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INTRODUCTION

It is well known that a diffeomorphism which exhibits an orbit of tangency between the stable and unstable manifolds of periodic orbits is not structurally stable. The same situation is observed for flows X_t generated by a vectorfield X. In fact there are real invariants for topological equivalence as we point out in §1, so that we have nondenumerable classes of equivalence in any neighborhood of f or X.

Even in this case, if it is possible to parameterize the classes of equivalence by finitely many parameters, we can get a nice description of the dynamical systems near f or X. When this happens we say that f or X have finite modulus of stability. Of course, a structurally stable vectorfield has zero modulus of stability.

Bifurcations of real dynamical systems are related to this subject as in [1], [2], [3], [4], [5], [6], [7], [8], [18], [23] as well as holomorphic vectorfields [9], [10].

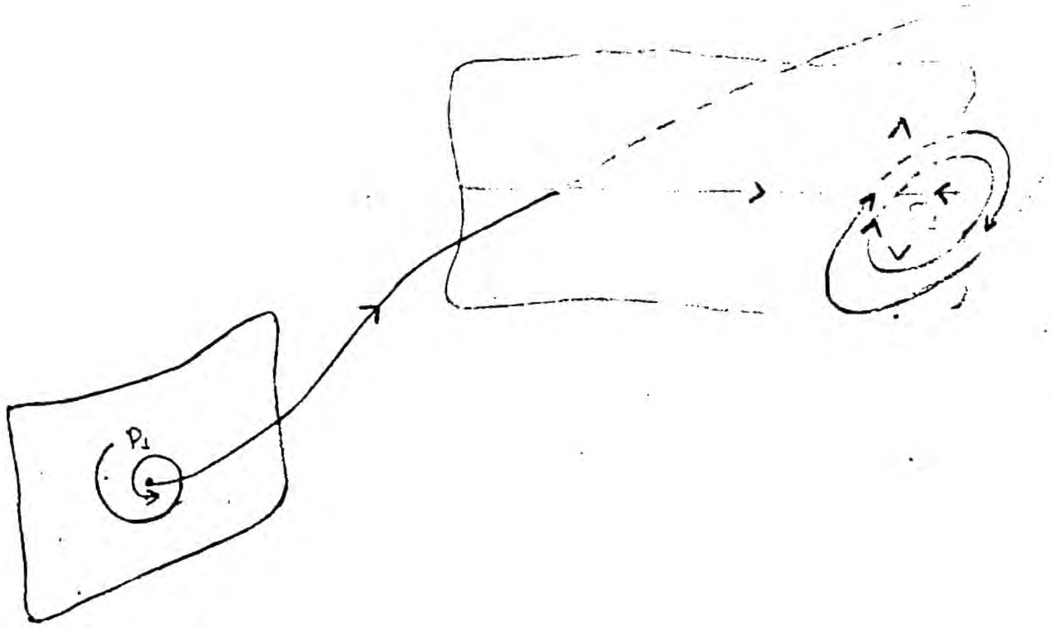
Namely, bifurcations of one-parameter families X_μ of vectorfields with simple recurrence occur generically (i.e. on a residual set) for vectorfields X_{μ_0} that exhibit a quasi-hyperbolic critical element or else have a quasi-transversal saddle connection [1], [3],[6], [7], [11], [23].

Here we examine the modulus of stability for vectorfields on a compact 3-dimensional manifold M that exhibits a quasi-transversal saddle connection between the unstable manifold $W^u(o_1)$ of a singularity p_1 and the stable manifold of a periodic orbit σ_1.

We shall need a few definitions in order to state our results. We indicate [12], [22] and [25] as general references for basic facts.

Let X(M) be the space of C^∞ vectorfields endowed with the C^∞ Whitney topology.

Let X be a C^∞ field on M^3 which exhibits a singularity p_1 and a closed orbit σ_1, both of saddle type, hyperbolic and C^2-locally linearizable. These two last conditions are open and dense ([12], [13]). So from the generic point of view they are not restrictive. Suppose $\alpha_1 \pm i\alpha_2$, $\alpha_1 < 0, \alpha_2, \alpha_3 > 0$ are the eigenvalues associated to p_1 (then $\dim W^s(o_1) = 2$) and $0 < \beta_1 < 1 < \beta_2$ are those corresponding to σ_1 which has period τ. Furthermore, $W^u(p_1) \subset W^s(\sigma_1)$. We shall call D the set of these fields.



A topological equivalence between two vectorfields X, X' on M is an homeomorphism $h:M \to M$ such that h sends orbits of X into orbits of X', preserving time orientation. If in addition h preserves time, that is $h X_t = X'_t h$ holds, then h is called a conjugation. A vectorfield X is called structurally stable if it is equivalent to any nearby vectorfield;

A semilocal equivalence between X and $X' \in D$ shall be an equivalence defined from a neighboorhood of $\overline{W^u(p_1)}$ onto a neighborhood of $\overline{W^u(p_1')}$

$\Omega = \Omega(X)$ is the set of non-wandering points, that is $\sigma \in \Omega$ if for every neighborhood U of σ and $t_0 > 0$ there exists $t_1 > t_0$ such that $X_{t_1}(U) \cap U \neq \emptyset$. Here Ω will have finitely many critical elements (singularities and periodic orbits), also called trivial recurrences.

An m-chain (for X) is an m+1-tuple $(\sigma_1, \ldots, \sigma_{m+1})$ of critical elements of X such that $W^u(\sigma_i) - \sigma_i \cap W^s(\sigma_{i+1}) - \sigma_{i+1} \cap \neq \emptyset (1 < i < m)$. In this case we say that the chain begins at σ_1 and ends at σ_{m+1}. An m-chain for (σ', σ'') is an m-chain such that there exists j, $1 \leq j \leq m$ for which $\sigma_j = \sigma'$ and $\sigma_{j+1} = \sigma''$. An m-cycle is an m-chain for which $\sigma_1 = \sigma_{j+1}$. The behaviour $\mathrm{beh}(\sigma_1, \sigma_2)$ of σ_1 and σ_2 is the cardinality of the longest chain which begins at σ_1 and ends at σ_2, whenever it exists.

The relevant chains for us are the "m-chains for (p_1, σ_1)". We will refer to them simply as "m-chains". Any other case shall be specified.

Let $X \in A_1 \subset D$ if:

a) $\Omega(X)$ has finitely many orbits.

b) $\Omega(X)$ has only trivial recurrences, all of them hyperbolic.

c) all invariant manifolds meet transversally except for $W^u(p_1)$ and $W^s(\sigma_1)$.

An X-orbit σ is a saddle connection if both $\alpha(\sigma)$ and $\omega(\sigma)$ are saddle type critical elements. In this case

$$\sigma \subset W^u(\alpha(\sigma)) \subset W^s(\omega(\sigma))$$

Observe that even if $X \in A_1$ it could have infinitely many saddle conections for which ω were $W^s(p_1)$ or α were $W^u(\sigma_1)$. For example if X exhibits a chain $(p_1,\sigma_1,\sigma_2,\sigma_3)$ where σ_2 and σ_3 are saddle type periodic orbits and all the corresponding invariant manifolds meet transversally. We can now state

We can now state

<u>Theorem A:</u> Let $X \in A_1$, and have finite modulus of equivalence. Then

a) There are finitely many saddle connections in $W^s(p_1)$ or $W^u(\sigma_1)$.

b) Beh $(\sigma,p_1) \leq 1$ and $\mathrm{beh}(\sigma_1,\sigma') \leq 1$ for σ,σ' saddle type critical elements.

c) There are no cycles for p_1 or σ_1.

d) Any chain for (p_1,σ_1) has at most 6 elements. Moreover if

$$(p_3,p_2,p_1,\sigma_1,\sigma_2,\sigma_3)$$

is one of these chains, then either p_2 or σ_2 are not periodic orbits.

