

# Journal Pre-proof

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Daniel Felipe Castro Ovalle, Ivan Shestakov

PII: S0021-8693(22)00111-9

DOI: <https://doi.org/10.1016/j.jalgebra.2022.03.012>

Reference: YJABR 18502

To appear in: *Journal of Algebra*

Received date: 2 May 2021

Please cite this article as: D.F.C. Ovalle, I. Shestakov, Composition Color Algebras, *J. Algebra* (2022), doi: <https://doi.org/10.1016/j.jalgebra.2022.03.012>.

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## COMPOSITION COLOR ALGEBRAS

DANIEL FELIPE CASTRO OVALLE AND IVAN SHESTAKOV

ABSTRACT. This work aims to define the composition color algebras and classify them. For this, we use the Cayley-Dickson process and the construction of the “canonical” basis of a split Cayley algebra as a reference.

## 1. INTRODUCTION

The role of composition algebras is difficult to overestimate. The quaternion and octonion algebras were some of the first known examples of non-commutative and non-associative algebras, respectively; in fact, the theory of algebras was initiated with these algebras. Composition algebras are a fundamental part of the classification of simple Jordan algebras and exceptional simple Lie algebras (“Freudenthal’s magic square”).

With the development of the theory of superalgebras, it was natural to see if the notion of composition algebra could be generalized to the super-analogue. In [She97], Shestakov classified the simple alternative superalgebras, and he noted that in characteristic 3 appeared two alternative superalgebras,  $B(1, 2)$  and  $B(4, 2)$ . Later, Elduque and Okubo [EO02] defined and classified the composition superalgebras with identity element (Hurwitz superalgebras). They proved that the two algebras mentioned above are Hurwitz superalgebras and that if the characteristic is different from 2, these two superalgebras are the only non-trivial Hurwitz superalgebras (with non-zero odd part).

A generalization of Lie superalgebras was studied by Scheunert in [Sch79], called Lie color algebras, in which case the algebra is graded by an abelian group and not only by  $\mathbb{Z}_2$ . The main objective of this work is to introduce and classify composition color algebras. In addition, Scheunert showed that the study of Lie color algebras is closely related to the study of gradings on Lie superalgebras. Therefore, for our purposes, gradings on composition superalgebras are important. Gradings on composition algebras have been described in [Eld98] and [Eld09]. Gradings on composition algebras are useful for finding gradings on exceptional simple Lie algebras. On the other hand, gradings on composition superalgebras were described in [Ara15].

The Cayley-Dickson process is an important tool for the classification of composition algebras so, we will show that it can be generalized to composition color algebras and it plays

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1991 *Mathematics Subject Classification.* 17A70, 17A75.

*Key words and phrases.* Commutation factor, gradings on algebras, color composition algebra, color alternative algebra, Cayley-Dickson process.

a fundamental role in the classification of them. On the other hand, the construction of the “canonical” basis of a split Cayley-Dickson algebra is important for the study of the gradings on these algebras so, we will also use this construction for the classification of composition color algebras.

The structure of the paper is the following way: Section 2 contains some basic results concerning graded algebraic structures and commutation factors. We introduce composition color algebras and study some properties of these algebras in Section 3, and at the end of that section we define a color alternative algebra and we prove that every composition color algebra is a color alternative algebra. The theme of Section 4 is a generalization of the Cayley-Dickson process. The aim of Section 5 is to give the classification of composition color algebras. In Theorems 5.8, 5.21 and 5.27, we gave a complete classification of color composition algebras. Up to equivalence, we have several types of such algebras. Some of them are the classical composition algebras or the alternative superalgebras  $B(1, 2)$  or  $B(4, 2)$  equipped with various gradings. But there are also some completely new algebras that have not been known before. We have one of dimension 4 and several algebras of dimension 8, that are non-alternative superalgebras.

## 2. GRADED ALGEBRAIC STRUCTURES AND COMMUTATION FACTORS

In this section we present some definitions related to graded algebraic structures, see [EK13, Chapter I] and [Sch79].

Let  $G$  be a group and  $V$  be a vector space over a field  $F$ . Generally, we will use multiplicative notation for  $G$ , but for abelian groups we will change to additive notation. A  $G$ -grading on  $V$  is any decomposition of  $V$  into a direct sum of subspaces of  $V$  indexed by  $G$ ,

$$\Lambda : V = \bigoplus_{g \in G} V_g.$$

Here we allow some of the subspaces  $V_g$  to be zero. The set

$$\text{Supp } \Lambda = \text{Supp}_G(V) := \{g \in G \mid V_g \neq 0\},$$

will be called the *support* of  $\Lambda$ . The support of a  $G$ -grading is not in general a subgroup of  $G$ . The subspace  $V_g$  will be called the *homogeneous component* of degree  $g$ . If  $0 \neq v \in V_g$  then we will say that  $v$  is *homogeneous of degree  $g$* . If a grading  $\Lambda$  is fixed, then  $V$  will be referred to as a  *$G$ -graded vector space*. Any element  $v \in V$  can be uniquely written as  $\sum_{g \in G} v_g$  where  $v_g \in V_g$  and all but finitely many of  $v_g$  are zero. We will refer to  $v_g$  as the *homogeneous components of  $v$* .

A subspace  $U$  of  $V$  is said to be a  *$G$ -graded subspace* if  $U = \bigoplus_{g \in G} (U \cap V_g)$ . Equivalently,  $U$  is  $G$ -graded if for any element  $u \in U$ , all its homogeneous components (as an element of  $V$ ) are also in  $U$ .

Let  $A$  be an algebra (not necessarily associative). A *grading* on  $A$  by  $G$ , or a  *$G$ -grading* on  $A$  is a decomposition of  $A$  into a direct sum of subspaces  $\Lambda : A = \bigoplus_{g \in G} A_g$ , such that  $A_g A_h \subseteq A_{gh}$  for any  $g, h \in G$ . If such a decomposition is fixed, we will refer to  $A$  as a  *$G$ -graded*

*algebra.* A subalgebra  $B$  of  $A$  is said to be a  $G$ -graded subalgebra of  $A$  if it is graded as a subspace of  $A$ , i.e.  $B = \bigoplus_{g \in G} (A_g \cap B)$ . If  $A$  has unity  $1$  then  $1 \in A_0$ . A superalgebra is a  $\mathbb{Z}_2$ -graded algebra.

The following definition is well-known in the theory of graded algebras.

**Definition 2.1.** Let  $\Gamma$  be an abelian group. A *commutation factor* on  $\Gamma$ , with values in  $F^\times$ , is a mapping

$$\epsilon : \Gamma \times \Gamma \rightarrow F^\times$$

such that

- (1)  $\epsilon(\alpha, \beta)\epsilon(\beta, \alpha) = 1$ ,
- (2)  $\epsilon(\alpha, \beta + \gamma) = \epsilon(\alpha, \beta)\epsilon(\alpha, \gamma)$ ,
- (3)  $\epsilon(\alpha + \beta, \gamma) = \epsilon(\alpha, \gamma)\epsilon(\beta, \gamma)$

for all  $\alpha, \beta, \gamma \in \Gamma$ .

From (1), (2) and (3) we have  $\epsilon(\alpha, \alpha) = \pm 1$ ,  $\epsilon(\alpha, 0) = \epsilon(0, \alpha) = 1$  and  $\epsilon(\alpha, -\beta) = \epsilon(\beta, \alpha) = \epsilon(-\alpha, \beta)$  for all  $\alpha, \beta \in \Gamma$ .

Note that a commutation factor is just a skew-symmetric bicharacter.

**Example 2.2.** The following are examples of commutation factors:

- (1) Let  $\Gamma$  be an arbitrary abelian group. The trivial commutation factor is defined by  $\epsilon(\alpha, \beta) = 1$  for all  $\alpha, \beta \in \Gamma$ .
- (2) Let  $\mathbb{Z}_2 = \mathbb{Z}/2\mathbb{Z}$  be the additive group of integers modulo 2 and let  $\epsilon$  defined by  $\epsilon(\alpha, \beta) = (-1)^{\alpha\beta}$  for all  $\alpha, \beta \in \mathbb{Z}_2$ . It is easy to see that  $\epsilon$  is a commutation factor on  $\mathbb{Z}_2$ .

As a next step we would like to classify some commutation factors up to equivalence, in the sense of the following definition.

**Definition 2.3.** Let  $\Gamma$  be an abelian group. Two commutation factors  $\epsilon$  and  $\epsilon'$  on  $\Gamma$  are said to be equivalent if there exists an automorphism  $h$  of  $\Gamma$  such that  $\epsilon'(\alpha, \beta) = \epsilon(h(\alpha), h(\beta))$  for all  $\alpha, \beta \in \Gamma$ .

We will recall some particular cases. First, when  $\Gamma = \mathbb{Z}_p^n$ , with  $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$  and  $p$  is a prime number. If we assume that on  $\Gamma = \mathbb{Z}_p^n$  there is a nontrivial commutation factor then  $\text{char}(F) \neq p$ .

**Proposition 2.4.** ([Sch79]) Let  $p$  be a prime number,  $E_p \in F$  be a nontrivial  $p$ -th root of unity and  $n \geq 1$  be an integer. For any element  $\alpha \in \mathbb{Z}_p^n$  the coordinates of  $\alpha$  with respect to the canonical basis of  $\mathbb{Z}_p^n$  will be denoted by  $\alpha_1, \alpha_2, \dots, \alpha_n$ .

Let  $r$  be an integer such that  $0 \leq 2r \leq n$ . We define an alternating bilinear form  $\psi_r$  on  $\mathbb{Z}_p^n$  by

$$\psi_r(\alpha, \beta) = \sum_{i=1}^r (\alpha_{2i-1}\beta_{2i} - \alpha_{2i}\beta_{2i-1}) \quad \text{for all } \alpha, \beta \in \mathbb{Z}_p^n.$$

Let  $s$  be an integer such that  $1 \leq s \leq n$ . We define a symmetric bilinear form  $\phi_s$  on  $\mathbb{Z}_p^n$  by

$$\phi_s(\alpha, \beta) = \sum_{i=1}^s \alpha_i \beta_i \quad \text{for all } \alpha, \beta \in \mathbb{Z}_p^n.$$

Then,

- (a) If  $p = 2$ , then any commutation factor on  $\mathbb{Z}_2^n$  is equivalent to  $(-1)^\psi$  with  $\psi$  equal to one of the forms  $\psi_r$  or  $\phi_s$ .
- (b) If  $p \geq 3$ , then any commutation factor on  $\mathbb{Z}_p^n$  is equivalent to  $E_p^\psi$  with  $\psi$  equal to one of the forms  $\psi_r$ .

Another particular case, which is important for our results, is when  $\Gamma = \mathbb{Z}_m \oplus \mathbb{Z}_n$ , where  $m, n \in \mathbb{Z}_{\geq 2} \cup \{0\}$ .

**Proposition 2.5.** *Let  $\Gamma = \mathbb{Z}_m \oplus \mathbb{Z}_n$ ,  $m, n \in \mathbb{Z}_{\geq 2} \cup \{0\}$ , and let  $\epsilon$  be a commutation factor on  $\Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  for all  $\alpha \in \Gamma$ . Then  $\epsilon$  is equivalent to one of the following commutation factors:*

- (a) If  $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$ , let  $\lambda \in F^\times$ , then

$$\epsilon((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \lambda^{\alpha_1 \beta_2 - \alpha_2 \beta_1}.$$

- (b) If  $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_n$ ,  $n \geq 2$ . Let  $E_n \in F^\times$  be a  $n$ -th root of unity. Then

$$\epsilon((\alpha_1, \bar{\alpha}_2), (\beta_1, \bar{\beta}_2)) = E_n^{\alpha_1 \bar{\beta}_2 - \bar{\alpha}_2 \beta_1}.$$

- (c) If  $\Gamma = \mathbb{Z}_m \oplus \mathbb{Z}_n$ ,  $m, n \geq 2$ . Let  $d = \gcd(m, n)$ , the greatest common divisor of  $m$  and  $n$ , and  $E_d \in F^\times$  be a  $d$ -th root of unity. Then

$$\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = E_d, \quad \epsilon((\bar{0}, \bar{1}), (\bar{1}, \bar{0})) = E_d^{-1}, \quad \epsilon(\alpha, \alpha) = 1$$

for all  $\alpha \in \mathbb{Z}_m \oplus \mathbb{Z}_n$

*Proof.* On the one hand, we have  $\epsilon(\alpha, \alpha) = 1$  for all  $\alpha \in \Gamma$  and on the other hand by (1),  $\epsilon(\beta, \alpha) = \epsilon(\alpha, \beta)^{-1}$ . Then, we only need to find  $\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1}))$ .

- (a) If  $\Gamma = \mathbb{Z} \oplus \mathbb{Z}$  then  $\epsilon((1, 0), (0, 1)) = \lambda$  for some  $\lambda \in F^\times$ . Therefore

$$\epsilon((\alpha_1, \alpha_2), (\beta_1, \beta_2)) = \lambda^{\alpha_1 \beta_2 - \alpha_2 \beta_1}.$$

- (b) If  $\Gamma = \mathbb{Z} \oplus \mathbb{Z}_n$ ,  $n \geq 2$ . By (2),

$$1 = \epsilon((1, \bar{0}), (0, \bar{n})) = \epsilon((1, \bar{0}), (0, \bar{1}))^n.$$

So,  $\epsilon((1, \bar{0}), (0, \bar{1}))$  must be a  $n$ -th root of unity. Hence

$$\epsilon((\alpha_1, \bar{\alpha}_2), (\beta_1, \bar{\beta}_2)) = E_n^{\alpha_1 \bar{\beta}_2 - \bar{\alpha}_2 \beta_1}$$

where  $E_n \in F^\times$  is a  $n$ -th root of unity.

- (c) If  $\Gamma = \mathbb{Z}_m \oplus \mathbb{Z}_n$ ,  $m, n \geq 2$ . Let  $d = \gcd(m, n)$ , the greatest common divisor of  $m$  and  $n$ . By (2) and (3)

$$1 = \epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{n})) = \epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1}))^n$$

$$1 = \epsilon((\bar{m}, \bar{0}), (\bar{0}, \bar{1})) = \epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1}))^m.$$

Then,  $\epsilon((1, \bar{0}), (0, \bar{1}))$  must be a  $d$ -th root of unity. Hence

$$\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = E_d$$

where  $E_d \in F^\times$  is a  $d$ -th root of unity. □

We recall that a *primitive root of unity* of order  $n$  in a field  $F$  is an element  $E_n \in F$  such that  $E_n^n = 1$  and  $E_n^r \neq 1$  for any positive integer  $r < n$ .

Let us see some considerations on the existence of nontrivial roots of unity.

**Remark 2.6.** Let  $F$  be an algebraically closed field with  $\text{char}(F) = p$ . If there exists a primitive root of unity of order  $n$  in  $F$  then  $n$  is relatively prime to  $p$ .  $F$  contains a primitive root of any order that is relatively prime to  $p$ . In the field of complex numbers, there are primitive roots of unity of every order.

Let  $E_n \in F^\times$  be a  $n$ -th root of unity. If there exists a prime number  $q \neq p$  such that  $q$  divides  $n$  then  $E_n$  can be different from 1.

Let  $d = \text{gcd}(m, n)$  be the greatest common divisor of  $m$  and  $n$ , and  $E_d \in F^\times$  be a  $d$ -th root of unity. If there exists a prime number  $q \neq p$  such that  $q$  divides  $d$  then  $E_d$  can be different from 1.

### 3. COLOR COMPOSITION ALGEBRAS

In this section we will introduce color composition algebras and discuss some of its fundamental properties. The constructions and proofs turn out to be completely analogous to those in the cases of composition algebras and composition superalgebras ([ZSSS82], [EO02]). We begin with the concept of *quadratic  $\epsilon$ -form*.

**Definition 3.1.** Let  $\Gamma$  be an abelian group and  $\epsilon$  be a commutation factor on  $\Gamma$ . A *quadratic  $\epsilon$ -form* on a  $\Gamma$ -graded vector space,  $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$  over a field  $F$ , is a pair  $q = (q_0, f)$  where

- i)  $q_0 : V_0 \rightarrow F$  is a usual quadratic form.
- ii)  $f : V \times V \rightarrow F$  is an  $\epsilon$ -symmetric even bilinear form. This is,  $f|_{(\sum_{i \in \mathbb{Z}} V_{i\alpha}) \times (\sum_{i \in \mathbb{Z}} V_{i\alpha})}$  is symmetric for all  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$ ,  $f|_{(V_\alpha + V_{-\alpha}) \times (V_\alpha + V_{-\alpha})}$  is skew-symmetric for all  $\alpha \in \Gamma$  with  $\epsilon(\alpha, \alpha) = -1$ , and  $f(V_0, V_\alpha) = f(V_\alpha, V_0) = 0$  for all  $0 \neq \alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = -1$ .
- iii)  $f|_{V_0 \times V_0}$  is the polar of  $q_0$ , that is,  $f(a_0, b_0) = q_0(a_0 + b_0) - q_0(a_0) - q_0(b_0)$  for any  $a_0, b_0 \in V_0$ .

The quadratic  $\epsilon$ -form  $q = (q_0, f)$  is called *nondegenerate* if for all  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  the symmetric form  $f|_{(\sum_{i \in \mathbb{Z}} V_{i\alpha}) \times (\sum_{i \in \mathbb{Z}} V_{i\alpha})}$  is nondegenerate (let  $a \in \sum_{i \in \mathbb{Z}} V_{i\alpha}$ , if  $f(a, x) = 0$  for all  $x \in \sum_{i \in \mathbb{Z}} V_{i\alpha}$  then  $a = 0$ ) and for all  $\alpha \in \Gamma$  that satisfies  $\epsilon(\alpha, \alpha) = -1$  the skew-symmetric form  $f|_{(V_\alpha + V_{-\alpha}) \times (V_\alpha + V_{-\alpha})}$  is nondegenerate.

**Definition 3.2.** Let  $\Gamma$  be an abelian group and  $\epsilon$  be a commutation factor on  $\Gamma$ . Let  $A$  be a  $\Gamma$ -graded algebra over a field  $F$ ,  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ , with identity element  $1 \in A_0$ , endowed with a nondegenerate quadratic  $\epsilon$ -form  $q = (q_0, f) : A \rightarrow F$  called the norm.  $A$  is called a *composition  $\epsilon$ -algebra* (also called color composition algebra) if the following identities are satisfied:

$$(4) \quad q_0(a_0 b_0) = q_0(a_0) q_0(b_0),$$

$$(5) \quad f(a_0 b_\beta, a_0 c_\gamma) = q_0(a_0) f(b_\beta, c_\gamma) = f(b_\beta a_0, c_\gamma a_0),$$

$$(6) \quad f(a_\alpha b_\beta, c_\gamma d_\xi) + \epsilon(\gamma, \xi) \epsilon(\beta, \xi) \epsilon(\beta, \gamma) f(a_\alpha d_\xi, c_\gamma b_\beta) = \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(b_\beta, d_\xi),$$

for all  $a_0, b_0 \in A_0$  and homogeneous elements  $a_\alpha \in A_\alpha$ ,  $b_\beta \in A_\beta$ ,  $c_\gamma \in A_\gamma$ ,  $d_\xi \in A_\xi$ .

If  $A$  is a composition  $\epsilon$ -algebra with quadratic  $\epsilon$ -form  $q = (q_0, f)$  then  $A_0$  is a composition algebra with quadratic form  $q_0$ .

**Remark 3.3.** Let  $\epsilon$  be the commutation factor on  $\mathbb{Z}_2$  defined by  $\epsilon(\bar{1}, \bar{1}) = -1$ . If  $\text{char}(F) \neq 2$  then a  $\mathbb{Z}_2$ -graded composition  $\epsilon$ -algebra is a Hurwitz superalgebra, see [EO02, Definition 2.2]. This due to, if  $\text{char}(F) \neq 2$  a bilinear form over  $F$  is alternating if and only if it is skew-symmetric.

**Lemma 3.4.** *Let  $A$  be a composition  $\epsilon$ -algebra with norm  $q = (q_0, f)$ . Then, for any homogeneous elements  $a_\alpha, b_\beta, c_\gamma, d_\delta$ :*

$$(7) \quad f(b_\beta, d_\delta) = \epsilon(\beta, \delta) f(d_\delta, b_\beta),$$

$$(8) \quad (1 + \epsilon(\beta, \beta)) f(a_\alpha b_\beta, c_\gamma b_\beta) = \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(b_\beta, b_\beta),$$

$$(9) \quad f(a_\alpha, c_\gamma^2) + \epsilon(\gamma, \gamma) f(a_\alpha c_\gamma, c_\gamma) - f(a_\alpha, c_\gamma) f(1, c_\gamma) = 0,$$

$$(10) \quad f(a_\alpha, d_\delta) = f(a_\alpha, 1) f(1, d_\delta) - f(a_\alpha d_\delta, 1).$$

*Proof.* Since  $A_0$  is a composition algebra then  $q_0(1) = 1$  and  $f(1, 1) = 2$ .

From (6), with  $a_\alpha = c_\gamma = 1$  ( $\alpha = \gamma = 0$ ) we have

$$f(b_\beta, d_\delta) + \epsilon(0, \delta) \epsilon(\beta, \delta) \epsilon(\beta, 0) f(d_\delta, b_\beta) = \epsilon(\beta, 0) f(1, 1) f(b_\beta, d_\delta).$$

Therefore

$$f(b_\beta, d_\delta) = \epsilon(\beta, \delta) f(d_\delta, b_\beta).$$

By (6), for  $d_\delta = b_\beta$  ( $\delta = \beta$ ) we obtain

$$f(a_\alpha b_\beta, c_\gamma b_\beta) + \epsilon(\gamma, \beta) \epsilon(\beta, \beta) \epsilon(\beta, \gamma) f(a_\alpha b_\beta, c_\gamma b_\beta) = \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(b_\beta, b_\beta).$$

Thus

$$(1 + \epsilon(\beta, \beta)) f(a_\alpha b_\beta, c_\gamma b_\beta) = \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(b_\beta, b_\beta).$$

From (6) with  $b_\beta = 1$  ( $\beta = 0$ ),  $d_\delta = c_\gamma$  ( $\delta = \gamma$ ) we have

$$f(a_\alpha, c_\gamma c_\gamma) + \epsilon(\gamma, \gamma) \epsilon(0, \gamma) \epsilon(0, \gamma) f(a_\alpha c_\gamma, c_\gamma) = \epsilon(0, \gamma) f(a_\alpha, c_\gamma) f(1, c_\gamma).$$

Therefore

$$f(a_\alpha, c_\gamma^2) + \epsilon(\gamma, \gamma) f(a_\alpha c_\gamma, c_\gamma) - f(a_\alpha, c_\gamma) f(1, c_\gamma) = 0.$$

By (6) for  $b_\beta = c_\gamma = 1$  ( $\beta = \gamma = 0$ ) we obtain

$$f(a_\alpha, d_\delta) + \epsilon(0, \delta) \epsilon(0, \delta) \epsilon(0, 0) f(a_\alpha d_\delta, 1) = \epsilon(0, 0) f(a_\alpha, 1) f(1, d_\delta).$$

Thus

$$f(a_\alpha, d_\delta) = f(a_\alpha, 1) f(1, d_\delta) - f(a_\alpha d_\delta, 1).$$

□

Let  $\Gamma$  be an abelian group and  $V = \bigoplus_{\gamma \in \Gamma} V_\gamma$  be a  $\Gamma$ -graded vector space. Henceforth we will say that a bilinear form  $f$  on  $V$  is  $\epsilon$ -symmetric if it satisfies the identity (7).

**Definition 3.5.** Let  $A$  be a  $\Gamma$ -graded algebra and  $\epsilon$  be a commutation factor on  $\Gamma$ . An endomorphism  $\rho$  on the vector space  $A$  is called an  $\epsilon$ -involution of the algebra  $A$  if,  $\rho(\rho(a)) = a$  and  $\rho(a_\alpha b_\beta) = \epsilon(\alpha, \beta)\rho(b_\beta)\rho(a_\alpha)$  for any  $a \in A$ ,  $a_\alpha \in A_\alpha$  and  $b_\beta \in A_\beta$ .

Note that an  $\epsilon$ -involution does not need to be an endomorphism of  $\Gamma$ -graded vector space.

Given a composition  $\epsilon$ -algebra  $A$  with quadratic  $\epsilon$ -form  $q = (q_0, f)$ , we shall prove that the function  $A \rightarrow A : a \mapsto \bar{a} = f(a, 1)1 - a$  is an  $\epsilon$ -involution of the algebra  $A$ . But first we need some results.

**Lemma 3.6.** Let  $A$  be a composition  $\epsilon$ -algebra,  $\Gamma$ -graded, with quadratic  $\epsilon$ -form  $q = (q_0, f)$  and let  $0 \neq \gamma \in \Gamma$ . Then  $f(c_\gamma, 1) = 0$  and  $\bar{c}_\gamma = -c_\gamma$  for all  $c_\gamma \in A_\gamma$ . Therefore, for all  $a_\alpha \in A_\alpha$ ,  $\alpha \in \Gamma$ ,  $\bar{a}_\alpha$  is a homogeneous element of  $A$ ,  $\bar{a}_\alpha \in A_\alpha$ . Furthermore,  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ . Also  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is nondegenerate for all  $\alpha \in \Gamma$ .

*Proof.* Let  $\gamma \neq 0$  and  $c_\gamma \in A_\gamma$ . If  $\epsilon(\gamma, \gamma) = -1$  or  $\text{char}(F) = 2$  ( $\epsilon(\gamma, \gamma) = 1 = -1$ ), by definition  $f(A_0, A_\gamma) = 0$ , thus  $f(c_\gamma, 1) = 0$ . Moreover, since  $q = (q_0, f)$  is nondegenerate, by (8) we have that  $f(c_\gamma, c_\gamma) = 0$ . Then, by (7)

$$\begin{aligned} f(c_\gamma + d_{-\gamma}, c_\gamma + d_{-\gamma}) &= f(c_\gamma, c_\gamma) + f(c_\gamma, d_{-\gamma}) + f(d_{-\gamma}, c_\gamma) + f(d_{-\gamma}, d_{-\gamma}) \\ &= f(c_\gamma, d_{-\gamma}) + \epsilon(-\gamma, \gamma)f(c_\gamma, d_{-\gamma}) = 0, \end{aligned}$$

for every  $c_\gamma \in A_\gamma$  and  $d_{-\gamma} \in A_{-\gamma}$ . Hence  $f|_{(A_\gamma + A_{-\gamma}) \times (A_\gamma + A_{-\gamma})}$  is alternating. On the other hand, if  $\epsilon(\gamma, \gamma) = 1$  and  $\text{char}(F) \neq 2$ , by (9) and (8) we have:

$$\begin{aligned} 0 &= f(a_\alpha, c_\gamma^2) + f(a_\alpha c_\gamma, c_\gamma) - f(a_\alpha, c_\gamma)f(1, c_\gamma) \\ &= f(a_\alpha, c_\gamma^2) + \frac{\epsilon(\gamma, 0)}{2}f(a_\alpha, 1)f(c_\gamma, c_\gamma) - f(a_\alpha, c_\gamma)f(1, c_\gamma) \\ &= f(a_\alpha, c_\gamma^2 + \frac{1}{2}f(c_\gamma, c_\gamma)1 - f(1, c_\gamma)c_\gamma). \end{aligned}$$

Then

$$(11) \quad c_\gamma^2 - f(1, c_\gamma)c_\gamma + \frac{1}{2}f(c_\gamma, c_\gamma)1 = 0,$$

since  $f|_{(\sum_{i \in \mathbb{Z}} A_{i\gamma}) \times (\sum_{i \in \mathbb{Z}} A_{i\gamma})}$  is nondegenerate. Hence  $f(c_\gamma, 1) = 0$ . Also  $f(c_\gamma, c_\gamma) = 0$  whenever  $2\gamma \neq 0$ .

