A REFINEMENT OF HARRIS-FKG INEQUALITY
FOR ORIENTED PERCOLATION

by

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Summary. For a set $A \subset 2\mathbb{Z}$, let $\nu(A)$ be the extinction probability for oriented bond percolation in two dimensions starting from A. We show that for any finite sets A and $B \subset 2\mathbb{Z}$,

$$\nu(A \cap B)\nu(A \cup B) \geq \nu(A)\nu(B)$$

This refines the classical Harris-FKG inequality which says that $\nu(A \cup B) \geq \nu(A)\nu(B)$. As an application of our result we give an upper bound on $\nu(\{0\})$.

1. Introduction

Let (V, E) be a directed graph, where $V = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even }\}$ is the set of

vertices and $E = \{((m,n),(m \pm 1, n+1)) : (m,n) \in V\}$, the set of oriented bonds. Under this graph, from each $(m,n) \in V$ there are two directed bonds, one pointing to $(m-1, n+1)$ and the other one pointing to $(m+1, n+1)$. A function $\beta : E \to \{0,1\}$ will be called a configuration of open and closed bonds of E; by $\beta(e) = 1$ we mean that the bond $e \in E$ is open, while $\beta(e) = 0$ means it is closed. On $\{0,1\}^E$ we define the measure $\mathbb{P}$ by postulating that each bond is open with probability p and independently of the other bonds. $(\{0,1\}^E, \mathbb{P})$ is called the oriented bond percolation model in two dimensions (see Durrett (1984, 1988) and Grimmett (1989) for detailed discussion of this model).

For $x, y \in V$, we say there is an open path from x to y in $\beta \in \{0,1\}^E$ if either $x = y$ or there is a sequence of vertices $x_0 = x, \ldots, x_n = y$ in V such that for each $1 \leq i \leq n$, (x_{i-1}, x_i) is a bond and it is open in β. For an $A \subset 2\mathbb{Z}$ define

$$C(A) = \{y \in V \; : \; \text{there is a } m \in A \text{ such that there is an open path from } (m,0) \text{ to } y\}$$

$C(A)$ is a random set of points of V which is called a cluster connected to A. The study of random clusters is one of the central problems in percolation models; a particular interest is devoted to $\mathbb{P}(|C(A)| < \infty)$ which is called the extinction probability ($|\cdot|$ means here the cardinality of a set). In the abstract we used the notation $\nu(A) = \mathbb{P}(|C(A)| < \infty)$. It has been proven by Harris (1960) that for any finite sets $A, B \subset 2\mathbb{Z}$,

$$\mathbb{P}(|C(A)| < \infty, |C(B)| < \infty) = \mathbb{P}(|C(A \cup B)| < \infty) \geq \mathbb{P}(|C(A)| < \infty) \, \mathbb{P}(|C(B)| < \infty).$$
$$(1.1)$$

The above inequality was also derived by Fortuin, Kasteleyn and Ginibre (1971). For this reason we call it the Harris-FKG inequality. See also Durrett (1984), page 1028. Inequality (1.1) holds for oriented bond percolation models on quite general graphs. For (V, E) defined above, the Harris-FKG inequality is refined by the following

Theorem 1. *For arbitrary finite $A, B \subset 2\mathbb{Z}$, it holds that*

$$\mathbb{P}(|C(A \cup B)| < \infty) \, \mathbb{P}(|C(A \cap B)| < \infty) \geq \mathbb{P}(|C(A)| < \infty) \, \mathbb{P}(|C(B)| < \infty) \qquad (1.2)$$

For a set $M \subset 2\mathbb{Z}$ we define the process $\{\xi_n^M\}_{n \geq 0}$ by putting $\xi_n^M := \{k \in \mathbb{Z} : \text{there exists an } m \in M \text{ such that there is an open path from } (m,0) \text{ to } (k,n)\}$. This process relates to

