_{y_j}(\omega x^\varepsilon)\right|^2 x^{n-2} dx dy = 0,$$

$$\text{(iii)} \quad \lim_{\varepsilon\to 0} \int_0^1 \int_{\partial\mathbb{B}} |x\partial_x(\omega) - x\partial_x(\omega x^\varepsilon)|^2 \, x^{n-2} dx dy = 0.$$

Again (i) follows directly from the dominated convergence theorem, (ii) is identically zero as the functions do not depend on y, and the integral in (iii) is smaller than two times

$$\int_0^1 \int_{\partial\mathbb{B}} |x\partial_x(\omega) - x^\varepsilon x\partial_x(\omega)|^2 \, x^{n-2} dx dy + \int_0^1 \int_{\partial\mathbb{B}} |\omega x\partial_x(x^\varepsilon)|^2 \, x^{n-2} dx dy.$$

The first term can be dealt again by dominated convergence. For the second one, note that

$$\int_0^1 |\omega x\partial_x(x^\varepsilon)|^2 \, x^{n-2} dx \le \frac{\varepsilon^2}{2\varepsilon + n - 1},$$

and the last term goes to zero, as ε goes to zero.
Step 4: If $\beta < 1$, then $H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{1,\beta}(\mathbb{B})$. By density, it is enough to show that there is a constant $C > 0$ such that $\|u\|_{\mathcal{H}^{1,\beta}(\mathbb{B})} \le C \|u\|_{H^1(\mathbb{B})}$, for all $u \in C_c^\infty(\mathbb{B}^\circ)$.
If $k + |\alpha| = 1$, then, in local coordinates on $[0,1) \times \partial\mathcal{B}$, we have

$$\left|x^{\frac{n+1}{2}-\beta}(x\partial_x)^k \partial_y^\alpha(\omega(x)u(x,y))\right| \le \left|x^{\frac{n-1}{2}}(x\partial_x)^k \partial_y^\alpha(\omega(x)u(x,y))\right|.$$

If $k + |\alpha| = 0$, then as

$$\omega(x)u(x,y) = -\int_x^1 \frac{\partial}{\partial s}(\omega(s)u(s,y))ds,$$

we have

$$\begin{aligned}
&\int_{\partial\mathbb{B}} \int_0^1 \left|x^{\frac{n+1}{2}-\beta}\omega(x)u(x,y)\right|^2 \frac{dx}{x} dy \\
&= \int_{\partial\mathbb{B}} \int_0^1 x^{n-2\beta} \left|\int_x^1 s^{-\frac{n}{2}} s^{\frac{n}{2}} \frac{\partial}{\partial s}(\omega(s)u(s,y))\, ds\right|^2 dx dy
\end{aligned}$$

$$\leq \int_{\mathbb{B}} \int_0^1 x^{n-2\beta} \left(\int_x^1 s^{-n} ds \right) \left(\int_x^1 s^n \left| \frac{\partial}{\partial s} (\omega(s) u(s,y)) \right|^2 ds \right) dx dy$$

$$= \int_0^1 x^{n-2\beta} \left(\int_x^1 s^{-n} ds \right) \left(\int_{\partial\mathbb{B}} \int_x^1 \left| s^{\frac{n-1}{2}} \left(s \frac{\partial}{\partial s} \right) (\omega(s) u(s,y)) \right|^2 \frac{ds}{s} dy \right) dx$$

$$\leq \int_0^1 x^{n-2\beta} \left(\int_x^1 s^{-n} ds \right) dx \, \|u\|^2_{H^1(\mathbb{B})},$$

where we have used Cauchy-Schwarz in the first inequality. The last integral is finite for $n \geq 1$ and $\beta < 1$. $\square$

For functions in $H^1(\mathbb{B})$, we define

$$(u,v)_{H_0^1(\mathbb{B})} = \int_{\mathbb{B}} \langle \nabla u, \nabla \overline{v} \rangle \, d\mu_g \quad \text{and} \quad \|u\|^2_{H_0^1} = \int_{\mathbb{B}} \langle \nabla u, \nabla \overline{u} \rangle \, d\mu_g.$$

In particular,

$$(u,v)_{H^1(\mathbb{B})} := (u,v)_{\mathcal{H}^{0,0}(\mathbb{B})} + (u,v)_{H_0^1(\mathbb{B})} \tag{2.3}$$

and

$$\|u\|_{H^1(\mathbb{B})} = \sqrt{\|u\|^2_{\mathcal{H}^{0,0}(\mathbb{B})} + \|u\|^2_{H_0^1(\mathbb{B})}}. \tag{2.4}$$

Moreover, whenever $u \in L^1(\mathbb{B})$, we define $(u)_{\mathbb{B}} := \fint_{\mathbb{B}} u dx = |\mathbb{B}|^{-1} \int_{\mathbb{B}} u d\mu_g$, where $|\mathbb{B}| = \int_{\mathbb{B}} d\mu_g$ is the area of $\mathbb{B}$.

Lemma 7 *(Poincaré-Wirtinger inequality). There is a constant $C > 0$ such that*

$$\|u - (u)_{\mathbb{B}}\|_{\mathcal{H}^{0,0}(\mathbb{B})} \leq C \|u\|_{H_0^1(\mathbb{B})}, \quad \forall u \in H^1(\mathbb{B}).$$

Proof. The proof follows the same argument as in the proof of [6, Theorem 1 of Section 5.8], using the fact that $H^1(\mathbb{B})$ is compactly embedded in $\mathcal{H}^{0,0}(\mathbb{B})$. $\square$

Definition 8. Denote by $H_0^1(\mathbb{B})$ the space of all $u \in H^1(\mathbb{B})$ such that $(u)_{\mathbb{B}} = 0$.

It is clear that $H^1(\mathbb{B}) = H_0^1(\mathbb{B}) \oplus \mathbb{C}$, where $\mathbb{C}$ is identified with the set of constant functions. Moreover, applying Lemma 7 with $(u)_{\mathbb{B}} = 0$, we see that the map $H_0^1(\mathbb{B}) \ni u \mapsto \|u\|_{H_0^1(\mathbb{B})} \in \mathbb{R}$ is equivalent to the $H^1(\mathbb{B})$ norm.

Definition 9. We denote by $H^{-1}(\mathbb{B})$ the dual space of $H^1(\mathbb{B})$ and by $H_0^{-1}(\mathbb{B}) \subset H^{-1}(\mathbb{B})$ its subspace defined by

$$H_0^{-1}(\mathbb{B}) := \{u \in H^{-1}(\mathbb{B}) : \langle u, 1 \rangle_{H^{-1}(\mathbb{B}) \times H^1(\mathbb{B})} = 0\}.$$

Using the fact that $H^1(\mathbb{B}) = H_0^1(\mathbb{B}) \oplus \mathbb{C}$, we can see that the map $H_0^{-1}(\mathbb{B}) \ni u \mapsto u|_{H_0^1(\mathbb{B})} \in \mathcal{L}(H_0^1(\mathbb{B}), \mathbb{C})$ is bijective, that is, $H_0^{-1}(\mathbb{B})$ can be identified with the dual of $H_0^1(\mathbb{B})$.

Proposition 10. *Let $u \in \mathcal{H}^{0,\beta}(\mathbb{B})$, for some $\beta > -1$. Then $T_u : H^1(\mathbb{B}) \to \mathbb{C}$ and $\widetilde{T}_u : H^1(\mathbb{B}) \to \mathbb{C}$ defined by*

$$T_u(v) = \int_{\mathbb{B}} uv d\mu_g,$$

$$\widetilde{T}_u(v) = \int_{\mathbb{B}} (u - (u)_{\mathbb{B}}) v d\mu_g$$

are continuous. Moreover the functional $\widetilde{T}_u$ belongs to $H_0^{-1}(\mathbb{B})$ and $\widetilde{T}_u|_{H_0^1(\mathbb{B})} = T_u|_{H_0^1(\mathbb{B})}$.

Proof. Since $\beta > -1$, we have the inclusion $\mathcal{H}^{0,\beta}(\mathbb{B}) \hookrightarrow L^1(\mathbb{B})$. In fact,

$$\int_{\mathbb{B}} |u| \, d\mu_g = \int_{\mathbb{B}} \mathsf{x}^{-\beta} |u| \, \mathsf{x}^{\beta} d\mu_g \leq \left(\int_{\mathbb{B}} \mathsf{x}^{2\beta} d\mu_g \right)^{1/2} \|u\|_{\mathcal{H}^{0,\beta}(\mathbb{B})},$$

due to Remark 3. Note that $\int_{\mathbb{B}} \mathsf{x}^{2\beta} d\mu_g$ is finite, as $\int_0^1 x^{n+2\beta} dx < \infty$. The fact that $\mathcal{H}^{0,\beta}(\mathbb{B}) \hookrightarrow L^1(\mathbb{B})$ ensures that $(u)_{\mathbb{B}}$ is well defined.

In order to prove that T_u is continuous, let us assume, without loss of generality, that $-1 < \beta \leq 0$. We denote by $\mathcal{I}_{-\beta} : H^1(\mathbb{B}) \to \mathcal{H}^{0,-\beta}(\mathbb{B})$ the continuous inclusion from Proposition 6. Then, we have that

$$T_u(v) = \int_{\mathbb{B}} uv d\mu_g = \langle u, \mathcal{I}_{-\beta}(v) \rangle_{\mathcal{H}^{0,\beta}(\mathbb{B}) \times \mathcal{H}^{0,-\beta}(\mathbb{B})}.$$

Therefore T_u is continuous as it is the composition of continuous functions. The continuity of $\widetilde{T}_u$ follows similarly. The fact that $\widetilde{T}_u|_{H_0^1(\mathbb{B})} = T_u|_{H_0^1(\mathbb{B})}$ follows from the fact that the integral of v is equal to zero if $v \in H_0^1(\mathbb{B})$. □

A version of Gauss theorem can also be proved for $H^1(\mathbb{B})$. It simplifies and improves [11, Lemma 4.3].

Theorem 11 *(Gauss theorem). Let u and v belong to $H^1(\mathbb{B})$ and $\Delta v \in \mathcal{H}^{0,\gamma}(\mathbb{B})$, for some $\gamma > -1$. Then*

$$\int_{\mathbb{B}} \langle \nabla u, \nabla v \rangle \, d\mu_g = -\int_{\mathbb{B}} u \Delta v d\mu_g.$$

In particular, if $u \in H^1(\mathbb{B})$ is such that $\Delta u \in \mathcal{H}^{0,\gamma}(\mathbb{B})$, for some $\gamma > -1$, then $\int_{\mathbb{B}} \Delta u \, d\mu_g = 0$.

Proof. Without loss of generality, we assume that $-1 < \gamma \leq 0$. First we note that for v and u in $C_c^\infty(\mathbb{B}^\circ)$, we have

$$\int_{\mathbb{B}} \langle \nabla u, \nabla v \rangle \, d\mu_g = -\int_{\mathbb{B}} u \Delta v d\mu_g = -\langle \Delta v, u \rangle_{\mathcal{D}'(\mathbb{B}^\circ) \times C_c^\infty(\mathbb{B}^\circ)}, \tag{2.5}$$

where $\mathcal{D}'(\mathbb{B}^\circ)$ stands for the dual space of $C_c^\infty(\mathbb{B}^\circ)$.