Besides that, since  $f(c_\gamma, 1) = 0$  for all  $c_\gamma \in A_\gamma$  if  $\gamma \neq 0$ , by the identity (10) we have that  $f(a_\alpha, b_\beta) \neq 0$  implies that  $\alpha + \beta = 0$ . Thus, if  $\alpha + \beta \neq 0$  then  $f(A_\alpha, A_\beta) = 0$ . Moreover, since  $q = (q_0, f)$  is nondegenerate, it follows that  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is nondegenerate for all  $\alpha \in \Gamma$ .  $\square$

**Remark 3.7.** Three facts to keep in mind and that we will often use are:  $\bar{c}_\gamma \in A_\gamma$  for all  $c_\gamma \in A_\gamma$  and  $\gamma \in \Gamma$  ( $f(1, c_\gamma) = 0$  if  $\gamma \neq 0$ ),  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$  and,  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is nondegenerate for all  $\alpha \in \Gamma$ .

**Remark 3.8.** Let  $A$  be a finite dimensional composition  $\epsilon$ -algebra with norm  $q = (q_0, f)$ , then  $f$  is nondegenerate as a bilinear map. Indeed, since  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ , and  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is nondegenerate for all  $\alpha \in \Gamma$ .

**Remark 3.9.** Let  $\Gamma$  an abelian group and  $\epsilon$  the trivial commutation factor on  $\Gamma$ , and let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a composition  $\epsilon$ -algebra over a field  $F$  with norm  $q = (q_0, f)$ . Then  $f$  is a nondegenerate symmetric bilinear form. If  $\text{char}(F) \neq 2$  then  $q(a) = \frac{f(a,a)}{2}$ , for all  $a \in A$ , is multiplicative by (6). Therefore,  $A$  is a composition algebra with quadratic form  $q(x)$ .

The next proposition gives some important identities which satisfy the composition color algebras.

**Proposition 3.10.** *Let  $A$  be a composition  $\epsilon$ -algebra,  $\Gamma$ -graded, with norm  $q = (q_0, f)$ . Then, for any homogeneous elements  $a_\alpha, b_\beta, c_\gamma \in A$  and  $d_0 \in A_0$  :*

- i)  $f(a_\alpha b_\beta, c_\gamma) = \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma \overline{b_\beta}) = \epsilon(\alpha, \beta) f(b_\beta, \overline{a_\alpha} c_\gamma)$ ,
- ii)  $f(a_\alpha b_\beta, c_\gamma) = \epsilon(\alpha, \beta + \gamma) \epsilon(\beta, \gamma) f(c_\gamma \overline{b_\beta}, a_\alpha) = \epsilon(\beta, \gamma) f(\overline{a_\alpha} c_\gamma, b_\beta)$ ,
- iii)  $d_0 \overline{d_0} = \overline{d_0} d_0 = q_0(d_0)1$ ,
- iv)  $a_\alpha \overline{b_\beta} + \epsilon(\alpha, \beta) b_\beta \overline{a_\alpha} = f(a_\alpha, b_\beta)1 = \overline{a_\alpha} b_\beta + \epsilon(\alpha, \beta) \overline{b_\beta} a_\alpha$ ,
- v)  $\overline{d_0}(d_0 c_\gamma) = q_0(d_0) c_\gamma = (c_\gamma d_0) \overline{d_0}$ ,
- vi)  $\overline{a_\alpha}(b_\beta c_\gamma) + \epsilon(\alpha, \beta) \overline{b_\beta}(a_\alpha c_\gamma) = f(a_\alpha, b_\beta) c_\gamma = (c_\gamma a_\alpha) \overline{b_\beta} + \epsilon(\alpha, \beta) (c_\gamma b_\beta) \overline{a_\alpha}$ .

*Proof.* It is easy to see that iii) and iv) are implied by v) and vi) respectively, taking  $c_\gamma = 1$ . So, it is enough to prove i), ii), v) and vi).

i) By (6)

$$\begin{aligned} f(a_\alpha, c_\gamma \overline{b_\beta}) &= f(a_\alpha, c_\gamma (f(b_\beta, 1)1 - b_\beta)) = f(a_\alpha, c_\gamma) f(1, b_\beta) - f(a_\alpha \cdot 1, c_\gamma b_\beta) \\ &= \epsilon(\gamma, \beta) f(a_\alpha b_\beta, c_\gamma). \end{aligned}$$

According to (6) and (7)

$$\begin{aligned} f(b_\beta, \overline{a_\alpha} c_\gamma) &= f(b_\beta, (f(a_\alpha, 1)1 - a_\alpha) c_\gamma) = f(b_\beta, c_\gamma) f(1, a_\alpha) - f(1 \cdot b_\beta, a_\alpha c_\gamma) \\ &= f(b_\beta, c_\gamma) f(1, a_\alpha) - [\epsilon(\beta, \alpha) f(1, a_\alpha) f(b_\beta, c_\gamma) - \epsilon(\beta, \alpha) \epsilon(\alpha + \beta, \gamma) f(c_\gamma, a_\alpha b_\beta)] \\ &= (1 - \epsilon(\beta, \alpha)) f(1, a_\alpha) f(b_\beta, c_\gamma) + \epsilon(\beta, \alpha) f(a_\alpha b_\beta, c_\gamma) \\ &= \epsilon(\beta, \alpha) f(a_\alpha b_\beta, c_\gamma). \end{aligned}$$

The last equality is true because  $(1 - \epsilon(\beta, \alpha)) f(1, a_\alpha) = 0$ . Indeed, if  $\epsilon(\alpha, \beta) \neq 1$  then  $\alpha \neq 0$ , hence  $f(1, a_\alpha) = 0$  for all  $a_\alpha \in A_\alpha$ .

ii) Follows by i) and (7).

v) By i) and (5)

$$f(a_\alpha, \overline{d_0}(d_0 c_\gamma)) = f(d_0 a_\alpha, d_0 c_\gamma) = q_0(d_0) f(a_\alpha, c_\gamma) = f(a_\alpha, q_0(d_0) c_\gamma).$$

Since  $f|_{(A_\gamma + A_{-\gamma}) \times (A_\gamma + A_{-\gamma})}$  is nondegenerate for all  $\gamma \in \Gamma$ ,  $\overline{d_0}(d_0 c_\gamma) = q_0(d_0) c_\gamma$ . Similarly we can prove that

$$(c_\gamma d_0) \overline{d_0} = q_0(d_0) c_\gamma.$$

vi) By *ii*), (6) and (7)

$$\begin{aligned}
f(\overline{a_\alpha}(b_\beta c_\gamma), d_\xi) &= \epsilon(\beta + \gamma, \xi) f(a_\alpha d_\xi, b_\beta c_\gamma) \\
&= \epsilon(\beta + \gamma, \xi) [\epsilon(\xi, \beta) f(a_\alpha, b_\beta) f(d_\xi, c_\gamma) - \epsilon(\beta, \gamma) \epsilon(\xi, \beta + \gamma) f(a_\alpha c_\gamma, b_\beta d_\xi)] \\
&= f(a_\alpha, b_\beta) f(c_\gamma, d_\xi) - \epsilon(\beta, \gamma) [\epsilon(\alpha + \gamma, \beta + \xi) f(b_\beta d_\xi, a_\alpha c_\gamma)] \\
&= f(a_\alpha, b_\beta) f(c_\gamma, d_\xi) - \epsilon(\alpha, \beta) \epsilon(\alpha + \gamma, \xi) [\epsilon(\xi, \alpha + \gamma) f(\overline{b_\beta}(a_\alpha c_\gamma), d_\xi)] \\
&= f(f(a_\alpha, b_\beta) c_\gamma - \epsilon(\alpha, \beta) \overline{b_\beta}(a_\alpha c_\gamma), d_\xi).
\end{aligned}$$

Since  $f|_{(A_\delta + A_{-\delta}) \times (A_\delta + A_{-\delta})}$  is nondegenerate for all  $\delta \in \Gamma$  and  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ , we have that  $\overline{a_\alpha}(b_\beta c_\gamma) = f(a_\alpha, b_\beta) c_\gamma - \epsilon(\alpha, \beta) \overline{b_\beta}(a_\alpha c_\gamma)$ .

On the other hand, by *i*), (6) and (7)

$$\begin{aligned}
f(d_\xi, (c_\gamma a_\alpha) \overline{b_\beta}) &= \epsilon(\alpha + \gamma, \beta) f(d_\xi b_\beta, c_\gamma a_\alpha) \\
&= \epsilon(\alpha + \gamma, \beta) [\epsilon(\beta, \gamma) f(d_\xi, c_\gamma) f(b_\beta, a_\alpha) - \epsilon(\gamma, \alpha) \epsilon(\beta, \alpha + \gamma) f(d_\xi a_\alpha, c_\gamma b_\beta)] \\
&= f(a_\alpha, b_\beta) f(d_\xi, c_\gamma) - \epsilon(\gamma, \alpha) f(d_\xi a_\alpha, c_\gamma b_\beta) \\
&= f(a_\alpha, b_\beta) f(d_\xi, c_\gamma) - \epsilon(\gamma, \alpha) \epsilon(\alpha, \gamma + \beta) f(d_\xi, (c_\gamma b_\beta) \overline{a_\alpha}) \\
&= f(d_\xi, f(a_\alpha, b_\beta) c_\gamma - \epsilon(\alpha, \beta) (c_\gamma b_\beta) \overline{a_\alpha}).
\end{aligned}$$

Since  $f|_{(A_\delta + A_{-\delta}) \times (A_\delta + A_{-\delta})}$  is nondegenerate for all  $\delta \in \Gamma$  and  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ , therefore  $(c_\gamma a_\alpha) \overline{b_\beta} = f(a_\alpha, b_\beta) c_\gamma - \epsilon(\alpha, \beta) (c_\gamma b_\beta) \overline{a_\alpha}$ . □

We will identify  $F$  with the subalgebra  $F \cdot 1$  of the algebra  $A$ .

**Lemma 3.11.** *Let  $A$  be a composition  $\epsilon$ -algebra,  $\Gamma$ -graded, with quadratic  $\epsilon$ -form  $q = (q_0, f)$ . The map  $a \mapsto \overline{a} = f(a, 1) - a$  is an  $\epsilon$ -involution of the algebra  $A$  which leaves all the elements of the field  $F$  fixed. Also, the elements  $a + \overline{a}$  and  $a_\alpha \overline{b_\beta} + \epsilon(\alpha, \beta) b_\beta \overline{a_\alpha}$  lie in  $F$  for all  $a \in A$  and any homogeneous elements  $a_\alpha, b_\beta \in A$ .*

*Proof.* We will verify each of the properties for an  $\epsilon$ -involution:

- (a)  $\overline{a + b} = \overline{a} + \overline{b}$ . We have  $\overline{a + b} = f(a + b, 1) - (a + b) = f(a, 1) - a + f(b, 1) - b = \overline{a} + \overline{b}$ .
- (b) Let  $r \in F$ . Then  $\overline{ra} = f(ra, 1) - ra = r(f(a, 1) - a) = r\overline{a}$ .
- (c)  $\overline{\overline{a}} = a$ . We have  $\overline{\overline{a}} = f(\overline{a}, 1) - \overline{a} = f(f(a, 1) - a, 1) - f(a, 1) + a = f(1, 1)f(a, 1) - 2f(a, 1) + a = a$ , since  $f(1, 1) = 2$ .
- (d)  $\overline{a_\alpha b_\beta} = \epsilon(\alpha, \beta) \overline{b_\beta} \overline{a_\alpha}$ . By Proposition 3.10 *i*) and *iv*)

$$\overline{a_\alpha b_\beta} = f(a_\alpha b_\beta, 1) - a_\alpha b_\beta = f(a_\alpha, \overline{b_\beta}) - a_\alpha \overline{\overline{b_\beta}} = \epsilon(\alpha, \beta) \overline{b_\beta} \overline{a_\alpha}.$$

Furthermore, if  $r \in F$  then  $\overline{r} = f(r, 1) - r = rf(1, 1) - r = r$ . Finally,  $a + \overline{a} = f(a, 1) \in F$  and by Proposition 3.10 *iv*)  $a_\alpha \overline{b_\beta} + \epsilon(\alpha, \beta) b_\beta \overline{a_\alpha} = f(a_\alpha, b_\beta) \in F$ . This proves the lemma. □

Let  $A$  be some algebra over an arbitrary field  $F$  and  $a, b, c \in A$ . We denote by  $(a, b, c) = (ab)c - a(bc)$  the *associator* of the elements  $a, b, c$ .

Now, we will define an alternative  $\epsilon$ -algebra (or color alternative algebra) and we are going to prove that every composition  $\epsilon$ -algebra is an alternative  $\epsilon$ -algebra.

**Definition 3.12.** Let  $\Gamma$  be an abelian group,  $\epsilon$  be a commutation factor on  $\Gamma$  and  $A$  be an  $\Gamma$ -graded algebra over a field  $F$ ,  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ .  $A$  is called an *alternative  $\epsilon$ -algebra* (or, color alternative algebra) if

$$(12) \quad (a_\alpha, b_\beta, c_\gamma) + \epsilon(\beta, \gamma)(a_\alpha, c_\gamma, b_\beta) = 0,$$

$$(13) \quad (a_\alpha, b_\beta, c_\gamma) + \epsilon(\alpha, \beta)(b_\beta, a_\alpha, c_\gamma) = 0,$$

$$(14) \quad (d_0, d_0, a_\alpha) = 0$$

for any  $d_0 \in A_0$  and homogeneous elements  $a_\alpha, b_\beta, c_\gamma \in A$ .

By (12), (13) and (14) we have the following identities

$$(d_0, a_\alpha, d_0) = (a_\alpha, d_0, d_0) = 0,$$

$$(15) \quad (a_\alpha, b_\beta, c_\gamma) + \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)(c_\gamma, b_\beta, a_\alpha) = 0.$$

**Proposition 3.13.** *Let  $A$  be a composition  $\epsilon$ -algebra with norm  $q = (q_0, f)$ . Then  $A$  is an alternative  $\epsilon$ -algebra.*

*Proof.* Taking into account that  $a = f(1, a) - \bar{a}$  for every  $a \in A$ . Let us prove that  $(a_\alpha, b_\beta, c_\gamma) + \epsilon(\beta, \gamma)(a_\alpha, c_\gamma, b_\beta) = 0$ . By Proposition 3.10 items *iv*) and *vi*), we have

$$\begin{aligned} (a_\alpha, b_\beta, c_\gamma) &= (a_\alpha b_\beta) c_\gamma - a_\alpha (b_\beta c_\gamma) = (a_\alpha b_\beta)(f(1, c_\gamma) - \bar{c}_\gamma) - a_\alpha [b_\beta (f(1, c_\gamma) - \bar{c}_\gamma)] \\ &= -(a_\alpha b_\beta) \bar{c}_\gamma + a_\alpha (b_\beta \bar{c}_\gamma) \\ &= \epsilon(\beta, \gamma)(a_\alpha c_\gamma) \bar{b}_\beta - f(b_\beta, c_\gamma) a_\alpha + a_\alpha [f(b_\beta, c_\gamma) - \epsilon(\beta, \gamma) c_\gamma \bar{b}_\beta] \\ &= \epsilon(\beta, \gamma)(a_\alpha c_\gamma)(f(1, b_\beta) - b_\beta) - \epsilon(\beta, \gamma) a_\alpha [c_\gamma (f(1, b_\beta) - b_\beta)] \\ &= -\epsilon(\beta, \gamma)[(a_\alpha c_\gamma) b_\beta - a_\alpha (c_\gamma b_\beta)] = -\epsilon(\beta, \gamma)(a_\alpha, c_\gamma, b_\beta). \end{aligned}$$

Let us show that  $(a_\alpha, b_\beta, c_\gamma) + \epsilon(\alpha, \beta)(b_\beta, a_\alpha, c_\gamma) = 0$ . By Proposition 3.10 *iv*) and *vi*)

$$\begin{aligned} (a_\alpha, b_\beta, c_\gamma) &= (a_\alpha b_\beta) c_\gamma - a_\alpha (b_\beta c_\gamma) = [(f(1, a_\alpha) - \bar{a}_\alpha) b_\beta] c_\gamma - (f(1, a_\alpha) - \bar{a}_\alpha)(b_\beta c_\gamma) \\ &= -(\bar{a}_\alpha b_\beta) c_\gamma + \bar{a}_\alpha (b_\beta c_\gamma) \\ &= -[f(a_\alpha, b_\beta) - \epsilon(\alpha, \beta) \bar{b}_\beta a_\alpha] c_\gamma + f(a_\alpha, b_\beta) c_\gamma - \epsilon(\alpha, \beta) \bar{b}_\beta (a_\alpha c_\gamma) \\ &= \epsilon(\alpha, \beta) (\bar{b}_\beta a_\alpha) c_\gamma - \epsilon(\alpha, \beta) \bar{b}_\beta (a_\alpha c_\gamma) \\ &= \epsilon(\alpha, \beta) [(f(1, b_\beta) - b_\beta) a_\alpha] c_\gamma - \epsilon(\alpha, \beta) (f(1, b_\beta) - b_\beta) (a_\alpha c_\gamma) \\ &= \epsilon(\alpha, \beta) [-(b_\beta a_\alpha) c_\gamma + b_\beta (a_\alpha c_\gamma)] = -\epsilon(\alpha, \beta)(b_\beta, a_\alpha, c_\gamma). \end{aligned}$$

Let us prove that  $(d_0, d_0, a_\alpha) = 0$ . By Proposition 3.10 *iii*) and *v*)

$$\begin{aligned} (d_0, d_0, a_\alpha) &= (d_0 d_0) a_\alpha - d_0 (d_0 a_\alpha) = [(f(1, d_0) - \bar{d}_0) d_0] a_\alpha - (f(1, d_0) - \bar{d}_0)(d_0 a_\alpha) \\ &= -(\bar{d}_0 d_0) a_\alpha + \bar{d}_0 (d_0 a_\alpha) = -q_0(d_0) a_\alpha + q_0(d_0) a_\alpha = 0. \end{aligned}$$

□

## 4. GENERALIZED CAYLEY-DICKSON PROCESS

Let  $\Gamma$  be a finitely generated abelian group,  $\epsilon$  be a commutation factor on  $\Gamma$  and  $A$  be a  $\Gamma$ -graded algebra,  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ , over a field  $F$  with identity element 1. Let  $a \rightarrow \bar{a}$  be an  $\epsilon$ -involution of the algebra  $A$  verifying the following conditions:

- $c_0 + \bar{c}_0, c_0 \bar{c}_0 \in F$  for any  $c_0 \in A_0$ .
- If  $\alpha \neq 0$  then  $\bar{a}_\alpha = -a_\alpha$  for any homogeneous element  $a_\alpha \in A$ .
- $a_\alpha \bar{b}_\beta + \epsilon(\alpha, \beta) b_\beta \bar{a}_\alpha \in F$  for any homogeneous elements  $a_\alpha, b_\beta \in A$ .
- If  $\alpha + \beta \neq 0$  then  $a_\alpha \bar{b}_\beta + \epsilon(\alpha, \beta) b_\beta \bar{a}_\alpha = 0$  for any homogeneous elements  $a_\alpha, b_\beta \in A$ .

Let  $f : A \times A \rightarrow F$  be the bilinear form on  $A$ , defined in the homogeneous components of  $A$  by  $f(a_\alpha, b_\beta) = a_\alpha \bar{b}_\beta + \epsilon(\alpha, \beta) b_\beta \bar{a}_\alpha$ ,  $a_\alpha \in A_\alpha$ ,  $b_\beta \in A_\beta$ , and let  $q_0(c_0) = c_0 \bar{c}_0$ ,  $c_0 \in A_0$ . Let us see that  $q = (q_0, f)$  is a quadratic  $\epsilon$ -form on  $A$ .

**Lemma 4.1.** *In the same conditions as above, we have that  $q = (q_0, f)$  is a quadratic  $\epsilon$ -form on  $A$ .*

*Proof.* We will verify each of the properties for a quadratic  $\epsilon$ -form.

a)  $q_0 : V_0 \rightarrow F$  is a usual quadratic form and  $f|_{A_0 \times A_0}$  is the polar of  $q_0$ . Indeed

$$\begin{aligned} q_0(a_0 + b_0) - q_0(a_0) - q_0(b_0) &= (a_0 + b_0)(\overline{a_0 + b_0}) - a_0 \bar{a}_0 - b_0 \bar{b}_0 \\ &= a_0 \bar{b}_0 + b_0 \bar{a}_0 = f(a_0, b_0). \end{aligned}$$

b)  $f : V \times V \rightarrow F$  is an  $\epsilon$ -symmetric even bilinear form. Let us see it.

If  $\epsilon(\alpha, \beta) = 1$  then

$$f(a_\alpha, b_\beta) = a_\alpha \bar{b}_\beta + b_\beta \bar{a}_\alpha = f(b_\beta, a_\alpha)$$

for any homogeneous elements  $a_\alpha, b_\beta \in A$ . Therefore  $f|_{(\sum_{i \in \mathbb{Z}} A_{i\alpha}) \times (\sum_{i \in \mathbb{Z}} A_{i\alpha})}$  is symmetric for all  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$ .

If  $\epsilon(\alpha, \beta) = -1$  then

$$f(a_\alpha, b_\beta) = a_\alpha \bar{b}_\beta - b_\beta \bar{a}_\alpha = -f(b_\beta, a_\alpha)$$

for any homogeneous elements  $a_\alpha, b_\beta \in A$ . Hence  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is skew-symmetric for all  $\alpha \in \Gamma$  with  $\epsilon(\alpha, \alpha) = -1$ .

If  $0 \neq \alpha \in \Gamma$  then

$$f(a_\alpha, b_0) = a_\alpha \bar{b}_0 + \epsilon(\alpha, 0) b_0 \bar{a}_\alpha = 0$$

for any homogeneous elements  $a_\alpha, b_0 \in A$ . Thus  $f(A_0, A_\alpha) = f(A_\alpha, A_0) = 0$ .

□

The *Generalized Cayley-Dickson process* consists of building a new algebra  $\Gamma \times \mathbb{Z}_2$ -graded with an  $\tilde{\epsilon}$ -involution (where  $\tilde{\epsilon}$  is a commutation factor on  $\Gamma \times \mathbb{Z}_2$ ) which contains  $A$  as a  $\Gamma \times \mathbb{Z}_2$ -graded subalgebra. In addition, if the dimension of the algebra  $A$  is  $m$ , then the dimension of the new algebra will be  $2m$ .

By the fundamental theorem of the finitely generated abelian groups we can suppose that

$$\Gamma = \mathbb{Z}_{2^{s_1}} \oplus \mathbb{Z}_{2^{s_2}} \oplus \dots \oplus \mathbb{Z}_{2^{s_l}} \oplus \mathbb{Z}_{p_1^{t_1}} \oplus \mathbb{Z}_{p_2^{t_2}} \oplus \dots \oplus \mathbb{Z}_{p_m^{t_m}} \oplus \mathbb{Z}^n$$

for some non-negative integers  $l, m, n, s_1, s_2, \dots, s_l, t_1, t_2, \dots, t_m$  and prime numbers, not necessarily different,  $p_1, p_2, \dots, p_m$  with  $p_i \neq 2$  for all  $1 \leq i \leq m$ .

Thus

$$\Gamma = \Gamma_1 \oplus \Gamma_2 \oplus \dots \oplus \Gamma_{l+m+n},$$

where  $\Gamma_i = \mathbb{Z}_{2^{s_i}}$  if  $1 \leq i \leq l$ ,  $\Gamma_{l+i} = \mathbb{Z}_{p_i^{t_i}}$  if  $1 \leq i \leq m$ , and  $\Gamma_{l+m+i} = \mathbb{Z}$  if  $1 \leq i \leq n$ . For each  $1 \leq i \leq l+m+n$  we can choose a generator  $\xi_i$  of the group  $\Gamma_i$ . A natural choice for  $\xi_i$  is the residue class of 1 modulo  $2^{s_i}$  if  $1 \leq i \leq l$ , the residue class of 1 modulo  $p_{i-l}^{t_{i-l}}$  if  $l+1 \leq i \leq l+m$  and 1 if  $l+m+1 \leq i \leq l+m+n$ .

Before building the new algebra we will define the commutation factors  $\tilde{\epsilon}$  on  $\Gamma \times \mathbb{Z}_2$ . Let  $\epsilon$  be a commutation factor on  $\Gamma$  and let  $i, j$  be integers such that  $1 \leq i, j \leq l+m+n$ . We define the following commutation factors  $\tilde{\epsilon}$  on  $\Gamma \times \mathbb{Z}_2$ :

$$\begin{aligned} \tilde{\epsilon}((0, \bar{1}), (0, \bar{1})) &= 1, \\ \tilde{\epsilon}((\xi_i, \bar{0}), (\xi_j, \bar{0})) &= \epsilon(\xi_i, \xi_j), \\ \tilde{\epsilon}((\xi_i, \bar{0}), (0, \bar{1})) &= \begin{cases} 1, & \text{if } l+1 \leq i \leq l+m \\ \pm 1, & \text{if } 1 \leq i \leq l \text{ or } l+m+1 \leq i \leq l+m+n. \end{cases} \end{aligned}$$

Note that we have  $2^{l+n}$  commutation factors of the type  $\tilde{\epsilon}$  defined on  $\Gamma \times \mathbb{Z}_2$ . Also, it is easy to see that

$$(16) \quad \tilde{\epsilon}((\xi, \bar{k}), (0, \bar{1})) = \tilde{\epsilon}((0, \bar{1}), (\xi, \bar{k}))$$

for any  $(\xi, \bar{k}) \in \Gamma \times \mathbb{Z}_2$ .

We fix  $\mu \in F^\times$  and a commutation factor  $\tilde{\epsilon}$  on  $\Gamma \times \mathbb{Z}_2$ . We denote by  $GCD(A, \mu, \tilde{\epsilon})$  the collection of all ordered pairs  $(a, b)$  where  $a, b \in A$ , with operations of componentwise addition, scalar multiplication and the multiplication on ordered pairs of homogeneous elements given by

$$(17) \quad (a_\alpha, b_\beta)(c_\gamma, d_\delta) = (a_\alpha c_\gamma + \tilde{\epsilon}((\beta, \bar{0}), (\delta, \bar{1}))\mu d_\delta \bar{b}_\beta, \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))\bar{a}_\alpha d_\delta + \tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{0}))c_\gamma b_\beta).$$

It is easy to see that  $GCD(A, \mu, \tilde{\epsilon})$  is an algebra over  $F$ . The element  $(1, 0)$  is an identity of the algebra  $GCD(A, \mu, \tilde{\epsilon})$ . The set  $A' = \{(a, 0) | a \in A\}$  is a  $\Gamma \times \mathbb{Z}_2$ -graded subalgebra of the algebra  $GCD(A, \mu, \tilde{\epsilon})$  which is equivalent to the algebra  $A$ . Let  $v = (0, 1)$  then  $v^2 = \mu(1, 0)$ , and  $GCD(A, \mu, \tilde{\epsilon})$  is the direct sum of the vector spaces  $A'$  and  $vA'$ . If we identify  $A'$  with  $A$ , the elements of the algebra  $GCD(A, \mu, \tilde{\epsilon})$  are represented in the form  $x = a + vb$ , where the elements  $a, b \in A$  are uniquely determined by the element  $x$ , and the multiplication in  $GCD(A, \mu, \tilde{\epsilon})$  is given by

$$(18) \quad (a_\alpha + vb_\beta)(c_\gamma + vd_\delta) = a_\alpha c_\gamma + \tilde{\epsilon}((\beta, \bar{0}), (\delta, \bar{1}))\mu d_\delta \bar{b}_\beta + v[\tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))\bar{a}_\alpha d_\delta + \tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{0}))c_\gamma b_\beta].$$

$GCD(A, \mu, \tilde{\epsilon})$  has a  $\Gamma \times \mathbb{Z}_2$ -grading given by

$$GCD(A, \mu, \tilde{\epsilon}) := \tilde{A} = \bigoplus_{(\gamma, \bar{k}) \in \Gamma \times \mathbb{Z}_2} \tilde{A}_{(\gamma, \bar{k})}$$

where  $\tilde{A}_{(\gamma, \bar{0})} = A_\gamma$  and  $\tilde{A}_{(\gamma, \bar{1})} = vA_\gamma$ . Since  $v \in \tilde{A}_{(0, \bar{1})}$ , in accordance with the notation  $v := v_{(0, \bar{1})}$ , but for short it will only be used  $v$ .