$C(M)$ as below:

$$C(M) = \bigcup_{n \geq 0} \{(m,n) : m \in \xi_n^M\}$$

It is well known (and will also follow from the above relation and a Markovian construction of $C(M)$ to be presented in Section 3) that $\{\xi_n^M\}_{n \geq 0}$ is a Markov process. Therefore $\{\xi_{2n}^{2Z}\}_{n \geq 0}$ is also a Markov process on $\{0,1\}^{2Z}$; we select even times in order to get the same state space at all times. Arguments similar to those applied to the contact process in Chapter 6 of Liggett (1985), imply that ξ_{2n}^{2Z} converges weakly, as $n \to \infty$, to a measure $\bar{\nu}$ on $\{0,1\}^{2Z}$. This measure is called the upper invariant measure because it is invariant for $\{\xi_{2n}\}_{n \geq 0}$ and may be different from the so-called lower invariant measure $\underline{\nu} = \delta_{\emptyset}$, the measure that gives mass one to the empty configuration. Duality as discussed in Section 8 of Durrett (1984) implies that

$$\mathbb{P}(|C(M)| < \infty) = \int_{\{0,1\}^{2Z}} H_M(\alpha) \hat{\nu}(d\alpha) =: \bar{\nu} H_M$$

where $H_M(\alpha) := \prod_{y \in M}(1 - \alpha(y))$ for any $\alpha \in \{0,1\}^{2Z}$. Set $A' := A \setminus B$, $B' := B \setminus A$, and $C' := A \cap B$. Then upon dividing (1.2) by $(\bar{\nu} H_{C'})^2$, we obtain the following equivalent inequality. For finite pairwise disjoint A', B' and C',

$$\bar{\nu}\left(H_{A'} H_{B'} \mid H_{C'} = 1\right) \geq \bar{\nu}\left(H_{A'} \mid H_{C'} = 1\right) \bar{\nu}\left(H_{B'} \mid H_{C'} = 1\right) \tag{1.3}$$

The above is a kind of Harris-FKG inequality for the measure $\bar{\nu}$ conditioned on having zeroes in the sites of C'. This is interesting because Liggett (1995) proved for the Contact Process that (the analogous to) $\bar{\nu}$ conditioned on having ones in C' does not satisfy the Harris-FKG inequality. We thank Tom Liggett for pointing out this fact to us.

This paper is organized as follows. In Section 2 we show how Theorem 1 follows almost immediately from two properties of $\mathbb{F}$ which are stated respectively in Lemma 2 and Proposition 3. In Section 3 we present a Markovian construction of the oriented bond percolation, which is then used to prove Lemma 2 and Lemma 4, the key element to show Proposition 3. In the remark after Lemma 4, we explain why our argument works only for the oriented percolation in two dimensions. In Section 4, we apply Theorem 1 to derive an upper bound on $\mathbb{P}(|C(\{0\})| < \infty)$ which improves the existing ones, to the best of our knowledge.

2. Proof of Theorem 1

We assume throughout that $A \cap B \neq \emptyset$, because for $A \cap B = \emptyset$ we have that the inequality we want to prove is just Harris-FKG inequality. For any finite $M \subset 2\mathbb{Z}$, let $\Gamma(M)$ be the set of all finite clusters that can be connected to M, that is,

$$\Gamma(M) = \{\gamma \subset V : \mathbb{P}(C(M) = \gamma) > 0\}$$

If $|\gamma| = \infty$, the probability that $C(M) = \gamma$ is zero, hence $\Gamma(M)$ contains only finite clusters. Let W_M denote the event $|C(M)| < \infty$. From these definitions, we have

$$\mathbb{P}(|C(M)| < \infty) = \sum_{\gamma \in \Gamma(A \cap B)} \mathbb{P}(W_{M \setminus (A \cap B)} \cap \{C(A \cap B) = \gamma\}) \tag{2.1}$$

In the sequel we will write $\sum_{\gamma}$ for $\sum_{\gamma \in \Gamma(A \cap B)}$ and use the shorthand notation introduced in the display below, where W and U are any events and $\gamma \in \Gamma(A \cap B)$ is arbitrary:

$$\begin{aligned}
\mathbb{P}(\gamma) &= \mathbb{P}(C(A \cap B) = \gamma), \\
\mathbb{P}(WU) &= \mathbb{P}(W \cap U), \\
\mathbb{P}(W|\gamma) &= \mathbb{P}(W \cap \{C(A \cap B) = \gamma\})/\mathbb{P}(C(A \cap B) = \gamma).
\end{aligned} \tag{2.2}$$

Using (2.1) and (2.2), the inequality (1.2) can be rewritten as

$$\sum_{\gamma} \sum_{\gamma'} \mathbb{P}(\gamma)\mathbb{P}(\gamma')\mathbb{P}(W_{A \setminus B} W_{B \setminus A}|\gamma) \geq \sum_{\gamma} \sum_{\gamma'} \mathbb{P}(\gamma)\mathbb{P}(\gamma')\mathbb{P}(W_{A \setminus B}|\gamma)\mathbb{P}(W_{B \setminus A}|\gamma') \tag{2.3}$$