Let $A_2 \subset A_1$ be the set of fields X such that:

a) $\Omega(X)$ has no cycles. (Hence it is Ω-stable by [21])

b) Any chain for (p_1, σ_1) has at most 5 elements.

c) X is C^2 linearizable on neighborhoods of the critical elements in chains for (p_1, σ_1).

We now state

<u>Theorem B</u>: If $X \in A_2$ then X has finite modulus of equivalence.

<u>Remark</u>: as it will be seen along the proof, the existence of a 6-chain $(p_3, p_2, p_1, \sigma_1, \sigma_2, \sigma_3)$ determines the infiniteness of the modulus of stability by conjugation. The finiteness of the modulus of stability in case A)d) remains undetermined, but in proposition 8 we see that we can C^r approximate X by a field Y with infinitely many tangencies between $W^u(p_2)$ and $W^s(\sigma_2)$.

§ 1 - THE SEMILOCAL INVARIANT μ.

A fence F is a surface transversal to $W^s(p_1)$ and to the flow, such that $F \cap W^s(p_1)$ is a fundamental domain of $W^s(p_1)$. In the locally linearizing coordinates, a cilynder will be a fence. For R a curve in F, and Σ_1 a section transversal to the flow at σ_1, a spiral E is any connected component of $\Sigma_1 \cap \{X_t(R)\ ,\ t \geq 0\}$ (the positive saturated set of R). When R is a line, we call E a linear spiral. Any connected component S of $E - W^s(\sigma_1)$ is called a sector. Let $\pi_2 : \Sigma_1 \longrightarrow W^u(\sigma_1)$ be the projection on $W^u(\sigma_1)$ in the linearizing coordinates (y_1, y_2) of Σ_1. To each "upper" sector S_j in E we associate its maximum e_j, that is, the maximum of $\pi_2(S_j)$. This induces a cannonical order between the "upper" sectors. We now establish the main invariant by semilocal equivalence.

Notation:

$$\text{For } X \in D \quad \text{let } \mu = \mu(X) = \frac{2\pi\alpha_1}{\alpha_2 \ln \beta_2}$$

Proposition 1: Let $E \subset \Sigma_1$ be a linear spiral, $\{S_i\}_{i \in N}$ the ordered sectors of E, e_i the maximum of each sector,

$$\lambda = e^{2\pi\alpha_1/\alpha_2} .$$

Then

$$\lim_{(m,n) \to +\infty} e_m \lambda^m / e_n \lambda^n = 1.$$

Proof: Consider $\phi: \Lambda \longrightarrow \Sigma_1$ the Poincaré diffeomorphism determined by the flow, where Λ is a plane parallel to $W^s(p_1)$ in the linearized domain of p_1. We choose coordinates (x_1,x_2) in Λ such that $x_1 \leqq 0$ corresponds to $\Lambda \cap W^s(\sigma_1)$ and $x_2 = k_1$ is perpendicular to $x_1 = k_2$. If $\phi = (\phi_1,\phi_2)$ in the coordinates of Σ_1 we have

$$\frac{\partial\phi_2}{\partial x_1}(0,0) = 0$$

and hence

$$\frac{\partial\phi_2}{\partial x_2}(0,0) \neq 0.$$

The equation of S_n being

$$\ln\rho-\alpha_1\Theta/\alpha_2 = \ln\rho_0 - \alpha_1 \ /\alpha_2 \quad ; \quad ((n-1) < \Theta < n\pi)$$

The equation of S_m shall be

$$\ln\rho-\alpha_1\Theta/\alpha_2 = \ln\rho_0-\alpha_1\Theta_0/\alpha_2+2\pi(m-n)\alpha_1/\alpha_2 .$$

in the polar coordinates of Λ.

A little calculation expressing ϕ in polar coordinates finishes the proof. □

The next Proposition proves the invariance of μ under semilocal equivalences.

Proposition 2: Take $X, X' \in D$ and h a semilocal equivalence between them

$$\mu(X) = \mu(X')$$

Proof: Let E be a linear spiral for the vectorfield X,

$$c \in W^s(\sigma_1) \cap \Sigma_1 . \quad \text{Let } \{S_n\}_{n \geq 1}$$

be the ordered sequence of the upper sectors of E and

$$\pi_2^{-1}(c) \cap S_n \neq \emptyset \ \forall n \in N.$$

Define

$$N_m(E,\pi_2^{-1}(c_1), S_p) = \#\{S_j / S_j \cap X_{-m\tau}(\pi_2^{-1}(c)) \neq \emptyset , \quad j \geq p\}$$

Easily we see that $N_m < +\infty$. It follows too that

$$N_m(E,\pi_2^{-1}(c_1), S_1) - N_m(E,\pi_2^{-1}(c_2) , S_p) = k_1 \ \forall m \in N.$$

By a similar argument

$$N_m(E_1,\pi_2^{-1}(c), S_1^1) - N_m(E_2,\pi_2^{-1}(c) S_1^2) = k_2 \ \forall m \in N,$$

where E_1 and E_2 are two different spirals.

We claim that

$$\lim_{m \longrightarrow +\infty} \frac{N_m(E,\pi_2^{-1}(c),S_1)}{m} = \frac{2\pi\alpha_1}{\alpha_2 \ln\beta_2} .$$

In fact, from Proposition 1 we derive

$$\pi_2(e_p)\lambda^{N_m+1} \leq c\beta_2^m \leq \pi_2(e_p)\lambda^{N_m}$$

for p big enough. Hence, taking logarithms

$$\frac{1}{m}\,[\ln(\pi_2(e_p)) + \ln\lambda - \ln c] \leq - \ln\lambda\frac{N_m}{m} + \ln\beta_2 \leq \frac{1}{m}[\ln\pi_2(e_p) - \ln c]$$

This proves that $\lim\limits_{m \to \infty} \dfrac{N_m(E,\pi_2^{-1}(c),S_p)}{m} = \dfrac{2\pi\alpha_1}{\alpha_2 \ln\beta_2}$.

As

$$\{N_m(E,\pi_2^{-1}(c),\ S_p\}_{m\in N} \quad \text{and} \quad \{N_m(E,\pi_2^{-1}(c),\ S_1)\}_{m\in N}$$

differ by a constant sequence, our claim holds.

We can also define N_m for $R \subset \Sigma_1$ a continuous curve transversal to $W^u(\sigma_1)$ by $N_m(E,R,S_p) = \#\{S_j/S_j \cap X_{-m\tau}(R) \neq \emptyset \ ,\ j \geq p\}$ and

$$N_m(E,R,S_p) - N_m(E,\pi_2^{-1}(c),\ S_p) = k \quad \text{for} \quad m \geq m_0.$$

For $E_2' = h(E)$ we take two linear spirals E_1' and E_3' such that the associated fibers in the fence F' (a cilynder) intersect $D^s(p_1')$ along nearby points.

In the same way, we choose $c_1, c_2 \in W^u(\sigma_1')$ such that

$$c_1 \geq \pi_2'(y_1',y_2') > c_2 > 0 \quad \text{for} \quad (y_1',y_2') \in h(R) \text{ sufficiently near } W^u(\sigma_1').$$

Then the following inequality holds:

$$N'(E_2',\pi_2'^{-1}(c_2),\ S_p'^1) \geqq N_m'(E_2',h(R),\ S_p'^2) \geqq N_m'(E_3',\pi_2'^{-1}(c_1),\ S_p'^3) \quad \text{and}$$

consequently

$$\lim_{m\to\infty} \frac{N_m'(E_2',h(R),S_p'^2)}{m} = \lim_{m\to\infty} \frac{N_m'(E_3',\pi_2'^{-1}(c_1),\ S_p'^3)}{m} = \mu'.$$

But

$$N_m(E,R,S_p) = N_m'(h(E),h(R),S_p')$$

Hence

$$\mu = \lim_{m\to\infty} \frac{N_m}{m} = \lim_{m\to\infty} \frac{N_m'}{m} = \mu' \qquad \square$$

§ 2 - THE RIGIDITY OF THE EQUIVALENCE

Propositions 3 and 4 in this section show that a semilocal equivalence between two vectorfields in D must be quite rigid when the invariant μ of § 1 is irrational.