For an arbitrary element  $x = a + vb \in GCD(A, \mu, \tilde{\epsilon})$ , we set  $\bar{x} = \bar{a} - vb$ . Also, we define  $\tilde{f}(x_{(\gamma, \bar{i})}, y_{(\delta, \bar{j})}) := x_{(\gamma, \bar{i})}\overline{y_{(\delta, \bar{j})}} + \tilde{\epsilon}((\gamma, \bar{i}), (\delta, \bar{j}))y_{(\delta, \bar{j})}\overline{x_{(\gamma, \bar{i})}}$  and  $\tilde{q}_0(z_{(0, \bar{0})}) = z_{(0, \bar{0})}\overline{z_{(0, \bar{0})}}$  for all  $z_{(0, \bar{0})} \in \tilde{A}_{(0, \bar{0})}$  and any homogeneous elements  $x_{(\gamma, \bar{i})}, y_{(\delta, \bar{j})} \in \tilde{A}$ . Extending by linearity we obtain a bilinear map  $\tilde{f} : \tilde{A} \times \tilde{A} \rightarrow \tilde{A}$ .

**Lemma 4.2.** *The map  $x \rightarrow \bar{x}$  is an  $\tilde{\epsilon}$ -involution of the algebra  $GCD(A, \mu, \tilde{\epsilon})$ . In addition,  $x + \bar{x} \in F$  for all  $x \in GCD(A, \mu, \tilde{\epsilon})$  and  $\tilde{f}(x_{(\gamma, \bar{i})}, y_{(\delta, \bar{j})}) = x_{(\gamma, \bar{i})}\overline{y_{(\delta, \bar{j})}} + \tilde{\epsilon}((\gamma, \bar{i}), (\delta, \bar{j}))y_{(\delta, \bar{j})}\overline{x_{(\gamma, \bar{i})}} \in F$  for any homogeneous elements  $x_{(\gamma, \bar{i})}, y_{(\delta, \bar{j})} \in GCD(A, \mu, \tilde{\epsilon})$ .  $\tilde{q} = (\tilde{q}_0, \tilde{f})$  is a quadratic  $\tilde{\epsilon}$ -form on  $GCD(A, \mu, \tilde{\epsilon})$ . If the quadratic  $\epsilon$ -form  $q = (q_0, f)$  is nondegenerate on  $A$  then the quadratic  $\tilde{\epsilon}$ -form  $\tilde{q} = (\tilde{q}_0, \tilde{f})$  is nondegenerate on  $GCD(A, \mu, \tilde{\epsilon})$ . Moreover,  $\bar{x} = \tilde{f}(x, 1) - x$  for all  $x \in GCD(A, \mu, \tilde{\epsilon})$ .*

*Proof.* It is clear that the map  $x \rightarrow \bar{x}$  is linear and  $\bar{\bar{x}} = x$  for all  $x \in GCD(A, \mu, \tilde{\epsilon})$ . We will show that  $\overline{x_{(\gamma, \bar{i})}y_{(\delta, \bar{j})}} = \tilde{\epsilon}((\gamma, \bar{i}), (\delta, \bar{j}))\overline{y_{(\delta, \bar{j})}} \cdot \overline{x_{(\gamma, \bar{i})}}$  with which it will be proved that  $x \rightarrow \bar{x}$  is an  $\tilde{\epsilon}$ -involution of the algebra  $GCD(A, \mu, \tilde{\epsilon})$ .

Let  $a_\alpha, b_\beta, c_\gamma$  and  $d_\delta$  homogeneous elements in  $A$ . By (18) we have

$$\overline{a_\alpha c_\gamma} = \epsilon(\alpha, \gamma)\overline{c_\gamma a_\alpha} = \tilde{\epsilon}((\alpha, \bar{0}), (\gamma, \bar{0}))\overline{c_\gamma a_\alpha}.$$

$$\begin{aligned} \tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))\overline{(vd_\delta)\overline{a_\alpha}} &= -\tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))\overline{(vd_\delta)\overline{a_\alpha}} = -\tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))v(\epsilon(\delta, \alpha)\overline{a_\alpha}d_\delta) \\ &= -\tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))v(\overline{a_\alpha}d_\delta) = \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))v(\overline{a_\alpha}d_\delta) \\ &= \overline{a_\alpha(vd_\delta)}. \end{aligned}$$

$$\begin{aligned} \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\overline{c_\gamma(vb_\beta)} &= -\tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\overline{c_\gamma(vb_\beta)} = -\tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\tilde{\epsilon}((\gamma, \bar{0}), (0, \bar{1}))v(c_\gamma b_\beta) \\ &= -\epsilon(\beta, \gamma)v(c_\gamma b_\beta) = \epsilon(\beta, \gamma)v(\overline{c_\gamma b_\beta}) = \overline{(vb_\beta)c_\gamma}. \end{aligned}$$

$$\begin{aligned} \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))\overline{(vd_\delta)\overline{(vb_\beta)}} &= \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))\overline{(vd_\delta)(vb_\beta)} = \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))\tilde{\epsilon}((\delta, \bar{0}), (\beta, \bar{1}))\mu b_\beta \overline{d_\delta} \\ &= \tilde{\epsilon}((\beta, \bar{0}), (\delta, \bar{1}))\mu[\epsilon(\delta, \beta)b_\beta \overline{d_\delta}] = \tilde{\epsilon}((\beta, \bar{0}), (\delta, \bar{1}))\overline{\mu d_\delta b_\beta} \\ &= \overline{(vb_\beta)(vd_\delta)}. \end{aligned}$$

Furthermore, for  $x = a + vb$  we have that  $x + \bar{x} = a + \bar{a} \in F$ .

Now, we will see how is the behavior of  $\tilde{f}$  in the different homogeneous components. By (18):

$$\tilde{f}(a_\alpha, c_\gamma) = a_\alpha \overline{c_\gamma} + \tilde{\epsilon}((\alpha, \bar{0}), (\gamma, \bar{0}))c_\gamma \overline{a_\alpha} = a_\alpha \overline{c_\gamma} + \epsilon(\alpha, \gamma)c_\gamma \overline{a_\alpha} = f(a_\alpha, c_\gamma).$$

$$\begin{aligned} \tilde{f}(a_\alpha, vd_\delta) &= a_\alpha \overline{(vd_\delta)} + \tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))\overline{(vd_\delta)\overline{a_\alpha}} \\ &= -a_\alpha(vd_\delta) + \tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))\epsilon(\delta, \alpha)v(\overline{a_\alpha}d_\delta) \\ &= -\tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))v(\overline{a_\alpha}d_\delta) + \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))v(\overline{a_\alpha}d_\delta) = 0. \end{aligned}$$

$$\begin{aligned}
\tilde{f}(vb_\beta, c_\gamma) &= (vb_\beta)\overline{c_\gamma} + \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))c_\gamma\overline{(vb_\beta)} \\
&= \epsilon(\beta, \gamma)v(\overline{c_\gamma}b_\beta) - \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))c_\gamma(vb_\beta) \\
&= \epsilon(\beta, \gamma)v(\overline{c_\gamma}b_\beta) - \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\tilde{\epsilon}((\gamma, \bar{0}), (0, \bar{1}))v(\overline{c_\gamma}b_\beta) \\
&= \epsilon(\beta, \gamma)v(\overline{c_\gamma}b_\beta) - \epsilon(\beta, \gamma)v(\overline{c_\gamma}b_\beta) = 0.
\end{aligned}$$

$$\begin{aligned}
\tilde{f}(vb_\beta, vd_\delta) &= (vb_\beta)\overline{(vd_\delta)} + \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))(vd_\delta)\overline{(vb_\beta)} \\
&= -(vb_\beta)(vd_\delta) - \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))(vd_\delta)(vb_\beta) \\
&= -\tilde{\epsilon}((\beta, \bar{0}), (\delta, \bar{1}))\mu d_\delta\overline{b_\beta} - \tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{1}))\tilde{\epsilon}((\delta, \bar{0}), (\beta, \bar{1}))\mu b_\beta\overline{d_\delta} \\
&= -\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu(b_\beta\overline{d_\delta}) + \epsilon(\beta, \delta)d_\delta\overline{b_\beta} \\
&= -\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta, d_\delta).
\end{aligned}$$

In short, the bilinear form  $\tilde{f} : \tilde{A} \times \tilde{A} \rightarrow F$  is defined as follows:

$$\begin{aligned}
\tilde{f}(a_{(\alpha, \bar{0})}, c_{(\gamma, \bar{0})}) &= f(a_\alpha, c_\gamma), \\
\tilde{f}(a_{(\alpha, \bar{0})}, d_{(\delta, \bar{1})}) &= \tilde{f}(b_{(\beta, \bar{1})}, c_{(\gamma, \bar{0})}) = 0, \\
\tilde{f}(b_{(\beta, \bar{1})}, d_{(\delta, \bar{1})}) &= -\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta, d_\delta),
\end{aligned}$$

where  $a_{(\alpha, \bar{0})} := a_\alpha$ ,  $b_{(\beta, \bar{1})} := vb_\beta$ ,  $c_{(\gamma, \bar{0})} := c_\gamma$  and  $d_{(\delta, \bar{1})} := vd_\delta$ .

Therefore  $\tilde{f}(x_{(\gamma, \bar{i})}, y_{(\delta, \bar{j})}) \in F$  and  $\tilde{q} = (\tilde{q}_0, \tilde{f})$  is a quadratic  $\tilde{\epsilon}$ -form on  $GCD(A, \mu, \tilde{\epsilon})$  by Lemma 4.1. In addition, if the quadratic  $\epsilon$ -form  $q = (q_0, f)$  is nondegenerate then the quadratic  $\tilde{\epsilon}$ -form  $\tilde{q} = (\tilde{q}_0, \tilde{f})$  is nondegenerate. Also  $\tilde{f}(x, 1) = x + \bar{x}$ .  $\square$

Now, let  $A$  be a composition  $\epsilon$ -algebra with norm  $q = (q_0, f)$ . By Lemma 3.11 there exists an  $\epsilon$ -involution  $a \rightarrow \bar{a}$  on  $A$  such that  $a_0 + \bar{a}_0$ ,  $a_0\bar{a}_0 \in F$ ,  $\bar{a}_\alpha = -a_\alpha$  if  $\alpha \neq 0$ ,  $a_\alpha\bar{b}_\beta + \epsilon(\alpha, \beta)b_\beta\bar{a}_\alpha \in F$  and  $a_\alpha\bar{b}_\beta + \epsilon(\alpha, \beta)b_\beta\bar{a}_\alpha = 0$  if  $\alpha + \beta \neq 0$ , for all  $a_\alpha, b_\beta \in A$ . This means that we can apply the generalized Cayley-Dickson process to  $A$ .

Below, we will find conditions where the obtained algebra  $GCD(A, \mu, \tilde{\epsilon})$ , with norm  $\tilde{q} = (q_0, \tilde{f})$  is a composition  $\tilde{\epsilon}$ -algebra. For this, first we shall prove some identities.

**Lemma 4.3.** *Let  $A$  be a composition  $\epsilon$ -algebra,  $\Gamma$ -graded. Then for any homogeneous elements  $a_\alpha, b_\beta, c_\gamma, d_\xi \in A$ :*

- i)  $f(\bar{a}_\alpha, \bar{c}_\gamma) = f(a_\alpha, c_\gamma)$ ,
- ii)  $f(b_\beta\bar{a}_\alpha, d_\xi\bar{c}_\gamma) = \epsilon(\beta, \alpha)\epsilon(\xi, \gamma)f(a_\alpha\bar{b}_\beta, c_\gamma\bar{d}_\xi)$ ,
- iii)  $\epsilon(\alpha, \beta)\epsilon(\gamma, \xi)f(b_\beta\bar{a}_\alpha, d_\xi\bar{c}_\gamma) + \epsilon(\alpha + \beta + \gamma, \xi)f(d_\xi\bar{a}_\alpha, b_\beta\bar{c}_\gamma) = \epsilon(\beta, \gamma)f(a_\alpha, c_\gamma)f(b_\beta, d_\xi)$ ,
- iv)  $\epsilon(\beta, \gamma)f(a_\alpha, c_\gamma)f(b_\beta, d_\xi) = \epsilon(\alpha, \beta + \xi)\epsilon(\gamma, \xi)f(a_\alpha, c_\gamma)f(b_\beta, d_\xi)$ .

*Proof.*

- i) By Proposition 3.10 i) and ii)

$$f(\bar{a}_\alpha, \bar{c}_\gamma) = f(\bar{a}_\alpha, 1 \cdot \bar{c}_\gamma) = \epsilon(0, \gamma)f(\bar{a}_\alpha c_\gamma, 1) = \epsilon(\gamma, 0)f(a_\alpha \cdot 1, c_\gamma) = f(a_\alpha, c_\gamma).$$

ii) By item i)

$$\begin{aligned} f(b_\beta \overline{a_\alpha}, d_\xi \overline{c_\gamma}) &= f(\overline{b_\beta a_\alpha}, \overline{d_\xi c_\gamma}) \\ &= f(\epsilon(\beta, \alpha) a_\alpha \overline{b_\beta}, \epsilon(\xi, \gamma) c_\gamma \overline{d_\xi}). \\ &= \epsilon(\beta, \alpha) \epsilon(\xi, \gamma) f(a_\alpha \overline{b_\beta}, c_\gamma \overline{d_\xi}). \end{aligned}$$

iii) By items i) and ii) and (6)

$$\begin{aligned} \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(b_\beta, d_\xi) &= \epsilon(\beta, \gamma) f(a_\alpha, c_\gamma) f(\overline{b_\beta}, \overline{d_\xi}) \\ &= f(a_\alpha \overline{b_\beta}, c_\gamma \overline{d_\xi}) + \epsilon(\beta + \gamma, \xi) \epsilon(\beta, \gamma) f(a_\alpha \overline{d_\xi}, c_\gamma \overline{b_\beta}) \\ &= \epsilon(\alpha, \beta) \epsilon(\gamma, \xi) f(b_\beta \overline{a_\alpha}, d_\xi \overline{c_\gamma}) + \epsilon(\alpha + \beta + \gamma, \xi) f(d_\xi \overline{a_\alpha}, b_\beta \overline{c_\gamma}). \end{aligned}$$

iv) Let us show that

$$[\epsilon(\beta, \gamma) - \epsilon(\alpha, \beta + \xi) \epsilon(\gamma, \xi)] f(a_\alpha, c_\gamma) f(b_\beta, d_\xi) = 0.$$

If  $f(a_\alpha, c_\gamma) f(b_\beta, d_\xi) \neq 0$  then  $\alpha + \gamma = 0$  and  $\beta + \xi = 0$ , hence for this case we have

$$\epsilon(\beta, \gamma) - \epsilon(\alpha, \beta + \xi) \epsilon(\gamma, \xi) = \epsilon(\beta, \gamma) - \epsilon(\alpha, 0) \epsilon(\gamma, -\beta) = \epsilon(\beta, \gamma) - \epsilon(\beta, \gamma) = 0.$$

Therefore equality is obtained. □

**Lemma 4.4.** *If  $A$  is a composition  $\epsilon$ -algebra,  $\Gamma$ -graded, then  $GCD(A, \mu, \tilde{\epsilon})$  is a composition  $\tilde{\epsilon}$ -algebra,  $\Gamma \times \mathbb{Z}_2$ -graded, if and only if the algebra  $A$  is associative*

*Proof.* The proof will be done by cases:

i) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. Let us prove that

$$\begin{aligned} \tilde{f}(a_\alpha(vb_\beta), c_\gamma(vd_\delta)) + \tilde{\epsilon}((\gamma, \overline{0}), (\delta, \overline{1})) \tilde{\epsilon}((\beta, \overline{1}), (\gamma + \delta, \overline{1})) \tilde{f}(a_\alpha(vd_\delta), c_\gamma(vb_\beta)) \\ = \tilde{\epsilon}((\beta, \overline{1}), (\gamma, \overline{0})) \tilde{f}(a_\alpha, c_\gamma) \tilde{f}(vb_\beta, vd_\delta). \end{aligned}$$

By (18), (6), (16) and lemma 4.3 i), we obtain

$$\begin{aligned} \tilde{f}(a_\alpha(vb_\beta), c_\gamma(vd_\delta)) + \tilde{\epsilon}((\gamma, \overline{0}), (\delta, \overline{1})) \tilde{\epsilon}((\beta, \overline{1}), (\gamma + \delta, \overline{1})) \tilde{f}(a_\alpha(vd_\delta), c_\gamma(vb_\beta)) \\ = \tilde{\epsilon}((\alpha + \gamma, \overline{0}), (0, \overline{1})) \tilde{f}(v(\overline{a_\alpha} b_\beta), v(\overline{c_\gamma} d_\delta)) \\ + \epsilon(\gamma, \delta) \epsilon(\beta, \gamma) \tilde{\epsilon}((\beta, \overline{1}), (\delta, \overline{1})) [\tilde{\epsilon}((\alpha + \gamma, \overline{0}), (0, \overline{1})) \tilde{f}(v(\overline{a_\alpha} d_\delta), v(\overline{c_\gamma} b_\beta))] \\ = \tilde{\epsilon}((\alpha + \gamma, \overline{0}), (0, \overline{1})) [-\tilde{\epsilon}((\alpha + \beta, \overline{0}), (0, \overline{1})) \mu f(\overline{a_\alpha} b_\beta, \overline{c_\gamma} d_\delta)] \\ + \epsilon(\gamma, \delta) \epsilon(\beta, \gamma) \tilde{\epsilon}((\beta, \overline{1}), (\delta, \overline{1})) \tilde{\epsilon}((\alpha + \gamma, \overline{0}), (0, \overline{1})) [-\tilde{\epsilon}((\alpha + \delta, \overline{0}), (0, \overline{1})) \mu f(\overline{a_\alpha} d_\delta, \overline{c_\gamma} b_\beta)] \\ = -\tilde{\epsilon}((\gamma + \beta, \overline{0}), (0, \overline{1})) \mu [f(\overline{a_\alpha} b_\beta, \overline{c_\gamma} d_\delta) + \epsilon(\gamma, \delta) \epsilon(\beta, \gamma + \delta) f(\overline{a_\alpha} d_\delta, \overline{c_\gamma} b_\beta)] \\ = -\tilde{\epsilon}((\gamma + \beta, \overline{0}), (0, \overline{1})) \mu \epsilon(\beta, \gamma) f(\overline{a_\alpha}, \overline{c_\gamma}) f(b_\beta, d_\delta) \\ = \tilde{\epsilon}((\beta, \overline{1}), (\gamma, \overline{0})) f(a_\alpha, c_\gamma) [-\tilde{\epsilon}((\beta, \overline{0}), (0, \overline{1})) \mu f(b_\beta, d_\delta)] \\ = \tilde{\epsilon}((\beta, \overline{1}), (\gamma, \overline{0})) \tilde{f}(a_\alpha, c_\gamma) \tilde{f}(vb_\beta, vd_\delta). \end{aligned}$$

ii) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. Let us show that

$$\begin{aligned} \tilde{f}(a_\alpha(vb_\beta), (vc_\gamma)d_\delta) + \tilde{\epsilon}((\gamma, \overline{1}), (\delta, \overline{0})) \tilde{\epsilon}((\beta, \overline{1}), (\gamma + \delta, \overline{1})) \tilde{f}(a_\alpha d_\delta, (vc_\gamma)(vb_\beta)) \\ = \tilde{\epsilon}((\beta, \overline{1}), (\gamma, \overline{1})) \tilde{f}(a_\alpha, vc_\gamma) \tilde{f}(vb_\beta, d_\delta) = 0 \end{aligned}$$

if and only if  $A$  is associative.

By (18), (16) and Proposition 3.10 i) and ii)

$$\begin{aligned}
& \tilde{f}(a_\alpha(vb_\beta), (vc_\gamma)d_\delta) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{0}))\tilde{\epsilon}((\beta, \bar{1}), (\gamma + \delta, \bar{1}))\tilde{f}(a_\alpha d_\delta, (vc_\gamma)(vb_\beta)) \\
&= \tilde{f}(\tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))v(\overline{a_\alpha b_\beta}), \epsilon(\gamma, \delta)v(d_\delta c_\gamma)) \\
&\quad + \epsilon(\gamma + \beta, \delta)\epsilon(\beta, \gamma)\tilde{\epsilon}((\beta + \gamma, \bar{0}), (0, \bar{1}))f(a_\alpha d_\delta, \tilde{\epsilon}((\gamma, \bar{0}), (\beta, \bar{1}))\mu b_\beta \overline{c_\gamma}) \\
&= \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))\epsilon(\gamma, \delta)[- \tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\mu f(\overline{a_\alpha b_\beta}, d_\delta c_\gamma)] \\
&\quad + \epsilon(\gamma + \beta, \delta)\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(a_\alpha d_\delta, b_\beta \overline{c_\gamma}) \\
&= \epsilon(\gamma, \delta)\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu[\epsilon(\beta, \delta)f(a_\alpha d_\delta, b_\beta \overline{c_\gamma}) - f(\overline{a_\alpha b_\beta}, d_\delta c_\gamma)] \\
&= \epsilon(\gamma, \delta)\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu[\epsilon(\beta, \delta)\epsilon(\beta, \gamma)f((a_\alpha d_\delta)c_\gamma, b_\beta) - \epsilon(\beta, \delta + \gamma)f(a_\alpha(d_\delta c_\gamma), b_\beta)] \\
&= \epsilon(\gamma, \delta)\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\epsilon(\beta, \delta + \gamma)\mu f((a_\alpha d_\delta)c_\gamma - a_\alpha(d_\delta c_\gamma), b_\beta).
\end{aligned}$$

The last term is 0 if and only if  $(a_\alpha d_\delta)c_\gamma - a_\alpha(d_\delta c_\gamma) = 0$  since  $f|_{(A_\beta + A_{-\beta}) \times (A_\beta + A_{-\beta})}$  is nondegenerate for all  $\beta \in \Gamma$ .

iii) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. We have that

$$\begin{aligned}
& \tilde{f}(a_\alpha b_\beta, (vc_\gamma)(vd_\delta)) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{1}))\tilde{\epsilon}((\beta, \bar{0}), (\gamma + \delta, \bar{0}))\tilde{f}(a_\alpha(vd_\delta), (vc_\gamma)b_\beta) \\
&= \tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{1}))\tilde{f}(a_\alpha, vc_\gamma)\tilde{f}(b_\beta, vd_\delta) = 0
\end{aligned}$$

if and only if  $A$  is associative.

The proof is analogous to the item ii).

iv) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. We see that

$$\begin{aligned}
& \tilde{f}((va_\alpha)b_\beta, c_\gamma(vd_\delta)) + \tilde{\epsilon}((\beta + \gamma, \bar{0}), (\delta, \bar{1}))\epsilon(\beta, \gamma)\tilde{f}((va_\alpha)(vd_\delta), c_\gamma b_\beta) \\
&= \epsilon(\beta, \gamma)\tilde{f}(va_\alpha, c_\gamma)\tilde{f}(b_\beta, vd_\delta) = 0
\end{aligned}$$

if and only if  $A$  is associative.

The proof is similar to item ii).

v) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. We note that

$$\begin{aligned}
& \tilde{f}((va_\alpha)(vb_\beta), c_\gamma d_\delta) + \epsilon(\gamma, \delta)\tilde{\epsilon}((\beta, \bar{1}), (\delta, \bar{0}))\tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\tilde{f}((va_\alpha)d_\delta, c_\gamma(vb_\beta)) \\
&= \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{0}))\tilde{f}(va_\alpha, c_\gamma)\tilde{f}(vb_\beta, d_\delta) = 0
\end{aligned}$$

if and only if  $A$  is associative.