We prove in Lemma 2 below that for any $\gamma \in \Gamma(A \cap B)$, we have

$$\mathbb{P}(W_{A \setminus B} W_{B \setminus A}|\gamma) \geq \mathbb{P}(W_{A \setminus B}|\gamma)\mathbb{P}(W_{B \setminus A}|\gamma). \tag{2.4}$$

From (2.4) and (2.3), to prove (1.2) it suffices to establish that

$$\sum_{\gamma} \sum_{\gamma'} \mathbb{P}(\gamma)\mathbb{P}(\gamma')\mathbb{P}(W_{A \setminus B}|\gamma)\mathbb{P}(W_{B \setminus A}|\gamma) \geq \sum_{\gamma} \sum_{\gamma'} \mathbb{P}(\gamma)\mathbb{P}(\gamma')\mathbb{P}(W_{A \setminus B}|\gamma)\mathbb{P}(W_{B \setminus A}|\gamma') \tag{2.5}$$

To rewrite (2.5) in a somewhat "familiar" form, we introduce two functions f and g : $\Gamma(A \cap B) \to \mathbb{R}$ and a measure μ on $\Gamma(A \cap B)$:

$$f(\gamma) := \mathbb{P}(W_{A \setminus B}|\gamma), \quad g(\gamma) := \mathbb{P}(W_{B \setminus A}|\gamma)$$

$$\mu(\gamma) := \mathbb{P}(C(A \cap B) = \gamma)/\mathbb{P}(|C(A \cap B)| < \infty)$$

Then, dividing by $(\mathbb{P}(|C(A\cap B)|<\infty))^2=\sum_\gamma\mathbb{F}(\gamma)\sum_{\gamma'}\mathbb{F}(\gamma')$, the inequality (2.5) becomes

$$\sum_\gamma f(\gamma)g(\gamma)\mu(\gamma)\geq\left(\sum_\gamma f(\gamma)\mu(\gamma)\right)\left(\sum_\gamma g(\gamma)\mu(\gamma)\right)\qquad(2.6)$$

With respect to the partial order $\subset$ on $\Gamma(A\cap B)$, f and g are increasing functions. Thus the inequality (2.6) follows from Proposition 3 below which says that the measure μ has positive correlations, that is to say that (2.6) holds if both f and g are either increasing or decreasing functions on $\Gamma(A\cap B)$ (see Definition 2.11, page 77, Liggett(1985)). This completes the proof of the theorem. $\qquad\square$

3. Markov construction.

We now present a Markovian construction of $C(M)$ that will be used in the proofs of the results stated above.

Let $V_n=\{x\in V:\text{there exists }m\text{ with }(m,n)=x\}$. For each $x\in V\setminus V_0$, let $D(x):=\{z\in V:(z,x)\in E\}$ denote the set of vertices from which emanate the bonds pointing to x. Given β on $\{0,1\}^E$ and $M\subset 2\mathbb{Z}$, define $C(M,0,\beta)=\{(m,0):m\in M\}$ and, iteratively define $C(M,n+1,\beta)$ by

$$1\{x\in C(M,n+1,\beta)\}=1-\prod_{z\in D(x)}(1-\beta(z,x))^{1\{z\in C(M,n,\beta)\}},\quad\forall x\in V_{n+1}\qquad(3.1)$$

where, $\beta(z,x)$ means the value of $\beta\in\{0,1\}^E$ on the bond that points from $z\in V$ to $x\in V$. It is easy to see that for each β,

$$\bigcup_{n\geq 0}C(M,n,\beta)=C(M,\beta)\qquad(3.2)$$

where $C(M,\beta)$ is the cluster connected to M for the bond configuration β.

