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Proofs of conjectures about
singular riemannian foliations

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PROOFS OF CONJECTURES ABOUT SINGULAR RIEMANNIAN FOLIATIONS

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ABSTRACT. A singular foliation on a complete riemannian manifold M is said to be riemannian if each geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets. We prove that if the distribution of normal spaces to the regular leaves is integrable, then each leaf of this normal distribution can be extended to be a complete immersed totally geodesic submanifold (called section), which meets every leaf orthogonally. In addition the set of regular points is open and dense in each section. This result generalizes a result of Boualem and solves a problem inspired by a remark of Palais and Terng about polar actions.

We also study the singular holonomy of a singular riemannian foliation with sections (s.r.f.s. for short) and in particular the tranverse orbit of the closure of each leaf. Furthermore we prove that the closure of the leaves of a s.r.f.s on M form a partition of M which is a singular riemannian foliation. This result proves partially a conjecture of Molino.

1. INTRODUCTION

In this section we will recall some definitions and state our main results as Theorem 1.4 and Theorem 1.7.

We start by recalling the definition of a singular riemannian foliation (see the book of P. Molino [9]).

Definition 1.1. A partition \mathcal{F} of a complete riemannian manifold M by connected immersed submanifolds (the *leaves*) is called a *singular foliation* of M if it verifies condition (1) and *singular riemannian foliation* if it verifies condition (1) and (2):

- (1) \mathcal{F} is *singular*, i.e., the module $\mathcal{X}_{\mathcal{F}}$ of smooth vector fields on M that are tangent at each point to the corresponding leaf acts transitively on each leaf. In other words, for each leaf L and each $v \in TL$ with foot point p , there is $X \in \mathcal{X}_{\mathcal{F}}$ with $X(p) = v$.
- (2) The partition is *transnormal*, i.e., every geodesic that is perpendicular at one point to a leaf remains perpendicular to every leaf it meets.

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Let \mathcal{F} be a singular riemannian foliation on a complete riemannian manifold M . A leaf L of \mathcal{F} (and each point in L) is called *regular* if the dimension of L is maximal, otherwise L is called *singular*. Let L be an immersed submanifold of a riemannian manifold M . A section ξ of the normal bundle νL is said to be a *parallel normal field* along L if $\nabla^\nu \xi \equiv 0$, where ∇^ν is the normal connection. L is said to have a *globally flat normal bundle*, if the holonomy of the normal bundle νL is trivial, i.e., if any normal vector can be extended to a globally defined parallel normal field.

Now we recall the concept of singular riemannian foliation with section, which was introduced in [1], [2] and continued to be studied by D. Töben and I in [16], [3], [4] and recently by A. Lytchak and G. Thorbergsson in [8].

Definition 1.2 (s.r.f.s.). Let \mathcal{F} be a singular riemannian foliation on a complete riemannian manifold M . \mathcal{F} is said to be a *singular riemannian foliation with sections* (s.r.f.s. for short) if for each regular point p , the set $\Sigma := \exp_p(\nu_p L_p)$ is a complete immersed submanifold that meets each leaf orthogonally. Σ is called a *section*.

Remark 1.3. In [2] it was also assumed that the set of regular points is open and dense in Σ . As we will see in Theorem 1.4 this is a consequence of the definition above. We will explain in Remark 1.6 the importance of the density of the set of regular points in a section.

Let $p \in M$ and $\text{Tub}(P_p)$ be a tubular neighborhood of a plaque P_p that contains p . Then the connected component of $\exp_p(\nu P_p) \cap \text{Tub}(P_p)$ that contains p is called a *slice* at p . Let S_p denote it. Now consider a section Σ , a point $p \in \Sigma$ and a convex normal neighborhood B of p in M . Then a connected component of $B \cap \Sigma$ that contains p is called *local section* (centered at p). These two concepts are related to each other. Indeed, the slice at a singular point is the union of the local sections that contain this singular point (see Theorem 2.4).

A typical example of a s.r.f.s is the partition formed by the orbits of a polar action.

An isometric action of a compact Lie group G on a complete riemannian manifold M is called *polar* if there exists a complete immersed submanifold Σ of M that meets all G -orbits orthogonally and whose dimension is equal to the codimension of the regular G -orbits. We call Σ a *section*. A typical example of a polar action is a compact Lie group with a biinvariant metric that acts on itself by conjugation. In this case the maximal tori are the sections. One can find a large number of examples of polar action on symmetric spaces in the work of Kollross [7] and Podestà and Thorbergsson [11].

