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SHAPE IDENTITIES IN BERNSTEIN ALGEBRAS

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Abstract

We study the shape identities arising in the theory of Bernstein algebras. We determine all shape identities of minimal degree for two subclasses of Bernstein algebras, namely, normal Bernstein algebras and exceptional Bernstein algebras.

1 Bernstein algebras

Let F be a field of characteristic not 2, A a nonassociative commutative algebra over F. If $\omega : A \to F$ is a nonzero homomorphism, the ordered pair (A,ω) is called a *baric algebra* over F, ω is its *weight function* and, for $x \in A$, $\omega(x)$ is the *weight* of x. If $N = \ker\omega$ and $\omega(a) = 1$ then $A = Fa \oplus N$. A *Bernstein algebra* is a baric algebra (A,ω) such that the following identity holds:

$$x^2x^2 = \omega(x)^2x^2. \tag{1}$$

Bernstein algebras always have idempotents of weight 1 (for example, $e = a^2$ with $w(a) = 1$). An idempotent e of weight 1 determines the Peirce decomposition $A = Fe \oplus U \oplus V$, where $U = \{n \in N : 2en = n\}$ and $V = \{n \in N : en = 0\}$.

It is well-known that if $U^2 = 0$ then the same equality holds for any other choice of idempotent. The condition $U^2 = 0$ is then invariant and defines a subclass of the class of all Bernstein algebras, the so-called *exceptional* (or exclusive in the language of Lyubich [8]). As shown by Correa and Labra [3], a Bernstein algebra is exceptional if and only if it satisfies the identity

$$2(xy)(zt) = \omega(xy)zt + \omega(zt)xy. \tag{2}$$

The conditions $UV = 0$ and $V^2 = 0$ define the subclass of *normal* Bernstein algebras. It can be proved that A is normal if and only if A satisfies the identity

$$x^2y = \omega(x)xy. \tag{3}$$

The class of all Bernstein algebras is not a variety of algebras as the equation (1) involves variable coefficients $\omega(x)$. The same is true for the subclasses. A natural question if to find polynomial identities of lowest degree which hold for such classes. The variety defined by these identities is refered as the *minimal variety* containing these classes. The minimal variety containing the class of all Bernstein algebras or some of its most interesting subclasses were determined in Correa, Hentzel and Peresi [5], and Correa and Suazo [4]. For algebras satisfying $x^2 = \omega(x)x$ (also a subclass of Bernstein algebras), the determination of this minimal variety appears in Peresi ([9], Theorem 1). In the same paper, Theorem 2 finds the mininal variety containing all the Bernstein-Jordan algebras, and other results in this direction involving some other classes of baric algebras are given by Theorem 4. In all these calculations, it is used the technique of "representing identities by matrices", which is fully explained in [5]. Notations and terminology used here are the same as in [5].

2 Shape identities

The concept of shape polynomials was introduced in Costa [6] based on ideas already present in Etherington [7]. We review here some of the basic properties used in the following sections. We consider the free F-algebra $F[x_1, x_2, \ldots]$ generated by an infinite number of nonassociative noncommutative variables $x_1, x_2, \ldots$ Any word of the form $x_1x_2\ldots x_k$, with some arrangement of parenthesis, is called a *shape monomial* of degree k. The number of such monomials is $k' = \frac{1}{k}\binom{2k-2}{k-1}$. There are 5 shape monomials of degree 4, namely,

$$\mu_1 = (x_1x_2)x_3.x_4, \quad \mu_2 = x_1(x_2x_3).x_4, \quad \mu_3 = (x_1x_2)(x_3x_4),$$
$$\mu_4 = x_1.(x_2x_3)x_4, \quad \mu_5 = x_1.x_2(x_3x_4). \tag{4}$$