<u>Proposition 3</u>: Let $X, X' \in D$ and h be a semilocal equivalence between them: If $\mu (= \mu') \notin Q$ then

$$h/W^u(\sigma_1) \cap \Sigma_1 \quad \text{is logarithmically linear.}$$

<u>Proof:</u> Take a linear spiral $E \subset \Sigma_1$, and S_n $n \geq 1$ the sequence of its ordered upper sectors. We identify e_n and $h(e_n)$ with their images by Π_2 and Π_2'. Let e_n and e_n^1 be the absolute maxima of S_n and $S_n' = h(S_n)$. We claim that if

$$e_{n_j} \cdot \beta_2^{m_j} \underset{j \to +\infty}{\to} z \qquad \text{then} \quad e'_{n_j} \cdot \beta_2'^{m_j} \underset{j \to +\infty}{\to} h(z)$$

This happens because

$$h(e_{n_j}) \leq e'_{n_j} \qquad \text{and}$$

$$h(e_{n_j}) \beta_2'^{m_j} \leq e'_{n_j} \beta_2'^{m_j}$$

so that $h(z) \leq \underline{\lim}_{j \to \infty} e'_{n_j} \beta_2'^{m_j}$

$h/W^u(\sigma_1) \cap \Sigma_1$ being a conjugation, we get

$$h\left(\frac{z}{\beta_2^i}\right) = \frac{1}{\beta_2'^{\,i}}\ h(z) \quad \text{and combining our equations,}$$

$$h\left(\frac{z}{\lambda^j \beta_2^i}\right) = \frac{h(z)}{\lambda'^{\,j} \beta_2'^{\,i}}$$

If $\mu \notin Q$ then $\forall\ r \in \mathbb{R}\ \exists$ a sequence $(m_j(r),\ n_j(r))$ such that

$\lambda^{m_j(r)} \beta_2^{n_j(r)} \to r$ therefore $h\left(\frac{z_0}{r}\right) = \frac{h(z_0)}{r^\beta}$

where $\beta = \mathrm{Ln}\beta_2 / \mathrm{Ln}\beta_2'$; and h is logarithmically linear $\square$

In order to get the rigidity of an equivalence h on D^s, we observe that in the linearized cilyndrical coordinates the orbits of $W^s(p_1)$ are given by

$$\ln\rho - \frac{\alpha_1}{\alpha_2}\Theta = \ln\rho_0 - \frac{\alpha_1}{\alpha_2}\Theta_0 = k_0$$

Then to each orbit in $W^s(p_1)$ we associate the angle Θ_0 (mod 2π) with which it intersects $\{\rho = 1\}$. In this way, a semilocal equivalence h induces a map

$$\tilde{h}: S^1 \text{---} S^1$$

where the orbit passing through $(1,\Theta) \in W^s(p_1)$ (polar coordinates) corresponds by h to the orbit passing through $\tilde{h}\,(1,\Theta) = (1,\ \tilde{h}(\Theta))$.

For the same reasons $h^{-1}(\overline{\lim}\, e'_{n_j} . \beta_2'^{\,m_j}) \leq z$

and consequently $h(z) = \underline{\lim}_{j\to+\infty} e'_{n_j} . \beta_2'^{\,m_j} = \overline{\lim}_{j\to+\infty} e'_{n_j} . \beta_2'^{\,m_j}$

Let $e_{n_{i,j}}$ be the maximum of $S_{i+n_{1,j}}$

Suppose now $e_{n_{1,j}} . \beta_2^{m_{1,j}} \longrightarrow z$ which implies

$$e'_{n_{1,j}} \ \beta_2'^{\,m_{1,j}} \longrightarrow z = h(z)$$

For j big enough, we have $e_{n_{i,j}} \cong e_{n_{1,j}} \ \lambda^i$.

Then $e_{n_{i,j}} \ \beta_2^{m_{1,j}} \to z/\lambda^i$ and a routine calculation shows that

$$h\left(\frac{z}{\lambda^i}\right) = \frac{h(z)}{\lambda'^i}$$

In fact

$$\lambda'^i h(z) = \lambda'^i \lim_{j\to+\infty} e'_{n_{1,j}} \ \beta_2'^{\,m_{1,j}} = \lim_{j\to+\infty} e'_{n_{1,j}} \ \beta_2'^{\,m_{1,j}} = h(\lim_{j\to+\infty} e_{n_{i,j}} \ \beta_2^{m_{1,j}}) =$$

$$= h(\lambda^i \lim_{j\to+\infty} e_{n_{1,j}} \ \beta_2^{m_{1,j}}) = h(\lambda^i z)$$

Proposition 4: In this case, $\tilde{h}$ is a rotation.

Proof: Take polar coordinates in Λ, as in Prop. 1, with $\{\Theta = 0\} = \{x_1 = 0\}$. With the same calculation, we can see the maxima in of the spiral E approximatedly occur along the same direction, i.e., arctg $(-\alpha_1/\alpha_2)$.

Therefore, if E_1 and E_2 are two linear spirals given by

$$\ln\rho - \frac{\alpha_1}{\alpha_2}\Theta = k_i \mod (2\Pi\alpha_{1/\alpha_2}) \qquad (i = 1,2)$$

we get $\lim\limits_{n\to+\infty} \Pi_2(e_n^1)/\Pi_2(e_n^2) = e^{k_1-k_2}$ for e_n^i the maxima of the "upper" sectors of $E_i (i=1,2)$ supposing $e_n^1 \geqq e_n^2 \geqq e_{n+1}^1$

Now take sequences

$$\{e^i_{m_j}\}_{j\in\mathbb{N}} \quad , \quad \{n_j\}_{j\in\mathbb{N}} \qquad\qquad (i = 1,2)$$

such that

$$e^i_{m_j}\beta_2^{n_j} \xrightarrow[j\to+\infty]{} w_i \qquad\qquad \text{which implies}$$

$$e'^i_{m_j}\beta_2'^{n_j} \xrightarrow[j\to+\infty]{} w_i$$

From $\dfrac{e^{i}_{m_j}}{\lambda^{m_j}} \underset{j\to\infty}{\to} e^{k_i}$ we derive $\lambda^{m_j}\beta_2^{n_j} \longrightarrow \dfrac{w_i}{e^{k_i}}$.

and $\lambda'^{m_j}\beta_2'^{n_j} \to \dfrac{w_i'}{e^{k_i'}}$. From this, we get that

$$\left(\frac{w_i'}{e^{k_i'}}\right)^{\beta} = \frac{w_j}{e^{k_j}} \quad (i,j = 1,2) \quad \text{Hence,} \quad \frac{w_i'^{\beta}}{w_i} = e^{\beta K_i' - K_i} .$$

But, as $h/w^u(\sigma_1) \cap \Sigma_1$ is logarithmically linear $\dfrac{w_1'^{\beta}}{w_1} = \dfrac{w_2'^{\beta}}{w_2}$

so that $e^{k_1'-k_1} = e^{k_2'-k_2}$.

And replacing this equality in the equations of the spirals, we are done □

§ 3 - Restrictions on behaviour: Theorem A

We can derive several relevant consequences from the rigidity properties established in § 2.

In fact, suppose $\mu \notin Q$.