Let M be finite and $\gamma\in\Gamma(M)$ be arbitrary. Set

$$\bar\gamma:=\{x\in V\setminus V_0:\text{there is a }z\in\gamma\text{ such that }(z,x)\in E\}$$

to be the set of vertices which are the extremes of the bonds emanating from γ. Observe that if $D(x)\cap C(M,n)=\emptyset$ then x cannot belong to $C(M,n+1)$. Thus, if $C(M,n,\beta)=\gamma\cap V_n$, then the indicator function of the event $C(M,n+1,\beta)=\gamma\cap V_{n+1}$ is

$$\prod_{x\in\bar\gamma\cap V_{n+1}}\left[\gamma(x)\left(1-\prod_{y\in D(x)}(1-\beta(y,x))^{\gamma(y)}\right)+(1-\gamma(x))\prod_{y\in D(x)}(1-\beta(y,x))^{\gamma(y)}\right],$$

where $\gamma(x) = 1\{x \in \gamma\}$. Since from the definitions, $C(M, 0, \beta) = \gamma \cap V_0$, then using the above expression and the Markov property (3.1) and (3.2), we conclude that $1\{C(M, \beta) = \gamma\} = \prod_{n \geq 0} 1\{C(M, n, \beta) = \gamma \cap V_n\}$ has the following expression:

$$\prod_{x \in \gamma} \left[\gamma(x) \left(1 - \prod_{y \in D(x)} (1 - \beta(y, x))^{\gamma(y)} \right) + (1 - \gamma(x)) \prod_{y \in D(x)} (1 - \beta(y, x))^{\gamma(y)} \right] \tag{3.3}$$

Taking expectations and employing the independence of $\beta(e), e \in E$, we derive that

$$\mathbb{P}(C(M) = \gamma) = \prod_{x \in \bar{\gamma}} \pi(x, \gamma), \tag{3.4}$$

where $\pi(x, \gamma)$ is the expected value of the factor corresponding to x in (3.3):

$$\pi(x, \gamma) = \left[\gamma(x) \left(1 - (1 - p)^{|D(x) \cap \gamma|} \right) + (1 - \gamma(x))(1 - p)^{|D(x) \cap \gamma|} \right] \tag{3.5}$$

Lemma 2. *For any $\gamma \in \Gamma(A \cap B)$, where $A, B \subset 2\mathbb{Z}$ are finite, we have that*

$$\mathbb{P}(W_{A \setminus B} W_{B \setminus A} | \gamma) \geq \mathbb{P}(W_{A \setminus B} | \gamma) \mathbb{P}(W_{B \setminus A} | \gamma).$$

Proof. Fix an arbitrary $\gamma \in \Gamma(A \cap B)$. Let

$$F(\gamma) := \{(x, y) \in E : x \in \gamma\}$$

be the set of the bonds emanating from the points of γ. For finite $M \subset 2\mathbb{Z}$, define

$$\tilde{C}(M) := \{x \in V : \text{there exists } m \in M \text{ such that there is an open path from } (m, 0) \text{ to } x$$

that does not contain any bond of $F(\gamma)\}$$

From the above definition, the event $\tilde{W}_M := \{|\tilde{C}(M)| < \infty\}$ does not depend on closeness/openness of any bond from $F(\gamma)$. It is straightforward to see that $1\{\tilde{W}_M\}$ is a non-increasing function on $\{0, 1\}^{E \setminus F(\gamma)}$.

Recall that (3.3) expresses $1\{C(A \cap B) = \gamma\}$ when $\gamma \in \Gamma(A \cap B)$ and $M = A \cap B$. It follows immediately from this expression that the event $C(A \cap B) = \gamma$ depends only on the bonds in $F(\gamma)$. Hence $\mathbb{P}(\cdot \,|\, C(A \cap B) = \gamma)$ is a product measure on $\{0, 1\}^{E \setminus F(\gamma)}$. By Durrett (1988), pages 129,130 (see also Liggett (1985) page 78), this measure has positive correlations.

Combining the conclusions of the above two paragraphs, we get that

$$\mathbb{P}(\tilde{W}_{A\setminus B}\tilde{W}_{B\setminus A}\,|\,C(A\cap B)=\gamma)\geq \mathbb{P}(\tilde{W}_{A\setminus B}\,|\,C(A\cap B)=\gamma)\mathbb{P}(\tilde{W}_{B\setminus A}\,|\,C(A\cap B)=\gamma) \quad (3.6)$$

The derivation of (3.6) is exactly the same as that of the Harris inequality in the cited above Durrett's book, page 130. But since γ is finite then

$$\{|\tilde{C}(M)|<\infty\}\cap\{C(A\cap B)=\gamma\}=\{|C(M)|<\infty\}\cap\{C(A\cap B)=\gamma\}, \text{ for any finite } M\subset 2\mathbb{Z}.$$