For $v \in H^1(\mathbb{B})$, we can choose a sequence of functions in $C_c^\infty(\mathbb{B}^\circ)$ that converge to v in $H^1(\mathbb{B})$ and, therefore, also in $\mathcal{D}'(\mathbb{B}^\circ)$. Hence the equality between the first and the third term of (2.5) still holds for all $v \in H^1(\mathbb{B})$ and $u \in C_c^\infty(\mathbb{B}^\circ)$.

Moreover if $v \in H^1(\mathbb{B})$ and $\Delta v \in \mathcal{H}^{0,\gamma}(\mathbb{B}) \subset L^2_{\mathrm{loc}}(\mathbb{B}^\circ) \subset L^1_{\mathrm{loc}}(\mathbb{B}^\circ)$, we have again

$$\langle \Delta v, u\rangle_{\mathcal{D}'(\mathbb{B}^\circ)\times C_c^\infty(\mathbb{B}^\circ)} = \int_{\mathbb{B}} u\Delta v d\mu_g, \tag{2.6}$$

for all $u \in C_c^\infty(\mathbb{B}^\circ)$.

Finally, if $v \in H^1(\mathbb{B})$, $\Delta v \in \mathcal{H}^{0,\gamma}(\mathbb{B})$ and $u \in H^1(\mathbb{B})$, we take a sequence in $C_c^\infty(\mathbb{B}^\circ)$ that converges to u in $H^1(\mathbb{B})$. As $H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{0,-\gamma}(\mathbb{B})$, the sequence will also converge to u in $\mathcal{H}^{0,-\gamma}(\mathbb{B})$. Using the duality of $\mathcal{H}^{0,-\gamma}(\mathbb{B})$ and $\mathcal{H}^{0,\gamma}(\mathbb{B})$, (2.5) and (2.6), we obtain our result. □

Proposition 12. *The operator $\Delta : H_0^1(\mathbb{B}) \to H_0^{-1}(\mathbb{B})$ is well-defined, continuous and bijective.*

Note that $\Delta u \in \mathcal{D}'(\mathbb{B}^\circ)$ is always well defined. As $C_c^\infty(\mathbb{B}^\circ)$ is dense in $H^1(\mathbb{B})$, $H_0^{-1}(\mathbb{B})$ can be easily identified, by restricting to $C_c^\infty(\mathbb{B}^\circ)$, with a subset of $\mathcal{D}'(\mathbb{B}^\circ)$.

Proof. By Riesz theorem, there is a bijective map $\mathcal{R} : H_0^1(\mathbb{B}) \to H_0^{-1}(\mathbb{B})$ such that

$$\langle \mathcal{R}u, \overline{v}\rangle_{H_0^{-1}(\mathbb{B})\times H_0^1(\mathbb{B})} = (u, v)_{H_0^1(\mathbb{B})} = \int_{\mathbb{B}} \langle \nabla u, \nabla \overline{v}\rangle \, d\mu_g.$$

Then, the operator Δ can be easily identified with $-\mathcal{R}$. □

Corollary 13. *There is a constant $C > 0$ such that*

$$\|u\|_{H_0^{-1}(\mathbb{B})} \le C \|\Delta u\|_{H_0^{-1}(\mathbb{B})}, \quad \forall u \in H_0^1(\mathbb{B}).$$

Proof. This follows from the continuous inclusion $H_0^1(\mathbb{B}) \hookrightarrow H_0^{-1}(\mathbb{B})$, provided by Propositions 6 and 10, and Proposition 12. □

2.1. Sobolev immersions

In this section, we prove some embeddings concerning Mellin-Sobolev spaces.

Proposition 14. *Suppose that $p \in [2, \infty)$, if $n = 1$, and $p \in [2, (2n + 2)/(n - 1)]$, if $n \ge 2$. Then, for each $\gamma \in \mathbb{R}$, we have $\mathcal{H}^{1,\gamma}(\mathbb{B}) \hookrightarrow \mathsf{x}^{\gamma-(n+1)(1/2-1/p)}L^p(\mathbb{B})$. In particular, if $\gamma \ge (n + 1)(1/2 - 1/p)$, then $\mathcal{H}^{1,\gamma}(\mathbb{B}) \hookrightarrow L^p(\mathbb{B})$.*

Proof. First, we note that

$$\mathcal{H}^{1,\gamma}(\mathbb{B}) \subset H^1_{\mathrm{loc}}(\mathbb{B}^\circ) \subset L^p_{\mathrm{loc}}(\mathbb{B}^\circ).$$

Therefore, it is enough to understand the behavior of elements of $\mathcal{H}^{1,\gamma}(\mathbb{B})$ in the neighborhood of the conical tip. We have $H^1(\mathbb{R}^{n+1}) \hookrightarrow L^p(\mathbb{R}^{n+1})$. Hence

$$\begin{aligned} &\left\|(x, y) \mapsto e^{(\gamma-\frac{n+1}{2})x}\omega(e^{-x})\phi_j(y)u(e^{-x}, y)\right\|_{L^p(\mathbb{R}^{n+1})} \\ &\le C \left\|(x, y) \mapsto e^{(\gamma-\frac{n+1}{2})x}\omega(e^{-x})\phi_j(y)u(e^{-x}, y)\right\|_{H^1(\mathbb{R}^{n+1})} \le C \|u\|_{\mathcal{H}^{1,\gamma}(\mathbb{B})}, \end{aligned} \tag{2.7}$$

where we recall that $\{\phi_j\}_{j\in J}$ is a partition of unity of $\partial\mathcal{B}$ and ω is as Definition 2. A change of variables $e^{-x} \mapsto x$ in (2.7) implies

$$\left\| \mathsf{x}^{(n+1)(\frac{1}{2}-\frac{1}{p})-\gamma} u \right\|_{L^p(\mathbb{B})} \leq C \|u\|_{\mathcal{H}^{1,\gamma}(\mathbb{B})},$$

which shows the claim. □

Corollary 15. *The following continuous inclusions hold:*

(i) *If* $\dim(\mathbb{B}) \in \{2,3\}$*, then* $H^1(\mathbb{B}) \hookrightarrow L^4(\mathbb{B})$.

(ii) *If* $\dim(\mathbb{B}) = 2$*, then* $H^1(\mathbb{B}) \hookrightarrow L^6(\mathbb{B})$.

(iii) *If* $\dim(\mathbb{B}) = 3$ *and* $\alpha > 0$*, then* $H^1(\mathbb{B}) \hookrightarrow \mathsf{x}^{-\alpha} L^6(\mathbb{B})$.

Proof. We have seen that $H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{1,\beta}(\mathbb{B})$, for all $\beta < 1$. For $\dim(\mathbb{B}) = 2$, if we choose $1/2 \leq \beta < 1$, we have $\mathcal{H}^{1,\beta}(\mathbb{B}) \hookrightarrow L^4(\mathbb{B})$. For $\dim(\mathbb{B}) = 3$, if we choose $3/4 \leq \beta < 1$, we have $\mathcal{H}^{1,\beta}(\mathbb{B}) \hookrightarrow L^4(\mathbb{B})$. For $\dim(\mathbb{B}) = 2$, if we choose if $2/3 \leq \beta < 1$, then $\mathcal{H}^{1,\beta}(\mathbb{B}) \hookrightarrow L^6(\mathbb{B})$. Finally, for $\dim(\mathbb{B}) = 3$ and $\alpha > 0$, then choose $1 - \alpha \leq \beta < 1$ so that $\mathcal{H}^{1,\beta}(\mathbb{B}) \hookrightarrow \mathsf{x}^{-\alpha} L^6(\mathbb{B})$. □

3. Realizations of the Laplacian and bi-Laplacian

We start with some basic concepts on analytic semigroup theory.

Definition 16. Let X be a complex Banach space and $A : \mathcal{D}(A) \to X$ be a densely defined closed operator in X. We say that A is a *negative generator of an analytic semigroup* if for some $\delta, C > 0$ we have

$$\{\lambda \in \mathbb{C} : \operatorname{Re}(\lambda) > -\delta\} \subset \rho(A) \quad \text{and} \quad \left\|(\lambda - A)^{-1}\right\|_{\mathcal{L}(X)} \leq C/|\lambda|, \;\; \operatorname{Re}(\lambda) > -\delta.$$

The semigroup associated to a negative generator A is denoted $e^{tA} \in \mathcal{L}(X)$, see e.g. [1, Chapter I.1.2]. Both semigroup and the complex powers $(-A)^z : \mathcal{D}((-A)^z) \to X$, $z \in \mathbb{C}$, can be defined by Cauchy's integral formula, see e.g. [1, Theorem III.4.6.5]. In the case of $(-A)^{it} \in \mathcal{L}(X)$ for all $t \in \mathbb{R}$ and $\left\|(-A)^{it}\right\| \leq Me^{\phi|t|}$, for some $M > 0$ and $\phi \geq 0$, we say that $-A$ has *bounded imaginary powers* and denote by $-A \in \mathcal{BIP}(\phi)$, see e.g. [1, Chapter III.4.7]. Recall that if $-A \in \mathcal{BIP}$, then $[X, \mathcal{D}(A)]_\theta = \mathcal{D}((-A)^\theta)$, see e.g. [12, Theorem 4.17.].

Next we recall some basic facts from the *cone calculus*, for more details we refer to [5,8,10,7,21,20,22]. The Laplacian Δ, as a cone differential operator, acts naturally on scales of Mellin-Sobolev spaces. Let us consider it as an unbounded operator in $\mathcal{H}^{s,\gamma}(\mathbb{B})$, $s, \gamma \in \mathbb{R}$, with domain $C_c^\infty(\mathbb{B}^\circ)$. Denote by $\Delta_{s,\min}$ the minimal extension (i.e. the closure) of Δ and by $\Delta_{s,\max}$ the maximal extension, defined as usual by

$$\mathcal{D}(\Delta_{s,\max}) = \left\{ u \in \mathcal{H}^{s,\gamma}(\mathbb{B}) \,|\, \Delta u \in \mathcal{H}^{s,\gamma}(\mathbb{B}) \right\}.$$

An important result in the field of cone differential operators tells us that those two domains differ in general, unlike the case of closed manifolds. In particular, there exists an s-independent finite-dimensional space $\mathcal{E}_{\max,\Delta,\gamma}$, that is called *asymptotics space*, such that

$$\mathcal{D}(\Delta_{s,\max}) = \mathcal{D}(\Delta_{s,\min}) \oplus \mathcal{E}_{\max,\Delta,\gamma}. \tag{3.1}$$

More precisely, $\mathcal{E}_{\max,\Delta,\gamma}$ consists of linear combinations of smooth functions in $\mathbb{B}^\circ$. Those functions vanish outside the collar neighborhood and, in local coordinates $(x, y) \in [0, 1) \times \partial\mathcal{B}$, can be written as $\omega(x)c(y)x^{-\rho}\log^k(x)$. The function ω is the cut-off function defined on Section 2, $c \in C^\infty(\partial\mathbb{B})$,

$\rho \in \{z \in \mathbb{C} \,|\, \mathrm{Re}(z) \in [\frac{n-3}{2} - \gamma, \frac{n+1}{2} - \gamma)\}$ and $k \in \{0,1\}$. Here, the metric $h(\cdot)$ determines explicitly the exponents ρ. As for the minimal domain, it can be proved that $\mathcal{D}(\Delta_{s,\min}) = \mathcal{H}^{s+2,\gamma+2}(\mathbb{B})$, whenever

$$\frac{n-3}{2} - \gamma \notin \Big\{\frac{n-1}{2} \pm \sqrt{\Big(\frac{n-1}{2}\Big)^2 - \lambda_j} : j \in \mathbb{N}\Big\}.$$

A suitable choice of the domain of the Laplacian is given below. For this, we denote by $\mathbb{C}_\omega$ the finite dimensional space of functions that are equal to zero outside the collar neighborhood and that close to the singularities are expressed by $\sum_{j=1}^{N} c_j\omega_j$, where N is the number of connected components of $\partial\mathcal{B}$ and ω_j are the restrictions of ω to these components. Also denote by $\mathbb{R}_\omega$ the subspace of $\mathbb{C}_\omega$ such that $c_j \in \mathbb{R}$.