There are 14 shape monomials of degree 5, namely,

$$\nu_1 = (x_1x_2.x_3)x_4.x_5, \quad \nu_2 = (x_1.x_2x_3)x_4.x_5, \quad \nu_3 = x_1(x_2x_3.x_4).x_5,$$
$$\nu_4 = x_1(x_2.x_3x_4).x_5, \quad \nu_5 = x_1.(x_2x_3.x_4)x_5, \quad \nu_6 = x_1.(x_2.x_3x_4)x_5,$$
$$\nu_7 = x_1.x_2(x_3x_4.x_5), \quad \nu_8 = x_1.x_2(x_3.x_4x_5), \quad \nu_9 = (x_1x_2.x_3x_4)x_5, \tag{5}$$
$$\nu_{10} = x_1(x_2x_3.x_4x_5), \quad \nu_{11} = (x_1x_2.x_3).x_4x_5, \quad \nu_{12} = (x_1.x_2x_3).x_4x_5,$$
$$\nu_{13} = x_1x_2.(x_3x_4.x_5), \quad \nu_{14} = x_1x_2.(x_3.x_4x_5).$$

In general, the k' shape monomials of degree k generate a vector subspace of $F[x_1, x_2, \ldots]$ of dimension k'. Any nonzero element of this subspace is called a *shape polynomial of degree* k and the corresponding variety of F-algebras is called a *shape variety of degree* k. Given a F-algebra A, the *level* of A is the degree of the shape polynomials of smallest degree vanishing on A (if these polynomials exist). It is easily seen that if a shape variety defined by $S(x_1, \ldots, x_k) = \sum \alpha_\mu \mu$, where μ runs over the set of shape monomials of degree k and $\alpha_\mu \in F$, contains an algebra with an idempotent, then necessarily $\sum \alpha_\mu = 0$. So, in particular, a shape polynomial of degree 3 must be a scalar multiple of the associator $(x_1, x_2, x_3) = (x_1x_2)x_3 - x_1(x_2x_3)$. This says that algebras of level 3 are associative. In [1], Baeza-Vega and Costa stated some properties of shape identities of degree 4 and 5. According to [1], Proposition 2, the variety defined by the shape polynomial

$$S(x_1, x_2, x_3, x_4) = -2\mu_1 + \mu_2 + 2\mu_3 + \mu_4 - 2\mu_5 \tag{6}$$

is the only one of interest among all varieties of degree 4 (for our purposes). In case the degree is 5, the situation is more involved but some of these varieties are more natural in our context of genetic algebras theory.

In [6] it was proved that the gametic algebra for simple Mendelian inheritance, denoted by $G(n+1,2)$, satisfies the shape identity (6) and, being nonassociative, it has level 4. Also from [6] its commutative duplicate $Z(n+1,2)$ satisfies the degree 5 shape identity $S(x_1,x_2,x_3,x_4)x_5 = 0$. It is well known that $G(n+1,2)$ satisfies the equation $x^2 = \omega(x)x$ and so $x^2x^2 = \omega(x)^2x^2$. It is also known that $Z(n+1,2)$ is a Bernstein algebra. As a consequence, it seems natural to study the occurence of shape identities in the class of all Bernstein algebras. In particular, to find the shape variety of smallest degree containing all these algebras. As the polynomial identities of minimal degree satisfied by all Bernstein algebras have degree 6 ([5], Theorem 5), the degree of this minimal shape variety should be at least seven.

Some of the results of this paper were obtained "by hand". For instance, the fact that the shape polynomial $(x_1x_2,(x_3,x_4,x_5),x_6)$ is an identity for all exceptional Bernstein algebras. But, it was unknown wether this was the only shape identity of degree 6 for these algebras. The use of more sophisticated techniques, using computer, seemed to be a good way towards the solution of the problem. This was done mainly by L. A. Peresi, using methods developed by I. R. Hentzel already mentioned in section 1.

3 Representation of the symmetric group

Let S_n denote the symmetric group and FS_n the corresponding group algebra. When char(F) is 0 or greater than n, FS_n is semisimple and, therefore, it is isomorphic to a direct sum of matrices. Clifton [2] gives an algorithm to construct the irreducible representations of S_n. To construct a representation, Clifton first associates to $\pi \in S_n$ a matrix A_π and then the representation is given by $\pi \in S_n \to A_I^{-1}A_\pi$. In this paper we use the maps $\pi \to (A_\pi)^t$ to construct an isomorphism between FS_n and a direct sum of matrices. We call each direct summand a representation although it is not necessarily a representation in the usual sense for $n > 4$. For instance, if $n = 4$ we have five representations and the isomorphism of FS_4 into the direct sum of matrices is given by

$$(12) \to [1] \oplus \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix} \oplus [-1],$$

$$(1234) \to [1] \oplus \begin{bmatrix} -1 & -1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \oplus \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} \oplus \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \oplus [-1]. \qquad (7)$$

In the next sections, to verify if there is or there is not a shape identity of degree n we use the representations (in our sense) of the group algebra FS_n. *Therefore, we assume that the characteristic of the field is zero, or greater than the degree of the shape identity.*

4 Normal Bernstein algebras

In this section, we find all shape identities of minimal degree satisfied by all normal Bernstein algebras. In the first part we show that no such identities of degree four exist.