Thus the lemma follows from (3.6). □

Remark. The fact that $\mathbb{P}(\cdot\,|\,C(A\cap B)=\gamma)$ is a product measure on $\{0,1\}^{E\setminus F(\gamma)}$ has appeared before in the literature. Aizenmann and Newman (1984) show that for the regular bond percolation model, the cluster of the origin is "self determined". Chayes et al (1989) used this idea to perform a "duplication trick". We thank Roberto Schonmann for pointing out these references. Notice that in the oriented percolation case the proof is more direct than in the unoriented case because of the explicit expression (3.3) for the indicator function of $C(M)=\gamma$.

Proposition 3. *For any finite $M\subset 2\mathbb{Z}$, the measure μ defined on $\Gamma(M)$ by*

$$\mu(\gamma)=\frac{\mathbb{P}(C(M)=\gamma)}{\mathbb{P}(|C(M)|<\infty)} \tag{3.7}$$

has positive correlations.

Proof. Set $\Gamma=\Gamma(M)$. Let f and g be increasing functions. We want to prove (2.6) which is equivalent to

$$\sum_{\gamma\in\Gamma}f(\gamma)\mu_2(\gamma)\geq\sum_{\gamma\in\Gamma}f(\gamma)\mu_1(\gamma) \tag{3.8}$$

where $\mu_1=\mu$ and μ_2, defined by

$$\mu_2(\gamma)=\frac{g(\gamma)}{\sum_{\gamma}g(\gamma)\mu_1(\gamma)}\mu_1(\gamma)$$

are probability measures on Γ giving positive mass to each cluster of Γ. Since f is increasing, Theorem 2.4 of Liggett (1985) says that to prove (3.8) it suffices to show that there exists a measure $\tilde{\mu}$ on $\Gamma\times\Gamma$ with marginals μ_1 and μ_2 such that

$$\tilde{\mu}\{(\zeta,\gamma):\zeta\subset\gamma\}=1. \tag{3.9}$$

In order to do this we introduce Markov processes $\eta_i(t)$, $i = 1, 2$, on Γ with the following transition rates: for all $\gamma \in \Gamma$ and all $x \in V \setminus V_0$ such that $x \notin \gamma$ and $\gamma^x = \gamma \cup \{x\}$ in Γ,

$$Q_i(\gamma, \gamma^x) = 1$$
$$Q_i(\gamma^x, \gamma) = \frac{\mu_i(\gamma)}{\mu_i(\gamma^x)} \tag{3.10}$$

and all other transitions have rate 0. Since $\mu_i(\gamma) > 0$ for any $\gamma \in \Gamma$ then these rates are finite. Also, it is easy to see that both processes $\eta_1(t)$ and $\eta_2(t)$ are ergodic and have respectively measures μ_1 and μ_2 as reversible. This implies that μ_i is the unique invariant measure for $\eta_i(t)$ and that, starting from any configuration $\gamma \in \Gamma$, the process $\eta_i(t)$ converges to the measure μ_i. Hence, to show (3.9) it is sufficient to find a process $\tilde{\eta}(t)$ on $\Gamma \times \Gamma$ whose marginals are $\eta_1(t)$ and $\eta_2(t)$ such that the following holds:

$$\eta_1(0) \subset \eta_2(0) \text{ implies } \eta_1(t) \subset \eta_2(t), \quad \text{for all } t \geq 0.$$

The non-zero rates for this process are given by $\tilde{Q}$ defined below. For all $\zeta, \gamma \in \Gamma$ and all $x \in V \setminus V_0$ such that $\zeta \subset \gamma$, $x \notin \zeta$, $x \notin \gamma$, $\gamma^x \in \Gamma$, we put

$$\tilde{Q}((\zeta, \gamma^x), (\zeta^x, \gamma^x)) = 1 \quad \text{if } \zeta^x \in \Gamma$$
$$\tilde{Q}((\zeta, \gamma), (\zeta^x, \gamma^x)) = 1 \quad \text{if } \zeta^x \in \Gamma$$
$$\tilde{Q}((\zeta, \gamma), (\zeta, \gamma^x)) = 1 \quad \text{if } \zeta^x \notin \Gamma$$
$$\tilde{Q}((\zeta, \gamma^x), (\zeta, \gamma)) = \frac{\mu_2(\gamma)}{\mu_2(\gamma^x)} \tag{3.11}$$
$$\tilde{Q}((\zeta^x, \gamma^x), (\zeta, \gamma)) = \frac{\mu_2(\gamma)}{\mu_2(\gamma^x)} \quad \text{if } \zeta^x \in \Gamma$$
$$\tilde{Q}((\zeta^x, \gamma^x), (\zeta, \gamma^x)) = \frac{\mu_1(\zeta)}{\mu_1(\zeta^x)} - \frac{\mu_2(\gamma)}{\mu_2(\gamma^x)} \quad \text{if } \zeta^x \in \Gamma$$