The proof is analogous to the item ii).

vi) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous elements. Let us prove that

$$\begin{aligned}
& \tilde{f}((va_\alpha)b_\beta, (vc_\gamma)d_\delta) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{0}))\epsilon(\beta, \delta)\tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{1}))\tilde{f}((va_\alpha)d_\delta, (vc_\gamma)b_\beta) \\
&= \tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{1}))\tilde{f}(va_\alpha, vc_\gamma)\tilde{f}(b_\beta, d_\delta).
\end{aligned}$$

By (7), (6) and Lemma 4.3 iv) we have

$$\begin{aligned}
& \tilde{f}((va_\alpha)b_\beta, (vc_\gamma)d_\delta) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{0}))\epsilon(\beta, \delta)\tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{1}))\tilde{f}((va_\alpha)d_\delta, (vc_\gamma)b_\beta) \\
&= \tilde{f}(\epsilon(\alpha, \beta)v(b_\beta a_\alpha), \epsilon(\gamma, \delta)v(d_\delta c_\gamma)) \\
&\quad + \epsilon(\beta + \gamma, \delta)\epsilon(\beta, \gamma)\tilde{\epsilon}((\beta + \delta, \bar{0}), (0, \bar{1}))\tilde{f}(\epsilon(\alpha, \delta)v(d_\delta a_\alpha), \epsilon(\gamma, \beta)v(b_\beta c_\gamma)) \\
&= \epsilon(\alpha, \beta)\epsilon(\gamma, \delta)[- \tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta a_\alpha, d_\delta c_\gamma)] \\
&\quad + \epsilon(\alpha + \beta + \gamma, \delta)\tilde{\epsilon}((\beta + \delta, \bar{0}), (0, \bar{1}))[- \tilde{\epsilon}((\alpha + \delta, \bar{0}), (0, \bar{1}))\mu f(d_\delta a_\alpha, b_\beta c_\gamma)] \\
&= -\epsilon(\alpha, \beta)\epsilon(\gamma, \delta)\tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta a_\alpha, d_\delta c_\gamma) \\
&\quad - \epsilon(\alpha + \beta + \gamma, \delta)\tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\mu[\epsilon(\alpha + \delta, \beta + \gamma)f(b_\beta c_\gamma, d_\delta a_\alpha)] \\
&= -\tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\epsilon(\alpha, \beta)\epsilon(\gamma, \delta)\mu[f(b_\beta a_\alpha, d_\delta c_\gamma) + \epsilon(\delta, \gamma)\epsilon(\alpha, \gamma + \delta)f(b_\beta c_\gamma, d_\delta a_\alpha)] \\
&= -\tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\epsilon(\alpha, \beta)\epsilon(\gamma, \delta)\mu[\epsilon(\alpha, \delta)f(b_\beta, d_\delta)f(a_\alpha, c_\gamma)] \\
&= -\tilde{\epsilon}((\alpha + \beta, \bar{0}), (0, \bar{1}))\epsilon(\alpha, \beta)\epsilon(\gamma, \delta)\mu[\epsilon(\beta, \alpha + \gamma)\epsilon(\delta, \gamma)f(b_\beta, d_\delta)f(a_\alpha, c_\gamma)] \\
&= \epsilon(\beta, \gamma)\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))[- \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))\mu f(a_\alpha, c_\gamma)]f(b_\beta, d_\delta) \\
&= \tilde{\epsilon}((\beta, \bar{0}), (\gamma, \bar{1}))\tilde{f}(va_\alpha, vc_\gamma)f(b_\beta, d_\delta).
\end{aligned}$$

vii) Let  $a_\alpha, b_\beta, c_\gamma, d_\delta \in A$  be homogeneous. Let us show that

$$\begin{aligned}
& \tilde{f}((va_\alpha)(vb_\beta), (vc_\gamma)(vd_\delta)) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{1}))\tilde{\epsilon}((\beta, \bar{1}), (\gamma + \delta, \bar{0}))\tilde{f}((va_\alpha)(vd_\delta), (vc_\gamma)(vb_\beta)) \\
&= \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{1}))\tilde{f}(va_\alpha, vc_\gamma)\tilde{f}(vb_\beta, vd_\delta).
\end{aligned}$$

Using the Lemma 4.3 iii) we obtain

$$\begin{aligned}
& \tilde{f}((va_\alpha)(vb_\beta), (vc_\gamma)(vd_\delta)) + \tilde{\epsilon}((\gamma, \bar{1}), (\delta, \bar{1}))\tilde{\epsilon}((\beta, \bar{1}), (\gamma + \delta, \bar{0}))\tilde{f}((va_\alpha)(vd_\delta), (vc_\gamma)(vb_\beta)) \\
&= \tilde{f}(\tilde{\epsilon}((\alpha, \bar{0}), (\beta, \bar{1}))\mu b_\beta \bar{a}_\alpha, \tilde{\epsilon}((\gamma, \bar{0}), (\delta, \bar{1}))\mu d_\delta \bar{c}_\gamma) \\
&\quad + \epsilon(\gamma + \beta, \delta)\epsilon(\beta, \gamma)\tilde{f}(\tilde{\epsilon}((\alpha, \bar{0}), (\delta, \bar{1}))\mu d_\delta \bar{a}_\alpha, \tilde{\epsilon}((\gamma, \bar{0}), (\beta, \bar{1}))\mu b_\beta \bar{c}_\gamma) \\
&= \tilde{\epsilon}((\alpha + \gamma, \bar{0}), (0, \bar{1}))\mu^2[\epsilon(\alpha, \beta)\epsilon(\gamma, \delta)f(b_\beta \bar{a}_\alpha, d_\delta \bar{c}_\gamma) + \epsilon(\alpha + \beta + \gamma, \delta)f(d_\delta \bar{a}_\alpha, b_\beta \bar{c}_\gamma)] \\
&= \tilde{\epsilon}((\alpha + \gamma, \bar{0}), (0, \bar{1}))\mu^2\epsilon(\beta, \gamma)f(a_\alpha, c_\gamma)f(b_\beta, d_\delta) \\
&= \epsilon(\beta, \gamma)\tilde{\epsilon}((\beta + \gamma, \bar{0}), (0, \bar{1}))[- \tilde{\epsilon}((\alpha, \bar{0}), (0, \bar{1}))\mu f(a_\alpha, c_\gamma)][- \tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta, d_\delta)] \\
&= \tilde{\epsilon}((\beta, \bar{1}), (\gamma, \bar{1}))\tilde{f}(va_\alpha, vc_\gamma)\tilde{f}(vb_\beta, vd_\delta).
\end{aligned}$$

viii) Let  $a_0, b_\beta, d_\delta \in A$  be homogeneous elements. Let us see that

$$\tilde{f}(a_0(vb_\beta), a_0(vd_\delta)) = q_0(a_0)\tilde{f}(vb_\beta, vd_\delta).$$

By (5)

$$\begin{aligned}
\tilde{f}(a_0(vb_\beta), a_0(vd_\delta)) &= \tilde{f}(v(\bar{a}_0 b_\beta), v(\bar{a}_0 d_\delta)) = -\tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(\bar{a}_0 b_\beta, \bar{a}_0 d_\delta) \\
&= q_0(\bar{a}_0)[- \tilde{\epsilon}((\beta, \bar{0}), (0, \bar{1}))\mu f(b_\beta, d_\delta)] = q_0(a_0)\tilde{f}(vb_\beta, vd_\delta).
\end{aligned}$$

ix) Let  $a_0, b_\beta, d_\delta \in A$  be homogeneous elements. Let us prove that

$$\tilde{f}((vb_\beta)a_0, (vd_\delta)a_0) = q_0(a_0)\tilde{f}(vb_\beta, vd_\delta).$$

By (5)

$$\begin{aligned}\tilde{f}((vb_\beta)a_0, (vd_\delta)a_0) &= \tilde{f}(v(a_0b_\beta), v(a_0d_\delta)) = -\tilde{\epsilon}((\beta, \bar{0})(0, \bar{1}))\mu f(a_0b_\beta, a_0d_\delta) \\ &= q_0(a_0)[-\tilde{\epsilon}((\beta, \bar{0})(0, \bar{1}))\mu f(b_\beta, d_\delta)] = q_0(a_0)\tilde{f}(vb_\beta, vd_\delta).\end{aligned}$$

The other cases are easy to verify. Therefore,  $GCD(A, \mu, \tilde{\epsilon})$  is a composition  $\tilde{\epsilon}$ -algebra if and only if  $A$  is associative.  $\square$

**Examples 4.5.** The following examples are composition  $\epsilon$ -algebras.

In the first 4 examples we assume that  $F$  is a field of characteristic different from 2.

- I.  $\Gamma = \mathbb{Z}_1$  the trivial group,  $A = A_0 = F$  and  $q_0(a) = a^2$  for all  $a \in F$ .  $A$  is a composition algebra.
- II.  $\tilde{K}(\mu) = GCD(F, \mu, \epsilon) = F \oplus Fe_1$  where  $e_1^2 = \mu$ ,  $\mu \in F^\times$  and  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_2$ .  $\tilde{K}(\mu)$  is a composition algebra.  $\tilde{K}(\mu) = \bigoplus_{\gamma \in \mathbb{Z}_2} \tilde{K}(\mu)_\gamma$  where  $\tilde{K}(\mu)_{\bar{0}} = F$  and  $\tilde{K}(\mu)_{\bar{1}} = Fe_1$ . The bilinear form  $f : \tilde{K}(\mu) \times \tilde{K}(\mu) \rightarrow F$  is defined as follows:  $f(1, 1) = 2$ ,  $f(1, e_1) = f(e_1, 1) = 0$  and  $f(e_1, e_1) = -2\mu$ . The algebra  $\tilde{K}(\mu)$  is associative and commutative.
- III.  $Q^{(l)}(\mu, \psi) = GCD(\tilde{K}(\mu), \psi, \epsilon_l)$  with  $\psi \neq 0$ ,  $l \in \{1, -1\}$  and  $\epsilon_l$  the commutation factor on  $\mathbb{Z}_2^2$  defined by  $\epsilon_l((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = l$ ,  $\epsilon_l((\bar{1}, \bar{0}), (\bar{1}, \bar{0})) = \epsilon_l((\bar{0}, \bar{1}), (\bar{0}, \bar{1})) = 1$ .  
 $Q^{(l)}(\mu, \psi) = \bigoplus_{\gamma \in \mathbb{Z}_2^2} Q^{(l)}(\mu, \psi)_\gamma$  with

$$\begin{aligned}Q^{(l)}(\mu, \psi)_{(\bar{0}, \bar{0})} &= F, & Q^{(l)}(\mu, \psi)_{(\bar{1}, \bar{0})} &= Fe_1, \\ Q^{(l)}(\mu, \psi)_{(\bar{0}, \bar{1})} &= Fe_2, & Q^{(l)}(\mu, \psi)_{(\bar{1}, \bar{1})} &= Fe_3,\end{aligned}$$

where  $e_2 := v_{(\bar{0}, \bar{1})}$  and  $e_3 := v_{(\bar{0}, \bar{1})}e_1$ . The multiplication in  $Q^{(l)}(\mu, \psi)$  is given by the following table:

$\cdot$	1	$e_1$	$e_2$	$e_3$
1	1	$e_1$	$e_2$	$e_3$
$e_1$	$e_1$	$\mu$	$-le_3$	$-l\mu e_2$
$e_2$	$e_2$	$e_3$	$\psi$	$\psi e_1$
$e_3$	$e_3$	$\mu e_2$	$-l\psi e_1$	$-l\mu\psi$

The bilinear form  $f : Q^{(l)}(\mu, \psi) \times Q^{(l)}(\mu, \psi) \rightarrow F$  is defined as follows:  $f(1, 1) = 2$ ,  $f(e_1, e_1) = -2\mu$ ,  $f(e_2, e_2) = -2\psi$ ,  $f(e_3, e_3) = 2l\mu\psi$ , and  $f(e_i, e_j) = f(1, e_j) = 0$  for  $i \neq j$ ,  $i, j = 1, 2, 3$ .  $Q^{(1)}(\mu, \psi)$  is the algebra of generalized quaternions and  $Q^{(-1)}(\mu, \psi) \cong F[x, y]/\langle x^2 - \mu, y^2 - \psi \rangle$ .  $Q^{(l)}(\mu, \psi) = GCD(K(\mu), \psi, \epsilon_l)$  is associative but not  $\epsilon_l$ -commutative.

- IV.  $C^{(l, j)}(\mu, \psi, \lambda) = GCD(Q^{(l)}(\mu, \psi), \lambda, \tilde{\epsilon}_{(l, j)})$  with  $\lambda \neq 0$ ,  $j = (j_1, j_2)$ ,  $j_1, j_2 \in \{1, -1\}$  and  $\tilde{\epsilon}_{(l, j)}$  the commutation factor on  $\mathbb{Z}_2^3$  defined by:

$$\tilde{\epsilon}_{(l, j)}((\bar{1}, \bar{0}, \bar{0}), (\bar{0}, \bar{1}, \bar{0})) = l, \quad \tilde{\epsilon}_{(l, j)}((\bar{1}, \bar{0}, \bar{0}), (\bar{0}, \bar{0}, \bar{1})) = j_1, \quad \tilde{\epsilon}_{(l, j)}((\bar{0}, \bar{1}, \bar{0}), (\bar{0}, \bar{0}, \bar{1})) = j_2$$

and  $\tilde{\epsilon}_{(l,j)}(\alpha, \alpha) = 1$  for all  $\alpha \in \mathbb{Z}_2^3$ .

$C^{(l,j)}(\mu, \psi, \lambda) = \bigoplus_{\gamma \in \mathbb{Z}_2^3} C^{(l,j)}(\mu, \psi, \lambda)_\gamma$  with

$$\begin{aligned} C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{0}, \bar{0})} &= F, & C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{0}, \bar{0})} &= Fe_1, & C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{1}, \bar{0})} &= Fe_2, \\ C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{1}, \bar{0})} &= Fe_3, & C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{0}, \bar{1})} &= Fe_4, & C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{0}, \bar{1})} &= Fe_5, \\ C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{1}, \bar{1})} &= Fe_6, & C^{(l,j)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{1}, \bar{1})} &= Fe_7, \end{aligned}$$

where  $e_4 := v_{(\bar{0}, \bar{0}, \bar{1})}$ ,  $e_5 := v_{(\bar{0}, \bar{0}, \bar{1})}e_1$ ,  $e_6 := v_{(\bar{0}, \bar{0}, \bar{1})}e_2$  and  $e_7 := v_{(\bar{0}, \bar{0}, \bar{1})}e_3$ . The multiplication in  $C^{(l,j)}(\mu, \psi, \lambda)$  is given by the following table:

$\cdot$	1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
1	1	$e_1$	$e_2$	$e_3$	$e_4$	$e_5$	$e_6$	$e_7$
$e_1$	$e_1$	$\mu$	$-le_3$	$-l\mu e_2$	$-j_1e_5$	$-j_1\mu e_4$	$lj_1e_7$	$lj_1\mu e_6$
$e_2$	$e_2$	$e_3$	$\psi$	$\psi e_1$	$-j_2e_6$	$-j_2e_7$	$-j_2\psi e_4$	$-j_2\psi e_5$
$e_3$	$e_3$	$\mu e_2$	$-l\psi e_1$	$-l\mu\psi$	$-j_1j_2e_7$	$-j_1j_2\mu e_6$	$lj_1j_2\psi e_5$	$lj_1j_2\mu\psi e_4$
$e_4$	$e_4$	$e_5$	$e_6$	$e_7$	$\lambda$	$\lambda e_1$	$\lambda e_2$	$\lambda e_3$
$e_5$	$e_5$	$\mu e_4$	$le_7$	$l\mu e_6$	$-j_1\lambda e_1$	$-j_1\mu\lambda$	$-lj_1\lambda e_3$	$-lj_1\mu\lambda e_2$
$e_6$	$e_6$	$-e_7$	$\psi e_4$	$-\psi e_5$	$-j_2\lambda e_2$	$j_2\lambda e_3$	$-j_2\psi\lambda$	$j_2\psi\lambda e_1$
$e_7$	$e_7$	$-\mu e_6$	$l\psi e_5$	$-l\mu\psi e_4$	$-j_1j_2\lambda e_3$	$j_1j_2\mu\lambda e_2$	$-lj_1j_2\psi\lambda e_1$	$lj_1j_2\mu\psi\lambda$

The bilinear form  $f : C^{(l,j)}(\mu, \psi, \lambda) \times C^{(l,j)}(\mu, \psi, \lambda) \rightarrow F$  is defined as follows:

$$\begin{aligned} f(1, 1) &= 2, & f(e_1, e_1) &= -2\mu, & f(e_2, e_2) &= -2\psi, \\ f(e_3, e_3) &= 2l\mu\psi, & f(e_4, e_4) &= -2\lambda, & f(e_5, e_5) &= 2j_1\mu\lambda, \\ f(e_6, e_6) &= 2j_2\psi\lambda, & f(e_7, e_7) &= -2lj_1j_2\mu\psi\lambda \end{aligned}$$

and  $f(e_i, e_j) = f(1, e_j) = 0$  for  $1 \leq i \neq j \leq 7$ .

$C^{(1, (1,1))}(\mu, \psi, \lambda)$  is the Cayley-Dickson algebra.  $C^{(1,j)}(\mu, \psi, \lambda)$  is not alternative if  $j = (j_1, j_2) \in \{(1, -1), (-1, 1), (-1, -1)\}$ , for example:  $(e_1 + e_4, e_1 + e_4, e_2) = 2(1 - j_1)e_7$  or  $(e_2 + e_4, e_2 + e_4, e_1) = 2(1 - j_2)e_7$  is different from zero.  $C^{(-1,j)}(\mu, \psi, \lambda)$  is not alternative, for example:  $(e_1 + e_2, e_1 + e_2, e_4) = -4j_1j_2e_7 \neq 0$ .

In the following examples we will consider the field  $F$  of arbitrary characteristic.

- V.  $\Gamma = \mathbb{Z}_1$  the trivial group.  $A = A_0 = K(\mu) = F + Fv_1$  where  $v_1^2 = v_1 + \mu$  and  $4\mu + 1 \neq 0$ , the involution is  $\overline{r + sv_1} = (r + s) - sv_1$  and the quadratic form is  $q_0(a) = a\bar{a}$ . If the polynomial  $x^2 - x - \mu$  is irreducible in  $F[x]$  then the algebra  $K(\mu)$  is a field (a separable quadratic extension of the field  $F$ ), otherwise  $K(\mu) = F \oplus F$ .
- VI.  $\tilde{Q}(\mu, \psi) = GCD(K(\mu), \psi, \epsilon) = K(\mu) \oplus v_2K(\mu)$  with  $\psi \neq 0$ ,  $v_2^2 = \psi$  and  $\epsilon$  the trivial commutation factor on  $\mathbb{Z}_2$ .  $\tilde{Q}(\mu, \psi)$  is the algebra of generalized quaternions.  $\tilde{Q}(\mu, \psi) = \bigoplus_{\gamma \in \mathbb{Z}_2} \tilde{Q}(\mu, \psi)_\gamma$  where  $\tilde{Q}(\mu, \psi)_{\bar{0}} = K(\mu) = F + Fv_1$ , and  $\tilde{Q}(\mu, \psi)_{\bar{1}} = v_2K(\mu) = v_2(F + Fv_1) = Fv_2 + Fv_2v_1$  and  $u_2 := v_2v_1$ . The multiplication in  $\tilde{Q}(\mu, \psi)$  is given by the following table:

$\cdot$	1	$v_1$	$v_2$	$u_2$
1	1	$v_1$	$v_2$	$u_2$
$v_1$	$v_1$	$v_1 + \mu$	$v_2 - u_2$	$-\mu v_2$
$v_2$	$v_2$	$u_2$	$\psi$	$\psi v_1$
$u_2$	$u_2$	$u_2 + \mu v_2$	$\psi(1 - v_1)$	$-\mu\psi$

The bilinear form  $f : \tilde{Q}(\mu, \psi) \times \tilde{Q}(\mu, \psi) \rightarrow F$  is defined as follows:

$$\begin{aligned} f(1, 1) &= 2, & f(v_1, 1) &= 1, & f(v_1, v_1) &= -2\mu, \\ f(v_2, v_2) &= -2\psi, & f(v_2, u_2) &= -\psi, & f(u_2, u_2) &= 2\mu\psi \end{aligned}$$

and  $f(\tilde{Q}(\mu, \psi)_{\bar{0}}, \tilde{Q}(\mu, \psi)_{\bar{1}}) = 0$ . It is easy to see that  $\tilde{Q}(\mu, \psi)$  is associative but not commutative.

VII.  $C^{(l)}(\mu, \psi, \lambda) = GCD(\tilde{Q}(\mu, \psi), \lambda, \tilde{\epsilon}_l)$  with  $\lambda \neq 0$ ,  $l \in \{1, -1\}$  and  $\tilde{\epsilon}_l$  the commutation factor on  $\mathbb{Z}_2^2$  defined by:  $\tilde{\epsilon}_l((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = l$  and  $\tilde{\epsilon}_l(\alpha, \alpha) = 1$  for all  $\alpha \in \mathbb{Z}_2^2$ .

$C^{(l)}(\mu, \psi, \lambda) = \bigoplus_{\gamma \in \mathbb{Z}_2^2} C^{(l)}(\mu, \psi, \lambda)_\gamma$  with

$$\begin{aligned} C^{(l)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{0})} &= F + Fv_1, & C^{(l)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{0})} &= Fv_2 + Fu_2, \\ C^{(l)}(\mu, \psi, \lambda)_{(\bar{0}, \bar{1})} &= Fv_3 + Fu_3, & C^{(l)}(\mu, \psi, \lambda)_{(\bar{1}, \bar{1})} &= Fv_4 + Fu_4, \end{aligned}$$

where  $v_3 := v_{(\bar{0}, \bar{1})}$ ,  $u_3 := v_{(\bar{0}, \bar{1})}v_1$ ,  $v_4 := v_{(\bar{0}, \bar{1})}v_2$  and  $u_4 := v_{(\bar{0}, \bar{1})}u_2$ . The multiplication in  $C^{(l)}(\mu, \psi, \lambda)$  is given by the following table:

$\cdot$	1	$v_1$	$v_2$	$u_2$	$v_3$	$u_3$	$v_4$	$u_4$
1	1	$v_1$	$v_2$	$u_2$	$v_3$	$u_3$	$v_4$	$u_4$
$v_1$	$v_1$	$v_1 + \mu$	$v_2 - u_2$	$-\mu v_2$	$v_3 - u_3$	$-\mu v_3$	$u_4$	$u_4 + \mu v_4$
$v_2$	$v_2$	$u_2$	$\psi$	$\psi v_1$	$-lv_4$	$-lu_4$	$-l\psi v_3$	$-l\psi u_3$
$u_2$	$u_2$	$u_2 + \mu v_2$	$\psi(1 - v_1)$	$-\mu\psi$	$-lu_4$	$-l(u_4 + \mu v_4)$	$l\psi(u_3 - v_3)$	$l\mu\psi v_3$
$v_3$	$v_3$	$u_3$	$v_4$	$u_4$	$\lambda$	$\lambda v_1$	$\lambda v_2$	$\lambda u_2$
$u_3$	$u_3$	$u_3 + \mu v_3$	$u_4$	$u_4 + \mu v_4$	$\lambda(1 - v_1)$	$-\mu\lambda$	$\lambda(v_2 - u_2)$	$-\mu\lambda v_2$
$v_4$	$v_4$	$v_4 - u_4$	$\psi v_3$	$\psi(v_3 - u_3)$	$-l\lambda v_2$	$l\lambda(u_2 - v_2)$	$-l\psi\lambda$	$l\psi\lambda(v_1 - 1)$
$u_4$	$u_4$	$-\mu v_4$	$\psi u_3$	$-\mu\psi v_3$	$-l\lambda u_2$	$l\mu\lambda v_2$	$-l\psi\lambda v_1$	$l\mu\psi\lambda$

The bilinear form  $f : C^{(l)}(\mu, \psi, \lambda) \times C^{(l)}(\mu, \psi, \lambda) \rightarrow F$  is defined as follows:

$$\begin{aligned} f(1, 1) &= 2, & f(1, v_1) &= 1, & f(v_1, v_1) &= -2\mu, \\ f(v_2, v_2) &= -2\psi, & f(v_2, u_2) &= -\psi, & f(u_2, u_2) &= 2\mu\psi, \\ f(v_3, v_3) &= -2\lambda, & f(v_3, u_3) &= -\lambda, & f(u_3, u_3) &= 2\mu\lambda, \\ f(v_4, v_4) &= 2l\psi\lambda, & f(v_4, u_4) &= l\psi\lambda, & f(u_4, u_4) &= -2l\mu\psi\lambda \end{aligned}$$

and  $f(C^{(l)}(\mu, \psi, \lambda)_\alpha, C^{(l)}(\mu, \psi, \lambda)_\beta) = 0$  if  $\alpha \neq \beta$ .

$C^{(1)}(\mu, \psi, \lambda)$  is a Cayley-Dickson algebra. If the characteristic of  $F$  is different from 2 then  $C^{(-1)}(\mu, \psi, \lambda)$  is not alternative, for example:  $(v_2 + v_3, v_2 + v_3, v_1) = 2v_4 - 4u_4 \neq 0$ .

$C^{(-1)}(\mu, \psi, \lambda)$  is not associative, for example:  $(v_1, u_2, v_3) = 2l\mu v_4 + lu_4 \neq 0$ .

VIII.  $\Gamma = \mathbb{Z}_1$  the trivial group.  $A = A_0 = Q(\mu, \psi) = CD(K(\mu), \psi)$ , with  $\psi \neq 0$ , is the algebra of generalized quaternions that is obtained by applying the Cayley-Dickson process to  $K(\mu)$ .

IX.  $\tilde{C}(\mu, \psi, \lambda) = GCD(Q(\mu, \psi), \lambda, \epsilon) = Q(\mu, \psi) \oplus Q(\mu, \psi)v_3$ , with  $\lambda \neq 0$ ,  $v_3^2 = \lambda$  and  $\epsilon$  the trivial commutation factor on  $\mathbb{Z}_2$ .  $\tilde{C}(\mu, \psi, \lambda)$  is a Cayley-Dickson algebra.  $\tilde{C}(\mu, \psi, \lambda) =$

$\bigoplus_{\gamma \in \mathbb{Z}_2} \tilde{C}(\mu, \psi, \lambda)_\gamma$ , where  $\tilde{C}(\mu, \psi, \lambda)_{\bar{0}} = Q(\mu, \psi)$  and  $\tilde{C}(\mu, \psi, \lambda)_{\bar{1}} = Q(\mu, \psi)v_3$ . It is easy to see that  $\tilde{C}(\mu, \psi, \lambda)$  is not associative.

- X.  $\Gamma = \mathbb{Z}_1$  the trivial group.  $A = A_0 = C(\mu, \psi, \lambda) = CD(Q(\mu, \psi), \lambda)$ , with  $\lambda \neq 0$ , is the Cayley-Dickson algebra that is obtained by applying the Cayley-Dickson process to  $Q(\mu, \psi)$ .
- XI.  $Q_n(0, 1) = \bigoplus_{\gamma \in \mathbb{Z}_n} Q_n(0, 1)_\gamma$  is a split generalized quaternions algebra with  $\mathbb{Z}_n$ -grading given by  $Q_n(0, 1)_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $Q_n(0, 1)_{\bar{1}} = Fu_1$ ,  $Q_n(0, 1)_{-\bar{1}} = Fv_1$ , with  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$ . Let  $\epsilon$  be the trivial commutation factor on  $\mathbb{Z}_n$ . The multiplication table of  $Q_n(0, 1)$  is as follows:

$\cdot$	$e_1$	$e_2$	$u_1$	$v_1$
$e_1$	$e_1$	$0$	$u_1$	$0$
$e_2$	$0$	$e_2$	$0$	$v_1$
$u_1$	$0$	$u_1$	$0$	$-e_1$
$v_1$	$v_1$	$0$	$-e_2$	$0$

The bilinear form  $f : Q_n(0, 1) \times Q_n(0, 1) \rightarrow F$  is defined as follows:  $f(1, 1) = 2$ ,  $f(e_i, 1) = f(e_1, e_2) = 1$ ,  $f(e_i, e_i) = 0$ ,  $f(u_1, v_1) = 1$ ,  $f(e_i, u_1) = f(e_i, v_1) = 0$  and  $f(u_1, u_1) = f(v_1, v_1) = 0$  for  $i \in \{1, 2\}$ . It is clear that  $Q_n(0, 1)$  is a composition  $\epsilon$ -algebra.  $Q_n(0, 1)$  is associative but not commutative.

- XII.  $D_n^{(l)}(0, 1, -1) = GCD(Q_n(0, 1), -1, \epsilon_l)$  with  $l \in \{1, -1\}$  if  $n$  is even, or  $l = 1$  if  $n$  is odd, and  $\epsilon_l$  the commutation factor on  $\mathbb{Z}_n \times \mathbb{Z}_2$  defined by:

$$\epsilon_l((\bar{1}, \bar{0}), (\bar{1}, \bar{0})) = \epsilon_l((\bar{0}, \bar{1}), (\bar{0}, \bar{1})) = 1, \quad \epsilon_l((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = l.$$

$D_n^{(l)}(0, 1, -1) = \bigoplus_{\gamma \in \mathbb{Z}_n \times \mathbb{Z}_2} D_n^{(l)}(0, 1, -1)_\gamma$ , where

$$\begin{aligned} D_n^{(l)}(0, 1, -1)_{(\bar{0}, \bar{0})} &= Fe_1 + Fe_2, & D_n^{(l)}(0, 1, -1)_{(\bar{1}, \bar{0})} &= Fu_1, \\ D_n^{(l)}(0, 1, -1)_{(-\bar{1}, \bar{0})} &= Fv_1, & D_n^{(l)}(0, 1, -1)_{(\bar{0}, \bar{1})} &= Fu_2 + Fv_2, \\ D_n^{(l)}(0, 1, -1)_{(\bar{1}, \bar{1})} &= Fv_3, & D_n^{(l)}(0, 1, -1)_{(-\bar{1}, \bar{1})} &= Fu_3, \end{aligned}$$

where  $u_2 := -v_{(\bar{0}, \bar{1})}e_2$ ,  $v_2 := -v_{(\bar{0}, \bar{1})}e_1$ ,  $u_3 := v_{(\bar{0}, \bar{1})}v_1$  and  $v_3 := v_{(\bar{0}, \bar{1})}u_1$ . The multiplication in  $D_n^{(l)}(0, 1, -1)$  is given by the following table:

$\cdot$	$e_1$	$e_2$	$u_1$	$v_1$	$u_2$	$v_2$	$u_3$	$v_3$
$e_1$	$e_1$	$0$	$u_1$	$0$	$u_2$	$0$	$u_3$	$0$
$e_2$	$0$	$e_2$	$0$	$v_1$	$0$	$v_2$	$0$	$v_3$
$u_1$	$0$	$u_1$	$0$	$-e_1$	$lv_3$	$0$	$-lv_2$	$0$
$v_1$	$v_1$	$0$	$-e_2$	$0$	$0$	$lu_3$	$0$	$-lu_2$
$u_2$	$0$	$u_2$	$-v_3$	$0$	$0$	$-e_1$	$v_1$	$0$
$v_2$	$v_2$	$0$	$0$	$-u_3$	$-e_2$	$0$	$0$	$u_1$
$u_3$	$0$	$u_3$	$v_2$	$0$	$-lv_1$	$0$	$0$	$-le_1$
$v_3$	$v_3$	$0$	$0$	$u_2$	$0$	$-lu_1$	$-le_2$	$0$

The bilinear form  $f : D_n^{(l)}(0, 1, -1) \times D_n^{(l)}(0, 1, -1) \rightarrow F$  is defined as follows:

$$\begin{aligned} f(e_1, e_2) &= 1, & f(e_i, e_i) &= 0, & f(u_1, v_1) &= f(u_2, v_2) = 1, \\ f(u_3, v_3) &= l, & f(u_j, u_k) &= 0, & f(v_j, v_k) &= 0, \\ f(e_i, u_j) &= 0, & f(e_i, v_j) &= 0, & f(u_j, v_k) &= 0 \text{ if } j \neq k, \end{aligned}$$

for all  $i \in \{1, 2\}$ ,  $j \in \{1, 2, 3\}$  and  $k \in \{1, 2, 3\}$ .