Degree four. Linearizing the identity (3) we obtain $f(x_1, x_2, x_3) = 2(x_1x_2)x_3 - \omega(x_1)x_2x_3 - \omega(x_2)x_1x_3 = 0$. Using this equation we obtain the following identities of degree four:

$$f(x_1, x_2, x_3)x_4 = 0, \qquad \omega(x_4)f(x_1, x_2, x_3) = 0,$$
$$f(x_1x_4, x_2, x_3) = 0, \qquad f(x_1, x_2, x_3x_4) = 0. \qquad (8)$$

These identities are expressed in terms of the following association types:

$$T_1 = \omega(R)(RR)R, \quad T_2 = \omega(RR)RR, \quad T_3 = (RR)R.R, \quad T_4 = (RR)(RR).$$

Here R has no meaning. It is just a convenient way to say how the asssociation types look like. Since we are interested in shape identities (which are polynomial identities) it is important to put the association types not involving the weight function ω at the end of the list of association types.

For each association type we have to consider also the identities implied by the commutative law. They are:

$$\begin{gathered}\omega(x_1)(x_2x_3)x_4-\omega(x_1)(x_3x_2)x_4=0,\quad \omega(x_1x_2)x_3x_4-\omega(x_2x_1)x_3x_4=0,\\ \omega(x_1x_2)x_3x_4-\omega(x_1x_2)x_4x_3=0,\quad (x_1x_2)x_3.x_4-(x_2x_1)x_3.x_4=0,\\ (x_1x_2)(x_3x_4)-(x_2x_1)(x_3x_4)=0,\quad (x_1x_2)(x_3x_4)-(x_3x_4)(x_1x_2)=0.\end{gathered}\tag{9}$$

Considering how the positions of the variables x_1, x_2, x_3, x_4 are changed in the terms of the identities (8) and (9) we represent each one of them by an element of the direct sum $FS_4 \oplus FS_4 \oplus FS_4 \oplus FS_4$, where each summand corresponds to each association type. For example, the identity $f(x_1x_4, x_2, x_3) = 0$ expands to $2(x_1x_4)x_2.x_3 - \omega(x_1x_4)x_2x_3 - \omega(x_2)(x_1x_4)x_3$ and is represented by $-(12)(34) \oplus -(234) \oplus 2(234) \oplus 0$. Now, using (7) we represent the identities (8) and (9) by block matrices. After reducing to the row canonical form we obtain:

Representation 1

$$\begin{array}{cccc} T_1 & T_2 & T_3 & T_4 \end{array}$$

$$\left[\begin{array}{c|c|c|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{array}\right]\tag{10}$$

Representation 2

$$\begin{array}{c} \qquad T_1 \qquad\qquad T_2 \qquad\qquad T_3 \qquad\qquad T_4 \\ \left[\begin{array}{ccc|ccc|ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \end{array} \tag{11}$$

Representation 3

$$\begin{array}{c} \qquad T_1 \qquad\quad T_2 \qquad\quad T_3 \qquad\quad T_4 \\ \left[\begin{array}{cc|cc|cc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{array}\right] \end{array} \tag{12}$$

The block matrices corresponding to representations 4 and 5 are formed by identity matrices.

A shape polynomial of degree four is a linear combination $\alpha_1\,\mu_1 + \ldots + \alpha_5\,\mu_5$ of the shape monomials (4). Using the commutative law we write μ_2, μ_4 and μ_5 as $\mu_2' = (x_2x_3)x_1.x_4$, $\mu_4' = (x_2x_3)x_4.x_1$ and $\mu_5' = (x_3x_4)x_2.x_1$. Thus, the shape polynomial can be written in terms of T_3 and T_4 as

$$[\alpha_1\,\mu_1 + \alpha_2\,\mu_2' + \alpha_4\,\mu_4' + \alpha_5\,\mu_5'] + \alpha_3\,\mu_3. \tag{13}$$