Theorem 17. *[21, Theorem 6.7] Let*

$$\frac{n-3}{2} < \gamma < \min\Big\{-1 + \sqrt{\Big(\frac{n-1}{2}\Big)^2 - \lambda_1}, \frac{n+1}{2}\Big\}, \tag{3.2}$$

where λ_1 is the greatest non-zero eigenvalue of the boundary Laplacian $\Delta_{h(0)}$ on $(\partial\mathcal{B}, h(0))$. Then for every $c, \phi > 0$, the operator $\Delta - c : \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega \to \mathcal{H}^{s,\gamma}(\mathbb{B})$ is a negative generator of an analytic semigroup such that $-\Delta + c \in \mathcal{BIP}(\phi)$. The Laplacian $\Delta : \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega \to \mathcal{H}^{s,\gamma}(\mathbb{B})$ will be denoted by Δ_s.

Similarly to the Laplacian, the bi-Laplacian has a minimal extension $\Delta^2_{s,\min}$, which is the closure of $\Delta^2 : C_c^\infty(\mathbb{B}^\circ) \to \mathcal{H}^{s,\gamma}(\mathbb{B})$ satisfying $\mathcal{H}^{s+4,\gamma+4}(\mathbb{B}) \hookrightarrow \mathcal{D}(\Delta^2_{s,\min}) \hookrightarrow \cap_{\varepsilon>0}\mathcal{H}^{s+4,\gamma+4-\varepsilon}(\mathbb{B})$, and a maximal extension $\Delta^2_{s,\max}$, whose domain is $\{u \in \mathcal{H}^{s,\gamma}(\mathbb{B}) : \Delta^2 u \in \mathcal{H}^{s,\gamma}(\mathbb{B})\}$. They are related by $\mathcal{D}(\Delta^2_{s,\max}) = \mathcal{D}(\Delta^2_{s,\min}) \oplus \mathcal{E}_{\max,\Delta^2,\gamma}$, where $\mathcal{E}_{\max,\Delta^2,\gamma}$ is a finite dimensional space consisting of linear combinations of functions of the form $\omega(x)c(y)x^{-\rho}\log^k(x)$, with $k \in \{0,1,2,3\}$ and $\rho \in \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\frac{n-7}{2} - \gamma, \frac{n+1}{2} - \gamma)\}$.

We will choose a bi-Laplacian domain $\mathcal{D}(\Delta^2_s)$ satisfying $\mathcal{D}(\Delta^2_{s,\min}) \hookrightarrow \mathcal{D}(\Delta^2_s) \hookrightarrow \mathcal{D}(\Delta^2_{s,\max})$. The definition and the properties of the operator $\Delta^2_s : \mathcal{D}(\Delta^2_s) \to \mathcal{H}^{s,\gamma}(\mathbb{B})$ are explained in the following corollary of Theorem 17.

Corollary 18 *(bi-Laplacian). Consider the bi-Laplacian $\mathcal{D}(\Delta^2_s) = \{u \in \mathcal{D}(\Delta_s) : \Delta_s u \in \mathcal{D}(\Delta_s)\}$, where Δ_s is as in Theorem 17. Then, there exists an s-independent finite dimensional space $\mathcal{E}_{\Delta^2,\gamma}$ contained in $\mathcal{H}^{\infty,\gamma+2+\alpha_0}(\mathbb{B}) \cap \mathcal{E}_{\max,\Delta^2,\gamma}$, for some $\alpha_0 > 0$, such that (1.6) holds. In particular, $\mathcal{D}(\Delta^2_s) \hookrightarrow \mathcal{H}^{s+4.\gamma+4-\varepsilon} \oplus \mathbb{C}_\omega \oplus \mathcal{E}_{\Delta^2,\gamma}$, for all $\varepsilon > 0$. The space $\mathcal{E}_{\Delta^2,\gamma}$ consists of $C^\infty(\mathbb{B}^\circ)$-functions, which in local coordinates $(x,y) \in [0,1) \times \partial\mathcal{B}$, are of the form $\omega(x)c(y)x^{\rho}\ln^k(x)$, where $c \in C^\infty(\partial\mathbb{B})$, $\rho \in \{z \in \mathbb{C} : \mathrm{Re}(z) \in [\frac{n-7}{2} - \gamma, \frac{n-3}{2} - \gamma)\}$ and $k \in \{0,1,2,3\}$; for more details we refer to [11, Section 3.2].*

Moreover, for every $\phi > 0$, the operator $A_s := -(1 - \Delta_s)^2 : \mathcal{D}(\Delta^2_s) \to \mathcal{H}^{s,\gamma}(\mathbb{B})$ is a negative generator of an analytic semigroup such that $(1 - \Delta_s)^2 \in \mathcal{BIP}(\phi)$, see [11, Proposition 3.6].

For $\alpha \in [0,2]$, we define $X^s_\alpha := \mathcal{D}((-A_s)^{\alpha/2}) = [\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{D}(\Delta^2_s)]_{\alpha/2}$. Notice that $X^s_2 = \mathcal{D}(\Delta^2_s)$ and $X^s_1 = \mathcal{D}(\Delta_s)$. We also denote $X^\infty_1 := \cap_{s\geq 0} X^s_1$ and $X^\infty_2 := \cap_{s\geq 0} X^s_2$. We close this section with some facts about the complex interpolation of the domain of the Laplacian and bi-Laplacian.

Proposition 19. *Let $\alpha \in (0,1)$ and assume that γ satisfies (3.2).*

(i) *If $\alpha \notin \{\frac{1-\gamma}{2} \pm \frac{1}{2}\sqrt{(\frac{n-1}{2})^2 - \lambda_j} : j \in \mathbb{N}\}$, then $[\mathcal{H}^{s,\gamma}(\mathbb{B}), X^s_1]_\alpha = \mathcal{H}^{s+2\alpha,\gamma+2\alpha}(\mathbb{B}) \oplus \mathbb{C}_\omega$.*

(ii) *We always have $\mathcal{H}^{s+4\alpha,\gamma+4\alpha}(\mathbb{B}) \oplus \mathbb{C}_\omega \oplus \mathcal{E}_{\Delta^2,\gamma} \subset [\mathcal{H}^{s,\gamma}(\mathbb{B}), X^s_2]_\alpha$.*

(iii) *If $2\alpha \notin \{\frac{1-\gamma}{2} \pm \frac{1}{2}\sqrt{(\frac{n-1}{2})^2 - \lambda_j} : j \in \mathbb{N}\}$, then*

$$[\mathcal{H}^{s,\gamma}(\mathbb{B}), X^s_2]_\alpha \hookrightarrow \cap_{\varepsilon>0}\mathcal{H}^{s+4\alpha,\gamma+4\alpha-\varepsilon} \oplus \mathbb{C}_\omega \oplus \underline{\mathcal{E}}_{\Delta^2,\gamma+4(\alpha-1)},$$

where $\underline{\mathcal{E}}_{\Delta^2,\gamma+4(\alpha-1)} \subset \mathcal{H}^{\infty,\gamma+2}(\mathbb{B})$ *is a subspace of the asymptotic space* $\mathcal{E}_{\max,\Delta^2,\gamma+4(\alpha-1)}$. *We note that the sums above are not necessarily direct.*

Proof. (i) It follows from [11, Lemma 4.5 (i)].
(ii) It follows from the inclusions $\mathbb{C}_\omega \oplus \mathcal{E}_{\Delta^2,\gamma} \hookrightarrow \mathcal{D}(\Delta_s^2) = X_2^s$ and

$$\mathcal{H}^{s+4\alpha,\gamma+4\alpha}(\mathbb{B}) = [\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{H}^{s+4,\gamma+4}(\mathbb{B})]_\alpha \hookrightarrow [\mathcal{H}^{s,\gamma}(\mathbb{B}), X_2^s]_\alpha,$$

where in the first equality we have used [11, Lemma 3.3 (iii)].
(iii) *Case* $\alpha \in (0, 1/2]$. We have

$$[\mathcal{H}^{s,\gamma}(\mathbb{B}), X_2^s]_\alpha = \mathcal{D}((-A_s)^\alpha) = \mathcal{D}((1-\Delta_s)^{2\alpha}) = [\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{D}(\Delta_s)]_{2\alpha} = \mathcal{H}^{s+4\alpha,\gamma+4\alpha}(\mathbb{B}) \oplus \mathbb{C}_\omega.$$

In the first and third equalities we have used the $\mathcal{BIP}$ property of $1-\Delta_s$ and $-A_s$. In the last equality, we have used (i).
Case $\alpha \in (1/2, 1)$. We first note that

$$\Delta^2[\mathcal{H}^{s,\gamma}(\mathbb{B}), \mathcal{D}(\Delta_s^2)]_\alpha \hookrightarrow [\mathcal{H}^{s-4,\gamma-4}(\mathbb{B}), \mathcal{H}^{s,\gamma}(\mathbb{B})]_\alpha = \mathcal{H}^{s+4(\alpha-1),\gamma+4(\alpha-1)}(\mathbb{B}),$$

where [11, Lemma 3.3 (iii)] was used in the last equality. Hence, we have

$$[\mathcal{H}^{s,\gamma}(\mathbb{B}), X_2^s]_\alpha \hookrightarrow \mathcal{D}(\Delta^2_{s+4(\alpha-1),\max}) \subset \mathcal{H}^{s+4\alpha,\gamma+4\alpha-\varepsilon}(\mathbb{B}) \oplus \mathcal{E}_{\max,\Delta^2,\gamma+4(\alpha-1)}. \tag{3.3}$$

Moreover,

$$[\mathcal{H}^{s,\gamma}(\mathbb{B}), X_2^s]_\alpha = \mathcal{D}((1-\Delta_s)^{2\alpha}) \hookrightarrow \mathcal{D}(\Delta_s) = \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega, \tag{3.4}$$

where in the first equality we have used the $\mathcal{BIP}$ property of $-A_s$. Now let $u \in [\mathcal{H}^{s,\gamma}(\mathbb{B}), X_2^s]_\alpha$. By (3.3), we have that $u = v + w$, where $v \in \mathcal{H}^{s+4\alpha,\gamma+4\alpha-\varepsilon}(\mathbb{B})$ and $w \in \mathcal{E}_{\max,\Delta^2,\gamma+4(\alpha-1)}$. Hence, by (3.4) and for sufficiently small $\varepsilon > 0$, we have

$$w = u - v \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega + \mathcal{H}^{s+4\alpha,\gamma+4\alpha-\varepsilon}(\mathbb{B}) = \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega.$$

Therefore

$$w \in \left(\mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega\right) \cap \mathcal{E}_{\max,\Delta^2,\gamma+4(\alpha-1)} \subset \mathbb{C}_\omega \oplus \left(\mathcal{E}_{\max,\Delta^2,\gamma+4(\alpha-1)} \cap \mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\right),$$

which concludes the proof. □

4. Existence and regularity of the global attractors

For the rest of the paper γ is fixed and satisfies (1.2). The constants $C > 0$ may change along the computations.