$D_n^{(1)}(0, 1, -1)$  is the split Cayley-Dickson algebra. If the characteristic of  $F$  is different from 2 then  $D_n^{(-1)}(0, 1, -1)$  is not alternative, for example:  $(u_2 + u_3, u_2 + u_3, v_2) = l(1 - l)u_3 = -2u_3 \neq 0$ .  $D_n^{(l)}(0, 1, -1)$  is not associative, for example:  $(u_1, v_1, u_3) = -u_3 \neq 0$ .

## 5. CLASSIFICATION OF COLOR COMPOSITION ALGEBRAS

The aim of this section is to give a characterization of color composition algebras without restriction in the characteristic of the field. As a reference, we will use the generalized Cayley-Dickson process and the construction of the canonical basis of the split Cayley algebra, see Chapter 4 above and [EK13, Chapter 4] respectively.

Let us recall some important facts. Let  $\Gamma$  be an abelian group and  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a  $\Gamma$ -graded algebra. Then the set

$$\text{Supp}_\Gamma(A) := \{\alpha \in \Gamma \mid A_\alpha \neq 0\}$$

is said the *support* of the grading. We note that  $A = \bigoplus_{\gamma \in \langle \text{Supp}_\Gamma(A) \rangle} A_\gamma$ , where  $\langle \text{Supp}_\Gamma(A) \rangle$  is the subgroup of  $\Gamma$  generated by  $\text{Supp}_\Gamma(A)$ . From now on, it will be always assumed that  $\langle \text{Supp}_\Gamma(A) \rangle = \Gamma$ .

We observe that every composition  $\epsilon$ -algebra  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  has a natural  $\mathbb{Z}_2$ -grading. In fact, it follows from (1) that the function

$$\Gamma \rightarrow F^\times, \quad \alpha \rightarrow \epsilon(\alpha, \alpha)$$

is a homomorphism of groups and that  $\epsilon(\alpha, \alpha) = \pm 1$  for all  $\alpha \in \Gamma$ . We define the following sets  $\Gamma_0 = \{\alpha \in \Gamma \mid \epsilon(\alpha, \alpha) = 1\}$  and  $\Gamma_1 = \{\alpha \in \Gamma \mid \epsilon(\alpha, \alpha) \neq 1\}$ . Then,  $\Gamma_0$  is a subgroup of  $\Gamma$  and we have either that  $\Gamma_0 = \Gamma$  (and hence  $\Gamma_1 = \emptyset$ ) or that  $\Gamma_0$  is a subgroup of index 2 in  $\Gamma$  and,  $\Gamma_0$  and  $\Gamma_1$  are the two residue classes modulo  $\Gamma_0$ .

We now set

$$A^{(i)} := \bigoplus_{\gamma \in \Gamma_i} A_\gamma, \quad \text{for } i=0,1,$$

and consider the indices 0, 1 as integers modulo 2; then it follows from the above that the decomposition  $A = A^{(0)} \oplus A^{(1)}$  is a  $\mathbb{Z}_2$ -grading of  $A$ .

We want to classify color composition algebras up to equivalence, in the sense of the following definition.

**Definition 5.1.** Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a  $\Gamma$ -graded composition  $\epsilon$ -algebra and  $B = \bigoplus_{\gamma' \in \Gamma'} B_{\gamma'}$  be a  $\Gamma'$ -graded composition  $\epsilon'$ -algebra. We say that  $A$  and  $B$  are *equivalent* if there exists an isomorphism of algebras  $g : A \rightarrow B$  and an isomorphism of groups  $\theta : \Gamma \rightarrow \Gamma'$  such that  $g(A_\gamma) = B_{\theta(\gamma)}$  for all  $\gamma \in \Gamma$  and  $\epsilon' \circ (\theta \times \theta) = \epsilon$ .

**Definition 5.2.** We say that two composition  $\epsilon$ -algebras,  $\Gamma$ -graded,  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  and  $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$  are *isomorphic* if there exists an isomorphism of algebras  $\varphi : A \rightarrow B$  such that  $\varphi(A_\gamma) = B_\gamma$  for all  $\gamma \in \Gamma$ .

On the one hand, a composition algebra  $C$  with norm  $q$  has zero divisors if and only if  $q(x) = 0$  for some  $x \neq 0$  in  $C$ , see [ZSSS82, Lemma 9, Chapter 2]. When this happens we say that  $C$  is a *split* composition algebra. On the other hand, we say that  $C$  is a *not split* composition algebra if  $C$  does not have zero divisors.

The aim of this chapter is to classify the composition  $\epsilon$ -algebras up to equivalence. For this, we divide the problem into three cases: the first case is  $A^{(1)} = 0$  and  $A_0$  is not split, the second case is  $A^{(1)} = 0$  and  $A_0$  is split and the third case is  $A^{(1)} \neq 0$ . In the first two cases  $A = A^{(0)}$  since we are under the condition that the subgroup of  $\Gamma$  generated by the set  $\text{Supp}_\Gamma(A)$  coincides with  $\Gamma$ .

Before analyzing the different cases we will see some results and definitions that will be useful to achieve our goal.

Let  $f(x, y)$  be the bilinear form associated with a composition  $\epsilon$ -algebra  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$ . Thus  $f$  is  $\epsilon$ -symmetric, that is,  $f|_{(A_\alpha + A_{-\alpha}) \times (A_\alpha + A_{-\alpha})}$  is symmetric if  $\epsilon(\alpha, \alpha) = 1$  and skew-symmetric if  $\epsilon(\alpha, \alpha) = -1$ , for all  $\alpha \in \Gamma$ .

Let  $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$  be a  $\Gamma$ -graded subalgebra of  $A$ . It will be denote by  $(B_\gamma + B_{-\gamma})^\perp$  the orthogonal complement of the subspace  $B_\gamma + B_{-\gamma}$  of  $A_\gamma + A_{-\gamma}$  with respect to the bilinear form  $f|_{(A_\gamma + A_{-\gamma}) \times (A_\gamma + A_{-\gamma})}$ , that is

$$(B_\gamma + B_{-\gamma})^\perp := \{a \in A_\gamma + A_{-\gamma} \mid f|_{(A_\gamma + A_{-\gamma}) \times (A_\gamma + A_{-\gamma})}(a, B_\gamma + B_{-\gamma}) = 0\}.$$

We shall denote by  $B^\perp$  the orthogonal complement of the subspace  $B$  of  $A$  with respect to the bilinear form  $f(x, y)$ , this is

$$B^\perp := \{a \in A \mid f(a, B) = 0\}.$$

Since  $f(A_\alpha, A_\beta) = 0$ , if  $\alpha + \beta \neq 0$ , and  $B$  is a  $\Gamma$ -graded subalgebra of  $A$ , it is easy to check that

$$B^\perp = \sum_{\gamma \in \Gamma} (B_\gamma + B_{-\gamma})^\perp.$$

**Proposition 5.3.** *Let  $A$  be a composition  $\epsilon$ -algebra and  $B$  be a finite dimensional  $\Gamma$ -graded subalgebra of  $A$ , such that  $f|_{B \times B}$  is nondegenerate. Then  $A = B \oplus B^\perp$ .*

*Proof.* Let  $B$  be a finite dimensional  $\Gamma$ -graded subalgebra of  $A$ ,  $B = \bigoplus_{\gamma \in \Gamma} B_\gamma$ , such that  $f|_{B \times B}$  is nondegenerate. Since  $f|_{B \times B}$  is nondegenerate and  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ , it follows from this that  $f|_{(B_\gamma + B_{-\gamma}) \times (B_\gamma + B_{-\gamma})}$  is nondegenerate for all  $\gamma \in \Gamma$ . Moreover, since

$f|_{(A_\gamma + A_{-\gamma}) \times (A_\gamma + A_{-\gamma})}$  is symmetric or skew-symmetric and  $B_\gamma + B_{-\gamma}$  is a finite dimensional subspace of  $A_\gamma + A_{-\gamma}$ , by a well-known theorem we obtain

$$A_\gamma + A_{-\gamma} = (B_\gamma + B_{-\gamma}) \oplus (B_\gamma + B_{-\gamma})^{\perp\gamma}.$$

Therefore

$$\begin{aligned} A &= \sum_{\gamma \in \Gamma} (A_\gamma + A_{-\gamma}) = \sum_{\gamma \in \Gamma} (B_\gamma + B_{-\gamma}) \oplus (B_\gamma + B_{-\gamma})^{\perp\gamma} \\ &= \sum_{\gamma \in \Gamma} (B_\gamma + B_{-\gamma}) \oplus \sum_{\gamma \in \Gamma} (B_\gamma + B_{-\gamma})^{\perp\gamma} = B \oplus B^\perp. \end{aligned}$$

□

**Proposition 5.4.** *Let  $A$  be a composition  $\epsilon$ -algebra and  $B$  be a  $\Gamma$ -graded subalgebra of  $A$ . Then  $B^\perp$  is a  $\Gamma$ -graded vector subspace of  $A$  and  $B^\perp B + B B^\perp \subseteq B^\perp$ . Moreover, when  $B$  is finite-dimensional and contains the identity element  $1$  of the algebra  $A$  and  $f|_{B \times B}$  is nondegenerate, if  $B^\perp \neq 0$  then  $B$  is associative.*

*Proof.* It is clear that  $B^\perp$  is a vector subspace of  $A$ . To show that it is  $\Gamma$ -graded, as  $B^\perp = \sum_{\gamma \in \Gamma} (B_\gamma + B_{-\gamma})^{\perp\gamma}$ , we only need to show that for all  $x = u_\gamma + v_{-\gamma} \in (B_\gamma + B_{-\gamma})^{\perp\gamma}$  with  $\gamma \neq -\gamma$  we have that  $u_\gamma, v_{-\gamma} \in (B_\gamma + B_{-\gamma})^{\perp\gamma}$ . Let  $b_\gamma \in B_\gamma$ , since  $f(A_\gamma, A_\gamma) = 0$  ( $\gamma + \gamma \neq 0$ ) we have

$$0 = f(x, b_\gamma) = f(u_\gamma + v_{-\gamma}, b_\gamma) = f(u_\gamma, b_\gamma) + f(v_{-\gamma}, b_\gamma) = f(v_{-\gamma}, b_\gamma).$$

Thus  $f(v_{-\gamma}, B_\gamma) = 0$ . Since  $f(A_{-\gamma}, A_{-\gamma}) = 0$  ( $-\gamma - \gamma \neq 0$ ) we have  $f(v_{-\gamma}, B_{-\gamma}) = 0$ . Hence  $v_{-\gamma} \in (B_\gamma + B_{-\gamma})^{\perp\gamma}$ . Similarly, we obtain that  $u_\gamma \in (B_\gamma + B_{-\gamma})^{\perp\gamma}$ . Therefore  $B^\perp$  is a  $\Gamma$ -graded vector subspace of  $A$ .

By (6) we have

$$f(a_\alpha, v_\gamma b_\beta) = f(a_\alpha \cdot 1, v_\gamma b_\beta) = f(a_\alpha, v_\gamma) f(1, b_\beta) - \epsilon(\gamma, \beta) f(a_\alpha b_\beta, v_\gamma) = 0$$

for any  $a_\alpha \in B_\alpha$ ,  $b_\beta \in B_\beta$  and  $v_\gamma \in B^\perp$ , and analogously  $f(a_\alpha, b_\beta v_\gamma) = 0$ . Since the elements  $a_\alpha, b_\beta \in B$  and  $v_\gamma \in B^\perp$  are arbitrary, this means that  $B^\perp B + B B^\perp \subseteq B^\perp$ .

Now, suppose that  $B$  is finite-dimensional and contains the identity element  $1$  of the algebra  $A$  and  $f|_{B \times B}$  is nondegenerate. By Proposition 3.10 *vi*)

$$(19) \quad \overline{a_\alpha}(x_\xi b_\beta) + \epsilon(\alpha, \xi) \overline{x_\xi}(a_\alpha b_\beta) = f(a_\alpha, x_\xi) b_\beta = 0$$

for any  $a_\alpha \in B_\alpha$ ,  $b_\beta \in B_\beta$  and  $x_\xi \in B^\perp$ , and by Proposition 3.10 *iv*)

$$(20) \quad \overline{a_\alpha}(x_\xi b_\beta) + \epsilon(\alpha, \xi + \beta) \overline{(x_\xi b_\beta)} a_\alpha = f(a_\alpha, x_\xi b_\beta) = 0,$$

the last term is zero because  $B B^\perp \subseteq B^\perp$ . On the other hand, since  $1 \in B$  we have  $\overline{v_\gamma} = -v_\gamma$  for all  $v_\gamma \in B^\perp$ . Hence by (19) and (20)

$$\epsilon(\alpha, \xi) x_\xi (a_\alpha b_\beta) = \overline{a_\alpha}(x_\xi b_\beta) = -\epsilon(\alpha, \beta + \xi) \overline{(x_\xi b_\beta)} a_\alpha = \epsilon(\alpha, \xi) \epsilon(\alpha, \beta) (x_\xi b_\beta) a_\alpha.$$

Thus

$$(21) \quad x_\xi (a_\alpha b_\beta) = \epsilon(\alpha, \beta) (x_\xi b_\beta) a_\alpha.$$

If  $(a_\alpha, b_\beta, c_\gamma) = (a_\alpha b_\beta)c_\gamma - a_\alpha(b_\beta c_\gamma)$  is the associator of any three homogeneous elements  $a_\alpha, b_\beta, c_\gamma \in B$ , then by (21)

$$\begin{aligned} x_\xi(a_\alpha, b_\beta, c_\gamma) &= x_\xi[(a_\alpha b_\beta)c_\gamma] - x_\xi[a_\alpha(b_\beta c_\gamma)] \\ &= \epsilon(\alpha + \beta, \gamma)(x_\xi c_\gamma)(a_\alpha b_\beta) - \epsilon(\alpha, \beta + \gamma)[x_\xi(b_\beta c_\gamma)]a_\alpha \\ &= \epsilon(\alpha + \beta, \gamma)\epsilon(\alpha, \beta)[(x_\xi c_\gamma)b_\beta]a_\alpha - \epsilon(\alpha, \beta + \gamma)\epsilon(\beta, \gamma)[(x_\xi c_\gamma)b_\beta]a_\alpha = 0 \end{aligned}$$

for any  $x_\xi \in B^\perp$ . Therefore  $B^\perp(B, B, B) = 0$ .

Assume that  $B^\perp \neq 0$ . By Proposition 5.3  $f|_{B^\perp \times B^\perp}$  is nondegenerate. Hence, there exist  $x_\xi, y_{-\xi} \in B^\perp \setminus \{0\}$  such that  $f(x_\xi, y_{-\xi}) \neq 0$ . Let  $a_\alpha, b_\beta, c_\gamma \in B$ ,  $d_\sigma = (a_\alpha, b_\beta, c_\gamma)$  with  $\sigma = \alpha + \beta + \gamma$  and  $e_{-\sigma} \in A_{-\sigma}$ . Since  $B^\perp(B, B, B) = 0$  then  $x_\xi d_\sigma = 0$  and  $y_{-\xi} d_\sigma = 0$ . Therefore, by (6)

$$\epsilon(\sigma, -\xi)f(x_\xi, y_{-\xi})f(d_\sigma, e_{-\sigma}) = f(x_\xi d_\sigma, y_{-\xi} e_{-\sigma}) + \epsilon(-\xi, -\sigma)\epsilon(\sigma, -\xi - \sigma)f(x_\xi e_{-\sigma}, y_{-\xi} d_\sigma) = 0.$$

Since  $f|_{B \times B}$  is nondegenerate, it follows from this that  $d_\sigma = 0$ . Consequently  $(B, B, B) = 0$ . Therefore, if  $B^\perp \neq 0$  then  $B$  is associative.  $\square$

In particular, if  $A$  is a composition  $\epsilon$ -algebra such that  $A_\alpha \neq 0$  for some  $\alpha \in \Gamma \setminus \{0\}$  then  $A_0$  is associative ( $\dim(A_0) \leq 4$ ).

### 5.1. $A^{(1)} = 0$ and $A_0$ is a non-split composition algebra.

**Lemma 5.5.** *Let  $A$  be a composition  $\epsilon$ -algebra and  $B$  be a  $\Gamma$ -graded subalgebra of  $A$  which contains the identity element 1 of the algebra  $A$ . Then for every  $a_\alpha, b_\beta \in B$  and  $v_\gamma \in B^\perp$  homogeneous elements the following relations are valid:*

$$(22) \quad \overline{v_\gamma} = -v_\gamma, \quad a_\alpha v_\gamma = \epsilon(\alpha, \gamma)v_\gamma \overline{a_\alpha},$$

$$(23) \quad a_\alpha(v_\gamma b_\beta) = \epsilon(\alpha, \gamma)v_\gamma(\overline{a_\alpha} b_\beta), \quad (v_\gamma b_\beta)a_\alpha = \epsilon(\beta, \alpha)v_\gamma(a_\alpha b_\beta),$$

$$(24) \quad (v_\gamma a_\alpha)(v_\gamma b_\beta) = \epsilon(\alpha, \beta + \gamma)\epsilon(2\gamma, \beta)[b_\beta v_\gamma^2] \overline{a_\alpha}.$$

*Proof.* Let  $v_\gamma \in B^\perp$ , from the equality  $f(1, v_\gamma) = 0$  ( $1 \in B$ ), we obtain that  $v_\gamma + \overline{v_\gamma} = 0$ . Let  $a_\alpha \in B$ , by Proposition 3.10 *iv*), we have that  $a_\alpha \overline{v_\gamma} + \epsilon(\alpha, \gamma)v_\gamma \overline{a_\alpha} = f(a_\alpha, v_\gamma) = 0$ , whence it follows (22). Furthermore, since  $\overline{a_\alpha} \in B$  by Proposition 3.10 *vi*) we have

$$a_\alpha(v_\gamma b_\beta) = -\epsilon(\alpha, \gamma)\overline{v_\gamma}(\overline{a_\alpha} b_\beta) + f(\overline{a_\alpha}, v_\gamma)b_\beta = \epsilon(\alpha, \gamma)v_\gamma(\overline{a_\alpha} b_\beta).$$

Similarly, by Proposition 3.10 *vi*) and (22) we have

$$\begin{aligned} (v_\gamma b_\beta)a_\alpha &= (\epsilon(\gamma, \beta)\overline{b_\beta} v_\gamma)a_\alpha = \epsilon(\gamma, \beta)[f(v_\gamma, \overline{a_\alpha})\overline{b_\beta} - \epsilon(\gamma, \alpha)(\overline{b_\beta} \overline{a_\alpha})\overline{v_\gamma}] \\ &= \epsilon(\gamma, \alpha + \beta)(\overline{b_\beta} \overline{a_\alpha})v_\gamma = \epsilon(\gamma, \alpha + \beta)(\epsilon(\beta, \alpha)\overline{a_\alpha} \overline{b_\beta})v_\gamma \\ &= \epsilon(\gamma, \alpha + \beta)\epsilon(\beta, \alpha)\epsilon(\alpha + \beta, \gamma)v_\gamma(a_\alpha b_\beta) \\ &= \epsilon(\beta, \alpha)v_\gamma(a_\alpha b_\beta). \end{aligned}$$

Therefore, we have (23). Finally, by Proposition 3.10 *vi*), (22) and  $B^\perp B \subseteq B^\perp$ , we have

$$\begin{aligned}
(v_\gamma a_\alpha)(v_\gamma b_\beta) &= f(a_\alpha, \overline{v_\gamma b_\beta})v_\gamma - \epsilon(\alpha, \beta + \gamma)[v_\gamma(\overline{v_\gamma b_\beta})]\overline{a_\alpha} \\
&= -f(a_\alpha, v_\gamma b_\beta)v_\gamma + \epsilon(\alpha, \beta + \gamma)[v_\gamma(v_\gamma b_\beta)]\overline{a_\alpha} \\
&= -\epsilon(\alpha, \beta + \gamma)[\overline{v_\gamma(v_\gamma b_\beta)}]\overline{a_\alpha} = -\epsilon(\alpha, \beta + \gamma)[\overline{v_\gamma(\epsilon(\gamma, \beta)\overline{b_\beta}v_\gamma)}]\overline{a_\alpha} \\
&= -\epsilon(\alpha, \beta + \gamma)\epsilon(\gamma, \beta)[f(v_\gamma, \overline{b_\beta})v_\gamma - \epsilon(\gamma, \beta)b_\beta(v_\gamma v_\gamma)]\overline{a_\alpha} \\
&= \epsilon(\alpha, \beta + \gamma)\epsilon(2\gamma, \beta)[b_\beta v_\gamma^2]\overline{a_\alpha}.
\end{aligned}$$

□

**Lemma 5.6.** *Let  $A$  be a composition  $\epsilon$ -algebra such that  $A_0$  is non-split, that is,  $q_0(a_0) \neq 0$  for all  $0 \neq a_0 \in A_0$ . Let  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  and  $A_\alpha \neq 0$ , then  $a_\alpha \overline{a_\alpha} \in F \setminus \{0\}$  for all  $0 \neq a_\alpha \in A_\alpha$  ( $2\alpha = 0$ ).*

*Proof.* Let  $0 \neq a_0 \in A_0$ . Since  $A_0$  is non-split then  $a_0 \overline{a_0} = q_0(a_0) \in F \setminus \{0\}$ . Let  $0 \neq \alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  and  $A_\alpha \neq 0$ . We will see that  $a_\alpha \overline{a_\alpha} = -a_\alpha^2 \neq 0$  and  $a_\alpha \overline{a_\alpha} \in F$  for all  $0 \neq a_\alpha \in A_\alpha$ . Let  $0 \neq a_\alpha \in A_\alpha$ . As  $f$  is nondegenerate there exists  $b_{-\alpha} \in A_{-\alpha}$  such that  $f(a_\alpha, b_{-\alpha}) \neq 0$  ( $a_\alpha b_{-\alpha} \neq 0$ ). Since  $A_0$  is not split and  $f(A_\alpha, A_\beta) = 0$  if  $\alpha + \beta \neq 0$ , by Proposition 3.10 *vi*) and *i*) we have

$$\begin{aligned}
a_\alpha^2 b_{-\alpha}^2 &= \overline{a_\alpha^2}(b_{-\alpha} b_{-\alpha}) = f(a_\alpha^2, b_{-\alpha})b_{-\alpha} - \epsilon(2\alpha, -\alpha)\overline{b_{-\alpha}}(a_\alpha^2 b_{-\alpha}) \\
&= \overline{b_{-\alpha}}[(a_\alpha a_\alpha)\overline{b_{-\alpha}}] = \overline{b_{-\alpha}}[f(a_\alpha, b_{-\alpha})a_\alpha - \epsilon(\alpha, -\alpha)(a_\alpha b_{-\alpha})\overline{a_\alpha}] \\
&= f(a_\alpha, b_{-\alpha})\overline{b_{-\alpha}}a_\alpha + \overline{b_{-\alpha}}[(a_\alpha b_{-\alpha})a_\alpha] \\
&= -f(a_\alpha \overline{b_{-\alpha}}, 1)b_{-\alpha}a_\alpha + f(b_{-\alpha}, a_\alpha b_{-\alpha})a_\alpha - \epsilon(-\alpha, 0)\overline{a_\alpha b_{-\alpha}}(b_{-\alpha}a_\alpha) \\
&= [f(a_\alpha b_{-\alpha}, 1) - \overline{a_\alpha b_{-\alpha}}](b_{-\alpha}a_\alpha) \\
&= (a_\alpha b_{-\alpha})\overline{(a_\alpha b_{-\alpha})} = q_0(a_\alpha b_{-\alpha}) \neq 0.
\end{aligned}$$

Therefore  $a_\alpha^2 \neq 0$ . Now we will see that  $a_\alpha \overline{a_\alpha} = -a_\alpha^2 \in F$  ( $2\alpha = 0$ ). As  $A_\alpha \neq 0$  and  $\alpha \neq 0$ , by Proposition 5.4, we have that  $A_0$  is associative ( $\dim(A_0) \leq 4$ ). In this case, we will prove that  $A_0$  is a division algebra. To do this, we only need to show that every  $0 \neq a \in A_0$  has a multiplicative inverse. Let  $0 \neq a \in A_0$ , since  $a\overline{a} = \overline{a}a = q_0(a)$  and  $A_0$  is not split ( $q_0(a) \neq 0$ ), it follows that  $a$  has a multiplicative inverse. Therefore  $A_0$  is a division algebra. Since  $f$  is nondegenerate,  $a_\alpha \neq 0$  and  $A_0$  is a division algebra there exists  $b_{-\alpha} \in A_{-\alpha}$  and  $u_0, d_0 \in A_0$  such that  $f(a_\alpha, b_{-\alpha}) \neq 0$ ,  $f(1, u_0) = 1$  and  $d_0(a_\alpha b_{-\alpha}) = u_0$ . Moreover, by Proposition 3.10 *vi*) we have

$$u_0 = d_0(a_\alpha b_{-\alpha}) = f(\overline{d_0}, a_\alpha)b_{-\alpha} - \epsilon(0, \alpha)\overline{a_\alpha}(\overline{d_0}b_{-\alpha}) = a_\alpha(\overline{d_0}b_{-\alpha}).$$

Let  $b'_{-\alpha} := \overline{d_0}b_{-\alpha}$ , then  $a_\alpha b'_{-\alpha} = u_0$  and  $b'_{-\alpha} a_\alpha = \overline{a_\alpha b'_{-\alpha}} = \overline{u_0}$ . In view of (10) we have

$$f(a_\alpha, b'_{-\alpha}) = f(a_\alpha, 1)f(b'_{-\alpha}, 1) - f(a_\alpha b'_{-\alpha}, 1) = -f(u_0, 1) = -1 \neq 0.$$

By Proposition 3.10 *vi*) and (14) we obtain

$$\begin{aligned}
f(a_\alpha, a_\alpha u_0) b'_{-\alpha} &= \overline{a_\alpha} [(a_\alpha u_0) b'_{-\alpha}] + \epsilon(\alpha, \alpha) \overline{(a_\alpha u_0)} (a_\alpha b'_{-\alpha}), \\
&= -a_\alpha [f(u_0, \overline{b'_{-\alpha}}) a_\alpha - (a_\alpha \overline{b'_{-\alpha}}) \overline{u_0}] - (a_\alpha u_0) u_0, \\
&= -a_\alpha [(a_\alpha b'_{-\alpha}) \overline{u_0}] - a_\alpha u_0^2, \\
&= -a_\alpha (u_0 \overline{u_0}) - a_\alpha (f(1, u_0) u_0 - q_0(u_0)), \\
&= -a_\alpha u_0.
\end{aligned}$$

Then

$$(25) \quad a_\alpha u_0 = -f(a_\alpha, a_\alpha u_0) b'_{-\alpha}$$

Since  $\alpha \neq 0$  then  $a_\alpha \in A_0^\perp$ . Hence, by Proposition 3.10 *v*), (24) and (25) we have

$$\begin{aligned}
q_0(u_0) a_\alpha^2 &= [a_\alpha^2 \overline{u_0}] u_0 = [(a_\alpha u_0) a_\alpha] u_0 = [(-f(a_\alpha, a_\alpha u_0) b'_{-\alpha}) a_\alpha] u_0 \\
&= -f(a_\alpha, a_\alpha u_0) \overline{u_0} u_0 = -f(a_\alpha, a_\alpha u_0) q_0(u_0).
\end{aligned}$$

Therefore  $a_\alpha \overline{a_\alpha} = -a_\alpha^2 = f(a_\alpha, a_\alpha u_0) \in F \setminus \{0\}$  and  $2\alpha = 0$ . □

**Lemma 5.7.** *Let  $A$  be a composition  $\epsilon$ -algebra such that  $A_0$  is non-split. Let  $\alpha, \beta \in \Gamma$ ,  $\alpha \neq 0$ , such that  $\epsilon(\alpha, \alpha) = \epsilon(\beta, \beta) = 1$  and  $A_\alpha, A_\beta \neq 0$ . Then, for all  $0 \neq a_\alpha \in A_\alpha$ , the operator of right multiplication  $R_{a_\alpha}^\beta : A_\beta \rightarrow A_{\alpha+\beta}$ ,  $b_\beta \mapsto b_\beta a_\alpha$  is bijective ( $A_{\alpha+\beta} \neq 0$ ). In particular,  $\dim(A_\alpha) = \dim(A_0)$  for all  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  and  $A_\alpha \neq 0$ .*

*Proof.* Since  $f$  is nondegenerate there exists  $u_0 \in A_0$  such that  $f(1, u_0) = 1$ . In the proof of the Lemma 5.6 we showed that  $f(a_\alpha, a_\alpha u_0) = a_\alpha \overline{a_\alpha} \neq 0$  for all  $0 \neq a_\alpha \in A_\alpha$  such that  $\epsilon(\alpha, \alpha) = 1$  and  $\alpha \neq 0$ . Furthermore, by (5)  $f(c_0, c_0 u_0) = q_0(c_0) f(1, u_0) = q_0(c_0) \neq 0$  for all  $0 \neq c_0 \in A_0$ , since  $A_0$  is not split.