In representation 1 the monomials μ_1, μ_2', μ_3, μ_4', μ_5' are represented by the same identity matrix $[1]$. In representation 2 they are represented, respectively, by the matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ -1 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix},$$

and in representation 3 by the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ -1 & -1 \end{bmatrix}.$$

Therefore, the shape polynomial (13) is represented by the block matrices:

Representation 1

$$\begin{array}{ccc} T_3 & & T_4 \\ [\alpha_1 + \alpha_2 + \alpha_4 + \alpha_5 & | & \alpha_3] \end{array} \tag{14}$$

Representation 2

$$\begin{array}{c} \qquad\qquad T_3 \qquad\qquad\qquad\qquad\qquad\qquad\qquad T_4 \end{array}$$

$$\left[\begin{array}{ccc|ccc} \alpha_1+\alpha_2 & \alpha_4 & \alpha_5 & \alpha_3 & 0 & 0 \\ -\alpha_5 & \alpha_1-\alpha_5 & \alpha_2+\alpha_4-\alpha_5 & 0 & \alpha_3 & 0 \\ -\alpha_2-\alpha_4 & -\alpha_2-\alpha_4+\alpha_5 & \alpha_1-\alpha_2-\alpha_4 & 0 & 0 & \alpha_3 \end{array}\right] \tag{15}$$

Representation 3

$$\begin{array}{c} \qquad T_3 \qquad\qquad\qquad\qquad\qquad T_4 \end{array}$$

$$\left[\begin{array}{cc|cc} \alpha_1-\alpha_4+\alpha_5 & \alpha_2-\alpha_4 & \alpha_3 & 0 \\ -\alpha_2-\alpha_5 & \alpha_1-\alpha_2+\alpha_4-\alpha_5 & 0 & \alpha_3 \end{array}\right] \tag{16}$$

The polynomial (13) is a shape identity for all normal Bernstein algebras if and only if it is a consequence of identities (8) and (9), and this happens if and only if the row space of the block matrix representing (13) is contained in the row space of the block matrix representing (8) and (9) for each representation.

We put the block matrices (14), (15), (16) at the botton of the matrices (10), (11), (12), respectively, and perform row reductions. Since the rank cannot increase, the coefficients α_i must satisfy the following constraints:

$$\begin{aligned} \alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 &= 0, \\ -2\alpha_1-2\alpha_2+\alpha_5 &= 0, \\ \alpha_2+\alpha_4+\alpha_5 &= 0, \\ \alpha_1+\alpha_2+\alpha_4 &= 0, \\ 2\alpha_1-\alpha_2-4\alpha_3-\alpha_4+2\alpha_5 &= 0, \\ -\alpha_1-\alpha_2+2\alpha_3-\alpha_4-\alpha_5 &= 0. \end{aligned}$$

Since the block matrices representing the identities (8) and (9) are formed by identity matrices in representations 4 and 5, they do not give any other constraints. Solving the above linear system of equations we obtain $\alpha_i = 0$ $(i = 1, \ldots, 5)$. Therefore there exists no shape identity of degree four satisfied by all normal Bernstein algebras.

Degree five. From the identity $f(x_1, x_2, x_3) = 0$ we obtain the following identities of degree five:

$$\begin{array}{ll} f(x_1,x_2,x_3)x_4.x_5 = 0, & f(x_1,x_2,x_3).x_4x_5 = 0, \\ \omega(x_5)f(x_1,x_2,x_3)x_4 = 0, & \omega(x_4x_5)f(x_1,x_2,x_3) = 0, \\ f(x_1x_4,x_2,x_3)x_5 = 0, & \omega(x_5)f(x_1x_4,x_2,x_3) = 0, \\ f(x_1,x_2,x_3x_4)x_5 = 0, & \omega(x_5)f(x_1,x_2,x_3x_4) = 0, \\ f(x_1x_4,x_2,x_3x_5) = 0, & f(x_1x_4,x_2x_5,x_3) = 0. \end{array} \tag{17}$$

These identities involve the following association types:

$$\omega(R)(RR.R)R, \quad \omega(R)RR.RR, \quad \omega(RR)RR.R, \quad \omega(RRR)RR,$$
$$(RR.R)R.R, \quad (RR.RR)R, \quad (RR.R).RR.$$

Using the representations of FS_5 these identities and those implied by the commutative law are represented by the block matrices:

(Here we write only the matrices corresponding to the last three association types.)