In this section we prove part (i) of Theorem 1. In the sequel all the spaces we use are the real parts of the ones defined previously. Recall that a global attractor for a semiflow $T : [0, \infty) \times X \to X$ defined on a Hilbert X is a compact set $\mathcal{A} \subset X$ such that $T(t)\mathcal{A} := \{T(t)x : x \in \mathcal{A}\} = \mathcal{A}$, for all $t \geq 0$, which, moreover, attracts all bounded sets $B \subset X$ in the following sense:

$$\lim_{t\to\infty} (\sup_{b\in B} \inf_{a\in\mathcal{A}} \|T(t)b - a\|_X) = 0.$$

If it exists, then it is unique.

As mentioned in the introduction, for any $s \geq 0$, we can define a semiflow T in X_1^s. It is convenient, however, to restrict T to a smaller space. For this reason, we first prove the following proposition.

Proposition 20. *Let $u_0 \in X_1^s$, then the function $(u)_{\mathbb{B}} : (0,\infty) \to \mathbb{R}$, defined by*

$$(u)_{\mathbb{B}}(t) := \frac{1}{|\mathbb{B}|} \int\limits_{\mathbb{B}} T(t)u_0 d\mu_g, \ t > 0,$$

is constant. Here $|\mathbb{B}| = \int_{\mathbb{B}} d\mu_g$.

Proof. The proof is similar to [26, Equation (4.61)], where we have to take into account (1.3) and Theorem 11. □

Consider the Hilbert space

$$X_{1,0}^s = \{u \in \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega : (u)_{\mathbb{B}} = 0\}.$$

By Proposition 20, $T(t)X_{1,0}^s \subset X_{1,0}^s$. Therefore, T restricts to a semiflow on $X_{1,0}^s$. Concerning the existence of global attractors, we recall the following result.

For two Hilbert spaces X and Y such that $Y \hookrightarrow X$ and a semiflow $T : [0,\infty) \times X \to X$, we define the ω-limit set $\omega_Z(B)$ of $B \subset X$, where $Z = X$ or Y, by

$$\omega_Z(B) = \Big\{ z \in Z : \exists \quad t_n \to \infty \quad \text{and} \quad \{x_n\}_{n\in\mathbb{N}} \subset B \quad \text{such that} \quad \lim_{n\to\infty} \|T(t_n)x_n - z\|_Z = 0 \Big\}. \qquad (4.1)$$

If $Z = Y$, then the above definition requires that $T(t_n)x_n \in Y$ for all $n \in \mathbb{N}$. In order to show existence and regularity of global attractors, we prove the following variation of [15, Theorem 10.5].

Theorem 21. *Let $Y \hookrightarrow X$ be Hilbert spaces, $T : [0,\infty) \times X \to X$ be a semiflow and $\mathcal{K} \subset Y$ be a compact set in Y. Assume that for all bounded sets $B \subset X$ there exists a constant $t_B > 0$ such that, if $t > t_B$, then $T(t)B \subset \mathcal{K}$. Under these conditions there exists a (unique) global connected attractor $\mathcal{A}$ for the semiflows T. Moreover, $\mathcal{A} = \omega_X(\mathcal{K}) = \omega_Y(\mathcal{K})$ is contained in Y and attracts bounded sets of X in Y in the following sense: for any bounded set $B \subset X$, the set $T(t)B$ is bounded in Y for large t and $\lim_{t\to\infty} \sup_{b\in B} \inf_{a\in\mathcal{A}} \|T(t)b - a\|_Y = 0$.*

Proof. First we show that $\omega_X(\mathcal{K}) = \omega_Y(\mathcal{K})$. Since $Y \hookrightarrow X$, the definition given by (4.1) implies that $\omega_Y(\mathcal{K}) \subset \omega_X(\mathcal{K})$. On the other hand, if $x \in \omega_X(\mathcal{K})$ and $\lim_{n\to\infty} \|T(t_n)x_n - x\|_X = 0$ for some $t_n \to \infty$ and $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{K}$, then $T(t_n)x_n \in \mathcal{K}$ for all $t_n > t_{\mathcal{K}}$. By the compactness of $\mathcal{K}$ in Y, some subsequence $\{T(t_{n_j})x_{n_j}\}_{j\in\mathbb{N}}$ converges in Y, which implies that $x \in \omega_Y(\mathcal{K})$.

Let $\mathcal{A} := \omega_X(\mathcal{K}) = \omega_Y(\mathcal{K})$. Using the notation $\overline{\mathcal{C}}^Y$ for the closure of $\mathcal{C}$ in Y, (4.1) says that

$$\omega_Y(\mathcal{K}) = \cap_{t>0} \overline{\cup_{s>t}(T(s)\mathcal{K} \cap Y)}^Y = \cap_{t>t_{\mathcal{K}}} \overline{\cup_{s>t} T(s)\mathcal{K}}^Y,$$

which implies that $\mathcal{A}$ is a non-empty compact set of Y - and also of X - since it is the intersection of decreasing non-empty compact sets. The invariance of $\mathcal{A}$ follows easily from (4.1), see also [4, Proposition 1.1.1].

Finally, the fact that $\mathcal{A}$ attracts bounded sets of X in Y - and in X as well - follows from the arguments of [15, Theorem 10.5], which we provide for completeness. Let us suppose that

$$\lim_{t\to\infty} \sup_{b\in B} \inf_{a\in\mathcal{A}} \|T(t)b - a\|_Y = 0$$

does not hold for some bounded set $B \subset X$. Then, we can find sequences $t_n \to \infty$, $\{b_n\}_{n\in\mathbb{N}} \subset B$ and $\varepsilon_0 > 0$ such that $\inf_{a\in\mathcal{A}} \|T(t_n)b_n - a\|_Y > \varepsilon_0$ for all n. But for $t_n > t_B$ and $b \in B$, then $T(t_n)b \in \mathcal{K}$ which is a compact set in Y. Hence a subsequence $\{T(t_{n_j})b_{n_j}\}_{j\in\mathbb{N}}$ converges in Y. As $T(t_{n_j})b_{n_j} = T(t_{n_j} - t_n)T(t_n)b_{n_j}$ and $T(t_n)b_{n_j} \in \mathcal{K}$, we conclude that the subsequence $\{T(t_{n_j})b_{n_j}\}$ converges to an element of $\omega_Y(\mathcal{K}) = \mathcal{A}$, which gives us a contradiction. Connectness follows exactly as in [15]. □

For our purposes we use the following consequence.

Corollary 22. *Let X be a Hilbert space, $A : \mathcal{D}(A) \subset X \to X$ be a negative generator of an analytic semigroup with compact resolvent and $F : X_\alpha = \mathcal{D}((-A)^\alpha) \to X$, $0 \le \alpha < 1$, be a locally Lipschitz function. Consider the following problem*

$$u'(t) = Au(t) + F(u(t)),$$
$$u(0) = u_0,$$

where $u_0 \in X_\alpha$. Assume that
(i) *There is a closed subspace $\widetilde{X}_\alpha \subset X_\alpha$ such that for all $u_0 \in \widetilde{X}_\alpha$, a global solution $u \in C^1((0,\infty), X) \cap C((0,\infty), \mathcal{D}(A)) \cap C([0,\infty), \widetilde{X}_\alpha)$ is defined.*
(ii) *There are Hilbert spaces Y and W such that $Y \overset{c}{\hookrightarrow} W \hookrightarrow X_\alpha$ and a constant $C_Y > 0$ with the following property: for every $R > 0$, there exists $t_R > 0$ such that if $u_0 \in \widetilde{X}_\alpha$ and $\|u_0\|_{X_\alpha} \le R$, then $u(t, u_0) \in Y$ and $\|u(t, u_0)\|_Y \le C_Y$, for all $t > t_R$.*

Then the semiflow $T : [0,\infty) \times \widetilde{X}_\alpha \to \widetilde{X}_\alpha$ defined by $T(t)u_0 = u(t)$ has a global connected attractor $\mathcal{A}$ that is contained in W and attracts bounded sets of $\widetilde{X}_\alpha$ in W.

Proof. We have to show that the conditions of Theorem 21 are satisfied. As $Y \overset{c}{\hookrightarrow} W$, the bounded set $\mathcal{K}_0 := \{x \in Y : \|x\|_Y \le C_Y\}$ is a relative compact subset of W. We define $\mathcal{K}$ to be the closure of $\mathcal{K}_0$ in W. Let $B \subset \widetilde{X}_\alpha$ be a bounded set, i.e. there exists $R > 0$ such that $\|u_0\|_{X_\alpha} \le R$, for all $u_0 \in B$. By our assumptions, there exists $t_R > 0$ such that if $t \ge t_R$, then $\|T(t)u_0\|_Y \le C_Y$. Therefore $T(t)B \subset \mathcal{K}_0 \subset \mathcal{K}$, if $t \ge t_R$. □

The above corollary will now be applied to the proof of the following theorem, from which part (i) of Theorem 1 will follow.

Theorem 23. *Let $0 < \varepsilon < (n+1)/16 - \gamma/8$. Then, for each $s \ge 0$ there is a constant $\varkappa_{s,\varepsilon} > 0$ with the following property: for every $R > 0$, there exists $\bar{t}_{R,s,\varepsilon} > 0$ such that if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t, u_0)\|_{\mathcal{D}((-A_s)^{1+\varepsilon})} \le \varkappa_{s,\varepsilon}$, for all $t > \bar{t}_{R,s,\varepsilon}$.*

The theorem will be proved in several steps. The first one follows directly from Temam [26]. We just highlight the necessary results for repeating the arguments.