Let  $a_\alpha, b_\beta \in A \setminus \{0\}$  homogeneous elements. By Proposition 3.10 *vi*) we obtain

$$(26) \quad (b_\beta u_0) a_\alpha = -f(u_0, a_\alpha) b_\beta + (b_\beta a_\alpha) \overline{u_0} = (b_\beta a_\alpha) \overline{u_0},$$

since  $\alpha \neq 0$ . Hence, by (6), (26) and Proposition 3.10 *i*) we have

$$\begin{aligned}
0 \neq \epsilon(\alpha, \beta) f(b_\beta, b_\beta u_0) f(a_\alpha, a_\alpha u_0) &= f(b_\beta a_\alpha, (b_\beta u_0)(a_\alpha u_0)) + f(b_\beta(a_\alpha u_0), (b_\beta u_0) a_\alpha) \\
&= f(b_\beta a_\alpha, (b_\beta u_0)(a_\alpha u_0)) + f(b_\beta(a_\alpha u_0), (b_\beta a_\alpha) \overline{u_0}) \\
&= f(b_\beta a_\alpha, (b_\beta u_0)(a_\alpha u_0)) + f([b_\beta(a_\alpha u_0)] u_0, b_\beta a_\alpha) \\
&= f(b_\beta a_\alpha, (b_\beta u_0)(a_\alpha u_0)) + [b_\beta(a_\alpha u_0)] u_0.
\end{aligned}$$

Then  $b_\beta a_\alpha \neq 0$ . Therefore the operator right multiplication  $R_{a_\alpha}^\beta$  is bijective. In particular, since  $A_\alpha \neq 0$  then  $A_{-\alpha} \neq 0$  ( $f$  is nondegenerate) and  $R_{b_{-\alpha}}^\alpha$  and  $R_{a_\alpha}^0$  are bijective for some  $0 \neq a_\alpha \in A_\alpha$  and some  $0 \neq b_{-\alpha} \in A_{-\alpha}$ . Hence  $\dim(A_\alpha) = \dim(A_0)$ . □

In the following theorem we will see that if  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  is a composition  $\epsilon$ -algebra such that  $A = A^{(0)}$  and  $A_0$  is not split then  $A$  is equivalent to one of the algebras of the Example 4.5.

**Theorem 5.8.** *Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a composition  $\epsilon$ -algebra such that  $A = A^{(0)}$  and  $A_0$  is not split. Then  $A$  is equivalent to one of the algebras of types I-X given in Examples 4.5.*

*Proof.* By the term “ $\Gamma$ -graded subalgebra” we shall mean a  $\Gamma$ -graded subalgebra containing the identity element 1. Since  $a + \bar{a} \in F$ , every such  $\Gamma$ -graded subalgebra is invariant under the involution.

Let  $B$  be a finite-dimensional  $\Gamma$ -graded subalgebra of the algebra  $A$  on which the restriction of the form  $f(x, y)$  is nondegenerate.  $B$  exists because we can just take  $B = A_0$ . By Corollary 5.3,  $A$  decomposes into a direct sum of subspaces  $A = B \oplus B^\perp$ . Moreover, the restriction of the form  $f(x, y)$  to  $B^\perp$  is also nondegenerate. We assume that  $B \neq A$ . Then we can find a homogeneous element  $0 \neq v_\gamma \in B^\perp$  ( $B^\perp$  is a  $\Gamma$ -graded vector space). Since  $A = A^{(0)}$  and  $A_0$  is not split, by Lemma 5.6, we have  $v_\gamma \bar{v}_\gamma = -\mu \in F \setminus \{0\}$  ( $2\gamma = 0$ ). Now, let  $a_\alpha, b_\beta \in B$ . From (24)

$$(27) \quad (v_\gamma a_\alpha)(v_\gamma b_\beta) = \epsilon(\alpha, \beta + \gamma)\epsilon(2\gamma, \beta)[b_\beta v_\gamma^2] \bar{a}_\alpha = \epsilon(\alpha, \beta + \gamma)\mu b_\beta \bar{a}_\alpha.$$

In particular, taking  $a_\alpha = 1 \in B$  we have that

$$(28) \quad v_\gamma(v_\gamma b_\beta) = \epsilon(0, \beta + \gamma)\mu b_\beta \bar{1} = \mu b_\beta.$$

Hence, by Proposition 3.10 *i*) and (28),

$$\begin{aligned} f(v_\gamma a_\alpha, v_\gamma b_\beta) &= -f(v_\gamma a_\alpha, \bar{v}_\gamma b_\beta) = -\epsilon(\alpha + \gamma, \gamma)f(v_\gamma(v_\gamma a_\alpha), b_\beta) \\ &= -\epsilon(\alpha, \gamma)f(\mu a_\alpha, b_\beta) = -\epsilon(\alpha, \gamma)\mu f(a_\alpha, b_\beta). \end{aligned}$$

Therefore

$$(29) \quad f(v_\gamma a_\alpha, v_\gamma b_\beta) = -\epsilon(\alpha, \gamma)\mu f(a_\alpha, b_\beta).$$

Since  $f$  is nondegenerate on  $B$ , it follows from this that the mapping  $a \rightarrow v_\gamma a$  of the subspace  $B$  onto  $v_\gamma B$  is one-to-one. Consequently,  $B$  and  $v_\gamma B$  have the same dimension. Furthermore, the relation (29) shows that the subspace  $v_\gamma B$  is nondegenerate with respect to  $f(x, y)$ . By Proposition 5.4 we have  $BB^\perp \subseteq B^\perp$ , then  $f(a_\alpha, v_\gamma b_\beta) = 0$ , this implies that the subspace  $B_1 = B + v_\gamma B$ , which is the orthogonal sum of two nondegenerate subspaces, is also nondegenerate with respect to  $f(x, y)$ . Relations (23) and (27) show that  $B_1 = B + v_\gamma B$  is the  $\Gamma$ -graded subalgebra of the algebra  $A$  obtained from  $B$  by means of the generalized Cayley-Dickson process,  $B_1 = GCD(B, \mu, \epsilon)$  (since  $2\gamma = 0$  and  $\epsilon(\alpha, \gamma) = \pm 1$  for all  $\alpha \in \Gamma$ ). As  $\bar{v}_\gamma = -v_\gamma$  and  $\overline{a_\alpha + v_\gamma b_\beta} = \bar{a}_\alpha - \epsilon(\gamma, \beta)\bar{b}_\beta v_\gamma = \bar{a}_\alpha - \epsilon(\gamma, \beta)\epsilon(\beta, \gamma)v_\gamma b_\beta = \bar{a}_\alpha - v_\gamma b_\beta$ , the involution induced on  $B_1$  by the involution on  $A$  coincides with the involution obtained in the generalized Cayley-Dickson process. Finally,  $f|_{B_1 \times B_1}$  is nondegenerate and  $B_1$  satisfies the same conditions as  $B$ . Therefore, we can repeat the same process with the algebra  $B_1$ .

We now return to the algebra  $A$  and consider separately four cases.

- (1)  $\dim(A_0) = 1$ . In this case the field  $F$  has characteristic  $\neq 2$ ,  $A_0 = F$ , the subalgebra  $A_0 = F$  is nondegenerate with respect to  $f(x, y)$ , and therefore we can set  $B = A_0 = F$ .  $B$  is a composition algebra of type I. By Lemma 5.7,  $\dim(A_\gamma) = \dim(A_0) = 1$  for all  $A_\gamma \neq 0$ . If  $A \neq F$  then, there is a  $0 \neq \alpha_1 \in \Gamma$  such that  $A_{\alpha_1} \neq 0$  ( $2\alpha_1 = 0$ ). Let  $\Gamma_1 = \langle \alpha_1 \rangle \cong \mathbb{Z}_2$ ,  $B_1 = A_0 \oplus A_{\alpha_1}$  and  $\epsilon_1 = \epsilon|_{\Gamma_1 \times \Gamma_1}$ .  $B_1$  is a composition  $\epsilon_1$ -algebra of type II. If  $B_1 \neq A$  then there is a  $\alpha_2 \in \Gamma \setminus \{0, \alpha_1\}$  such that  $A_{\alpha_2} \neq 0$  ( $2\alpha_2 = 0$ ). Let  $\Gamma_2 = \langle \alpha_1, \alpha_2 \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ ,  $B_2 = \bigoplus_{\gamma \in \Gamma_2} A_\gamma$  and  $\epsilon_2 = \epsilon|_{\Gamma_2 \times \Gamma_2}$ .  $B_2$  is a composition  $\epsilon_2$ -algebra of type III. Finally, if  $B_2 \neq A$ , there is a  $\alpha_3 \in \Gamma \setminus \{0, \alpha_1, \alpha_2\}$  such that  $A_{\alpha_3} \neq 0$  ( $2\alpha_3 = 0$ ). Let  $\Gamma_3 = \langle \alpha_1, \alpha_2, \alpha_3 \rangle \cong \mathbb{Z}_2^3$ ,  $B_3 = \bigoplus_{\gamma \in \Gamma_3} A_\gamma$  and  $\epsilon_3 = \epsilon|_{\Gamma_3 \times \Gamma_3}$ .  $B_3$  is a composition  $\epsilon_3$ -algebra of type IV. Since  $B_3$  is non-associative, by Lemma 4.3 the

process must finish. Consequently  $A = B_3$ ,  $\Gamma = \Gamma_3$  and  $\epsilon = \epsilon_3$ .

If  $\dim(A_0) \geq 2$ , the field  $F$  has arbitrary characteristic.

- (2)  $\dim(A_0) = 2$ . In this case  $A_0 = K(\mu) = F + Fv_1$  where  $v_1^2 = v_1 + \mu$  and  $4\mu + 1 \neq 0$  with involution  $\overline{r + sv_1} = (r + s) - sv_1$  and quadratic form  $q_0(a) = a\bar{a}$ . By Lemma 5.7,  $\dim(A_\gamma) = \dim(A_0) = 2$  for all  $A_\gamma \neq 0$ . Since the subalgebra  $A_0$  is nondegenerate with respect to  $f(x, y)$ , we can set  $B = A_0$ . The  $\Gamma$ -graded subalgebra  $B$  is a composition algebra of type V. If  $A \neq A_0$  then there is a  $0 \neq \alpha \in \Gamma$  such that  $A_\alpha \neq 0$  ( $2\alpha = 0$ ). Let  $\Gamma_1 = \langle \alpha \rangle \cong \mathbb{Z}_2$ ,  $B_1 = A_0 \oplus A_\alpha$  and  $\epsilon_1 = \epsilon|_{\Gamma_1 \times \Gamma_1}$ .  $B_1$  is a composition  $\epsilon_1$ -algebra of type VI. Finally, if  $B_1 \neq A$  then there is a  $\beta \in \Gamma \setminus \Gamma_1$  such that  $A_\beta \neq 0$  ( $2\beta = 0$ ). Let  $\Gamma_2 = \langle \alpha, \beta \rangle \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ ,  $B_2 = \bigoplus_{\gamma \in \Gamma_2} A_\gamma$  and  $\epsilon_2 = \epsilon|_{\Gamma_2 \times \Gamma_2}$ .  $B_2$  is a composition  $\epsilon_2$ -algebra of type VII. Since  $B_2$  is non-associative, by Lemma 4.3 the process must finish. Consequently  $A = B_2$ ,  $\Gamma = \Gamma_2$  and  $\epsilon = \epsilon_2$ .
- (3)  $\dim(A_0) = 4$ . In this case  $A_0 = Q(\mu, \psi) = CD(K(\mu), \psi)$  ( $\psi \neq 0$ ) is the algebra of generalized quaternions. By Lemma 5.7  $\dim(A_\gamma) = \dim(A_0) = 4$  for all  $A_\gamma \neq 0$ . Since the subalgebra  $A_0$  is nondegenerate with respect to  $f(x, y)$ , we can set  $B = A_0$ .  $B$  is a composition algebra of type VIII. If  $A \neq A_0$  then there is a  $0 \neq \alpha \in \Gamma$  such that  $A_\alpha \neq 0$  ( $2\alpha = 0$ ). Let  $\Gamma_1 = \langle \alpha \rangle \cong \mathbb{Z}_2$ ,  $B_1 = A_0 \oplus A_\alpha$  and  $\epsilon_1 = \epsilon|_{\Gamma_1 \times \Gamma_1}$ .  $B_1$  is a composition  $\epsilon_1$ -algebra of type IX. Since  $B_1$  is non-associative the process must finish, since otherwise by Lemma 4.3 there would be a subalgebra  $B_2$  of  $A$  which is not a composition  $\epsilon$ -algebra, and that is impossible. Consequently  $A = B_1$ .
- (4)  $\dim(A_0) = 8$ . In this case  $A_0 = C(\mu, \psi, \lambda) = CD(Q(\mu, \psi), \lambda)$  with  $\lambda \neq 0$ , is the Cayley-Dickson algebra.  $A_0$  is non-associative, therefore  $A = A_0$ . The  $\Gamma$ -graded algebra  $A$  is of type X.

□

With the notation of the Example 4.5, IV, we have the following result.

**Proposition 5.9.** *We have the following equivalences:*

- i)  $C^{(1, (1, -1))}(\mu, \psi, \lambda) \equiv C^{(1, (-1, 1))}(\psi, \mu, \lambda)$ .
- ii)  $C^{(1, (1, -1))}(\mu, \psi, \lambda) \equiv C^{(-1, (1, 1))}(\lambda, \psi, \mu)$ .
- iii)  $C^{(1, (-1, -1))}(\mu, \psi, \lambda) \equiv C^{(-1, (1, -1))}(\mu, \lambda, \psi)$ .
- iv)  $C^{(1, (-1, -1))}(\mu, \psi, \lambda) \equiv C^{(-1, (-1, 1))}(\lambda, \psi, \mu)$ .
- v)  $C^{(1, (1, -1))}(\mu, s, -rs) \equiv C^{(-1, (-1, -1))}(r\mu, s, -rs)$ ,

for any  $\mu, \psi, \lambda \in F^\times$  and  $r, s \in \{1, -1\}$ .

*Proof.* i) Let  $\theta : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$  be given by

$$\theta((\bar{1}, \bar{0}, \bar{0})) = (\bar{0}, \bar{1}, \bar{0}), \quad \theta((\bar{0}, \bar{1}, \bar{0})) = (\bar{1}, \bar{0}, \bar{0}), \quad \theta((\bar{0}, \bar{0}, \bar{1})) = (\bar{0}, \bar{0}, \bar{1}).$$

Let  $g : C^{(1, (1, -1))}(\mu, \psi, \lambda) \rightarrow C^{(1, (-1, 1))}(\psi, \mu, \lambda)$  be given by

$$\begin{aligned} g(e_1) &= 1, & g(e_1) &= e_2, & g(e_2) &= e_1, & g(e_3) &= -e_3, \\ g(e_4) &= e_4, & g(e_5) &= e_6, & g(e_6) &= e_5, & g(e_7) &= -e_7. \end{aligned}$$

It is easy to see that  $g$  is an isomorphism of algebras,  $\tilde{\epsilon}_{(1, (-1, 1))} = \tilde{\epsilon}_{(1, (1, -1))} \circ (\theta \times \theta)$  and  $g(C^{(1, (1, -1))}(\mu, \psi, \lambda)_\gamma) \subseteq C^{(1, (-1, 1))}(\psi, \mu, \lambda)_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_2^3$ . Therefore

$$C^{(1, (1, -1))}(\mu, \psi, \lambda) \equiv C^{(1, (-1, 1))}(\psi, \mu, \lambda).$$

ii) Let  $\theta : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$  be given by

$$\theta((\bar{1}, \bar{0}, \bar{0})) = (\bar{0}, \bar{0}, \bar{1}), \quad \theta((\bar{0}, \bar{1}, \bar{0})) = (\bar{0}, \bar{1}, \bar{0}), \quad \theta((\bar{0}, \bar{0}, \bar{1})) = (\bar{1}, \bar{0}, \bar{0}).$$

Let  $g : C^{(1,(1,-1))}(\mu, \psi, \lambda) \rightarrow C^{(-1,(1,1))}(\lambda, \psi, \mu)$  be given by

$$\begin{aligned} g(1) &= 1, & g(e_1) &= -e_4, & g(e_2) &= -e_2, & g(e_3) &= e_6, \\ g(e_4) &= e_1, & g(e_5) &= e_5, & g(e_6) &= e_3, & g(e_7) &= e_7. \end{aligned}$$

It is easy to see that  $g$  is an isomorphism of algebras,  $\tilde{\epsilon}_{(-1,(1,1))} = \tilde{\epsilon}_{(1,(1,-1))} \circ (\theta \times \theta)$  and  $g(C^{(1,(1,-1))}(\mu, \psi, \lambda)_\gamma) \subseteq C^{(-1,(1,1))}(\lambda, \psi, \mu)_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_2^3$ . Therefore

$$C^{(1,(1,-1))}(\mu, \psi, \lambda) \cong C^{(-1,(1,1))}(\lambda, \psi, \mu).$$

iii) Let  $\theta : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$  be given by

$$\theta((\bar{1}, \bar{0}, \bar{0})) = (\bar{1}, \bar{0}, \bar{0}), \quad \theta((\bar{0}, \bar{1}, \bar{0})) = (\bar{0}, \bar{0}, \bar{1}), \quad \theta((\bar{0}, \bar{0}, \bar{1})) = (\bar{0}, \bar{1}, \bar{0}).$$

Let  $g : C^{(1,(-1,-1))}(\mu, \psi, \lambda) \rightarrow C^{(-1,(1,-1))}(\mu, \lambda, \psi)$  be given by

$$\begin{aligned} g(1) &= 1, & g(e_1) &= e_1, & g(e_2) &= e_4, & g(e_3) &= e_5, \\ g(e_4) &= e_2, & g(e_5) &= e_3, & g(e_6) &= e_6, & g(e_7) &= e_7. \end{aligned}$$

It is easy to see that  $g$  is an isomorphism of algebras,  $\tilde{\epsilon}_{(-1,(1,-1))} = \tilde{\epsilon}_{(1,(-1,-1))} \circ (\theta \times \theta)$  and  $g(C^{(1,(-1,-1))}(\mu, \psi, \lambda)_\gamma) \subseteq C^{(-1,(1,-1))}(\mu, \lambda, \psi)_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_2^3$ . Therefore

$$C^{(1,(-1,-1))}(\mu, \psi, \lambda) \cong C^{(-1,(1,-1))}(\mu, \lambda, \psi).$$

iv) Let  $\theta : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$  be given by

$$\theta((\bar{1}, \bar{0}, \bar{0})) = (\bar{0}, \bar{0}, \bar{1}), \quad \theta((\bar{0}, \bar{1}, \bar{0})) = (\bar{0}, \bar{1}, \bar{0}), \quad \theta((\bar{0}, \bar{0}, \bar{1})) = (\bar{1}, \bar{0}, \bar{0}).$$

Let  $g : C^{(1,(-1,-1))}(\mu, \psi, \lambda) \rightarrow C^{(-1,(-1,1))}(\lambda, \psi, \mu)$  be given by

$$\begin{aligned} g(1) &= 1, & g(e_1) &= e_4, & g(e_2) &= -e_2, & g(e_3) &= -e_6, \\ g(e_4) &= e_1, & g(e_5) &= e_5, & g(e_6) &= e_3, & g(e_7) &= e_7. \end{aligned}$$

It is easy to see that  $g$  is an isomorphism of algebras,  $\tilde{\epsilon}_{(-1,(-1,1))} = \tilde{\epsilon}_{(1,(-1,-1))} \circ (\theta \times \theta)$  and  $g(C^{(1,(-1,-1))}(\mu, \psi, \lambda)_\gamma) \subseteq C^{(-1,(-1,1))}(\lambda, \psi, \mu)_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_2^3$ . Therefore

$$C^{(1,(-1,-1))}(\mu, \psi, \lambda) \cong C^{(-1,(-1,1))}(\lambda, \psi, \mu).$$

v) Let  $\theta : \mathbb{Z}_2^3 \rightarrow \mathbb{Z}_2^3$  be given by

$$\theta((\bar{1}, \bar{0}, \bar{0})) = (\bar{1}, \bar{1}, \bar{1}), \quad \theta((\bar{0}, \bar{1}, \bar{0})) = (\bar{0}, \bar{1}, \bar{0}), \quad \theta((\bar{0}, \bar{0}, \bar{1})) = (\bar{0}, \bar{0}, \bar{1}).$$

Let  $g : C^{(1,(1,-1))}(\mu, s, -rs) \rightarrow C^{(-1,(-1,-1))}(r\mu, s, -rs)$  be given by

$$\begin{aligned} g(1) &= 1, & g(e_1) &= e_7, & g(e_2) &= se_2, & g(e_3) &= e_5, \\ g(e_4) &= se_4, & g(e_5) &= -re_3, & g(e_6) &= e_6, & g(e_7) &= -re_1. \end{aligned}$$

It is easy to see that  $g$  is an isomorphism of algebras,  $\tilde{\epsilon}_{(-1,(-1,-1))} = \tilde{\epsilon}_{(1,(1,-1))} \circ (\theta \times \theta)$  and  $g(C^{(1,(1,-1))}(\mu, s, -rs)_\gamma) \subseteq C^{(-1,(-1,-1))}(r\mu, s, -rs)_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_2^3$ . Therefore

$$C^{(1,(1,-1))}(\mu, s, -rs) \cong C^{(-1,(-1,-1))}(r\mu, s, -rs).$$

□

**Corollary 5.10.** *Let  $A$  be a composition  $\epsilon$ -algebra of dimension 8 such that,  $A = A^{(0)}$  and  $A_0$  is not split. Then we have, up to equivalence, one of the following cases:*

- i)  $A = A_0 = C(\mu, \psi, \lambda)$  is a non-split Cayley-Dickson algebra.
- ii)  $A = \tilde{C}(\mu, \psi, \lambda)$  with  $A_0 = Q(\mu, \psi)$  a non-split quaternion algebra.
- iii)  $A$  is  $C^{(1)}(\mu, \psi, \lambda)$ , or  $C^{(-1)}(\mu, \psi, \lambda)$ , with  $A_0 = K(\mu)$  a non-split composition algebra.
- iv)  $A$  is  $C^{(1, (1, 1))}(\mu, \psi, \lambda)$ , or  $C^{(1, (1, -1))}(\mu_1, \psi_1, \lambda_1)$ , or  $C^{(1, (-1, -1))}(\mu_2, \psi_2, \lambda_2)$ , or  $C^{(-1, (-1, -1))}(\mu_3, \psi_3, \lambda_3)$ , with  $A_0 = F$  a field of characteristic different from 2.

5.2.  $A^{(1)} = 0$  and  $A_0$  is a split composition algebra. A composition algebra  $C$  is split if it has a nonzero isotropic element, that is, there is  $0 \neq x \in C$  such that  $q_0(x) = 0$ . Note that any composition algebra of dimension  $\geq 2$  over an algebraically closed field is split.

Let  $A$  be a composition  $\epsilon$ -algebra such that  $A^{(1)} = 0$  and  $A_0$  is a split composition algebra. To analyze this case we will be based on the construction of the ‘‘canonical’’ basis of a split Cayley algebra. Since  $A^{(1)} = 0$  ( $A = A^{(0)}$ ), we have that  $\epsilon(\gamma, \gamma) = 1$  for all  $\gamma \in \Gamma$ . Moreover, since  $A_0$  is a split composition algebra ( $\dim A_0 \geq 2$ ) and  $f|_{A_0 \times A_0}$  is nondegenerate then, there is an isotropic element  $0 \neq a_0 \in A_0$  ( $q_0(a_0) = 0$ ) and a  $b_0 \in A_0$  such that  $f(a_0 b_0, 1) = f(a_0, \overline{b_0}) = 1$ . Let  $e_1 := a_0 b_0$ , we have  $q_0(e_1) = q_0(a_0 b_0) = 0$  and  $f(e_1, 1) = 1$ , so  $e_1^2 = e_1$ . Let  $e_2 := \overline{e_1} = 1 - e_1$ , so  $q_0(e_2) = 0$ ,  $e_2^2 = e_2$ ,  $e_1 e_2 = e_2 e_1 = 0$  and  $f(e_1, e_2) = f(e_1, 1 - e_1) = f(e_1, 1) = 1$ . Then  $K := F e_1 \oplus F e_2$  is a composition subalgebra of  $A_0$ . By Corollary 5.3, we have that  $A = K \oplus K^\perp$  and  $K^\perp$  is a  $\Gamma$ -graded vector subspace of  $A$ .