Representation 1

$$\left[\begin{array}{c|c|c} 1 & 0 & -1 \\ 0 & 1 & -1 \end{array}\right] \tag{18}$$

Representation 2

$$\left[\begin{array}{cccc|cccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -11 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2
\end{array}\right] \tag{19}$$

Representation 3

$$\left[\begin{array}{ccccc|ccccc|ccccc}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 2
\end{array}\right] \tag{20}$$

The block matrices corresponding to representations 4, 5, 6 and 7 are formed by identity matrices.

A shape polynomial of degree five is a linear combination $\sum_{i=1}^{14} \beta_i \; \nu_i$ of the shape monomials (5) and it can be written in terms of the last three types as

$$[\sum_{i=1}^{8} \beta_i \; \nu'_i] + [\beta_9 \; \nu'_9 + \beta_{10} \; \nu'_{10}] + [\sum_{i=11}^{14} \beta_i \; \nu'_i] \qquad (21)$$

where

$$\begin{array}{lll}
\nu'_1 = (x_1 x_2 . x_3) x_4 . x_5, & \nu'_2 = (x_2 x_3 . x_1) x_4 . x_5, & \nu'_3 = (x_2 x_3 . x_4) x_1 . x_5, \\
\nu'_4 = (x_3 x_4 . x_2) x_1 . x_5, & \nu'_5 = (x_2 x_3 . x_4) x_5 . x_1, & \nu'_6 = (x_3 x_4 . x_2) x_5 . x_1, \\
\nu'_7 = (x_3 x_4 . x_5) x_2 . x_1, & \nu'_8 = (x_4 x_5 . x_3) x_2 . x_1, & \nu'_9 = (x_1 x_2 . x_3 x_4) x_5, \\
\nu'_{10} = (x_2 x_3 . x_4 x_5) x_1, & \nu'_{11} = (x_1 x_2 . x_3) . x_4 x_5, & \nu'_{12} = (x_2 x_3 . x_1) . x_4 x_5, \\
 & \nu'_{13} = (x_3 x_4 . x_5) . x_1 x_2, & \nu'_{14} = (x_4 x_5 . x_3) . x_1 x_2.
\end{array}$$

The polynomial (21) is represented by the block matrices:

Representation 1

$$[\sum_{i=1}^{8} \beta_i \quad | \quad \beta_9 + \beta_{10} \quad | \quad \sum_{i=11}^{14} \beta_i] \qquad (22)$$

Representation 2

$$[A_1 \quad | \quad A_2 \quad | \quad A_3] \qquad (23)$$

where

$$A_1 = \begin{bmatrix} \beta_1+\beta_2+\beta_3+\beta_4 & \beta_5+\beta_6 & \beta_7 & \beta_8 \\ -\beta_8 & \beta_1+\beta_2-\beta_8 & \beta_3+\beta_5-\beta_8 & \beta_4+\beta_6+\beta_7-\beta_8 \\ -\beta_4-\beta_6-\beta_7 & -\beta_4-\beta_6-\beta_7 & \beta_1-\beta_4-\beta_6- \beta_7+\beta_8 & \beta_2+\beta_3-\beta_4+ \beta_5-\beta_6-\beta_7 \\ -\beta_2-\beta_3-\beta_5 & -\beta_2-\beta_3-\beta_5+ \beta_7+\beta_8 & -\beta_2-\beta_3+\beta_4- \beta_5+\beta_6 & \beta_1-\beta_2-\beta_3-\beta_5 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} \beta_9 & \beta_{10} & 0 & 0 \\ 0 & \beta_9 & \beta_{10} & 0 \\ 0 & 0 & \beta_9 & \beta_{10} \\ -\beta_{10} & -\beta_{10} & -\beta_{10} & \beta_9-\beta_{10} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} \beta_{11}+\beta_{12} & 0 & \beta_{13} & \beta_{14} \\ -\beta_{14} & \beta_{11}+\beta_{12}-\beta_{14} & -\beta_{14} & \beta_{13}-\beta_{14} \\ -\beta_{13} & -\beta_{13} & \beta_{11}-\beta_{13}+\beta_{14} & \beta_{12}-\beta_{13} \\ -\beta_{12}+\beta_{13}+\beta_{14} & -\beta_{12} & -\beta_{12} & \beta_{11}-\beta_{12} \end{bmatrix}.$$