Proposition 24. *There is a constant κ with the following property: for every $R > 0$, there is a constant $t_R > 0$ such that if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t, u_0)\|_{H_0^1(\mathbb{B})} \le \kappa$, for all $t > t_R$.*

Proof. The proof is obtained by following the arguments in [26, Section 4.2.2]. First we prove the existence of $\kappa > 0$ with the following property: for every $R > 0$, there is a constant $t_R > 0$ such that if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t, u_0)\|_{H_0^{-1}(\mathbb{B})} \le \kappa$, for all $t > t_R$, see deductions of [26, Equations (4.89)-(4.90)]. For our situation, we only have to take into account the Sobolev immersion $H^1(\mathbb{B}) \hookrightarrow L^4(\mathbb{B})$ from Corollary 15 and Theorem 11, which allow the definition of the strict Lyapunov function [9, Definition 8.4.5] $\mathcal{L} : H^1(\mathbb{B}) \to \mathbb{R}$ by

$$\mathcal{L}(v) = \frac{1}{2}\int\limits_{\mathbb{B}} \langle \nabla v, \nabla v\rangle \, d\mu_g + \int\limits_{\mathbb{B}} \left(\frac{1}{4}v^4 - \frac{1}{2}v^2\right) d\mu_g, \tag{4.2}$$

see also [11, Section 4.2]. We also use Proposition 10 to identify elements of $\mathcal{H}^{0,\beta}(\mathbb{B})$, $\beta > -1$, with elements in $H^{-1}(\mathbb{B})$ for the computations. The rest of proof follows the deduction of [26, Equation (4.95)]. □

In order to proceed to the proof of Theorem 23, we write (1.1) as

$$u'(t) = A_s u + F(u), \tag{4.3}$$

where $A_s : X_2^s \to X_0^s$ is given by $A_s = -(1-\Delta_s)^2$ and $F : X_1^s \to X_0^s$ is given by $F(u) = \Delta_s(u^3 - 3u) + u$.

It is well known, see [14, Theorem 6.13], that for some $\delta > 0$ depending on A_s, the fractional powers satisfy

$$\left\|(-A_s)^\alpha e^{tA_s}\right\|_{\mathcal{B}(\mathcal{H}^{s,\gamma}(\mathbb{B}))} \le c_{\alpha,s} t^{-\alpha} e^{-\delta t}, \quad t > 0, \tag{4.4}$$

where $c_{\alpha,s} > 0$ only depends on $\alpha, s \ge 0$.

Lemma 25. *Let $0 \le \sigma \le \alpha \le \beta < 1$, $0 < \tilde{t} < t$, $\delta > 0$ be as in (4.4) and $u \in C([\tilde{t}, t], \mathcal{D}(A_s))$. Then*

$$\begin{aligned}\|(-A_s)^\alpha u(t)\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} &\le C_\sigma e^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\sigma}\left\|(-A_s)^{\alpha-\sigma}u(\tilde{t})\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + C_{\alpha,\beta}\int\limits_{\tilde{t}}^{t} e^{-\delta(t-s)}(t-s)^{-\beta} \\ &\times\left(\left\|(-A_s)^{\alpha-\beta+\frac{1}{2}}u^3(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + \left\|(-A_s)^{\alpha-\beta+\frac{1}{2}}u(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + \left\|(-A_s)^{\alpha-\beta}u(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\right) ds, \end{aligned} \tag{4.5}$$

for some constants C_σ, $C_{\alpha,\beta}$ only depending on α, β, σ and s.

Proof. We apply $(-A_s)^\alpha$ to the variation of constants formula

$$u(t) = e^{(t-\tilde{t})A_s}u(\tilde{t}) + \int\limits_{\tilde{t}}^{t} e^{(t-s)A_s}F(u(s))ds$$

to obtain

$$\begin{aligned}&\|(-A_s)^\alpha u(t)\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \\ &\le \left\|(-A_s)^\sigma e^{-A_s(t-\tilde{t})}\right\|_{\mathcal{B}(\mathcal{H}^{s,\gamma}(\mathbb{B}))}\left\|(-A_s)^{\alpha-\sigma}u(\tilde{t})\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + \int\limits_{\tilde{t}}^{t}\left\|(-A_s)^\beta e^{A_s(t-s)}\right\|_{\mathcal{B}(\mathcal{H}^{s,\gamma}(\mathbb{B}))} \\ &\quad\times\left\|(-A_s)^{\alpha-\beta}\left(\Delta(u^3(s) - 3u(s)) + u(s)\right)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} ds \\ &\overset{(1)}{\le} C_\sigma e^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\sigma}\left\|(-A_s)^{\alpha-\sigma}u(\tilde{t})\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + C_{\alpha,\beta}\int\limits_{\tilde{t}}^{t} e^{-\delta(t-s)}(t-s)^{-\beta} \\ &\quad\times\left(\left\|(-A_s)^{\alpha-\beta+\frac{1}{2}}u^3(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + \left\|(-A_s)^{\alpha-\beta+\frac{1}{2}}u(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} + \left\|(-A_s)^{\alpha-\beta}u(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\right) ds.\end{aligned}$$

In (1) we have used that

$$\left\|(-A_s)^{\alpha-\beta}\Delta v\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} = \left\|(1-\Delta_s)^{-1}\Delta_s(-A_s)^{\alpha-\beta+\frac{1}{2}}v\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} \le c_s\left\|(-A_s)^{\alpha-\beta+\frac{1}{2}}v\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}. \quad \square$$

Proposition 26. *There is a constant $\kappa_1 > 0$ with the following property: for every $R > 0$, there is a constant $t_{R,1} > 0$ such that if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t, u_0)\|_{X^0_1} \le \kappa_1$, for all $t > t_{R,1}$.*

Proof. *Step 1.* Let $\theta \in [1/2, 1)$. There exists $\kappa_\theta > 0$ with the following property: for every $R > 0$, there is a constant $t_{R,\theta} > 0$ such that if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t, u_0)\|_{X^0_\theta} \le \kappa_\theta$, for all $t > t_{R,\theta}$.

Let $u_0 \in X^0_{1,0}$ be such that $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, $u(t) = T(t)u_0$ and t_R, κ be as Proposition 24. If x is as in Remark 3, then we have

$$\|u(s)^3\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} \le C \left(\int_{\mathbb{B}} \left|\mathsf{x}^{-\gamma/3} u(s)\right|^6 d\mu_g \right)^{1/2} = C \|u\|^3_{\mathsf{x}^{\gamma/3} L^6(\mathbb{B})} \le C \|u\|^3_{H^1(\mathbb{B})}, \tag{4.6}$$

where we have used Corollary 15 and (1.2). Also, due to Proposition 6 and Proposition 19 (i), for suitable $0 \le \ell < 1$, we have $\gamma + \ell < 1$ and

$$H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{\ell,\gamma+\ell}(\mathbb{B}) = \mathcal{H}^{\ell,\gamma+\ell}(\mathbb{B}) \oplus \mathbb{C}_\omega = X^0_{\ell/2}.$$

Hence, using (4.5) with $\tilde{t} = t_R$, $\alpha = \frac{\theta}{2}$, $\sigma = \frac{\theta}{2} - \frac{1}{4} + \varepsilon$ for some $\varepsilon > 0$, $\beta = \frac{\theta}{2} + \frac{1}{2}$, (4.6) and Proposition 24, we obtain for $t > t_R$

$$\begin{aligned}
\left\|(-A_0)^{\frac{\theta}{2}} u(t)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} &\le Ce^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-(\frac{\theta}{2}-\frac{1}{4}+\varepsilon)} \left\|(-A_0)^{\frac{1}{4}-\varepsilon} u(\tilde{t})\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} \\
&\quad + C\int_{\tilde{t}}^{t} e^{-\delta(t-s)}(t-s)^{-(\frac{\theta}{2}+\frac{1}{2})} \left(\|u^3(s)\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} + \|u(s)\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} + \left\|(-A_0)^{-\frac{1}{2}} u(s)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} \right) ds \\
&\le Ce^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-(\frac{\theta}{2}-\frac{1}{4}+\varepsilon)} \|u(\tilde{t})\|_{H^1(\mathbb{B})} \\
&\quad + C\int_{\tilde{t}}^{t} e^{-\delta(t-s)}(t-s)^{-(\frac{\theta}{2}+\frac{1}{2})} \left(\|u(s)\|^3_{H^1(\mathbb{B})} + \|u(s)\|_{H^1(\mathbb{B})} \right) ds \\
&\le Ce^{-\delta(t-t_R)}(t-t_R)^{-(\frac{\theta}{2}-\frac{1}{4}+\varepsilon)} + C\int_0^\infty e^{-\delta s} s^{-\frac{\theta}{2}-\frac{1}{2}} ds,
\end{aligned}$$

where the constants C in the last line depend on κ, since $\|u(t)\|_{H^1(\mathbb{B})} \le \kappa$ for $t \ge t_R$.

Let us define $\kappa_\theta := C + C\int_0^\infty e^{-\delta s} s^{-\theta/2-1/2} ds$ and choose $t_{R,\theta} > t_R$ such that

$$e^{-\delta(t_{R,\theta}-t_R)}(t_{R,\theta}-t_R)^{-(\frac{\theta}{2}-\frac{1}{4}+\varepsilon)} < 1.$$

Then $\|(-A_0)^{\theta/2} u(t)\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} \le \kappa_\theta$, $\forall t > t_{R,\theta}$.

Step 2. Choose in (4.5) $\alpha = \frac{1}{2}$, $\sigma = \frac{1}{4}$ and β, such that $\frac{1}{2} < \beta < 1 + \frac{\gamma}{4} - \frac{n+1}{8}$. This is possible as $n \in \{1, 2\}$ and $\frac{n-3}{2} < \gamma \le 0$. Hence $\frac{n+1}{2} - \gamma < 2$, which implies that $\frac{\gamma}{4} - \frac{n+1}{8} > -\frac{1}{2}$. With this choice of β, we also have $\frac{1}{2} > 1 - \beta > \frac{n+1}{8} - \frac{\gamma}{4}$. Therefore, with a suitable choice of β, we have, according to Proposition 19 (i), that $X^0_{2(1-\beta)} = \mathcal{H}^{4(1-\beta),4(1-\beta)+\gamma}(\mathbb{B}) \oplus \mathbb{C}_\omega$ is an algebra as $4(1-\beta) + \gamma > \frac{n+1}{2}$ and $4(1-\beta) > \frac{n+1}{2}$, as $\gamma \le 0$.