Observe that in the bottom line of the above display we considered configurations satisfying

$$\zeta \subset \gamma, x \notin \gamma, x \notin \zeta, \quad \gamma^x = \gamma \cup \{x\} \in \Gamma, \quad \zeta^x = \zeta \cup \{x\} \in \Gamma \tag{3.12}$$

Therefore the numbers $\tilde{Q}$ corresponds to (non-negative) rates of a Markov process on $\Gamma \times \Gamma$ provided

$$(3.12) \Rightarrow \frac{\mu_1(\zeta)}{\mu_1(\zeta^x)} \geq \frac{\mu_2(\gamma)}{\mu_2(\gamma^x)}.$$

Due to the definition of μ_2, the above implication is equivalent to the following one:

$$(3.12) \Rightarrow \frac{\mu_1(\zeta)}{\mu_1(\zeta^x)} \geq \frac{g(\gamma)\mu_1(\gamma)}{g(\gamma^x)\mu_1(\gamma^x)}.$$

But since g is an increasing function and $\mu_1 = \mu$, the latter follows from Lemma 4 below. $\square$

Remark. The above argument repeats, with a slight modification (compare (3.10) and (3.11) with the rates of page 76 of Liggett's book) the proofs of Theorem 2.9 and Corollary 2.12, Chapter II, Liggett (1985). Corollary 2.12 can not be applied directly to our case because the set $\gamma \wedge \zeta$ defined by $(\gamma \wedge \zeta)(x) = \min(\gamma(x), \zeta(x))$ may not be a cluster. If this is the case, $\mu(\gamma \wedge \zeta) = 0$ and the inequality (2.13) in the hypothesis of that corollary is not satisfied by our μ. In our proof, the role of inequality (2.13) in Corollary 2.12 of the book is played by (3.13) below.

Lemma 4. *Let* $\zeta, \gamma \in \Gamma$ *and* $x \in V \setminus V_0$. *Then*

$$(3.12) \Rightarrow \frac{\mu(\zeta)}{\mu(\zeta^x)} \geq \frac{\mu(\gamma)}{\mu(\gamma^x)} \tag{3.13}$$

which is equivalent to the implication below, due to (3.7):

$$(3.12) \Rightarrow \frac{\mathbb{P}(C(M) = \zeta)}{\mathbb{P}(C(M) = \zeta^x)} \geq \frac{\mathbb{P}(C(M) = \gamma)}{\mathbb{P}(C(M) = \gamma^x)} \tag{3.14}$$

Proof. We prove (3.14). Applying (3.4) twice, once for γ and the other time for γ^x instead of γ (which is possible since $\gamma^x \in \Gamma$ by (3.12)), we conclude that

$$\frac{\mathbb{P}(C(M) = \gamma)}{\mathbb{P}(C(M) = \gamma^x)} = \frac{\pi(x, \gamma)}{\pi(x, \gamma^x)} \prod_{y:(x,y) \in E} \frac{\pi(y, \gamma)}{\pi(y, \gamma^x)} \tag{3.15}$$

By the same reasoning, the identity (3.15) holds with γ being substituted by ζ. Thus, (3.14) follows if we show that (3.12) implies that the value of (3.15) does not decrease if we substitute there γ by ζ. For this, we will show that (3.12) implies

$$\frac{\pi(x, \zeta)}{\pi(x, \zeta^x)} \geq \frac{\pi(x, \gamma)}{\pi(x, \gamma^x)}, \tag{3.16}$$

$$\frac{\pi(y, \zeta)}{\pi(y, \zeta^x)} \geq \frac{\pi(y, \gamma)}{\pi(y, \gamma^x)}, \quad \text{for all } y \text{ such that } (x, y) \in E. \tag{3.17}$$

From (3.5) and because $\gamma(x) = 0, \gamma^x(x) = 1$, we have that the lhs of (3.16) is

$$\frac{(1-p)^{|D(x) \cap \gamma|}}{1 - (1-p)^{|D(x) \cap \gamma|}} \tag{3.18}$$

Since $\zeta(x) = 1 - \zeta^x(x) = 0$ in the considered case, since $\zeta \subset \gamma$ yields that $|D(x) \cap \zeta| \leq |D(x) \cap \gamma|$, and since the function in (3.18) is decreasing in $|D(x) \cap \gamma|$, then (3.16) follows.