For any  $x_\alpha \in K^\perp$ ,  $\overline{x_\alpha} = -x_\alpha$  since  $1 = e_1 + e_2 \in K$ . By Proposition 3.10 i) we have  $x_\alpha e_1 + \overline{x_\alpha e_1} = f(x_\alpha e_1, 1) = f(x_\alpha, \overline{e_1}) = f(x_\alpha, e_2) = 0$ , so  $x_\alpha e_1 = -\overline{x_\alpha e_1} = -\epsilon(\alpha, 0) \overline{e_1} \overline{x_\alpha} = e_2 x_\alpha$ . Hence  $x_\alpha e_1 = e_2 x_\alpha$ . Analogously we can see that  $x_\alpha e_2 = e_1 x_\alpha$ . Moreover,  $x_\alpha = 1 \cdot x_\alpha = e_1 x_\alpha + e_2 x_\alpha$  and by Proposition 3.10 v) we have

$$\begin{aligned} e_2(e_1 x_\alpha) &= \overline{e_1}(e_1 x_\alpha) = q_0(e_1) x_\alpha = 0 \\ e_1(e_2 x_\alpha) &= \overline{e_2}(e_2 x_\alpha) = q_0(e_2) x_\alpha = 0. \end{aligned}$$

Also, by (14)

$$\begin{aligned} e_i(e_i x_\alpha) &= (e_i e_i) x_\alpha = e_i x_\alpha \\ (x_\alpha e_i) e_i &= x_\alpha (e_i e_i) = x_\alpha e_i. \end{aligned}$$

Therefore  $K^\perp = U \oplus V$  with

$$\begin{aligned} U &= \{x \in A : e_1 x = x = x e_2, e_2 x = 0 = x e_1\} = (e_1 A) e_2, \\ V &= \{x \in A : e_2 x = x = x e_1, e_1 x = 0 = x e_2\} = (e_2 A) e_1. \end{aligned}$$

Let  $a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n} \in U$  then

$$\begin{aligned} a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n} &= e_1(a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n}) = e_1 a_{\alpha_1} + e_1 a_{\alpha_2} + \dots + e_1 a_{\alpha_n} \\ a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n} &= (a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n}) e_2 = a_{\alpha_1} e_2 + a_{\alpha_2} e_2 + \dots + a_{\alpha_n} e_2 \\ 0 &= e_2(a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n}) = e_2 a_{\alpha_1} + e_2 a_{\alpha_2} + \dots + e_2 a_{\alpha_n} \\ 0 &= (a_{\alpha_1} + a_{\alpha_2} + \dots + a_{\alpha_n}) e_1 = a_{\alpha_1} e_1 + a_{\alpha_2} e_1 + \dots + a_{\alpha_n} e_1. \end{aligned}$$

Since  $e_1, e_2 \in A_0$  then

$$\begin{aligned} a_{\alpha_i} &= e_1 a_{\alpha_i} \\ a_{\alpha_i} &= a_{\alpha_i} e_2 \\ 0 &= e_2 a_{\alpha_i} \\ 0 &= a_{\alpha_i} e_1 \end{aligned}$$

for any  $i \in \{1, 2, \dots, n\}$ . Hence  $a_{\alpha_1}, a_{\alpha_2}, \dots, a_{\alpha_n} \in U$ . Therefore,  $U$  and  $V$  are  $\Gamma$ -graded subspaces of  $A$ . By (5), for any  $u_\alpha, u'_\beta \in U$ ,

$$f(u_\alpha, u'_\beta) = f(e_1 u_\alpha, e_1 u'_\beta) = q_0(e_1) f(u_\alpha, u'_\beta) = 0$$

so  $f(U, U) = 0$ . Analogously  $f(V, V) = 0$ . Also, by Proposition 3.10 i) we have

$$f(u_\alpha u'_\beta, K) \subseteq f(u_\alpha, K u'_\beta) \subseteq f(U, U) = 0$$

and for any  $v_\gamma \in V$  by (6) we have

$$f(u_\alpha u'_\beta, v_\gamma) = f(u_\alpha u'_\beta, e_2 v_\gamma) = \epsilon(\beta, 0) f(u_\alpha, e_2) f(u'_\beta, v_\gamma) - \epsilon(\beta, \gamma) f(u_\alpha v_\gamma, e_2 u'_\beta) = 0.$$

Hence  $U^2$  is orthogonal to  $K$  and  $V$ , so  $U^2 \subseteq V$  due to  $f(x, y)$  is nondegenerate. Analogously we have that  $V^2$  is orthogonal to  $K$  and  $U$ , then  $V^2 \subseteq U$ . Furthermore, by Proposition 3.10 i) we have

$$\begin{aligned} f(U, UV) &\subseteq f(U^2, V) \subseteq f(V, V) = 0, \\ f(UV, V) &\subseteq f(U, V^2) \subseteq f(U, U) = 0, \\ f(VU, U) &\subseteq f(V, U^2) \subseteq f(V, V) = 0, \\ f(V, VU) &\subseteq f(V^2, U) \subseteq f(U, U) = 0, \end{aligned}$$

so  $UV + VU \subseteq K$ . Moreover  $f(UV, e_1) \subseteq f(U, e_1 V) = 0$  and  $f(VU, e_2) \subseteq f(V, e_2 U) = 0$ , hence  $UV \subseteq Fe_1$  and  $VU \subseteq Fe_2$ . More precisely by Proposition 3.10 vi), for  $u_\alpha \in U$  and  $v_\gamma \in V$ ,  $f(u_\alpha, v_\gamma) e_1 = \overline{u_\alpha}(v_\gamma e_1) + \epsilon(\alpha, \gamma) \overline{v_\gamma}(u_\alpha e_1) = -u_\alpha v_\gamma$ . Then

$$(30) \quad u_\alpha v_\gamma = -f(u_\alpha, v_\gamma) e_1.$$

By Proposition 3.10 vi),  $f(u_\alpha, v_\gamma) e_2 = \overline{u_\alpha}(v_\gamma e_2) + \epsilon(\alpha, \gamma) \overline{v_\gamma}(u_\alpha e_2) = -\epsilon(\alpha, \gamma) v_\gamma u_\alpha$ . Hence

$$(31) \quad v_\gamma u_\alpha = -\epsilon(\gamma, \alpha) f(u_\alpha, v_\gamma) e_2 = -f(v_\gamma, u_\alpha) e_2.$$

Now let us see what happens with  $u_\alpha^2$  and  $v_\gamma^2$ : If  $\text{char}(F) \neq 2$ , since  $\epsilon(\alpha, \alpha) = 1$  for all  $\alpha \in \Gamma$  and  $f(U, U) = f(V, V) = f(U \oplus V, K) = 0$ , by (11) we have  $u_\alpha^2 = v_\gamma^2 = 0$ . If  $\text{char}(F) = 2$ . Take  $0 \neq u_\alpha \in U$  and  $v_{-\alpha} \in V$  with  $f(u_\alpha, v_{-\alpha}) = 1$  ( $f$  is nondegenerate). By Proposition 3.10 vi) and (31) we have

$$\begin{aligned} v_{-\alpha} u_\alpha^2 &= -\overline{v_{-\alpha}}(u_\alpha u_\alpha) = -f(v_{-\alpha}, u_\alpha) u_\alpha + \epsilon(-\alpha, \alpha) \overline{u_\alpha}(v_{-\alpha} u_\alpha) \\ &= -u_\alpha - \epsilon(\alpha, \alpha) u_\alpha (-f(v_{-\alpha}, u_\alpha) e_2) = -u_\alpha + u_\alpha e_2 = 0, \end{aligned}$$

so

$$(32) \quad v_{-\alpha} u_\alpha^2 = 0.$$

By Proposition 3.10 *vi*) and (30) we have

$$\begin{aligned} u_\alpha v_{-\alpha}^2 &= -\overline{u_\alpha}(v_{-\alpha}v_{-\alpha}) = -f(u_\alpha, v_{-\alpha})v_{-\alpha} + \epsilon(\alpha, -\alpha)\overline{v_{-\alpha}}(u_\alpha v_{-\alpha}) \\ &= -v_{-\alpha} - \epsilon(\alpha, \alpha)v_{-\alpha}(-f(u_\alpha, v_{-\alpha})e_1) = -v_{-\alpha} + v_{-\alpha}e_1 = 0, \end{aligned}$$

thus

$$(33) \quad u_\alpha v_{-\alpha}^2 = 0.$$

On the other hand, by Proposition 3.10 *vi*), (30) and (32) we obtain

$$\begin{aligned} u_\alpha^2 &= f(v_{-\alpha}, u_\alpha)u_\alpha^2 = \overline{v_{-\alpha}}(u_\alpha u_\alpha^2) + \epsilon(-\alpha, \alpha)\overline{u_\alpha}(v_{-\alpha}u_\alpha^2) \\ &= v_{-\alpha}[f(u_\alpha, u_\alpha^2)e_1] = f(u_\alpha, u_\alpha^2)v_{-\alpha}, \end{aligned}$$

then

$$(34) \quad u_\alpha^2 = f(u_\alpha, u_\alpha^2)v_{-\alpha}.$$

By Proposition 3.10 *vi*), (31) and (33) we have

$$\begin{aligned} v_{-\alpha}^2 &= f(u_\alpha, v_{-\alpha})v_{-\alpha}^2 = \overline{u_\alpha}(v_{-\alpha}v_{-\alpha}^2) + \epsilon(\alpha, -\alpha)\overline{v_{-\alpha}}(u_\alpha v_{-\alpha}^2) \\ &= u_\alpha[f(v_{-\alpha}, v_{-\alpha}^2)e_2] = f(v_{-\alpha}, v_{-\alpha}^2)u_\alpha, \end{aligned}$$

so

$$(35) \quad v_{-\alpha}^2 = f(v_{-\alpha}, v_{-\alpha}^2)u_\alpha.$$

By replacing (34) and (35) in (32) and (33), respectively, we obtain

$$(36) \quad f(u_\alpha^2, u_\alpha)v_{-\alpha}^2 = 0 \quad \text{and} \quad f(v_{-\alpha}^2, v_{-\alpha})u_\alpha^2 = 0.$$

Thus  $f(u_\alpha^2, u_\alpha) = 0$  or  $v_{-\alpha}^2 = 0$  and,  $f(v_{-\alpha}^2, v_{-\alpha}) = 0$  or  $u_\alpha^2 = 0$ . If  $f(u_\alpha^2, u_\alpha) = 0$  then  $u_\alpha^2 = 0$  by (34) and, if  $f(v_{-\alpha}^2, v_{-\alpha}) = 0$  then  $v_{-\alpha}^2 = 0$  by (35). Therefore we conclude that if  $\text{char}(F) = 2$  then  $u_\alpha^2 = 0$  or  $v_{-\alpha}^2 = 0$ . If  $v_{-\alpha}^2 \neq 0$  by (35) we have  $-2\alpha = \alpha$ , that is,  $3\alpha = 0$ . If  $u_\alpha^2 \neq 0$  by (34) we have  $2\alpha = -\alpha$ , that is,  $3\alpha = 0$ .

Now we are ready to show the first example for this case.

**Example 5.11.** (The  $\mathbb{Z}_3$ -graded algebra  $Q_{rs}$ ) Let  $Q_{rs} = \bigoplus_{\gamma \in \mathbb{Z}_3} (Q_{rs})_\gamma$  be a  $\mathbb{Z}_3$ -graded algebra,  $(Q_{rs})_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $(Q_{rs})_{\bar{1}} = Fu_{\bar{1}}$  and  $(Q_{rs})_{\bar{2}} = Fv_{\bar{2}}$ , with the following multiplication table:

$\cdot$	$e_1$	$e_2$	$u_{\bar{1}}$	$v_{\bar{2}}$
$e_1$	$e_1$	$0$	$u_{\bar{1}}$	$0$
$e_2$	$0$	$e_2$	$0$	$v_{\bar{2}}$
$u_{\bar{1}}$	$0$	$u_{\bar{1}}$	$rv_{\bar{2}}$	$-e_1$
$v_{\bar{2}}$	$v_{\bar{2}}$	$0$	$-e_2$	$su_{\bar{1}}$

where  $r, s \in F$ . Let  $q_0(l_1e_1 + l_2e_2) = l_1l_2$  be a norm, for any  $l_1, l_2 \in F$ , defined in  $(Q_{rs})_{\bar{0}}$  and  $f$  be a bilinear form on  $Q_{rs}$  defined as follows:  $f(e_1, e_2) = 1$ ,  $f(e_1, 1) = 1$ ,  $f(e_2, 1) = 1$ ,  $f(u_{\bar{1}}, v_{\bar{2}}) = 1$ ,  $f(u_{\bar{1}}, u_{\bar{1}}) = 0$ ,  $f(v_{\bar{2}}, v_{\bar{2}}) = 0$  and  $f((Q_{rs})_{\bar{0}}, (Q_{rs})_{\bar{1}} + (Q_{rs})_{\bar{2}}) = 0$ . If  $r = s = 0$  then  $Q_{00}$  is a composition algebra.

**Proposition 5.12.** Let  $\epsilon$  be the trivial commutation factor on  $\mathbb{Z}_3$ . If  $r \neq 0$  or  $s \neq 0$ ,  $Q_{rs}$  is a composition  $\epsilon$ -algebra if and only if  $r = 0$  or  $s = 0$ , and  $\text{char}(F) = 2$ .

*Proof.* We have

$$\begin{aligned}
f(e_2v_2, v_2v_2) + f(e_2v_2, v_2v_2) - f(e_2, v_2)f(v_2, v_2) &= 2f(v_2, su_1) = 2s, \\
f(v_2e_1, v_2v_2) + f(v_2v_2, v_2e_1) - f(v_2, v_2)f(e_1, v_2) &= 2f(v_2, su_1) = 2s, \\
f(v_2 \cdot 1, v_2v_2) + f(v_2v_2, v_2 \cdot 1) - f(v_2, v_2)f(1, v_2) &= f(v_2, su_1) + f(su_1, v_2) = 2s, \\
f(v_2v_2, u_1u_1) + f(v_2u_1, u_1v_2) - f(v_2, u_1)f(v_2, u_1) &= f(su_1, rv_2) + f(-e_2, -e_1) - 1 = rs, \\
f(e_2v_2, u_1e_1) + f(e_2e_2, u_1v_2) - f(e_2, u_1)f(v_2, e_2) &= f(v_2, u_1) + f(e_2, -e_1) - 0 = 0, \\
f(e_1u_1, u_1u_1) + f(e_1u_1, u_1u_1) - f(e_1, u_1)f(u_1, u_1) &= 2f(u_1, rv_2) = 2r, \\
f(u_1e_2, u_1u_1) + f(u_1u_1, u_1e_2) - f(u_1, u_1)f(e_2, u_1) &= 2f(u_1, rv_2) = 2r, \\
f(u_1 \cdot 1, u_1u_1) + f(u_1u_1, u_1 \cdot 1) - f(u_1, u_1)f(1, u_1) &= f(u_1, rv_2) + f(rv_2, u_1) = 2r.
\end{aligned}$$

The other cases are similar. Therefore, if  $r \neq 0$ ,  $Q_{rs}$  is a composition  $\epsilon$ -algebra if and only if  $s = 0$  and  $\text{char}(F) = 2$ . And, if  $s \neq 0$ ,  $Q_{rs}$  is a composition  $\epsilon$ -algebra if and only if  $r = 0$  and  $\text{char}(F) = 2$ .  $\square$

**Remark 5.13.** If  $s \neq 0$  then the algebra  $Q_{0s}$  is not power-associative, for example

$$(v_2, v_2, v_2) = (v_2v_2)v_2 - v_2(v_2v_2) = (su_1)v_2 - v_2(su_1) = -se_1 + se_2 \neq 0.$$

**Remark 5.14.**  $Q_{s0}$  and  $Q_{0s}$  are equivalents: Let  $g : Q_{s0} \rightarrow Q_{0s}$  be given by  $g(u_1) = v_2$ ,  $g(v_2) = u_1$ ,  $g(e_1) = e_2$  and  $g(e_2) = e_1$ . Let  $\theta : \mathbb{Z}_3 \rightarrow \mathbb{Z}_3$  be given by  $\theta(1) = 2$ . It is easy to see that  $g$  is an isomorphism of algebras and  $g((Q_{s0})_\gamma) = (Q_{0s})_{\theta(\gamma)}$  for all  $\gamma \in \mathbb{Z}_3$ .

**Remark 5.15.** If  $\dim(U) = \dim(V) = 1$  then  $A$  is equivalent to the algebra of matrices  $M_2(F)$ , or  $A = Q_{0s}$ , or  $A = Q_{r0}$ .

Now, let us see what happens when  $\dim(U) \geq 2$  or  $\dim(V) \geq 2$ . For that, we need the following proposition first.

**Proposition 5.16.** *If  $\dim(U) \geq 2$  or  $\dim(V) \geq 2$  then there exist  $u_\alpha, u'_\beta \in U$  and  $v_{-\alpha}, v'_{-\beta} \in V$  linearly independent satisfying that*

$$\begin{aligned}
f(u_\alpha, v_{-\alpha}) &= 1, & f(u'_\beta, v_{-\alpha}) &= 0, \\
f(u_\alpha, v'_{-\beta}) &= 0, & f(u'_\beta, v'_{-\beta}) &= 1.
\end{aligned}$$

*Proof.* This follows easily because the restriction of  $f$  to  $(U \cap A_\alpha) \times (V \cap A_{-\alpha})$  is nondegenerate for each  $\alpha$ .  $\square$

Returning to our algebra  $A$ . If  $\dim(U) \geq 2$  or  $\dim(V) \geq 2$  then, by the above proposition, there exists  $u_\alpha, u'_\beta \in U$  and  $v_{-\alpha}, v'_{-\beta} \in V$ , linearly independent, satisfying that

$$\begin{aligned}
f(u_\alpha, v_{-\alpha}) &= 1, & f(u'_\beta, v_{-\alpha}) &= 0, \\
f(u_\alpha, v'_{-\beta}) &= 0, & f(u'_\beta, v'_{-\beta}) &= 1.
\end{aligned}$$

Let us show that  $u_\alpha^2 = 0$ . By Proposition 3.10 *vi*), (34), (31) and (30), we obtain that

$$\begin{aligned}
u_\alpha^2 &= f(u'_\beta, v'_{-\beta})u_\alpha^2 = \overline{u'_\beta}[v'_{-\beta}u_\alpha^2] + \epsilon(\beta, -\beta)\overline{v'_{-\beta}}[u'_\beta u_\alpha^2] \\
&= u'_\beta[f(v'_{-\beta}, u_\alpha)u_\alpha + \epsilon(-\beta, \alpha)u_\alpha(v'_{-\beta}u_\alpha)] - v'_{-\beta}[u'_\beta(f(u_\alpha^2, u_\alpha)v_{-\alpha})] \\
&= \epsilon(-\beta, \alpha)u'_\beta[u_\alpha(-f(v'_{-\beta}, u_\alpha)e_2)] - f(u_\alpha^2, u_\alpha)v'_{-\beta}[-f(u'_\beta, v_{-\alpha})e_1] = 0.
\end{aligned}$$

Similarly we can prove that  $(u'_\beta)^2 = v_{-\alpha}^2 = (v'_{-\beta})^2 = 0$ .

Let us show that  $u_\alpha u'_\beta \neq 0$  and  $v_{-\alpha} v'_{-\beta} \neq 0$ . By Proposition 3.10 *vi*) and (30) we have

$$\begin{aligned} (u_\alpha u'_\beta) v_{-\alpha} &= -(u_\alpha u'_\beta) \overline{v_{-\alpha}} = -f(u'_\beta, v_{-\alpha}) u_\alpha + \epsilon(\beta, -\alpha) (u_\alpha v_{-\alpha}) \overline{u'_\beta} \\ &= \epsilon(\alpha, \beta) (f(u_\alpha, v_{-\alpha}) e_1) u'_\beta = \epsilon(\alpha, \beta) u'_\beta \neq 0. \end{aligned}$$

Consequently,  $u_\alpha u'_\beta \neq 0$ .

By Proposition 3.10 *vi*) and (31) we obtain

$$\begin{aligned} (v_{-\alpha} v'_{-\beta}) u_\alpha &= -(v_{-\alpha} v'_{-\beta}) \overline{u_\alpha} = -f(v'_{-\beta}, u_\alpha) v_{-\alpha} + \epsilon(-\beta, \alpha) (v_{-\alpha} u_\alpha) \overline{v'_{-\beta}} \\ &= \epsilon(\alpha, \beta) (f(v_{-\alpha}, u_\alpha) e_2) v'_{-\beta} = \epsilon(\alpha, \beta) v'_{-\beta} \neq 0. \end{aligned}$$

Therefore  $v_{-\alpha} v'_{-\beta} \neq 0$ .

On the other hand, by Proposition 3.10 *i*) we have

$$\begin{aligned} f(v_{-\alpha} v'_{-\beta}, v_{-\alpha}) &= \epsilon(-\alpha, -\beta) f(v'_{-\beta}, \overline{v_{-\alpha} v_{-\alpha}}) = -\epsilon(-\alpha, -\beta) f(v'_{-\beta}, v_{-\alpha}^2) = 0, \\ f(v_{-\alpha} v'_{-\beta}, v'_{-\beta}) &= \epsilon(-\beta, -\beta) f(v_{-\alpha}, \overline{v'_{-\beta} v'_{-\beta}}) = -\epsilon(-\beta, -\beta) f(v_{-\alpha}, (v'_{-\beta})^2) = 0, \\ f(u_\alpha u'_\beta, u_\alpha) &= \epsilon(\alpha, \beta) f(u'_\beta, \overline{u_\alpha u_\alpha}) = -\epsilon(\alpha, \beta) f(u'_\beta, u_\alpha^2) = 0, \\ f(u_\alpha u'_\beta, u'_\beta) &= \epsilon(\beta, \beta) f(u_\alpha, \overline{u'_\beta u'_\beta}) = -\epsilon(\beta, \beta) f(u_\alpha, (u'_\beta)^2) = 0. \end{aligned}$$

Therefore  $v_{-\alpha} v'_{-\beta}$ ,  $u_\alpha$  and  $u'_\beta$  are linearly independent and,  $u_\alpha u'_\beta$ ,  $v_{-\alpha}$  and  $v'_{-\beta}$  are also linearly independent. Thus we have  $\dim(U) = \dim(V) \geq 3$ . Consequently, there exist  $u_\alpha, u'_\beta, u''_\gamma \in U$  ( $\gamma = -\alpha - \beta$ ) linearly independent such that  $f(u_\alpha u'_\beta, u''_\gamma) = 1$ . Then  $v_{-\alpha} = u'_\beta u''_\gamma$ ,  $v'_{-\beta} = u''_\gamma u_\alpha$  and  $v''_{-\gamma} = u_\alpha u'_\beta$  are the dual elements of  $u_\alpha, u'_\beta$  and  $u''_\gamma$  relative to  $f$ , respectively. In particular, by Proposition 3.10 *i*),

$$\begin{aligned} f(u_\alpha, v_{-\alpha}) &= f(u_\alpha, u'_\beta u''_\gamma) = f(u_\alpha, \overline{u'_\beta u''_\gamma}) = f(u_\alpha, \epsilon(\beta, \gamma) \overline{u''_\gamma u'_\beta}) \\ &= -\epsilon(\beta, \gamma) f(u_\alpha, u''_\gamma u'_\beta) = \epsilon(\beta, \gamma) f(u_\alpha, u''_\gamma \overline{u'_\beta}) = f(u_\alpha u'_\beta, u''_\gamma) = 1. \\ f(u'_\beta, v'_{-\beta}) &= f(u'_\beta, u''_\gamma u_\alpha) = f(u'_\beta, \overline{u''_\gamma u_\alpha}) = f(u'_\beta, \epsilon(\gamma, \alpha) \overline{u_\alpha u''_\gamma}) = -\epsilon(\gamma, \alpha) f(u'_\beta, u_\alpha u''_\gamma) \\ &= \epsilon(\gamma, \alpha) f(u'_\beta, \overline{u_\alpha u''_\gamma}) = \epsilon(\gamma, \alpha) \epsilon(\beta, \alpha) f(u_\alpha u'_\beta, u''_\gamma) = \epsilon(-\alpha, \alpha) = 1. \\ f(u''_\gamma, v''_{-\gamma}) &= f(u''_\gamma, u_\alpha u'_\beta) = \epsilon(\gamma, \alpha + \beta) f(u_\alpha u'_\beta, u''_\gamma) = \epsilon(\gamma, -\gamma) = 1. \end{aligned}$$

By (30)

$$\begin{aligned} u_\alpha v_{-\alpha} &= -f(u_\alpha, v_{-\alpha}) e_1 = -e_1, \\ u'_\beta v'_{-\beta} &= -f(u'_\beta, v'_{-\beta}) e_1 = -e_1, \\ u''_\gamma v''_{-\gamma} &= -f(u''_\gamma, v''_{-\gamma}) e_1 = -e_1. \end{aligned}$$

By (31)

$$\begin{aligned} v_{-\alpha} u_\alpha &= -f(v_{-\alpha}, u_\alpha) e_2 = -e_2, \\ v'_{-\beta} u'_\beta &= -f(v'_{-\beta}, u'_\beta) e_2 = -e_2, \\ v''_{-\gamma} u''_\gamma &= -f(v''_{-\gamma}, u''_\gamma) e_2 = -e_2. \end{aligned}$$

By Proposition 3.10 *iv*)

$$\begin{aligned} u'_\beta u_\alpha &= -f(u'_\beta, u_\alpha) - \epsilon(\beta, \alpha)u_\alpha u'_\beta = -\epsilon(\beta, \alpha)v''_{-\gamma}, \\ u_\alpha u''_\gamma &= -f(u_\alpha, u''_\gamma) - \epsilon(\alpha, \gamma)u''_\gamma u_\alpha = -\epsilon(\alpha, -\alpha - \beta)v'_{-\beta} = -\epsilon(\beta, \alpha)v'_{-\beta}, \\ u''_\gamma u'_\beta &= -f(u''_\gamma, u'_\beta) - \epsilon(\gamma, \beta)u'_\beta u''_\gamma = -\epsilon(-\alpha - \beta, \beta)v_{-\alpha} = -\epsilon(\beta, \alpha)v_{-\alpha}. \end{aligned}$$

By Proposition 3.10 *vi*) and (31)

$$\begin{aligned} v_{-\alpha} v'_{-\beta} &= v_{-\alpha}(u''_\gamma u_\alpha) = -f(v_{-\alpha}, u''_\gamma)u_\alpha - \epsilon(-\alpha, \gamma)u''_\gamma(v_{-\alpha}u_\alpha) \\ &= -\epsilon(-\alpha, -\alpha - \beta)u''_\gamma(-e_2) = \epsilon(-\alpha, -\beta)u''_\gamma = \epsilon(\alpha, \beta)u''_\gamma, \\ v_{-\alpha} v''_{-\gamma} &= v_{-\alpha}(u_\alpha u'_\beta) = -f(v_{-\alpha}, u_\alpha)u'_\beta - \epsilon(-\alpha, \alpha)u_\alpha(v_{-\alpha}u'_\beta) \\ &= -u'_\beta + u_\alpha(f(v_{-\alpha}, u'_\beta)e_2) = -u'_\beta, \\ v'_{-\beta} v''_{-\gamma} &= v'_{-\beta}(u_\alpha u'_\beta) = -f(v'_{-\beta}, u_\alpha)u'_\beta - \epsilon(-\beta, \alpha)u_\alpha(v'_{-\beta}u'_\beta) \\ &= -\epsilon(\alpha, \beta)u_\alpha(-e_2) = \epsilon(\alpha, \beta)u_\alpha. \end{aligned}$$

The other products are calculated similarly. In summary, we have the following example.