Choosing $\tilde{t} = \max\{t_{R,2(1-\beta)}, t_{R,1/2}\}$ and using (4.5) with $t > \tilde{t}$, $\alpha = 1/2$, $\sigma = 1/4$ and β as above, we have

$$\left\|(-A_0)^{\frac{1}{2}} u(t)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} \le Ce^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\frac{1}{4}} \left\|(-A_0)^{\frac{1}{4}} u(\tilde{t})\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})} + C\int_{\tilde{t}}^{t} e^{-\delta(t-s)}(t-s)^{-\beta}$$

$$\times\left(\left\|(-A_0)^{1-\beta}u^3(s)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})}+\left\|(-A_0)^{1-\beta}u(s)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})}+\left\|(-A_0)^{\frac{1}{2}-\beta}u(s)\right\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})}\right)ds$$

$$\leq Ce^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\frac{1}{4}}\left\|u(\tilde{t})\right\|_{X^0_{1/2}}+C\int_{\tilde{t}}^{t}e^{-\delta(t-s)}(t-s)^{-\beta}$$

$$\times(\|u(s)\|^3_{X^0_{2(1-\beta)}}+\|u(s)\|_{X^0_{2(1-\beta)}})ds$$

$$\leq Ce^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\frac{1}{4}}\kappa_{1/2}+C\int_0^{\infty}e^{-\delta s}s^{-\beta}ds\left(\kappa^3_{2(1-\beta)}+\kappa_{2(1-\beta)}\right).$$

Let us define

$$\kappa_1:=C\kappa_{1/2}+C\left(\int_0^{\infty}e^{-\delta s}s^{-\beta}ds\right)\left(\kappa^3_{2(1-\beta)}+\kappa_{2(1-\beta)}\right)$$

and we choose $t_{R,1}$ such that $t_{R,1}>\tilde{t}$ and $C\kappa_{1/2}e^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\frac{1}{4}}<1$, for $t>t_{R,1}$. Therefore, we conclude that for $t>t_{R,1}$, we have $\|(-A_0)^{\frac{1}{2}}u(t)\|_{\mathcal{H}^{0,\gamma}(\mathbb{B})}\leq\kappa_1$. □

Proposition 27. *For every $s\geq 0$ and $\theta\in[1/2,1)$ there is a constant $\kappa_{s,\theta}>0$ with the following property: for every $R>0$ there is a constant $t_{R,s,\theta}>0$ such that, if $u_0\in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})}\leq R$, then $\|u(t,u_0)\|_{\mathcal{D}((-A_s)^{\theta})}\leq\kappa_{s,\theta}$, for all $t>t_{R,s,\theta}$.*

Proof. The result is a direct consequence of the following two claims.
First claim: Suppose that for some $s\geq 0$ there is a constant $\kappa_s>0$ with the following property: for every $R>0$ there exists $\tilde{t}_{R,s}>0$ such that, if $u_0\in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})}\leq R$, then $\|u(t,u_0)\|_{\mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega}\leq\kappa_s$, for all $t>\tilde{t}_{R,s}$. If such a constant $\kappa_s>0$ exists, then for each $\theta\in[1/2,1)$ there is also a constant $\kappa_{s,\theta}>0$ with the following property: for every $R>0$ there exists $t_{R,s,\theta}>0$ such that, if $u_0\in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})}\leq R$, then $\|u(t,u_0)\|_{\mathcal{D}((-A_s)^{\theta})}\leq\kappa_{s,\theta}$, for all $t>t_{R,s,\theta}$.
Proof of the first claim: We use (4.5) with $\alpha=\beta=\sigma=\theta\in[1/2,1)$, $\tilde{t}=\tilde{t}_{R,s}$ and $t>\tilde{t}$, to obtain

$$\left\|(-A_s)^{\theta}u(t)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\leq C_\sigma e^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\theta}\left\|u(\tilde{t})\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}+C_{\alpha,\beta}\int_{\tilde{t}}^{t}e^{-\delta(t-s)}(t-s)^{-\theta}$$

$$\times\left(\left\|(-A_s)^{\frac{1}{2}}u^3(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}+\left\|(-A_s)^{\frac{1}{2}}u(s)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}+\|u(s)\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\right)ds$$

$$\leq C_\sigma e^{-\delta(t-\tilde{t})}(t-\tilde{t})^{-\theta}\left\|u(\tilde{t})\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}+C_{\alpha,\beta}\int_{\tilde{t}}^{t}e^{-\delta(t-s)}(t-s)^{-\theta}$$

$$\times\left(\|u(s)\|^3_{\mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega}+\|u(s)\|_{\mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega}+\|u(s)\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\right)ds$$

$$\leq C_\sigma(t-\tilde{t}_{R,s})^{-\theta}e^{-\delta(t-\tilde{t}_{R,s})}\kappa_s+C_{\alpha,\beta}\left(\kappa_s^3+2\kappa_s\right)\int_0^{\infty}s^{-\theta}e^{-\delta s}ds.$$

Let us choose

$$\kappa_{s,\theta}:=C_\sigma\kappa_s+C_{\alpha,\beta}\left(\kappa_s^3+2\kappa_s\right)\int_0^{\infty}s^{-\theta}e^{-\delta s}ds$$

and $t_{R,s,\theta} > \tilde{t}_{R,s}$ such that $(t-\tilde{t}_{R,s})^{-\theta}e^{-\delta(t-\tilde{t}_{R,s})} \le 1$ when $t \ge t_{R,s,\theta}$. Then $\|u(t)\|_{\mathcal{D}((-A_s)^\theta)} \le \kappa_{s,\theta}$ for all $t > t_{R,s,\theta}$.

Second claim: For every $s \ge 0$ there is a constant $\kappa_s > 0$ with the following property: for every $R > 0$, there exists $\tilde{t}_{R,s} > 0$ such that, if $u_0 \in X^0_{1,0}$ and $\|u_0\|_{H_0^{-1}(\mathbb{B})} \le R$, then $\|u(t,u_0)\|_{\mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega} \le \kappa_s$, for all $t > \tilde{t}_{R,s}$.

Proof of the second claim: We have seen that this is true for $s = 0$ by Proposition 26. We now proceed by induction as follows: we prove that, if the property holds for some $s_0 \ge 0$, then it also holds for all $s \in [s_0, s_0+1]$. Indeed, let us suppose that it holds for some $s_0 \ge 0$. Taking $\theta > 3/4$, $\sigma \in [0,1]$, $s = s_0+\sigma$ and a suitable small $\varepsilon > 0$, Proposition 19 (iii) with $\alpha = 3/4+\varepsilon$ implies

$$\begin{aligned}
&\|u(t,u_0)\|_{\mathcal{H}^{s_0+\sigma,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega} \le \|u(t,u_0)\|_{\mathcal{H}^{s_0+3,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega} \le \|u(t,u_0)\|_{\mathcal{H}^{s_0+3,\gamma+3}(\mathbb{B})\oplus\mathbb{C}_\omega\oplus\underline{\mathcal{E}}_{\Delta^2,\gamma-1+4\varepsilon}}\\
&\le \|u(t,u_0)\|_{[\mathcal{H}^{s_0,\gamma}(\mathbb{B}),X_2^{s_0}]_{3/4+\varepsilon}} = \|u(t,u_0)\|_{\mathcal{D}((-A_{s_0})^{3/4+\varepsilon})} \le \|u(t,u_0)\|_{\mathcal{D}((-A_{s_0})^\theta)}\,.
\end{aligned}$$

By the induction hypothesis and the first claim, the last term is smaller or equal to $\kappa_{s_0,\theta}$ for all $t > t_{R,s_0,\theta}$. Hence the result follows for $\tilde{t}_{R,s_0+\sigma} := t_{R,s_0,\theta}$, for all $\sigma \in (0,1]$. □

Proof. (of Theorem 23) First we note that, choosing $\varepsilon > 0$ properly, we have

$$\begin{aligned}
\mathcal{D}((-A_s)^{2\varepsilon}) &= [\mathcal{H}^{s,\gamma}(\mathbb{B}),\mathcal{D}(-A_s)]_{2\varepsilon} = [\mathcal{H}^{s,\gamma}(\mathbb{B}),\mathcal{D}((-A_s)^{1/2})]_{4\varepsilon}\\
&= [\mathcal{H}^{s,\gamma}(\mathbb{B}),\mathcal{H}^{s+2,\gamma+2}(\mathbb{B})\oplus\mathbb{C}_\omega]_{4\varepsilon} \overset{(1)}{=} \mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B})\oplus\mathbb{C}_\omega = \mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B}),
\end{aligned} \tag{4.7}$$

where we have used Proposition 19 (i) in (1) and that $\gamma+8\varepsilon < (n+1)/2$ in the last equality. Moreover for suitable $0 < \tilde{\varepsilon} < 1/2-2\varepsilon$ we have

$$\begin{aligned}
\mathcal{D}((-A_s)^{1/2+2\varepsilon+\tilde{\varepsilon}}) &= [\mathcal{H}^{s,\gamma}(\mathbb{B}),\mathcal{D}(A_s)]_{1/2+2\varepsilon+\tilde{\varepsilon}}\\
&\hookrightarrow \mathcal{H}^{s+2+8\varepsilon,\gamma+2+8\varepsilon}(\mathbb{B})\oplus\mathbb{C}_\omega\oplus\underline{\mathcal{E}}_{\Delta^2,\gamma-2+8\varepsilon+4\tilde{\varepsilon}},
\end{aligned} \tag{4.8}$$

where we have used Proposition 19 (iii).

Let $t_{R,s,\theta} > 0$, $\theta = 1/2+2\varepsilon+\tilde{\varepsilon}$ be as in Proposition 27, and $u_0 \in X^0_{1,0}$. Then, applying formally $(-A_s)^{1+\varepsilon}$ to the variation of constants formula give us

$$(-A_s)^{1+\varepsilon}u(t) = (-A_s)^{1+\varepsilon}e^{(t-t_{R,s,\theta})A_s}u(t_{R,s,\theta}) + \int_{t_{R,s,\theta}}^{t} (-A_s)^{1-\varepsilon}e^{(t-s)A_s}(-A_s)^{2\varepsilon}F(u(s))ds. \tag{4.9}$$

Notice however that we do not know that $u(t) \in \mathcal{D}((-A_s)^{1+\varepsilon})$ a priori. This will follow by showing that the $\mathcal{H}^{s,\gamma}(\mathbb{B})$ norm of the integrand of (4.9) is integrable, see [2, Proposition 1.1.7], which is a consequence of the following computations, similar to Lemma 25.

$$\begin{aligned}
\|u(t)\|_{\mathcal{D}((-A_s)^{1+\varepsilon})} &= \left\|(-A_s)^{1+\varepsilon}u(t)\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\\
&\le C\left\|(-A_s)^{1+\varepsilon}e^{(t-t_{R,s,\theta})A_s}\right\|_{\mathcal{B}(\mathcal{H}^{s,\gamma}(\mathbb{B}))}\|u(t_{R,s,\theta})\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}\\
&\quad +C\int_{t_{R,s,\theta}}^{t}\left\|(-A_s)^{1-\varepsilon}e^{(t-s)A_s}\right\|_{\mathcal{B}(\mathcal{H}^{s,\gamma}(\mathbb{B}))}\left\|(-A_s)^{2\varepsilon}F(u(s))\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})}ds.
\end{aligned} \tag{4.10}$$

Notice that

$$
\begin{aligned}
\left\|(-A_s)^{2\varepsilon}F(u(s))\right\|_{\mathcal{H}^{s,\gamma}(\mathbb{B})} &\overset{(1)}{=} \left\|\Delta_s(u^3(s)-3u(s))+u(s)\right\|_{\mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B})} \\
&\overset{(2)}{\leq} C\left\|u^3(s)-3u(s)\right\|_{\mathcal{H}^{s+8\varepsilon+2,\gamma+8\varepsilon+2}(\mathbb{B})\oplus\mathbb{C}_\omega} + C\left\|u(s)\right\|_{\mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B})} \\
&\leq C(\|u(s)\|^3_{\mathcal{H}^{s+8\varepsilon+2,\gamma+8\varepsilon+2}(\mathbb{B})\oplus\mathbb{C}_\omega\oplus\underline{\mathcal{E}}_{\Delta^2,\gamma-2+8\varepsilon+4\tilde{\varepsilon}}} + \|u(s)\|_{\mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B})}) \\
&\overset{(3)}{\leq} C(\|u(s)\|^3_{\mathcal{D}((-A_s)^{1/2+2\varepsilon+\tilde{\varepsilon}})} + \|u(s)\|_{\mathcal{D}((-A_s)^{1/2+2\varepsilon+\tilde{\varepsilon}})}) \\
&\overset{(4)}{\leq} C(\kappa^3_{s,1/2+2\varepsilon+\tilde{\varepsilon}} + \kappa_{s,1/2+2\varepsilon+\tilde{\varepsilon}}),
\end{aligned} \tag{4.11}
$$

where we have used (4.7) in (1), the continuity of $\Delta : \mathcal{H}^{s+8\varepsilon+2,\gamma+8\varepsilon+2}(\mathbb{B}) \oplus \mathbb{C}_\omega \to \mathcal{H}^{s+8\varepsilon,\gamma+8\varepsilon}(\mathbb{B})$ in (2), (4.8) in (3) and Proposition 27 in (4). By (4.10) and (4.11), we find

$$
\|u(t)\|_{\mathcal{D}((-A_s)^{1+\varepsilon})} \leq C(t-t_{R,s,\theta})^{-(1+\varepsilon)}e^{-\delta(t-t_{R,s,\theta})}\kappa_{s,\theta} + C\left(\kappa^3_{s,\theta}+\kappa_{s,\theta}\right)\int_0^\infty s^{-(1-\varepsilon)}e^{-\delta s}ds.
$$