Take D^u a fundamental domain of $W^u(\sigma_1) \cap \Sigma_1$ and $\{z_{i,0},\dots,z_{i,m}\} \subset D^u$ points of the saddle connections between $W^u(\sigma_1)$ and $W^s(\sigma_i)$ $(i \neq 1)$. Consider the corresponding objects for a field X' equivalent to X under h. Then $h/W^u(\sigma_1) \cap \Sigma_1$ is logarithmically linear and we must have

$$\mu(i,j) = \left(\frac{z_{i,0}}{z_{i,j}}\right)^{1/\ln\beta_2} = \left(\frac{z'_{i,0}}{z'_{i,j}}\right)^{1/\ln\beta'_2} = \mu'(i,j) \qquad i \leq j \leq m$$

or

<u>Corollary 5</u>: If $\{z_{i,0},\dots,z_{i,m}\} \subset D^u$ are points of the saddle connections between $W^u(\sigma_1)$ and $W^s(\sigma_i)$ then there are at least m real invariants $\mu(i,j)$ for the existence of an equivalence.

In the same way, let D^s be a fundamental domain in $W^s(p_1)$, which we may assume to be the circle $\{\rho = 1\}$. Let $\{(1,\Theta_{i,0}),\dots,(1,\Theta_{i,m})\} \subset D^s$ be points of the saddle connections between $W^s(p_1)$ and $W^u(\sigma_i)$ $(i \geq 2)$. Consider the analogous objects for a field X' equivalent to X. Then if h is an equivalence it will induce a rotation between D^s and D'^s. Hence we must have:

$$\lambda(i,j) = \Theta_{i,j} - \Theta_{i,0} = \Theta'_{i,j} - \Theta'_{i,0} = \lambda'(i,j) \qquad 1 \leq j \leq m$$

<u>Corollary 6</u>: Let $\{(1,\Theta_{i,0}),\dots,(1,\Theta_{i,m})\} \subset D^s$ be as above. Then there are at least m real invariants for the existence of an equivalence,

namely

$$\lambda(i,j) = \lambda'(i,j) \qquad (1 \leq j \leq m)$$

Proof of Theorem A: We may assume, taking a small perturbation of the field if necessary, that the semi-local invariant μ of Section 1 is not rational. Thus by Corollaries 5 and 6 if $\mu \notin Q$, each new saddle connection gives rise to a new invariant, at least; so a) holds. if (x_1,x_2,p_1) is a chain, $W^u(x_1)$ intersects $W^s(p_1)$ along infinitely may orbits, because of the transversality between invariant manifolds. Therefore either x is a source or there are infinitely many moduli of equivalence. Similarly, for chains $(\sigma_1,x_1,x_2)x_2$ must be a sink. Now we prove c): if (p_1,σ_1,p_1) is a cycle, we get homoclinic points for σ_1. The existence of a cycle (x_1,p_1,σ_1,x_1) means the existence of homoclinic points for x_1. Longer cycles are forbidden on account of a) and b).

Let $X \in A_1$ and $(p_3,p_2,p_1,\sigma_1,\sigma_2,\sigma_3)$ be a 6 - chain for (p_1,σ_1). Necessarily p_3 and σ_3 cannot be saddles, if X has finite modulus. But

Prop 8: Let X be as before. Then X can be aproximated in the C^r topology by fields Z with infinitely many tangencies.

Proof: (part of this proof was suggested by F. Takens)

Let ψ: $R \to [0,1]$ C^∞· be a bump function $\|\psi\|_r \leq M$, $\psi \equiv 1$ on a neighborhood of 0, $\psi(x) \equiv 0$ for $|x| \geq 1$, the ball $B[0,1]$ being contained in the domain of linearization L of p_1. Call $\psi_1 = \psi$.

Define

$$Y_{\lambda,1}(x,y,z) = \lambda(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})(\psi_1(x)\psi_1(y)\psi_1(z))$$

a vectorfield on M^3, zero on $M\text{-}L$. Observe that $\Omega(X+Y_\lambda) = \Omega(X)$ and $W^u(p_1, X+Y_\lambda) \subset W^s(\sigma_1, X+Y_\lambda)$ because $< Y_\lambda, \frac{\partial}{\partial z} > = 0$.

As $W^s(\sigma_2, X+Y_\lambda) \cap \Sigma_1$ does not change with λ, we may take $z_1 \in W^s(\sigma_2) \cap W^u(\sigma_1) \cap \Sigma_1$ and a disk R of $W^s(\sigma_2) \cap \Sigma_1$ around z_1. Let $E(\lambda)$ be a spiral on Σ_1 defined by $W^u(p_2, X+Y_\lambda)$. Then we define $N_m(E(\lambda), R, S_1(\lambda))$ which varies with λ. For λ small enough, $S_1(\lambda)$ will be near $S_1(0)$. In this sense, a point of discontinuity of N_m shall be a point of tangency between $W^u(p_2, X+Y_\lambda)$ and $W^s(\sigma_2, X+Y_\lambda)$. These points exist for $\mu(X+Y_\lambda) = \frac{2\Pi\alpha_1}{(\alpha_2+\lambda)\ln\beta_2}$ changes with λ, and $\mu = \lim_{m\to+\infty}\frac{N_m}{m}$ as seen before. So some N_m must change. Accordingly take λ_1 such that $X+Y_{\lambda_1}$ has a tangency and $\lambda_1 M < \varepsilon/2$. Now our field $Z_1 = X + Y_{\lambda_1}$ is $\frac{\varepsilon}{2} - C^r$ near X and has a tangency between $W^u(p_2, Z_1)$ and $W^s(\sigma_2, Z_1)$.

Now let r be the distance between the orbit of tangency and p_1, and let $\psi_2(X) = \psi(\frac{2x}{r_1})$. Define

$$Y_{\lambda,2}(x,y,z) = \lambda\psi_2(x)\psi_2(y)\psi_2(z)(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y})$$

and consider $Z_1 + Y_{\lambda,2}$.

This field continues to exhibit the tangency of Z_1 as $Y_{\lambda,2}$ is zero on this orbit. Repeating our arguments, there exists λ_2 such that $Z + Y_{\lambda_2}$ has a new tangency and $\lambda_2 \frac{4M}{r_1^2} \leqq \varepsilon/4$.

Inductively, we construct fields $Z_n = Z_{n-1} + Y_{\lambda_n}$ which exhibit n tangencies at distances $r_1 > r_2 > \ldots > r_n$ from p_1. They have the same critical points and are identical on M-L. These $Z_n \in X(M)$ and converge in the C^r topology to a field $Z \in C^\infty$ on M, a fact that is easy to prove. Z has infinitely many tangencies betwen $W^s(\sigma_2, Z)$ and $W^u(p_2, Z)$ and is ε - C^r near X □

Note: *Observe that these tangencies are parabolic.*

These tangencies give rise to new moduli of equivalence, in case p_2 and σ_2 are closed orbits. In fact, the situation is anologous to the one treated in [8], §2, for diffeomorphisms.

So we get the following:

<u>Corollary 9:</u> Let $X \in A_1$ and have 6 - chain $(p_3,p_2,p_1,\sigma_1,\sigma_2,\sigma_3)$ where p_2 and σ_2 are closed orbits. Then X has infinite moduli of equivalence.

<u>Proof:</u> Let $Z \in X(M)$ be the ε - C^r near X field constructed in Prop 8 which exhibits infinitely many orbits of parabolic tangency between $W^u(p_2;Z)$ and $W^s(\sigma_2;Z)$.