To prove (3.17), we will consider three cases which exhaust all the possibilities for $\gamma(y), \zeta(y)$ which satisfy $\zeta \subset \gamma$:

Case 1. $\gamma(y) = 0$ and $\zeta(y) = 0$. In this case $\zeta^z(y) = 0, |D(y) \cap \zeta^z| = |D(y) \cap \zeta| + 1$, and thus, the lhs of (3.17) is

$$\frac{(1-p)^{|D(y)\cap\zeta|}}{(1-p)^{|D(y)\cap\zeta^z|}} = \frac{1}{1-p} \tag{3.19}$$

Similarly, $(1-p)^{-1}$ is the value of the rhs of (3.17).

Case 2. $\gamma(y) = 1, \zeta(y) = 0$. The lhs of (3.17) is (3.19). The rhs is

$$\frac{1 - (1-p)^{|D(y)\cap\gamma|}}{1 - (1-p)^{|D(y)\cap\gamma^z|}}$$

Since $(1-p)^{-1} \geq (1 - (1-p)^k)/(1 - (1-p)^{k+1})$ for all k and all $p \in [0,1]$ then (3.17) holds for this case.

Case 3. $\gamma(y) = \zeta(y) = 1$. In this case $\gamma^z(y) = \zeta^z(y) = 1$ and thus,

$$\text{the lhs of (3.17)} = \frac{1 - (1-p)^{|D(y)\cap\zeta|}}{1 - (1-p)^{|D(y)\cap\zeta^z|}}, \qquad \text{the rhs of (3.17)} = \frac{1 - (1-p)^{|D(y)\cap\gamma|}}{1 - (1-p)^{|D(y)\cap\gamma^z|}}$$

But since calling $q = 1 - p$,

$$\frac{1-q^k}{1-q^{k+1}} < \frac{1-q^{k+1}}{1-q^{k+2}}, \quad \forall k \quad \text{when } 0 < q < 1$$

then (3.17) is untrue unless

$$(3.12) \text{ and } y \in V \text{ is such that } (x,y) \in E \Rightarrow |D(y) \cap \zeta| = |D(y) \cap \gamma| \tag{3.20}$$

For the oriented bond percolation in two dimensions, (3.20) is satisfied. Indeed, let $y = (m,n)$ and $x = (m+1, n-1)$ (when $x = (m-1, n-1)$ the argument will be the same). Recall $x \notin \gamma$, $x \notin \zeta$. Thus, assuming that $(m-1, n-1) \in \gamma$, we conclude that $|D(y)\cap\zeta| < |D(y)\cap\gamma|$ requires that $(m-1, n-1) \notin \zeta$. But the latter means that $\zeta \notin \Gamma$, since $y \in \zeta$ in the considered case. If, to the contrary $(m-1, n-1) \notin \gamma$ then already $\gamma \notin \Gamma$. Observe (3.20) fails for the oriented bond percolation on any oriented graph with at least three bonds pointing to a site. $\square$

Remark. The above requirement (3.20) is the only point where our proof fails when we want to generalize Theorem 1 to an arbitrary graph.

4. Application

First we introduce some notation as follows:

$$\nu(\circ) := \mathbb{P}(|C(\{0\})| < \infty), \quad \nu(\circ\circ) := \mathbb{P}(|C(\{0,2\})| < \infty),$$

$$\nu(\circ\circ\circ) := \mathbb{P}(|C(\{-2,0,2\})| < \infty), \quad \nu(\circ \times \circ) := \mathbb{P}(|C(\{-2,2\})| < \infty).$$