**Example 5.17.** (The algebra  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$ ). Let  $\Gamma = \langle \alpha, \beta \rangle$  be the abelian group generated by  $\alpha$  and  $\beta$ ,  $\epsilon$  be a commutation factor on  $\Gamma$  such that  $\epsilon(\gamma, \gamma) = 1$  for all  $\gamma \in \Gamma$ , and  $B = K \oplus U \oplus V$  ( $K = Fe_1 \oplus Fe_2 \subseteq B_0$ ,  $U = Fu_\alpha \oplus Fu'_\beta \oplus Fu''_\gamma$ ,  $V = Fv_{-\alpha} \oplus Fv'_{-\beta} \oplus Fv''_{-\gamma}$ ,  $\gamma = -\alpha - \beta$ ) be a  $\Gamma$ -graded algebra, with multiplication table:

$\cdot$	$e_1$	$e_2$	$u_\alpha$	$u'_\beta$	$u''_\gamma$	$v_{-\alpha}$	$v'_{-\beta}$	$v''_{-\gamma}$
$e_1$	$e_1$	$0$	$u_\alpha$	$u'_\beta$	$u''_\gamma$	$0$	$0$	$0$
$e_2$	$0$	$e_2$	$0$	$0$	$0$	$v_{-\alpha}$	$v'_{-\beta}$	$v''_{-\gamma}$
$u_\alpha$	$0$	$u_\alpha$	$0$	$v''_{-\gamma}$	$-\epsilon(\beta, \alpha)v'_{-\beta}$	$-e_1$	$0$	$0$
$u'_\beta$	$0$	$u'_\beta$	$-\epsilon(\beta, \alpha)v''_{-\gamma}$	$0$	$v_{-\alpha}$	$0$	$-e_1$	$0$
$u''_\gamma$	$0$	$u''_\gamma$	$v'_{-\beta}$	$-\epsilon(\beta, \alpha)v_{-\alpha}$	$0$	$0$	$0$	$-e_1$
$v_{-\alpha}$	$v_{-\alpha}$	$0$	$-e_2$	$0$	$0$	$0$	$\epsilon(\alpha, \beta)u''_\gamma$	$-u'_\beta$
$v'_{-\beta}$	$v'_{-\beta}$	$0$	$0$	$-e_2$	$0$	$-u''_\gamma$	$0$	$\epsilon(\alpha, \beta)u_\alpha$
$v''_{-\gamma}$	$v''_{-\gamma}$	$0$	$0$	$0$	$-e_2$	$\epsilon(\alpha, \beta)u'_\beta$	$-u_\alpha$	$0$

The norm  $q = (q_0, f)$  is given by:  $q_0(l_1e_1 + le_2) = l_1l_2$  for any  $l_1, l_2 \in F$ ,  $f(e_1, e_2) = 1$ ,  $f(e_1, 1) = 1$ ,  $f(e_2, 1) = 1$ ,  $f(u_\alpha, v_{-\alpha}) = 1$ ,  $f(u'_\beta, v'_{-\beta}) = 1$ ,  $f(u''_\gamma, v''_{-\gamma}) = 1$ ,  $f(K \oplus V, V) = f(K \oplus U, U) = 0$  and  $f(u_\alpha, Fv'_{-\beta} + Fv''_{-\gamma}) = f(u'_\beta, Fv_{-\alpha} + Fv''_{-\gamma}) = f(u''_\gamma, Fv_\alpha + Fv'_{-\beta}) = 0$ . We denote the algebra  $B$  as  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$ .  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, 1)$  is the Cayley-Dickson algebra with a  $\langle \alpha, \beta \rangle$ -grading.

**Proposition 5.18.**  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$  is a composition  $\epsilon$ -algebra.

*Proof.* We have

$$\begin{aligned} f(u_\alpha e_2, u'_\beta u''_\gamma) + \epsilon(\beta, \gamma)\epsilon(0, \gamma)\epsilon(0, \beta)f(u_\alpha u''_\gamma, u'_\beta e_2) - \epsilon(0, \beta)f(u_\alpha, u'_\beta)f(e_2, u''_\gamma) \\ = f(u_\alpha, v_{-\alpha}) + \epsilon(\beta, -\alpha)f(-\epsilon(\beta, \alpha)v'_{-\beta}, u_\beta) = 1 - \epsilon(\beta, -\alpha)\epsilon(\beta, \alpha) = 0 \end{aligned}$$

$$\begin{aligned} f(e_1 u_\alpha, u'_\beta u''_\gamma) + \epsilon(\beta, \gamma)\epsilon(\alpha, \gamma)\epsilon(\alpha, \beta)f(e_1 u''_\gamma, u'_\beta \cdot u_\alpha) - \epsilon(\alpha, \beta)f(e_1, u'_\beta)f(u_\alpha, u''_\gamma) \\ = f(u_\alpha, v_{-\alpha}) + \epsilon(\beta, -\alpha)f(u''_\gamma, -\epsilon(\beta, \alpha)v''_{-\gamma}) = 1 - \epsilon(\beta, -\alpha)\epsilon(\beta, \alpha) = 0 \end{aligned}$$

$$\begin{aligned} & f(v_{-\alpha}e_1, v'_{-\beta}v''_{-\gamma}) + \epsilon(-\beta, -\gamma)f(v_{-\alpha}v''_{-\gamma}, v'_{-\beta}e_1) - f(v_{-\alpha}, v'_{-\beta})f(e_2, v''_{-\gamma}) \\ &= f(v_{-\alpha}, \epsilon(\alpha, \beta)u_\alpha) + \epsilon(-\beta, \alpha)f(-u'_\beta, v_{-\beta}) = \epsilon(\alpha, \beta) - \epsilon(-\beta, \alpha) = 0 \end{aligned}$$

$$\begin{aligned} & f(e_2v_{-\alpha}, v'_{-\beta}v''_{-\gamma}) + \epsilon(-\alpha, -\beta)f(e_2v''_{-\gamma}, v'_{-\beta}v_{-\alpha}) - \epsilon(-\alpha, -\beta)f(e_2, v'_{-\beta})f(v_{-\alpha}, v''_{-\gamma}) \\ &= f(v_{-\alpha}, \epsilon(\alpha, \beta)u_\alpha) + \epsilon(-\alpha, -\beta)f(v''_{-\gamma}, -u''_\gamma) = \epsilon(\alpha, \beta) - \epsilon(-\alpha, -\beta) = 0 \end{aligned}$$

$$\begin{aligned} & f(u_\alpha u'_\beta, v_{-\alpha}v'_{-\beta}) + \epsilon(-\alpha, -\beta)\epsilon(\beta, -\alpha)f(u_\alpha v'_{-\beta}, v_{-\alpha}u'_\beta) - \epsilon(\beta, -\alpha)f(u_\alpha, v_{-\alpha})f(u'_\beta, v'_{-\beta}) \\ &= f(v''_{-\gamma}, \epsilon(\alpha, \beta)u''_\gamma) - \epsilon(\beta, -\alpha) = \epsilon(\alpha, \beta) - \epsilon(\beta, -\alpha) = 0 \end{aligned}$$

$$\begin{aligned} & f(u_\alpha v_{-\alpha}, v'_{-\beta}u'_\beta) + \epsilon(-\alpha, \beta)\epsilon(-\alpha, -\beta)f(u_\alpha u'_\beta, v'_{-\beta}v_{-\alpha}) - \epsilon(-\alpha, -\beta)f(u_\alpha, v'_{-\beta})f(v_{-\alpha}, u'_{-\beta}) \\ &= f(-e_1, -e_2) + f(v''_{-\gamma}, -u''_\gamma) = 1 - 1 = 0 \end{aligned}$$

The other cases are similar.  $\square$

**Remark 5.19.** Let  $A$  be a composition  $\epsilon$ -algebra such that  $A^{(1)} = 0$ ,  $A_0$  is a split composition algebra and  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta)) \subseteq A$ . As  $(u_\alpha u'_\beta)e_1 - u_\alpha(u'_\beta e_1) = v''_{-\gamma}e_1 = v''_{-\gamma}$  then  $CG(\langle \alpha, \beta \rangle, \epsilon(\alpha, \beta))$  is non-associative. Since  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$  is a non-associative  $\Gamma$ -graded subalgebra of  $A$  then, by proposition 5.4,  $A = CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$ . On the other hand,  $CG(\langle \alpha, \beta \rangle, \alpha, \beta, \epsilon(\alpha, \beta))$  is not always an alternative algebra: If  $\epsilon(\beta, \alpha) \neq 1$  then

$$(u_\alpha + u'_\beta, u_\alpha + u'_\beta, e_1) = [1 - \epsilon(\beta, \alpha)]v''_{-\gamma} \neq 0.$$

**Remark 5.20.** Let  $\Gamma$  be an abelian group generated by two elements, without loss of generality, we have  $\Gamma = \mathbb{Z}_l$  or  $\Gamma = \mathbb{Z}_n \oplus \mathbb{Z}_m$ , with  $l \in \mathbb{Z}_{>0}$  and  $m, n \in \mathbb{Z}_{\geq 2} \cup \{0\}$ . Let  $\epsilon$  a commutation factor on  $\Gamma$  such that  $\epsilon(\alpha, \alpha) = 1$  for all  $\alpha \in \Gamma$ . If  $\Gamma = \mathbb{Z}_l$ ,  $l \in \mathbb{Z}_{\geq 0}$ , then  $\epsilon$  is the trivial commutation factor on  $\Gamma$  and if  $\Gamma = \mathbb{Z}_n \oplus \mathbb{Z}_m$ ,  $m, n \in \mathbb{Z}_{\geq 2} \cup \{0\}$ , then  $\epsilon$  is a commutation factor of Proposition 2.5.

To characterize color composition algebras in this case it is sufficient to know the gradings of composition algebras ([Eld98] and [Eld09]) and some commutation factors on abelian groups (see section 2). So we have the following result.

**Theorem 5.21.** Let  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a composition  $\epsilon$ -algebra over a field  $F$  such that  $A = A^{(0)}$  and  $A_0$  is split. Then  $A$  is equivalent to one of the following algebras.

- i)  $A = A_0$ . Thus,  $A$  is equivalent to either the algebra  $F \oplus F$ , or the algebra of matrices  $M_2(F)$ , or the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- ii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_2} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$  and  $A_{\bar{1}} = Fu_{\bar{1}} \oplus Fv_{\bar{1}}$ , and  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_2$ .  $A$  is equivalent to the algebra of matrices  $M_2(F)$ .
- iii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_{\bar{1}} = Fu_{\bar{1}}$ ,  $A_{-\bar{1}} = Fv_{-\bar{1}}$ ,  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_n$  and  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$ .  $A$  is equivalent to the algebra of matrices  $M_2(F)$ .
- iv)  $A = Q_{r0}$  for any  $r \in F$ ,  $\text{char}(F) = 2$ .
- v)  $A = \bigoplus_{\gamma \in \mathbb{Z}_2} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2 \oplus Fu_{\bar{0}} \oplus Fv_{\bar{0}}$ ,  $A_{\bar{1}} = Fu'_{\bar{1}} \oplus Fv'_{\bar{1}} \oplus Fv''_{\bar{1}} \oplus Fu''_{\bar{1}}$ , and  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_2$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .

- vi)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_{\bar{1}} = Fu_{\bar{1}} \oplus Fu'_{\bar{1}} \oplus Fu''_{\bar{1}}$ ,  $A_{-\bar{1}} = Fv_{-\bar{1}} \oplus Fv'_{-\bar{1}} \oplus Fv''_{-\bar{1}}$ ,  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_n$  and  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- vii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_4} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_{\bar{1}} = Fu_{\bar{1}} \oplus Fu'_{\bar{1}}$ ,  $A_{\bar{2}} = Fu''_{\bar{2}} \oplus Fv''_{\bar{2}}$ ,  $A_{\bar{3}} = Fv_{\bar{3}} \oplus Fv'_{\bar{3}}$ , and  $\epsilon$  trivial commutation factor on  $\mathbb{Z}_4$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- viii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2 \oplus Fu''_{\bar{0}} \oplus Fv''_{\bar{0}}$ ,  $A_{\bar{1}} = Fu_{\bar{1}} \oplus Fv'_{\bar{1}}$ ,  $A_{-\bar{1}} = Fu'_{-\bar{1}} \oplus Fv_{-\bar{1}}$ ,  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_n$  and  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- ix)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_{\bar{1}} = Fu_{\bar{1}} \oplus Fu'_{\bar{1}}$ ,  $A_{-\bar{1}} = Fv_{-\bar{1}} \oplus Fv'_{-\bar{1}}$ ,  $A_{\bar{2}} = Fv''_{\bar{2}}$ ,  $A_{-\bar{2}} = Fu''_{-\bar{2}}$ ,  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_n$  and  $n \in \mathbb{Z}_{\geq 5} \cup \{0\}$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- x)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$ , where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_\alpha = Fu_\alpha$ ,  $A_{-\alpha} = Fv_{-\alpha}$ ,  $A_\beta = Fu'_\beta$ ,  $A_{-\beta} = Fv'_{-\beta}$ ,  $A_\gamma = Fu''_\gamma$ ,  $A_{-\gamma} = Fv''_{-\gamma}$  ( $\gamma = -\alpha - \beta$ ),  $\alpha$  and  $\beta$  generate  $\mathbb{Z}_n$  and  $n \in \mathbb{Z}_{\geq 7} \cup \{0\}$ .  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_n$  and  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- xi)  $A = \bigoplus_{\gamma \in \mathbb{Z}_{2n}} A_\gamma$  where  $A_{\bar{0}} = Fe_1 \oplus Fe_2$ ,  $A_{\bar{1}} = Fu_{\bar{1}}$ ,  $A_{-\bar{1}} = Fv_{-\bar{1}}$ ,  $A_{\overline{n-1}} = Fu'_{\overline{n-1}}$ ,  $A_{\overline{1-n}} = Fv'_{\overline{1-n}}$ ,  $A_{\bar{n}} = Fu''_{\bar{n}} \oplus v''_{\bar{n}}$ ,  $\epsilon$  is the trivial commutation factor on  $\mathbb{Z}_{2n}$  and  $n \in \mathbb{Z}_{\geq 3}$ .  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$ .
- xii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_2 \oplus \mathbb{Z}_2} A_\gamma$  where  $A_{(\bar{0}, \bar{0})} = Fe_1 \oplus Fe_2$ ,  $A_{(\bar{1}, \bar{0})} = Fu_{(\bar{1}, \bar{0})} \oplus Fv_{(\bar{1}, \bar{0})}$ ,  $A_{(\bar{0}, \bar{1})} = Fu'_{(\bar{0}, \bar{1})} \oplus Fv'_{(\bar{0}, \bar{1})}$ ,  $A_{(\bar{1}, \bar{1})} = Fv''_{(\bar{1}, \bar{1})} \oplus Fu''_{(\bar{1}, \bar{1})}$ , and  $\epsilon$  is any commutation factor on  $\mathbb{Z}_2 \oplus \mathbb{Z}_2$  such that  $\epsilon(\alpha, \alpha) = 1$  for all  $\alpha \in \mathbb{Z}_2 \oplus \mathbb{Z}_2$ . If  $\epsilon$  is the trivial commutation factor then  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$  and if  $\epsilon$  is not the trivial commutation factor ( $\text{char}(F) \neq 2$  and  $\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = -1$ ) then  $A$  is equivalent to the algebra  $CG(\mathbb{Z}_2 \oplus \mathbb{Z}_2, (\bar{1}, \bar{0}), (\bar{0}, \bar{1}), -1)$ .
- xiii)  $A = \bigoplus_{\gamma \in \mathbb{Z}_l \oplus \mathbb{Z}_m} A_\gamma$  where  $A_{(\bar{0}, \bar{0})} = Fe_1 \oplus Fe_2$ ,  $A_\alpha = Fu_\alpha$ ,  $A_{-\alpha} = Fv_{-\alpha}$ ,  $A_\beta = Fu'_\beta$ ,  $A_{-\beta} = Fv'_{-\beta}$ ,  $A_\gamma = Fu''_\gamma$ ,  $A_{-\gamma} = Fv''_{-\gamma}$  ( $\gamma = -\alpha - \beta$ ),  $\alpha$  and  $\beta$  generate  $\mathbb{Z}_l \oplus \mathbb{Z}_m$ ,  $l, m \in \mathbb{Z}_{\geq 3} \cup \{0\}$ , and  $\epsilon$  is a commutation factor on  $\mathbb{Z}_l \oplus \mathbb{Z}_m$  such that  $\epsilon(\delta, \delta) = 1$  for all  $\delta \in \mathbb{Z}_l \oplus \mathbb{Z}_m$ . If  $\epsilon$  is the trivial commutation factor then  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$  and if  $\epsilon$  is not the trivial commutation factor ( $\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) \neq 1$ ) then  $A$  is equivalent to the algebra  $CG(\mathbb{Z}_l \oplus \mathbb{Z}_m, \alpha, \beta, \epsilon(\alpha, \beta))$ .
- xiv)  $A = \bigoplus_{\gamma \in \mathbb{Z}_n \oplus \mathbb{Z}_2} A_\gamma$  where  $A_{(\bar{0}, \bar{0})} = Fe_1 \oplus Fe_2$ ,  $A_\alpha = Fu_\alpha$ ,  $A_{-\alpha} = Fv_{-\alpha}$ ,  $A_\beta = Fu'_\beta$ ,  $A_{-\beta} = Fv'_{-\beta}$ ,  $A_\gamma = Fu''_\gamma \oplus Fv''_\gamma$ , ( $\gamma = -\alpha - \beta = \alpha + \beta = -\gamma$ ),  $\alpha$  and  $\beta$  generate  $\mathbb{Z}_n \oplus \mathbb{Z}_2$ ,  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$  and  $\epsilon$  is a commutation factor on  $\mathbb{Z}_n \oplus \mathbb{Z}_2$  such that  $\epsilon(\delta, \delta) = 1$  for all  $\delta \in \mathbb{Z}_n \oplus \mathbb{Z}_2$ . If  $\epsilon$  is the trivial commutation factor then  $A$  is equivalent to the Cayley-Dickson algebra  $\mathbf{C}(F)$  and if  $\epsilon$  is not the trivial commutation factor ( $n$  is an even non-negative integer,  $\text{char}(F) \neq 2$  and  $\epsilon((\bar{1}, \bar{0}), (\bar{0}, \bar{1})) = -1$ ) then  $A$  is equivalent to the algebra  $CG(\mathbb{Z}_n \oplus \mathbb{Z}_2, \alpha, \beta, -1)$ .

It is easy that an algebra of the item  $x$ ) is equivalent to an algebra of the item  $xiii$ ) if  $l$  and  $m$  are relatively prime and  $n = l \cdot m$ . Also, an algebra of the item  $xi$ ) is equivalent to an algebra of the item  $xiv$ ) in the case that  $n$  is odd.  $\epsilon$  is the trivial commutation factor in both cases.

5.3.  $A^{(1)} \neq 0$ . In this section we show that every color composition algebra  $A$  such that  $A^{(1)} \neq 0$  is equivalent to the alternative superalgebra  $B(1, 2)$  or  $B(4, 2)$ .

**Example 5.22.** (The superalgebra  $B(1, 2)$ , [She97]). Let  $F$  be a field of characteristic 3 and  $V$  be a vector space over  $F$  of dimension 2 with an alternating nondegenerate form  $(\cdot, \cdot)$ . Consider the superspace  $B(1, 2) = B(1, 2)_{\bar{0}} \oplus B(1, 2)_{\bar{1}}$  with  $B(1, 2)_{\bar{0}} = F1$ ,  $B(1, 2)_{\bar{1}} = V$ , and supercommutative multiplication given by  $1x = x1 = x$  and  $uv = (u, v)1$  for all  $x \in B(1, 2)$  and  $u, v \in V$ .  $B(1, 2)$  is a Hurwitz superalgebra ([EO02]) with the norm given by  $q_{\bar{0}}(1) = 1$ ,  $f(1, V) = 0$  and  $f(u, v) = (u, v)$  for all  $u, v \in V$ .

**Example 5.23.** (The superalgebra  $B(4, 2)$ , [She97]). Let  $F$  be a field of characteristic 3. Consider the superspace  $B(4, 2) = B(4, 2)_{\bar{0}} \oplus B(4, 2)_{\bar{1}}$  over  $F$ , with  $B(4, 2)_{\bar{0}} = M_2(F)$  the algebra of  $2 \times 2$  matrices over  $F$ , and  $B(4, 2)_{\bar{1}} = V = Fm_1 + Fm_2$  the 2-dimensional irreducible Cayley  $M_2(F)$ -bimodule, that is,  $M_2(F)$  acts on  $B(4, 2)_{\bar{1}}$  as follows:

$$e_{ij} \cdot m_k = \delta_{ik} m_j, \quad i, j, k \in \{1, 2\},$$

$$m \cdot a = \bar{a} \cdot m,$$

where  $a \in B(4, 2)_{\bar{0}}$ ,  $m \in B(4, 2)_{\bar{1}}$  and  $a \rightarrow \bar{a}$  is the symplectic involution in  $M_2(F)$  ( $a \mapsto \bar{a} := \text{tr}(a)1 - a$  for all  $a \in M_2(F)$ ). The odd multiplication on  $B(4, 2)_{\bar{1}}$  is defined by

$$m_1^2 = -e_{21}, \quad m_2^2 = e_{12}, \quad m_1 m_2 = e_{11} \quad \text{and} \quad m_2 m_1 = -e_{22}.$$

$B(4, 2)$  is a Hurwitz superalgebra ([EO02]) with norm given by  $q_{\bar{0}}(a) = \det(a)$  for all  $a \in M_2(F)$ ,  $f(M_2(F), V) = 0 = f(V, M_2(F))$ ,  $f(m_2, m_1) = -f(m_1, m_2) = 1$  and  $f(m_i, m_i) = 0$ .

**Lemma 5.24.** Let  $F$  be a field of characteristic different from 2 and  $A = \bigoplus_{\gamma \in \Gamma} A_\gamma$  be a composition  $\epsilon$ -algebra. If there exists  $\alpha \in \Gamma$  such that  $\epsilon(\alpha, \alpha) = -1$  and  $A_\alpha \neq 0$  then the superalgebra  $B = B^{(0)} \oplus B^{(1)}$  with  $B^{(0)} = \bigoplus_{i \in \mathbb{Z}} A_{2i\alpha}$  and  $B^{(1)} = \bigoplus_{i \in \mathbb{Z}} A_{(2i+1)\alpha}$ , is isomorphic to a Hurwitz superalgebra, that is,  $\text{char}(F) = 3$  and  $B \cong B(1, 2)$  or  $B \cong B(4, 2)$ . In addition  $\Gamma = \langle \alpha \rangle$ ,  $A = \bigoplus_{i \in \mathbb{Z}} A_{i\alpha}$  and  $A$  is isomorphic, as an algebra, to  $B(1, 2)$  or  $B(4, 2)$ .

*Proof.* Let  $\alpha \in \Gamma$  such that  $A_\alpha \neq 0$  and  $\epsilon(\alpha, \alpha) = -1$ . Since  $\epsilon(r\alpha, \alpha) = (-1)^r$  for any  $r \in \mathbb{Z}$  then  $|\langle \alpha \rangle| = 2n \geq 2$  or  $|\langle \alpha \rangle| = \infty$ . By Proposition 5.4,  $B^{(0)} := \bigoplus_{i \in \mathbb{Z}} A_{2i\alpha}$  is associative. Since  $\epsilon(2i\alpha, 2j\alpha) = 1$  for any  $i, j \in \mathbb{Z}$  and  $\text{char}(F) \neq 2$ , then  $B^{(0)}$  is a composition algebra with dimension  $\leq 4$ , due to the remark 3.9. Now let  $B^{(1)} = \bigoplus_{i \in \mathbb{Z}} A_{(2i+1)\alpha}$ ,  $B^{(1)} \neq 0$ . Since  $\epsilon((2i+1)\alpha, (2j+1)\alpha) = -1$ ,  $\epsilon(2i\alpha, (2j+1)\alpha) = \epsilon((2i+1)\alpha, 2j\alpha) = 1$  for all  $i, j \in \mathbb{Z}$ ,  $f(B^{(0)}, B^{(1)}) = 0$  and  $\text{char}(F) \neq 2$ , it follows from this that  $B = B^{(0)} \oplus B^{(1)}$  is a Hurwitz superalgebra. Hence  $B$  is isomorphic to  $B(1, 2)$  or  $B(4, 2)$ . Moreover, again by Proposition 5.4, we have that  $A = B = \bigoplus_{i \in \mathbb{Z}} A_{i\alpha}$ , due to  $B(1, 2)$  and  $B(4, 2)$  are non-associative algebras. In addition  $\Gamma = \langle \alpha \rangle$  and  $A$  is isomorphic, as an algebra, to  $B(1, 2)$  or  $B(4, 2)$ .  $\square$

Therefore in this case is enough to consider some gradings of  $B(1, 2)$  and  $B(4, 2)$ , which were studied in [Ara15].

**Example 5.25.** Gradings of  $B(1, 2)$ .

Let  $B(1, 2)_{\bar{0}} = F1$ ,  $B(1, 2)_{\bar{1}} = Fx + Fy$ . Consider the following grading of  $B(1, 2)$  :  $A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma$ , where  $n \in \mathbb{Z}_{\geq 3} \cup \{0\}$ ,

$$(37) \quad A_{\bar{0}} = F1, \quad A_\alpha = Fx, \quad A_{-\alpha} = Fy \quad \text{and} \quad \langle \alpha \rangle = \mathbb{Z}_n.$$

**Example 5.26.** Gradings of  $B(4, 2)$ .

Let  $B(4, 2)_{\bar{0}} = M_2(F)$  and  $B(4, 2) = Fm_1 \oplus Fm_2$ . Consider the following grading of  $B(4, 2)$  :

$$\text{i) } A = \bigoplus_{\gamma \in \mathbb{Z}_n} A_\gamma \text{ where } n \in \mathbb{Z}_{\geq 5} \cup \{0\},$$

$$(38) \quad A_{\bar{0}} = Fe_{11} \oplus Fe_{22}, \quad A_\alpha = Fm_1, \quad A_{-\alpha} = Fm_2, \quad A_{2\alpha} = Fe_{21}, \quad A_{-2\alpha} = Fe_{12}$$

$$\text{and } \langle \alpha \rangle = \mathbb{Z}_n.$$

$$\text{ii) } A = \bigoplus_{\gamma \in \mathbb{Z}_4} A_\gamma,$$

$$(39) \quad A_{\bar{0}} = Fe_{11} \oplus Fe_{22}, \quad A_{\bar{2}} = Fe_{12} \oplus Fe_{21}, \quad A_{\bar{1}} = Fm_1, \quad A_{\bar{3}} = Fm_2.$$

$$\text{iii) } A = \bigoplus_{\gamma \in \mathbb{Z}_3} A_\gamma,$$

$$(40) \quad A_{\bar{0}} = Fe_{11} \oplus Fe_{22}, \quad A_{\bar{1}} = Fe_{12} \oplus Fm_1, \quad A_{\bar{2}} = Fe_{21} \oplus Fm_2.$$

Since on  $\mathbb{Z}_3$  only exists the trivial commutation factor, the last example is not important in our work.

According to the Theorem 11 and 15 of [Ara15], we have the following result.

**Theorem 5.27.** *Let  $F$  be a field of characteristic  $\neq 2$  and  $A$  be a composition  $\epsilon$ -algebra over the field  $F$  such that  $A^{(1)} \neq 0$ . Then,  $\text{char}(F) = 3$  and  $A$  is equivalent to one of the following cases:*

- i)  $A = B(1, 2)$  or  $A = B(4, 2)$  with the main  $\mathbb{Z}_2$ -grading.
- ii)  $A$  is the  $\mathbb{Z}_n$ -grading of  $B(1, 2)$  given in (37) with  $n$  an even non-negative integer other than 2,  $\alpha = \bar{1}$  and the commutation factor on  $\mathbb{Z}_n$  defined by  $\epsilon(\bar{1}, \bar{1}) = -1$ .
- iii)  $A$  is the  $\mathbb{Z}_n$ -grading of  $B(4, 2)$  given in (38) with  $n$  an even non-negative integer greater than 4,  $\alpha = \bar{1}$  and the commutation factor on  $\mathbb{Z}_n$  defined by  $\epsilon(\bar{1}, \bar{1}) = -1$ .
- iv)  $A$  is the  $\mathbb{Z}_4$ -grading of  $B(4, 2)$  given in (39) and the commutation factor on  $\mathbb{Z}_4$  defined by  $\epsilon(\bar{1}, \bar{1}) = -1$ .

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## ACKNOWLEDGMENTS

The first author was supported by CNPq. The second author was supported by CNPq and the Brazilian FAPESP.

D. CASTRO OVALLE: DEPARTAMENTO DE CIENCIAS BÁSICA, UNIVERSIDAD ECCI, BOGOTÁ, COLOMBIA.

*Email address:* dcastroo@eccci.edu.co

I. SHESTAKOV: INSTITUTO DE MATEMÁTICA E ESTATÍSTICA, UNIVERSIDADE DE SÃO PAULO, SÃO PAULO, BRAZIL.

*Email address:* shestak@ime.usp.br

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