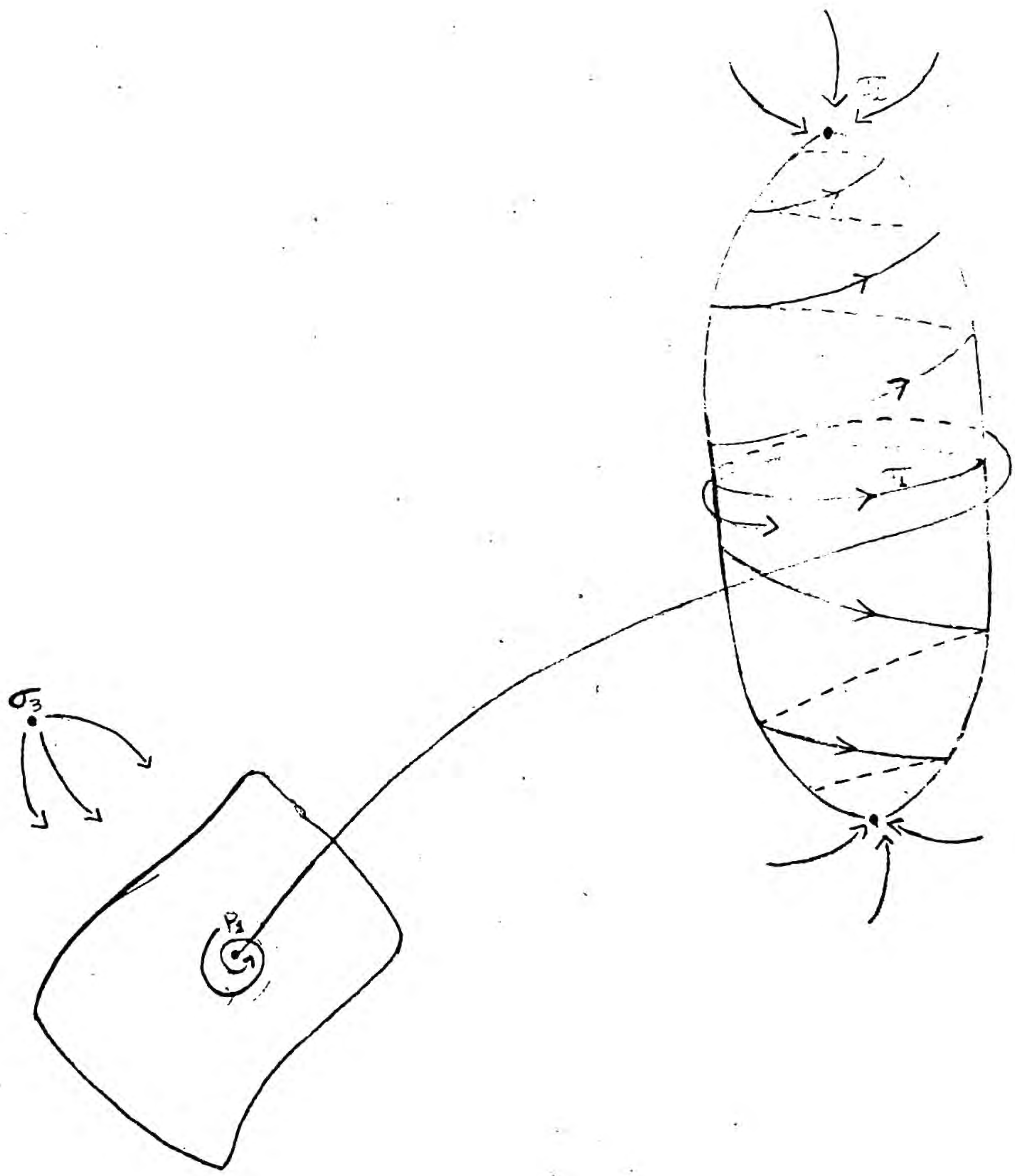
Take transversal sections $\Sigma(p_2)$ and $\Sigma(\sigma_2)$ invariant by the time one flow, reparameterizing, if necessary, X and all fields in a neighbourhood. Let $D^u(p_2) \subset W^u(p_2) \cap \Sigma(p_2)$ be a fundamental domain and $w_i \in D^u(p_2) \cap W^s(\sigma_2)$ be the points of tangency between $W^u(p_2)$ and $W^s(\sigma_2)$.

Then we define $T_2(p_2,\sigma_2,w_1,w_i)$ (the quotient of the normal derivatives). This number is an invariant by topological equivalence, or a modulus of equivalence, for each $i \in N$, $i \geq 2$. For details see [8] §2.

The existence of infinitely many tangencies arises the same number of moduli thus establishing the result $\square$

§ 4 - Sufficient conditions for finite modulus Fields with 4-chains

In this section we prove theorem B for fields $X \in D$ such that any chain for (p_1,σ_1) has length 4, that is, their expression is $(\sigma_3,p_1,\sigma_1,\sigma_2)$ where σ_3 is a source and σ_2 is a sink.



On account of the results of the last section, we recall the definition of $A_2 . X \in A_2 \subset A_1$ if

a) any chain for (p_1, σ_1) has at most 5 elements.

b) all the critical elements are C^2 linearizable.

Theorem 10: Let $X, X' \in A_2$ and

a) They are ε -C^r near;

b) $\mu(X) = \mu(X')$;

c) any chain for (p_1, σ_1) has at most 4 elements.

Proof: It is clear that if X exhibit only 4 - chains, there is a C^r neighborhood of it such that any nearby X' will be Morse-Smale or else belong to A_2. And if X, X' A_2, for analogous reasons their phase diagrams shall be isomorphic.

The proof shall begin by constructing a semilocal equivalence h in a neighborhood of $W^u(p_1)$. Then we shall extend h to all of M^3 using the methods in [20].

Take in L cylindrical coordinates (ρ, Θ, z) where $\{z = 0\} = W^s_{loc}(p_1)$, $\{\rho = 0\} = W^u_{loc}(p_1)$, $\{z > 0\} = W^u(p_1) \cap W^s(\sigma_1) \cap L$.

Suppose $\{\rho \leq 1, \ z \leq 1\} \subset L$ and $\Lambda_n = X_n(\{\rho \leq 1, z = 1\}) \subset \Sigma_1$ reparameterizing X if necessary.

Considerate

$$\Pi_1 : C = \{\rho = 1, \ z \leq 1\} \to \{\rho = 1\} \quad \text{given by}$$

$$\Pi_1^{-1}(\Theta_0) = \{(\rho,\Theta,z)/ \ \rho = 1, \ \Theta = \Theta_0, \ z < 1\} \quad \text{(trivial fibration)}$$

In Σ_1 take $\Pi_2 : \Sigma_1 \to W^u(\sigma_1) \cap \Sigma_1$ given by

$$\Pi_2^{-1}(w_0) = \{w = w_0\} \quad \text{(trivial fibration).}$$

Consider the analogous objects for X'.

We begin our proof by constructing a semilocal equivalence h. Then we shall extend h to all of M^3 approximatedly like in [20].
Like in Proposition 1, let $\Lambda = \{\rho \leq 1, \ z = 1\}$, with coordinates (x_1, x_2), (Θ, ρ) and consider

$\psi = (\psi_1, \psi_2) : \Lambda \to \Sigma_1$ induced by the flow. We want to see that spirals $X_t(\Pi_1^{-1}(\Theta_0)) \cap X_1(\Lambda)$ tangenciate the fibers of Π_2 along a curve C_1, differentiable and unique.