In this section, using Theorem 1, we will give an upper bound on the percolation probability from the origin, i.e., $\sigma(p)(= \mathbb{P}(|C(\{0\})| = \infty) = 1 - \nu(\circ))$. By translation invariance, we have

$$\nu(\circ) = p^2\nu(\circ\circ) + 2p(1-p)\nu(\circ) + (1-p)^2, \tag{4.1}$$

$$\nu(\circ\circ) = p^3(2-p)\nu(\circ\circ\circ) + p^2(1-p)^2\nu(\circ \times \circ)$$
$$+ 2p^2(1-p)(2-p)\nu(\circ\circ) + p(1-p)^2(4-3p)\nu(\circ) + (1-p)^4. \tag{4.2}$$

Applying Theorem 1 to $A = \{-2,0\}$ and $B = \{0,2\}$ gives

$$\nu(\circ)\nu(\circ\circ\circ) \geq \nu(\circ\circ)^2. \tag{4.3}$$

Note that this inequality refines $\nu(\circ\circ\circ) \geq \nu(\circ)\nu(\circ\circ)$ given by the Harris-FKG inequality (1.1) with $A = \{-2,0\}$ and $B = \{2\}$. Moreover, applying the Harris-FKG inequality (1.1) to $A = \{-2\}$ and $B = \{2\}$ gives

$$\nu(\circ \times \circ) \geq \nu(\circ)^2. \tag{4.4}$$

By using (4.1)-(4.4), we have the following inequality.

$$\Big[\nu(\circ) - 1\Big]\Big[p^4\nu^2(\circ) + \Big\{-2p^4 + 4p^3 - 2p^2 + 2p - 1\Big\}\nu(\circ) + p^4 - 4p^3 + 5p^2 - 2p\Big] \leq 0.$$

This inequality implies

Corollary 5. *For* $\sqrt{3}/3 \leq p \leq 1$,

$$\sigma(p) \leq (4p^3 - 2p^2 + 2p - 1 - \sqrt{D})/2p^4, \tag{4.5}$$

where

$$D = 4p^6 - 16p^5 + 24p^4 - 16p^3 + 8p^2 - 4p + 1.$$

Remarks. (i) The above upper bound on a survival probability $\sigma(p)$ is the best one to our knowledge. This result gives also a lower bound on a critical probability, $p_c = \inf\{p \in [0,1] :$

$\sigma(p) > 0\}$, as follows: $\sqrt{3}/3 \approx 0.577 \leq p_c$. However, this lower bound is not the best. See Section 6 in Durrett (1984), for example.

(ii) By (4.3) and a trivial inequality $\nu(\circ \times \circ) \geq \nu(\circ \circ \circ)$, we have

$$\nu(\circ)\nu(\circ \times \circ) \geq \nu(\circ\circ)^2. \tag{4.6}$$

So, instead of (4.4), we can choose (4.6). Similarly, we get

$$\sigma(p) \leq (p^4 - 4p^3 + 5p^2 - 1)/(p^4 - 4p^3 + 6p^2 - 2p), \tag{4.7}$$

for $p_* \leq p \leq 1$, where $p_* = \inf\{p \in [0,1] : p^4 - 4p^3 + 5p^2 - 1 > 0\}$. However, it can be checked that this bound is not better than that in (4.5).

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References

Aizenmann, M. and Newman, C. M. (1984) Three graph inequalities and critical behavior in percolation models. *J. Stat Phys.* **36** 107-143.

Chayes, J. T., Chayes, L., Grimmett, G. R., Kesten, H. and Schonmann, R. H. (1989). The correlation length for the high-density phase of Bernoulli percolation. *Ann. Probab.* **17** 4:1277-1302.

Durrett, R. (1984). Oriented percolation in two dimensions. *Ann. Probab.* **12** 999-1040.

Durrett, R. (1988). *Lecture Notes on Particle Systems and Percolation.* Wadsworth, Belmont, CA.

Fortuin, C. M., Kasteleyn, P. W. and Ginibre, J. (1971). Correlation inequalities on some partially ordered sets. *Comm. Math. Phys.* **22** 89-103.

Grimmett, G. (1989). *Percolation.* Springer, New York.

Harris, T.E. (1960). A lower bound for the critical probability in certain percolation processes. *P. Camb. Phyl. Soc.* **56** 13-20.

Liggett, T. M. (1985). *Interacting Particle Systems*. Springer, New York.

Liggett, T. M. (1995). Survival and coexistence in interacting particle systems. *Probability and Phase Transition* NATO ASI Series C: Mathematical and Physical Sciences Vol 420. 209-226. Kluwer Academic Publishers. Dordrecht (1994).

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