Let us choose

$$
\varkappa_{s,\varepsilon} := C\kappa_{s,\theta} + C\left(\kappa^3_{s,\theta}+\kappa_{s,\theta}\right)\int_0^\infty s^{-(1-\varepsilon)}e^{-\delta s}ds
$$

and $\bar{t}_{R,s,\varepsilon} > t_{R,s,\theta}$ such that $(t-t_{R,s,\theta})^{-(1+\varepsilon)}e^{-\delta(t-t_{R,s,\theta})} \leq 1$ for $t \geq \bar{t}_{R,s,\varepsilon}$. Then $\|u(t)\|_{\mathcal{D}((-A_s)^{1+\varepsilon})} \leq \varkappa_{s,\varepsilon}$ for all $t > \bar{t}_{R,s,\varepsilon}$. □

We are now finally in position to prove part (i) of Theorem 1.

Proof. (of part (i) of Theorem 1). We check the conditions of Corollary 22 for (4.3). Here we use $\alpha = 1/2$, so that $X_\alpha = X_1^s$, and $\widetilde{X}_\alpha = X_{1,0}^s$. For any $r \geq s$, we choose $W = \mathcal{D}(\Delta_r^2)$ and $Y = \mathcal{D}((-A_r)^{1+\varepsilon})$, where ε is as in Theorem 23. Condition (i) follows from (1.3)-(1.4) and Proposition 20. For condition (ii), we first note that $\mathcal{D}((-A_r)^{1+\varepsilon}) \overset{c}{\hookrightarrow} \mathcal{D}(-A_r) = \mathcal{D}(\Delta_r^2) \hookrightarrow \mathcal{H}^{s+2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega$. Moreover, if $u_0 \in X_{1,0}^s$ and $\|u_0\|_{X_1^s} \leq R$, then, as $X_1^s \hookrightarrow H^{-1}(\mathbb{B})$ and $\int_{\mathbb{B}} u_0 d\mu_g = 0$, we conclude that $\|u_0\|_{H_0^{-1}(\mathbb{B})} \leq \widetilde{R}$. Theorem 23 gives the necessary estimate of the second condition.

Corollary 22 implies the existence of a connected global attractor $\mathcal{A}^s$ for the semiflow $T : [0,\infty) \times X_{1,0}^s \to X_{1,0}^s$. By uniqueness of the global attractor, $\mathcal{A}^s$ does not depend on r. Hence $\mathcal{A}^s \subset \mathcal{D}(\Delta_r^2)$ for all $r > 0$ and it attracts bounded sets of $X_{1,0}^s$ in $\mathcal{D}(\Delta_r^2)$.

For the s-independence, let $s_1 > s_2 \geq 0$. As $X_1^{s_1} \hookrightarrow X_1^{s_2}$ is continuous and $\mathcal{A}^{s_1}$ is compact in $X_{1,0}^{s_1}$, we conclude that $\mathcal{A}^{s_1}$ is also compact in $X_{1,0}^{s_2}$. Consider now a bounded set $B \subset X_{1,0}^{s_2}$. Due to Theorem 23, there exists $\tilde{t} > 0$ such that the set $T(\tilde{t})B$ is a bounded set of $X_{1,0}^{s_1}$. Therefore, for $t \geq \tilde{t}$, we have

$$
\begin{aligned}
\sup_{b\in B}\inf_{a\in\mathcal{A}^{s_1}} \|T(t)b-a\|_{X_1^{s_2}} &\leq \sup_{b\in B}\inf_{a\in\mathcal{A}^{s_1}} \|T(t)b-a\|_{X_1^{s_1}} \\
&= \sup_{b\in B}\inf_{a\in\mathcal{A}^{s_1}} \left\|T(t-\tilde{t})T(\tilde{t})b-a\right\|_{X_1^{s_1}} \overset{t\to\infty}{\longrightarrow} 0.
\end{aligned}
$$

Finally, as $T(t)\mathcal{A}^{s_1} = \mathcal{A}^{s_1}$, we conclude that $\mathcal{A}^{s_1}$ is a global attractor for the semiflow T in $X_{1,0}^{s_2}$. By uniqueness of global attractors $\mathcal{A}^{s_1} = \mathcal{A}^{s_2}$. □

5. Convergence to the equilibrium

In this section, we prove part (ii) of Theorem 1. We first state an abstract result from [9]. Let V and H be real Hilbert spaces such that V is densely and continuously embedded to H. We recall that an element $x \in H$ defines a continuous linear functional in V by $y \in V \mapsto (y, x)_H \in \mathbb{R}$. Under this, we have $V \overset{i_{V,H}}{\hookrightarrow} H \overset{i_{H,V^*}}{\hookrightarrow} V^*$, where $i_{V,H}$ and i_{H,V^*} are continuous embeddings with dense image.

Theorem 28. *[9, Section 11.2] Let $E : V \to \mathbb{R}$ be a real analytic function such that $E(0) = 0 \in \mathbb{R}$, $DE(0) = 0 \in V^*$ and $A := D^2E(0) : V \to V^*$ is a Fredholm operator, where $DE : V \to V^*$ and $D^2E : V \to \mathcal{B}(V, V^*)$ are the first and second Fréchet derivatives. Then there exist $\theta \in (0, 1/2]$, $\sigma > 0$ and $c > 0$ such that*

$$|E(v)|^{1-\theta} \leq c \, \|DE(v)\|_{V^*} \quad \textit{for all } v \in V \textit{ satisfying } \|v\|_V < \sigma.$$

The above inequality is called the Lojasiewicz-Simon inequality at 0. In our application, the function E of Theorem 28 will be related to the Lyapunov (energy) functional defined for the Cahn-Hilliard equation. In this section, we always assume that $\dim(\mathbb{B}) \in \{2, 3\}$ and work with the subspaces of real functions. In order to apply the Theorem 28, we need the following technical lemma.

Lemma 29. *If $u \in H^1(\mathbb{B})$, then the linear operator $T_u : H^1(\mathbb{B}) \to H^{-1}(\mathbb{B})$ defined by*

$$\langle T_u(v), h \rangle_{H^{-1}(\mathbb{B}) \times H^1(\mathbb{B})} = \int_{\mathbb{B}} u^2 v h \, d\mu_g$$

is continuous and compact.

Proof. Let $\beta > 0$ and $\widetilde{T}_u : H^1(\mathbb{B}) \to \mathcal{H}^{0,-\beta}(\mathbb{B})$ be defined by $\widetilde{T}_u(v) = u^2 v$. This function is continuous. In fact,

$$\int_{\mathbb{B}} \mathsf{x}^{2\beta} \left|u^2 v\right|^2 d\mu_g = \int_{\mathbb{B}} \mathsf{x}^{\beta} u^4 \mathsf{x}^{\beta} v^2 d\mu_g \overset{(1)}{\leq} \|\mathsf{x}^{\frac{\beta}{4}} u\|^4_{L^6(\mathbb{B})} \|\mathsf{x}^{\frac{\beta}{2}} v\|^2_{L^6(\mathbb{B})} \overset{(2)}{\leq} C \, \|u\|^4_{H^1(\mathbb{B})} \, \|v\|^2_{H^1(\mathbb{B})} \, .$$

In (1) we have used Hölder inequality and in (2) Corollary 15. Therefore we have $\|\widetilde{T}_u(v)\|_{\mathcal{H}^{0,-\beta}(\mathbb{B})} \leq C \, \|u\|^2_{H^1(\mathbb{B})} \, \|v\|_{H^1(\mathbb{B})}$.

Let us fix $0 < \beta < \alpha < 1$. We observe that $H^1(\mathbb{B}) \hookrightarrow \mathcal{H}^{1,\alpha}(\mathbb{B}) \overset{c}{\hookrightarrow} \mathcal{H}^{0,\beta}(\mathbb{B}) \hookrightarrow \mathcal{H}^{0,0}(\mathbb{B})$. Therefore $\mathcal{H}^{0,\beta}(\mathbb{B})^* \overset{c}{\hookrightarrow} H^{-1}(\mathbb{B})$ is also compact. Denote by $i_{\mathcal{A},\mathcal{B}} : \mathcal{A} \to \mathcal{B}$ the inclusion map $\mathcal{A} \hookrightarrow \mathcal{B}$ and by $I : \mathcal{H}^{0,-\beta}(\mathbb{B}) \to \mathcal{H}^{0,\beta}(\mathbb{B})^*$ the usual identification induced by the inner product in $\mathcal{H}^{0,0}(\mathbb{B})$. The following map

$$i_{\mathcal{H}^{0,\beta}(\mathbb{B})^*, H^1(\mathbb{B})^*} \circ I \circ \widetilde{T}_u : H^1(\mathbb{B}) \to \mathcal{H}^{0,-\beta}(\mathbb{B}) \to \mathcal{H}^{0,\beta}(\mathbb{B})^* \to H^{-1}(\mathbb{B})$$

is continuous and compact, as $i_{\mathcal{H}^{0,\beta}(\mathbb{B})^*, H^{-1}(\mathbb{B})}$ is a compact operator. The result follows now by the equality $T_u = i_{\mathcal{H}^{0,\beta}(\mathbb{B})^*, H^{-1}(\mathbb{B})} \circ I \circ \widetilde{T}_u$. $\square$

As $H^1_0(\mathbb{B}) \hookrightarrow L^4(\mathbb{B})$, we can define the Lyapunov function $\mathcal{L} : H^1_0(\mathbb{B}) \to \mathbb{R}$ by (4.2). It is a real analytic function, as it is the composition of linear and multilinear functions. The following theorem is our main result of this section. Given $u_0 \in X^0_{1,0}$, we denote by $\omega(u_0)$ the ω-limit set $\omega_{X^0_{1,0}}(\{u_0\})$.

Theorem 30. *Let $u_0 \in X^0_{1,0}$. If $\varphi \in \omega(u_0)$, then there exist constants $c, \sigma > 0$ and $\theta \in (0, 1/2]$ such that the following inequality holds:*

$$|\mathcal{L}(v) - \mathcal{L}(\varphi)|^{1-\theta} \leq c \, \|D\mathcal{L}(v)\|_{H_0^{-1}(\mathbb{B})},$$

whenever $\|v - \varphi\|_{H_0^1(\mathbb{B})} < \sigma$.