Representation 3

$$[\,B_1 \quad | \quad B_2 \quad | \quad B_3\,] \tag{24}$$

where

$$B_1 = \begin{bmatrix} \beta_1+\beta_2-\beta_6+\beta_8 & \beta_3-\beta_7+\beta_8 & \beta_4-\beta_7 & \beta_5-\beta_7+\beta_8 & \beta_6-\beta_7 \\ -\beta_4-\beta_5 & \beta_1-\beta_4-\beta_8 & \beta_2+\beta_3-\beta_4+\beta_7-\beta_8 & -\beta_6-\beta_8 & \beta_5-\beta_6+\beta_7-\beta_8 \\ -\beta_2-\beta_3-\beta_8 & -\beta_2-\beta_3+\beta_4 & \beta_1-\beta_2-\beta_3 & -\beta_5+\beta_6+\beta_7 & -\beta_5+\beta_8 \\ -\beta_2+\beta_4+\beta_6+\beta_7 & -\beta_3+\beta_4-\beta_5+\beta_6+\beta_7 & -\beta_3-\beta_5+\beta_8 & \beta_1-\beta_3+\beta_4-\beta_5+\beta_6+\beta_7 & \beta_2-\beta_3-\beta_5+\beta_8 \\ -\beta_1-\beta_7 & -\beta_4-\beta_6 & \beta_3-\beta_4+\beta_5-\beta_6 & -\beta_2-\beta_4-\beta_6-\beta_8 & \beta_1-\beta_2+\beta_3-\beta_4+\beta_5-\beta_6+\beta_7-\beta_8 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} \beta_{11}+\beta_{12}+\beta_{14} & -\beta_{13}+\beta_{14} & -\beta_{13} & -\beta_{13}+\beta_{14} & -\beta_{13} \\ 0 & \beta_{11}-\beta_{14} & \beta_{12}+\beta_{13}-\beta_{14} & -\beta_{14} & \beta_{13}-\beta_{14} \\ -\beta_{12} & -\beta_{12}+\beta_{13} & \beta_{11}-\beta_{12}+\beta_{14} & 0 & 0 \\ -\beta_{12}+\beta_{13} & \beta_{13} & \beta_{14} & \beta_{11}+\beta_{13} & \beta_{12}+\beta_{14} \\ -\beta_{11}-\beta_{14} & -\beta_{14} & \beta_{13}-\beta_{14} & -\beta_{12} & \beta_{11}-\beta_{12} \end{bmatrix},$$

$$B_2 = \begin{bmatrix} \beta_9 & 0 & 0 & \beta_{10} & 0 \\ -\beta_{10} & \beta_9 & 0 & 0 & \beta_{10} \\ 0 & 0 & \beta_9 & -\beta_{10} & -\beta_{10} \\ 0 & -\beta_{10} & -\beta_{10} & \beta_9 - \beta_{10} & -\beta_{10} \\ -\beta_9 & 0 & \beta_{10} & 0 & \beta_9 + \beta_{10} \end{bmatrix}.$$

As before we put the block matrices (22), (23) and (24) at the botton of the block matrices (18), (19) and (20), respectively, and perform row reductions. We find a linear system of equations on the variables $\beta_1, \ldots, \beta_{14}$. The row canonical form of the coefficient matrix of this system is

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & -7 & -1 & -7 & 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 7 & 0 & 6 & 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -1 & 0 & -2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}.$$

It follows that the shape polynomial $\sum_{i=1}^{14} \beta_i \, \nu_i$ is an identity for all normal Bernstein algebras if and only if it is a linear combination of the following shape polynomials:

$$\begin{aligned} 2\nu_1 + \nu_2 - 7\nu_3 + 2\nu_4 - 2\nu_5 + 7\nu_6 - \nu_7 - 2\nu_8, \\ 2\nu_1 - \nu_2 - \nu_3 + 2\nu_4 - 2\nu_9, \\ 2\nu_1 + \nu_2 - 7\nu_3 + 2\nu_4 + 6\nu_6 - 2\nu_7 - 2\nu_{10}, \\ -\nu_2 + 3\nu_3 - 2\nu_4 + 2\nu_5 - 3\nu_6 + \nu_7 + \nu_{11} - 2\nu_{12} + 2\nu_{13} - \nu_{14}. \end{aligned} \tag{25}$$