As each spiral as given by $\ln \rho - \frac{\alpha_1}{\alpha_2}\Theta = a_0$ we examine the equation

$$0 = \frac{d}{ds}\left(a_0 e^{\alpha_1 s/\alpha_2} \cos s, \ a_0 e^{\alpha_1 s/\alpha_2} \operatorname{sen} s\right) =$$

$= a_0 e^{\alpha_1 s/\alpha_2}\left[\left(\frac{\alpha_1}{\alpha_2}\cos s - \operatorname{sen} s\right)\frac{\partial\psi_2}{\partial x_1} + \left(\frac{\alpha_1}{\alpha_2}\operatorname{sen} s + \cos s\right)\frac{\partial\psi_2}{\partial x_2}\right]$. Since $\frac{\partial\psi_2}{\partial x_1}(0,0) =$

$= 0, \frac{\partial\psi_2}{\partial x_2}(0,0) \neq 0$, the factor $\left(\frac{\alpha_1}{\alpha_2}\operatorname{sen} s + \cos s\right)$

must be approximatedly zero for the solutions of the equation, for

$\rho(s) = \alpha_0 e^{\alpha_1 s/\alpha_2}$ small enough, that is $s \sim \operatorname{arctg}(-\frac{\alpha_2}{\alpha_1})$. We calculate

$$\frac{d^2\psi_2}{ds^2}(\rho(s) \cos s, \rho(s) \operatorname{sen} s) =$$

$$= \rho(s) \left(\frac{\alpha_1}{\alpha_2} \cos s - \operatorname{sen} s\right) \frac{\partial\psi_2}{\partial x_2} + 0(\rho < s)) \neq 0$$

for points satisfying 10.1. This follows because if

$$\left(\frac{\alpha_1}{\alpha_2} \operatorname{sen} s + \cos s\right) \sim 0 \quad \text{then}$$

$$\left(\frac{\alpha_1}{\alpha_2} \cos s - \operatorname{sen} s\right) - \left(\frac{\alpha_1}{\alpha_2} + \frac{\alpha_2}{\alpha_1}\right) \frac{\alpha_1}{\sqrt{\alpha_1^2 + \alpha_2^2}} \neq 0 \quad \text{for the same } s.$$

As $\frac{d^2\psi_2}{ds^2} \neq 0$ and on account of the continuous dependence on a_0, it follows, by the Implicit function Theorem, that locally each curve of tangency is unique and differentiable.

The differentiability on (0,0) follows because $s \underset{\rho \to 0}{\to} \operatorname{arctg}(-\frac{\alpha_2}{\alpha_1})$.

The uniqueness of the curve is a consequence of the fact that on the "upper" sectors, the critical points of $\psi_2(\rho(s),s)$ are all extrema of the same type (maxima); hence unique.

Let us begin to define the homeo h.

For the sake of simplicity, we identify $\theta = 0$ and $\theta' = 0$. Then define $h(\rho = 1, \theta, 1) = (\rho' = 1, \theta' = \theta, z' = 1)$. This induces a correspondence between the spirals in Λ and Λ', which together with the condition $h(C_1) = C_1'$, define h uniquely along C_1.

For our purposes we need $h(\Lambda) \subset \Lambda$ and to preserve Π_2 and Π_2' in a neighborhood of C_1'. In order to do so, take $(y_1, y_2) \in C_1$ and associate the fiber

$$\Pi_2'^{-1} \; (\Pi_2' \, h(y_1, y_2))$$

to the fiber

$$\Pi_2^{-1}(y_2).$$

These 3 facts (preservation of spirals, curves of tangency and segments transversal to C_1, C_1') define h uniquely on a neighborhood U_1 of C_1. Extend h to $X_{-1}\,(U_1) = U_0$ and to $U_n = X_n\,(U_1) \cap \Sigma_1$ by conjugation.

We want to prove that $h(y_{1,n}, y_2)$ converges (to a logarithmically-linear appliccation). Therefore observe the second coordinate:

$$\Pi_2'(h(y_{1,n}, y_2)) = \beta_2'^{n} \psi_2'(h(\psi_2^{-1}\,(\frac{y_{1,n}}{\beta_1^n}\,,\,\frac{y_2}{\beta_2^n})))$$

A little calculation concludes that

$h(\rho, \theta) = (\rho^{\beta'}\, e^{\alpha_1'/\alpha_2' [\theta' - \theta]}, \theta')$ along C_1. As $\rho_n = \dfrac{y_2}{\beta_2^n \frac{\partial \psi_2}{\partial x_2}(0,0)} + 0\,(\frac{y_2}{\beta_2^n})$

We conclude

$$\Pi_2(h(y_{1,n}, y_2) \sim \frac{y_2^{\beta'} e^{\frac{\alpha_1}{\alpha_2'}[\theta_n' - \theta_n]} \frac{\partial\psi_2}{\partial x_2'}(0,0)}{[\frac{\partial\psi_2}{\partial x_2}(0,0)]^{\beta'}}$$

Hence $\lim_{n\to+\infty} \Pi_2'(h(y_{1,n}, y_2)) = A\, y_2^{\beta'}$. where $\beta' = \ln \beta_2'/\ln\beta_2$ and

$$A = e^{\frac{\alpha_1'}{\alpha_2'}[\theta'(0) - \theta(0)]} \frac{\partial\psi_2'}{\partial x_2'}(0,0)[\frac{\partial\psi_2}{\partial x_2}(0,0)]^{-\beta'}$$

, as we wished to prove.

Now we are concerned with a modification of Π_2' "between" U_1' and U_2'. Applying the same techniques of [2] and the last conclusion, we get a new Π_2' where each fiber is piecewise linear and

$$\Pi_2'(h(\Pi_2^{-1}(0, y_2))) = A\, y_2^{\beta'}$$

, i e, Π_2' is compatible with h and Π_2.

Let us return to L. Extend h to $\{\rho \leq 1, 0 \leq z \leq 1\}$ by arc length between U_0 and C, which is continuous by our previous constructions.

At this point we remember the techniques in [20], and the differentiability of h and h^{-1} on $C - W^s(P_1)$ and on $(W^u(\sigma_1) - \sigma_1) \cap \Sigma_1$ to end our proof

□

Now we prove

Theorem B: If $X \in A_2$ and every chain for (p_1, σ_1) has at most 4 elements, then it has finite modulus of equivalence.

Proof: There is a neighborhood U of X such that

a) X is Ω-equivalent to every $Y \in U$;

b) all critical elements $\sigma_1(Y)$ meets $W^u(\sigma_j(X)$ transversally then so do

$$W^s(\sigma_i(Y)) \quad \text{and} \quad W^u(\sigma_j(Y)).$$

Take a family $F(\lambda) = X + Y_\lambda$ as in proposition 8. Then for V open $\subset U$ small enough every $Y \in V$ either is equivalent to $F(\lambda_0)$ for some λ_0 (on account of the last theorem) or is Morse-Smale.

Suppose now Y is Morse-Smale. Then there are at most 2 possible equivalence classes in a neighborhood of X, as it is easily seen from the proof in [20].

Hence the equivalence classes in V are described by $F(\lambda)$ and two more vectorfields. $\square$

§ 5 - Sufficient conditions for finite modulus. Fields without 6-chains.

We proved Theorem B for fields with chains of length 4. In case X exhibits a 5-chain for (p_1,σ_1) the proof is more complicated. First, there shall appear new moduli. Second, the fibrations which we constructed along that proof shall be different. But except for these two modifications, the method shall be essentially the same.

We recall the invariants $\mu(i,j)$ and $\lambda(i,j)$ from corollaries 5 and 6. To be consistent with the notation for the invariants we introduce here, denote $\mu_1(i,j) = \mu(i,j)$, $\lambda_1(i,j) = \lambda(i,j)$

As the maximum length for chains for (p_1,σ_1) is five, there are two possibilities:

a) $\alpha(W^s(p_1) - p_1)$ is a source or else

b) $\omega(W^s(\sigma_1) - \sigma_1)$ is a sink or two.

Call $\sigma_i (2 \leq i \leq \ell_1)$ the saddle type singularities and $\sigma_i (\ell_i+1 \leq i \leq \ell_2)$ the saddle type periodic orbits such that $W^u(\sigma_1) \cap W^s(\sigma_i) \neq \emptyset$ (case a) or $W^s(p_1) \cap W^u(\sigma_i) \neq \emptyset$ (case b). In either case let r_i be the number of orbits in these intersections.