Proof. The argument is standard and can be found e.g. in [3] and [19]. We only stress here the necessary changes for the conical singularities situation.

We check the assumptions of Theorem 28. For this, we choose $V = H_0^1(\mathbb{B})$, $V^* = H_0^{-1}(\mathbb{B})$ and $H = \{u \in \mathcal{H}^{0,0}(\mathbb{B}); \int_{\mathbb{B}} u d\mu_g = 0\}$, and define the function $E : H_0^1(\mathbb{B}) \to \mathbb{R}$ by $E(v) = \mathcal{L}(v + \varphi) - \mathcal{L}(\varphi)$. It is clear that $E(0) = 0$. For the derivatives, for $v, h \in H_0^1(\mathbb{B})$ we have that

$$D\mathcal{L}(v) = -\Delta v + v^3 - v - \fint_{\mathbb{B}} v^3 d\mu_g \in H_0^{-1}(\mathbb{B}),$$

$$D^2\mathcal{L}(v)h = -\Delta h + (3v^2 - 1)h - 3\fint_{\mathbb{B}} v^2 h d\mu_g \in H_0^{-1}(\mathbb{B}).$$

The proof of the above expressions uses Theorem 11, the Mellin-Sobolev embeddings from Corollary 15 and the identification of Proposition 10.

In order to prove that $DE(0) = 0 \in V^*$, we note that $DE(v) = D\mathcal{L}(v + \varphi)$. Therefore

$$DE(0) = -\Delta\varphi + \varphi^3 - \varphi - (\varphi^3)_{\mathbb{B}}.$$

Since $\varphi \in \omega(u)$, we know that $\varphi \in H_0^1(\mathbb{B})$ is an equilibrium point [9, Theorem 8.4.6]. Hence $-\Delta\varphi + \varphi^3 - \varphi$ is constant. In fact, as $\varphi \in \mathcal{D}(\Delta_0^2)$ and $\frac{\partial \varphi}{\partial t} = 0$, Theorem 11 with $u = v = \Delta\varphi - \varphi^3 + \varphi$ and (1.1) shows that $\nabla(\Delta\varphi - \varphi^3 + \varphi) = 0$. This constant must be equal to $(\varphi^3)_{\mathbb{B}}$ by Theorem 11 and $\int_{\mathbb{B}} \varphi d\mu_g = 0$, which implies that $DE(0) = 0$.

For showing Fredholm property of $D^2E(0)$, we first note that

$$D^2E(0)h = -\Delta h + (3\varphi^2 - 1)h - 3\fint_{\mathbb{B}} \varphi^2 h d\mu_g \in H_0^{-1}(\mathbb{B}).$$

The inclusion $H^1(\mathbb{B}) \hookrightarrow H^{-1}(\mathbb{B})$ is compact and by Lemma 29 the map $h \in H^1(\mathbb{B}) \mapsto v^2 h \in H^{-1}(\mathbb{B})$ is also compact. In addition, the map $h \in H_0^1(\mathbb{B}) \mapsto -3\fint_{\mathbb{B}} v^2 h d\mu_g$ has finite rank and, therefore, it is also a compact operator from $H_0^1(\mathbb{B})$ to $H_0^{-1}(\mathbb{B})$. We conclude that $D^2E(0) : H_0^1(\mathbb{B}) \to H_0^{-1}(\mathbb{B})$ is a compact perturbation of the isomorphism $\Delta : H_0^1(\mathbb{B}) \to H_0^{-1}(\mathbb{B})$. □

As $\mathcal{L}$ is a Lyapunov function bounded from below and as $\omega(u_0)$ is compact in $\mathcal{H}^{2,\gamma+2}(\mathbb{B}) \oplus \mathbb{C}_\omega \hookrightarrow H^1(\mathbb{B})$, we conclude the following:

Corollary 31. *Let $u_0 \in X^0_{1,0}$. Then*
(i) *There is a constant $\mathcal{L}_\infty \in \mathbb{R}$ such that $\mathcal{L}(\varphi) = \mathcal{L}_\infty$, for all $\varphi \in \omega(u_0)$.*
(ii) *There is a neighborhood $U \subset H_0^1(\mathbb{B})$ of $\omega(u_0)$ and constants $C > 0$, $\theta \in (0, 1/2]$ such that*

$$|\mathcal{L}(v) - \mathcal{L}_\infty|^{1-\theta} \leq C \, \|D\mathcal{L}(v)\|_{H_0^{-1}(\mathbb{B})},$$

for all $v \in U$.

Having the Lojasiewicz-Simon inequality, we can prove the convergence theorem below.

Proposition 32. *Let* $u_0 \in X^0_{1,0}$ *and* u *be the solution of* (1.1). *Then there exists a* $u_\infty \in \omega(u_0)$ *such that* $\lim_{t\to\infty} \|u(t) - u_\infty\|_{H_0^{-1}(\mathbb{B})} = 0$.

Proof. The proof follows from the arguments in [3, Section 3]. We sketch here the steps for the convenience of the reader.

Let $\mathcal{L}_\infty = \lim_{t\to\infty} \mathcal{L}(u(t))$. Then $\mathcal{L}_\infty \leq \mathcal{L}(u(t))$ for all $t \in [0, \infty)$. We define the function $H : [0, \infty) \to \mathbb{R}$ by

$$H(t) = (\mathcal{L}(u(t)) - \mathcal{L}_\infty)^\theta,$$

where $\theta \in (0, 1/2]$ as in Corollary 31. The function H is non-negative and non-increasing, as

$$\frac{d}{dt}H(t) = \theta(\mathcal{L}(u(t)) - \mathcal{L}_\infty)^{\theta-1}\frac{d}{dt}\mathcal{L}(u(t)) \leq 0.$$

Moreover $\lim_{t\to\infty} H(t) = 0$. Let $U \subset H^1_0(\mathbb{B})$ be the open set of Corollary 31. Due to [9, Theorem 5.1.8], we have

$$\lim_{t\to\infty} \big(\inf_{v\in\omega(u_0)} \|T(t)u - v\|_{\mathcal{H}^{2,\gamma+2}(\mathbb{B})\oplus\mathbb{R}_\omega} \big) = 0. \tag{5.1}$$

Thus, there exists $t_0 > 0$ such that, for $t \geq t_0$, we have $u(t) \in U$. Hence, for $t \geq t_0$, we estimate

$$\begin{aligned}
-\frac{d}{dt}H(t) &= \theta(\mathcal{L}(u(t)) - \mathcal{L}_\infty)^{\theta-1}\left(-\frac{d}{dt}\mathcal{L}(u(t))\right) \\
&\overset{(1)}{\geq} C\frac{\int_{\mathbb{B}} \left\langle \nabla(-\Delta u + u^3 - u), \nabla(-\Delta u + u^3 - u)\right\rangle_g d\mu_g}{\|-\Delta u + u^3 - u - (u^3)_{\mathbb{B}}\|_{H_0^{-1}(\mathbb{B})}} \\
&\overset{(2)}{\geq} C\frac{\int_{\mathbb{B}} \left\langle \nabla(-\Delta u + u^3 - u), \nabla(-\Delta u + u^3 - u)\right\rangle d\mu_g}{\|\Delta(\Delta u - u^3 + u)\|_{H_0^{-1}(\mathbb{B})}} \\
&\overset{(3)}{\geq} C\frac{\left\|\Delta(-\Delta u + u^3 - u)\right\|^2_{H_0^{-1}(\mathbb{B})}}{\|\Delta(\Delta u - u^3 + u)\|_{H_0^{-1}(\mathbb{B})}} = C\left\|\Delta(-\Delta u + u^3 - u)\right\|_{H_0^{-1}(\mathbb{B})}.
\end{aligned}$$

In (1) we have used Theorem 11 and Corollary 31, in (2) Corollary 13 and in (3) the definition of $\|\cdot\|_{H_0^{-1}(\mathbb{B})}$ and the isomorphism $\Delta : H^1_0(\mathbb{B}) \to H_0^{-1}(\mathbb{B})$. By (1.1), we infer that

$$\left\|\frac{\partial u}{\partial t}\right\|_{H_0^{-1}(\mathbb{B})} = \left\|\Delta(-\Delta u + u^3 - u)\right\|_{H_0^{-1}(\mathbb{B})} \leq -C\frac{d}{dt}H(t).$$

Hence $\frac{\partial u}{\partial t} \in L^1(0, \infty; H_0^{-1}(\mathbb{B}))$ and $u_\infty := \lim_{t\to\infty} u(t) = u(0) + \int_0^\infty \frac{\partial u}{\partial t}(s)ds$, where the limit is taken in $H_0^{-1}(\mathbb{B})$.

It remains to prove that $u_\infty \in \omega(u_0)$. We know that $\omega(u_0) \subset \mathcal{H}^{2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega$ is compact. As $\mathcal{H}^{2,\gamma+2}(\mathbb{B}) \oplus \mathbb{R}_\omega \hookrightarrow H^{-1}(\mathbb{B})$, we conclude that $\omega(u_0)$ is also compact in $H_0^{-1}(\mathbb{B})$. Thus it is also closed. Equation (5.1) implies that

$$\lim_{t\to\infty} \big(\inf_{v\in\omega(u_0)} \|T(t)u - v\|_{H^{-1}(\mathbb{B})} \big) = 0$$

and, therefore, that $\inf_{v\in\omega(u_0)}\|u_\infty - v\|_{H^{-1}(\mathbb{B})} = 0$. As $\omega(u_0)$ is closed in $H_0^{-1}(\mathbb{B})$, we conclude that $u_\infty \in \omega(u_0)$. □

Finally, we prove part (ii) of Theorem 1.

Proof. (of part (ii) of Theorem 1) Recall first that $T(t)u_0 \in \cap_{r\geq 0}\mathcal{D}(\Delta_r^2)$ due to (1.3). Therefore the limit in $\mathcal{D}(\Delta_r^2)$ makes sense. We know by Theorem 23 that $\{T(t)u_0\}_{t\geq T}$ is precompact in $\mathcal{D}(\Delta_r^2)$ for some $T > 0$. Let $u_\infty \in \omega(u_0)$ be such that $\lim_{t\to\infty} T(t)u_0 = u_\infty$ in $H_0^{-1}(\mathbb{B})$. Note that since u_∞ is an equilibrium point [9, Theorem 8.4.6], $T(t)u_\infty = u_\infty$ for all $t > 0$ and, hence, $u_\infty \in \cap_{r\geq 0}\mathcal{D}(\Delta_r^2)$. Suppose that this limit does not hold in $\mathcal{D}(\Delta_r^2)$. Then, there exist an $\varepsilon_0 > 0$ and a sequence $t_k \to \infty$ such that $\|T(t_k)u_0 - u_\infty\|_{\mathcal{D}(\Delta_r^2)} \geq \varepsilon_0$. By compactness, there exists a subsequence $t_{k_j} \to \infty$ such that $T(t_{k_j})u_0$ converges to a function w in $\mathcal{D}(\Delta_r^2)$. This implies that $T(t_{k_j})u_0$ also converges to w in $H^{-1}(\mathbb{B})$. Therefore $w = u_\infty$ by uniqueness of the limit and we obtain a contradiction. □

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