As clear, the set of shape identities given by (25) is equivalent to the following set:

$$2\nu_1 - \nu_2 - \nu_3 + 2\nu_4 - 2\nu_9 = 0, \tag{26}$$

$$\nu_2 - 3\nu_3 + 3\nu_6 - \nu_7 + \nu_9 - \nu_{10} = 0, \tag{27}$$

$$2\nu_4 - 2\nu_5 - \nu_9 + \nu_{10} - \nu_{11} + 2\nu_{12} - 2\nu_{13} + \nu_{14} = 0, \tag{28}$$

$$2\nu_5 - \nu_6 - \nu_7 + 2\nu_8 - 2\nu_{10} = 0. \tag{29}$$

On the other hand, it is easy to verify that each one of these equations is an identity for all normal Bernstein algebras. As an example, we verify that (26) is an identity. Using the identity $f(x_1, x_2, x_3) = 0$ many times we obtain:

$$\begin{aligned} 2\nu_1 &= \omega(x_1x_2x_3)x_4x_5 + \frac{1}{2}\omega(x_1x_2x_4)x_3x_5 + \frac{1}{4}\omega(x_1x_3x_4)x_2x_5 \\ &+ \frac{1}{4}\omega(x_2x_3x_4)x_1x_5, \end{aligned}$$

$$\begin{aligned} -\nu_2 &= -\frac{1}{2}\omega(x_1x_2x_3)x_4x_5 - \frac{1}{4}\omega(x_2x_3x_4)x_1x_5 - \frac{1}{8}\omega(x_1x_2x_4)x_3x_5 \\ &- \frac{1}{8}\omega(x_1x_3x_4)x_2x_5, \end{aligned}$$

$$\begin{aligned} -\nu_3 &= -\frac{1}{2}\omega(x_2x_3x_4)x_1x_5 - \frac{1}{4}\omega(x_1x_2x_3)x_4x_5 - \frac{1}{8}\omega(x_1x_2x_4)x_3x_5 \\ &- \frac{1}{8}\omega(x_1x_3x_4)x_2x_5, \end{aligned}$$

$$\begin{aligned} 2\nu_4 &= \omega(x_2x_3x_4)x_1x_5 + \frac{1}{2}\omega(x_1x_3x_4)x_2x_5 + \frac{1}{4}\omega(x_1x_2x_3)x_4x_5 \\ &+ \frac{1}{4}\omega(x_1x_2x_4)x_3x_5, \end{aligned}$$

$$\begin{aligned} -2\nu_9 &= -\frac{1}{2}\omega(x_1x_2x_3)x_4x_5 - \frac{1}{2}\omega(x_1x_2x_4)x_3x_5 - \frac{1}{2}\omega(x_1x_3x_4)x_2x_5 \\ &- \frac{1}{2}\omega(x_2x_3x_4)x_1x_5. \end{aligned}$$

Now, we add these identities to obtain $2\nu_1 - \nu_2 - \nu_3 + 2\nu_4 - 2\nu_9 = 0$.

The results of this section can be summarized in the following theorem.

Theorem 1 *For the class of normal Bernstein algebras we have:*

(i) There are no shape identities of degree less or equal to four.

(ii) The degree five shape identities are linear combinations of identities (26), (27), (28) and (29).

Remark. In terms of associators identities (26), (27), (28) and (29) are given by:

$$\{2(x_1x_2, x_3, x_4) - (x_1, x_2x_3, x_4) - 2x_1(x_2, x_3, x_4)\}x_5 = 0,$$

$$\begin{aligned} \{(x_1, x_2x_3, x_4) + (x_1, x_2, x_3x_4) - 2x_1(x_2, x_3, x_4)\}x_5 + \\ x_1\{(x_2, x_3x_4, x_5) + (x_2x_3, x_4, x_5) - (x_2, x_3, x_4)x_5\} - \\ (x_1, x_2.x_3x_4, x_5) = 0, \end{aligned}$$