The new moduli $\mu_2(i,j)(\ell_1+1 \leq i \leq \ell_2, 1 \leq j \leq r_i)$ arise from the obstruction to the extension of h to a neighborhood of $\sigma_i(\ell_1+1 \leq i \leq \ell_2)$ that is, the saddle type periodic orbits. The reason is the differentiability of $h/W^u(\sigma_1) \cap \Sigma_1 - \sigma_1$ and $h/D^s(p_1)$ for $\mu \notin Q$.

From now on, we state and prove the results for case b), that is $\omega(W^u(\sigma_1) - \sigma_1)$ is a sink or two. For the other case the results are analogous and the proof of the theorem is slightly different.

In the first place a Proposition about necessity of the invariants $\mu_2(i,j)$,

Proposition: Suppose $X, X' \in A_1$. Let σ_i be a saddle type periodic orbit such that $W^s(p_1) \cap W^u(\sigma_i) \neq \emptyset$ and $\{w_1, \ldots, w_{r_i}\} = D^s(p_1) \cap W^u(\sigma_i)$. If h is an equivalence between X and X' then

a) $\mu_2(i,1) = \mu_2'(i,1)$ (the eigenvalues associated to $W^s(\sigma_i)$ and $W^s(\sigma_i'))$

b) $\mu_2(i,j) = T_2(\sigma_1, \sigma_i, w_1, w_j) = T_2(\sigma_1', \sigma_i', w_1', w_j') = \mu_2'(i,j)$ $(2 \leq j \leq r_i)$

(the "normal derivatives" associated to each new orbit in the intersection).

Proof: use the methods in §2,3 and [8]. □

Now we can state

Theorem 11: Let $X, X' \in A_2$ and be $\varepsilon - C^\infty$ near. Suppose:

a) $\mu = \mu'$

b) $\mu_1(i,j) = \mu_1'(i,j)$ $\quad 1 \leq i \leq \ell_2$, $1 \leq j \leq r_i$

c) $\mu_2(i,j) = \mu_2'(i,j)$ $\quad \ell_1 + 1 \leq i \leq \ell_2$, $1 \leq j \leq r_i$

d) their phase diagrams are isomorphic

Then they are topologically equivalent.

Proof: As in Theorem 10 we shall begin by constructing a semilocal equivalence h in a neighborhood of $W^u(p_1)$. Then we shall extend h to all of M^3 using the methods in [20].

We reparameterize C^2 all periodic orbits to period 1. Consider all singularities and periodic orbits C^2 linearized in a neighborhood. For the singularities σ_i we define fences Σ_i, transversal to the corresponding 2-invariant submanifold.

Again we reparameterize C^2 in order to get: If

$$x \in D_i^u \cap W^s(\sigma_j) - W^u(p_1)$$

there exists a neighborhood V of x, $V \subset \Sigma_i$ such that $X_1(V) \subset \Sigma_j$ for some fundamental domain D_i^u in $W^u(\sigma_i)$.

If $x \in D_i^s \cap W^u(\sigma_j) - W^u(p_1)$ there exists a neighborhood V of X, $V \subset \Sigma_i$ such that $X_{-1}(V) \subset \Sigma_j$ for some fundamental domain D_i^s in $W^s(\sigma_i)$.

Define trivial fibrations

$$\pi_i^u \colon V_i^u \to W^s(p_i)(\ell_1 \geqq i \geqq 2)$$

where V_i^u is a neighborhood of $\sigma_i(\ell_1 \geqq i \geqq 2)$ or V_i^u is a neighborhood of $\sigma_i \cap \Sigma_i$ in Σ_i $(\ell_2 \geqq i \geqq \ell_1 + 1)$. Then it is possible to define $\pi_0^u \colon F \to W^s(p_1) \in C^2$ compatible with all π_i^u $(\ell_2 \geqq i \geqq 2)$.

Again we shall prove that there is a unique curve of tangencies C_1 between the spirals defined by the saturation of $(\pi_0^u)^{-1}(x)$ and the trivial fibration $\pi_2 \colon V^s \cap \Sigma_1 \to W^u(\sigma_1) \cap \Sigma_1$. Using the same notation of theorem 10, the expression of π_1^u in the cilyndrical coordinates is

$$(\pi_1^u)^{-1}(x) = \{(1, \beta(s), s)\}$$

because $W^u(\sigma_i)$ intersects $W^s(p_1)$ transversally.

We examine $g(s) = \phi_2(s^{-\alpha_1/\alpha_3}, \beta(s) - \frac{\alpha_2}{\alpha_3} \ln s, 1)$ that is, the image of a spiral in Λ by $\phi = (\phi_1,\phi_2): \Lambda \to \Sigma_1$ the corresponding Poincaré transform.

The points of tangency satisfy the equation

$$(11.1) \qquad 0 = \frac{dg}{ds} = \frac{s^{\frac{-\alpha_1}{\alpha_3} - 1}}{\alpha_3} \left(r_1(s) \frac{\partial\phi_2}{\partial x_1} + r_2(s) \frac{\partial\phi_2}{\partial x_2}\right)$$

where $r_1(s) = \alpha_1 \cos s - \alpha_2 \operatorname{sen} s + \alpha_3 s \operatorname{sen} s \frac{d\beta}{ds}$.

$r_2(s) = \alpha_1 \operatorname{sen} s + \alpha_2 \cos s - \alpha_3 s \cos s \frac{d\beta}{ds}$.

For small ρ and hence small s we have

$$\frac{\partial\phi_2}{\partial x_1} \sim 0 \qquad \text{and} \qquad \frac{\partial\phi_2}{\partial x_2} \neq 0.$$

This means $r_2 \sim 0$ if $\frac{dg}{ds} = 0$. Consequently $\operatorname{tg} s \sim \frac{-\alpha_2}{\alpha_1}$ and $r_2(s) \sim -(\alpha_1^2 + \alpha_2^2) \cos s$ for points satisfying (11.1). Let us calculate $\frac{d^2g}{ds^2}$ on such points:

$$(11.2) \quad \frac{d^2g}{ds^2} = \frac{s^{\frac{-\alpha_1}{\alpha_3} - 2}}{\alpha_3^2} \left\{(\alpha_1 + \alpha_3)\left(r_1 \frac{\partial\phi_2}{\partial x_1} + r_2 \frac{\partial\phi_2}{\partial x_2}\right) - \alpha_2 r_2 \frac{\partial\phi_2}{\partial x_1} + \alpha_2 r_1 \frac{\partial\phi_2}{\partial x_2} + 0(s^{\kappa})\right\}$$

for $\kappa > 0$.

For points satisfying (11.1) the signal of $\frac{d^2g}{ds^2}$ depends only on $r_1\frac{\partial\phi_2}{\partial x_2}$, which is preserved.

Then all critical points of $g(s)$ are extrema of the same type, yielding unicity of the curve. The differentiability on $W^u(p_1)\cap\Sigma_1$ follows as in theorem 10.

The following step s consists on the semilocal definition of the homeomorphism. Assume we have performed the same constructions for X'.

Carry out the beginning of the definition by steps:

a) Define $h/W^s(p_1)\cap F$ as the unique possible rotation on account of prop. 4 and hypothesis b).

b) This definition induces a correspondence between fibers in F given by $h((\pi_0^u)^{-1}(x)) = (\pi_0'^u)^{-1}(h(x))$ $\forall\, x\in W^s(p_1)\cap F$ and consequently a correspondence between spirals in neighborhoods of C_1 and C_1'.

c) Now we require $h(C_1)\subset C_1'$ which completely defines h along C_1 on account of b).

d) The observation before induces a unique correspondence between the fibers of π_2 and π_2'. Namely, if $(y_1,y_2)\in C_1$ then

$$h(\pi_2^{-1}(0,y_2)) = (\pi_2')^{-1}(\pi_2'h(y_1,y_2)).$$

e) The two last considerations yield a definition of h on a neighborhood $U_1\subset\Sigma_1$ of C_1.

f) Extend h/U_1 to $U_0 = X_{-1}(U_1)\subset\Lambda$ and to $U_n = X_n(U_0)$ by conjugation.

g) Extend h/U_0 to the solid cilynder whose border is F, by arc length.