$$\begin{aligned} (x_1x_2, x_3x_4, x_5) + (x_1, x_2x_3, x_4x_5) + \\ 3x_1x_2.(x_3, x_4, x_5) + 3(x_1, x_2, x_3).x_4x_5 + \\ 2x_1.(x_2, x_3, x_4)x_5 - 2(x_1x_2, x_3, x_4x_5) - 2(x_1, x_2.x_3x_4, x_5) = 0, \end{aligned}$$

$$x_1\{2(x_2x_3, x_4, x_5) - (x_2, x_3x_4, x_5) - 2x_2(x_3, x_4, x_5)\} = 0.$$

Notice that the first identity is $S(x_1, x_2, x_3, x_4)x_5 = 0$ and the last identity is $x_1S(x_2, x_3, x_4, x_5) = 0$.

5 Exceptional Bernstein algebras

In the previous section we presented the full argument used to obtain the shape identities of normal Bernstein algebras. The argument we use to study the shape identities of exceptional Bernstein algebras is very similar. Therefore, in this section, we give the argument in the proof of the theorem in a brief manner.

Theorem 2 *For the class of exceptional Bernstein algebras we have:*
(i) There are no shape identities of degree less or equal to five.
(ii) The degree six shape identities are linear combinations of the identities

$$(x_1, x_2, x_3)(x_4, x_5, x_6) = 0, \quad (x_1x_2, (x_3, x_4, x_5), x_6) = 0,$$
$$(x_1, (x_2, x_3, x_4), x_5x_6) = 0.$$

Proof. Using identity (2), which we write as

$$g(x_1, x_2, x_3, x_4) = 2(x_1x_2)(x_3x_4) - \omega(x_1x_2)x_3x_4 - \omega(x_3x_4)x_1x_2 = 0,$$

we obtain

$$g(x_1, x_2, x_3, x_4)x_5 = 0, \quad \omega(x_5)g(x_1, x_2, x_3, x_4) = 0,$$
$$g(x_1x_5, x_2, x_3, x_4) = 0. \tag{30}$$

These identities are expressed in terms of five association types. The ones that do not involve the ω-function are $(RR.RR)R$ and $(RR.R).RR$. Thus, a polynomial that is a candidate for a shape identity in this case has the form

$$[\beta_9\, \nu_9' + \beta_{10}\, \nu_{10}'] + [\sum_{i=11}^{14} \beta_i\, \nu_i'] \tag{31}$$

Representing (30) (together with the identities implied by commutativity) and (31) by block matrices, and comparing them as in section 4 we get $\alpha_i = 0$ $(i = 9, \ldots 14)$. Therefore, there is no shape identity of degree five (or less).

In the degree six case there are nine identities, namely,

$$\begin{aligned} g(x_1,x_2,x_3,x_4)x_5.x_6 = 0, &\quad g(x_1,x_2,x_3,x_4).x_5x_6 &= 0,\\ g(x_1,x_2,x_3,x_4x_5)x_6 = 0, &\quad g(x_1,x_2,x_3,x_4x_5.x_6) &= 0,\\ g(x_1,x_2,x_3x_5,x_4x_6) = 0, &\quad g(x_1x_5,x_2,x_3x_6,x_4) &= 0,\\ \omega(x_6)g(x_1,x_2,x_3,x_4)x_5 = 0, &\quad \omega(x_6)g(x_1,x_2,x_3,x_4x_5) &= 0,\\ \omega(x_5x_6)g(x_1,x_2,x_3,x_4) = 0, & \end{aligned}$$

which are expressed in terms of eleven association types. The ones without a ω are $(RR.RR)R.R$, $(RR.RR).RR$, $(RR.R)R.RR$, $(RR.R)(RR.R)$ and $((RR.R).RR)R$. There are 42 shape monomials of degree six. We need to consider only 26 of them, that is, the ones that can be rewritten modulo the commutative law to fit in one of these five association types. Representing the identities and the linear combination of these 26 shape monomials by block matrices, we conclude that a shape identity of degree six for the exceptional Bernstein algebras has to be a linear combination of $(x_1,x_2,x_3)(x_4,x_5,x_6) = 0$, $(x_1x_2,(x_3,x_4,x_5),x_6) = 0$, $(x_1,(x_2,x_3,x_4),x_5x_6) = 0$. On the other hand, using $g(x_1,x_2,x_3,x_4) = 0$ it is straighforward to check that each one of these last three equations is a shape identity.

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