In order to extend the definition of h to all of Σ_1 it is necessary to change the fibers of π_2' as in theorem 10. They will be piecewise linear, and horizontal in neighborhoods of $X_n(C_j)$. We shall be allowed to carry out such a modification verifying that the applications $h_n = h/U_n$ converge to a (logarithmically linear) function defined along $W^u(\sigma_1) \cap \Sigma_1$. This will happen because π_0^u behaves as a trivial fibration; so h/F will define the germ of a log linear application on $W^u(p_1) \cap C_1$. Then h/C_1, when iterated by conjugation shall converge to the desired function.

Indeed the fibers of π_0^u are given by $(1,\eta(s,t_0))$ for $\eta\colon [-1,1] \times [0,2\pi] \to \mathbb{R} \in C^2$ where $t_0 = \eta(0,t_0)$ is the angle corresponding to the fiber on $W^s(p_1)$. The associated spirals of Λ satisfy the equation:

$$\ln \rho - \frac{\alpha_1}{\alpha_2}\,\theta = -\frac{\alpha_1}{\alpha_2}\,\eta(s,t_0).$$

Suppose $C_0 = X_{-1}(C_1)$ is given by $(\rho,\theta(\rho),1)$ where $\theta(\rho) \underset{\rho\to 0}{\to} \operatorname{arctg}(-\alpha_2/\alpha_1)$.

Compare the intersection of a spiral with C_0 and its image:

$$\frac{\alpha_2}{\alpha_1}\ln \rho - \frac{\alpha_2'}{\alpha_1'}\ln \rho' = \eta(s,t) - \eta'(s',t') + \theta(\rho) - \theta'(\rho') \tag{11.4}$$

observe that: a) $\theta(\rho) - \theta'(\rho') \to \operatorname{arctg} -\alpha_2/\alpha_1 - \operatorname{arctg} - \alpha_2'/\alpha_1'$

b) uniformly $\eta(s,t) - \eta'(s',t') \underset{s,s'\to 0}{\to} \xi_0$ the angle of the rotation given by $h/F \cap W^s(p_1)$.

Hence, for ρ small enough,

$$\rho'^{\alpha_2/\alpha_1} \sim K\rho^{\alpha_2'/\alpha_1'} .$$

Now take a sequence

$$h_n(y_2) = \pi_2'(h(y_{1,n},y_2)) \text{ where } (y_{1,n},y_2) \in U_n.$$

The notation is legitimate because each fiber is horizontal on U_n i.e., examine only the case in which $(y_{1,n},y_2) \in X_n(C_0)$. To end up the proof of the convergence of h_n proceed like in theorem 10.

Now we are able to see which are the classes of equivalence in a neighborhood of $X \in A_2$ where $\omega(W^u(\sigma_1(X)) - \sigma_1(X))$ has only sinks, and how we can parameterize them.

For the sake of simplicity we only examine the case for which $W^u(\sigma_i) \cap W^s(p_1)$ has exactly one orbit for all $\ell_2 \geq i \geq 2$.

Theorem B. If $X \in A_2$ (no 6-chains) then it has finite modulus of equivalence.

Proof: Take a neighborhood U of X where it is Ω-stable.

Using the same type of argument as in Prop 8 we define families of fields G, $F_{i,j}(\lambda)$, $H_{i,j}$, $M^1_{i,j}$, $M^2_{i,j}$ in a neigborhood of $X \in X(M)$.

I) $G \subset A_1$ is the family described by the parameters of theorem 11.

II) $F_{i,J}(\lambda)$ $(\ell_2 \geq i \geq \ell_1+1\ ,\quad J \in \mathbb{N}^{\ell_2-1})$ verifies

a) $\sigma_i(F_{i,J}(\lambda))$ is the periodic orbit corresponding to $\sigma_i(X)$.

b) There exist exactly j_p orbits of transversal intersection between $W^u(\sigma_p(F_{i,J}))$ and $W^s(\sigma_1(F_{i,J}))$ $(\ell_2 \geq p \geq 2)$.

c) There exists only one non-transversal orbit in $W^u(\sigma_i(F_{i,J})) \cap W^s(\sigma_1(F_{i,J})$ and it is quasi transversal

d) $\dfrac{\ln \beta_1(\sigma_i(F_{i,J}(\lambda)))}{\ln \beta_2(\sigma_1(F_{i,J}(\lambda)))} = \lambda$ where $\beta_1(\sigma_i)$ is the stable eigenvalue associated to σ_i.

III) $H_{i,J}(\ell_1 \geq i \geq 2\ ,\quad J \in \mathbb{N}^{\ell_2-1}$

a) There exist exactly j_p orbits of transversal intersection in

$W^u(\sigma_p(F_{i,J})) \cap W^s(\sigma_1(F_{i,J}))$ $(\ell_2 \geq p \geq 2)$.

b) There exists only one orbit of non transversal intersection in

$W^u(\sigma_i(F_{i,J})) \cap W^s(\sigma_1(F_{i,J}))$ and it is quasi transversal.

c) $\sigma_i(H_{i,J})$ is the singularity associated to $\sigma_i(X)$.

Observe that in cases II) and III) $|j_p - j_1| \leq 1$.

IV) $M^1_{i,J}$ is a family of only two non equivalent Morse-Smale vectorfields near $F_{i;J}(\lambda)$. They are non equivalent because the quasi-transversal intersection of $F_{i,J}(\lambda)$ was turned transversal either by avoiding $W^s(\sigma_i)$ or by intersecting it twice.

V) $M^2_{i,J}$ is also a family of only two Morse-Smale vectorfields, near $H_{i,J}$. They are non equivalent for the same reason as IV.

We claim that fields Y which are ε-C^∞ (actually ε-C^3) near X have at most one tangency and it is quasi-transversal.

In case $W^u(p_1(X')) \subset W^s(\sigma_1(X'))$ this is obvious. For the other case, express $W^s_{loc}(\sigma_1(X'))$ as the graph of a function F_1 of $W^s_{loc}(\sigma_1(X))$, $\| F_1 \|_r \leq \varepsilon$ and a sector of $W^u(\sigma_i)$ $(\ell_2 \geq i \geq 2)$ as the graph of a function F_2 of $W^s_{loc}(\sigma_1(X))$. The points of tangency must verify

$$\begin{cases} F_1(x) = F_2(x) \\ F_1'(x) = F_2'(x) \end{cases}$$

As in theorem 11, $(F_1 - F_2)''$ preserves its signal along these points and the same argument holds.

We use [8], [24], theorem 11, and the proof of [20] to prove that these families really exhaust the classes of equivalence around X.

Remark on fields with 6-chains

As we proved before, in order to classify the classes of equivalence around fields that exhibit a chain $(\sigma_3, p_1, \sigma_1, \sigma_2)$ where σ_2 and σ_3 are saddle type periodic orbits we shall need infinitely many real parameters.

(Using the same proof we see that any of these fields has infinite moduli of conjugation, σ_2 and σ_3 being periodic orbits or not)

For fields such that either $\alpha(W^s(p_1))$ or $\omega(W^u(\sigma_1))$ contain no saddle type periodic orbit there is at least one more parameter which we must take into account. In any case we stress that one more invariant must be taken into account. Namely, if $h/W^u(\sigma_1)\cap\Sigma_1$ is a fixed logarithmically linear application it shall induce a unique rotation on $h/D^s(p_1)$ whose angle is a new invariant. The proof is similar to the one concerning $\mu_1(i,j)$ or $\lambda_1(i,j)$ in §3.

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