In [10][page 96, Remark 5.6.8] Palais and Terng proposed a conjecture that can be formulated as follows: *Let G be an isometric action of a compact Lie group on M such that the distribution of the normal space to the regular orbit is integrable. Then there exists a complete totally geodesic immersed section for the action of G which meets all orbits perpendicularly.*

In [6] Heintze, Olmos and Liu proved that Palais and Terng were right in their conjecture. In particular they proved that the set of regular points is dense in each section (see Tebege [12] for an alternative proof of the density of regular points in a section).

In [5] Boualem proved the following result: *Let \mathcal{F} be a singular riemannian foliation such that the closure of each leaf is compact and the distribution of normal spaces to the regular leaves is integrable. Then each leaf of this normal distribution can be extended to be an immersed totally geodesic submanifold (called generalized section), which meets every leaf orthogonally.*

He did not prove that the so called generalized section is complete and hence could not give a complete answer to the conjecture of Palais and Terng for the case of singular riemannian foliation. Also the question of the density of the set of regular points in a section was not considered.

In this paper we prove the following result, which gives a complete answer to the conjecture of Palais and Terng for the case of singular riemannian foliations.

Theorem 1.4. *Let \mathcal{F} be a singular riemannian foliation on a complete manifold M . Suppose that the distribution of normal spaces of the regular leaves is integrable. Then \mathcal{F} is s.r.f.s. In addition the set of regular points is open and dense in each section.*

We note that Lytchak and Thorbergsson [8] have recently proved this result under the additional condition that singular riemannian foliation has no horizontal conjugate point.

Before we state the other main result of this paper, we must briefly recall the notion of isoparametric and equifocal submanifolds. These concepts are fundamental to understand local and global aspects of the theory of s.r.f.s.

A submanifold of an euclidean space is called *isoparametric* if its normal bundle is flat and if the principal curvatures along any parallel normal vector field are constant. The singular partition formed by the parallel submanifolds of an isoparametric submanifold in an euclidean space is a typical example of s.r.f.s. The history of isoparametric submanifolds and their generalizations can be found in the survey [14] of G. Thorbergsson (see also [15]).

According to the slice theorem (see Theorem 2.4) the restriction of a s.r.f.s to a slice at a singular point is a s.r.f.s which is diffeomorphic to an isoparametric foliation on an open set of an euclidean space. Hence the slice theorem give us a description of the plaques of a s.r.f.s. on a riemannian manifold. However, it does not guarantee that different plaques belong to different leaves. To obtain this kind of information, we need the concept of singular holonomy which is related to the concept of local equifocal submanifold reviewed below.

In [13] C.L. Terng and G. Thorbergsson introduced the concept of equifocal submanifold with flat sections in symmetric space in order to generalize

the definition of isoparametric submanifold in Euclidean space. Now we review the definition of equifocal submanifolds in a riemannian manifold.

Definition 1.5. A connected immersed submanifold L of a complete Riemannian manifold M is called *equifocal* if

- 0) the normal bundle $\nu(L)$ is globally flat,
- 1) L has sections, i.e., for each $p \in L$ the set $\Sigma := \exp_p(\nu_p L_p)$ is a complete, immersed, totally geodesic submanifold.
- 2) For each parallel normal field ξ along L , the derivative of the map $\eta_\xi : L \rightarrow M$, defined as $\eta_\xi(x) := \exp_x(\xi)$, has constant rank.

A connected immersed submanifold L is called *locally equifocal* if, for each $p \in L$, there exists a neighborhood $U \subset L$ of p in L such that U is an equifocal submanifold.

Remark 1.6 (Relation between s.r.f.s. and equifocal submanifolds). In [13] Terng and Thorbergsson proved that the singular partition formed by the parallel submanifolds of an equifocal submanifold L on M is a s.r.f.s, when M is a symmetric space of compact type and the sections are flat. In [2] I proved that the leaves of a s.r.f.s. on a riemannian manifold M are locally equifocal (see Theorem 2.1). In this proof the property that the set of regular points is dense in a section plays an important role. In [16] Töben used the blow up technique to study local equifocal submanifolds (which he called submanifold with parallel focal structure). He gave a necessary and sufficient condition for a closed embedded local equifocal submanifold to induce a s.r.f.s. Töben's Theorem (for the case of equifocal submanifold) follows from the next result, which is proved in [3]. *Let L be a closed embedded equifocal submanifold. Suppose that set of regular points of each section Σ is an open and dense set in Σ . Define Ξ as the set of all parallel normal fields along L . Then $\mathcal{F} := \{\eta_\xi(L)\}_{\xi \in \Xi}$ is a s.r.f.s.* Here a point of a section is called regular if there exists only one local section that contains p , i.e., if given two local sections σ and $\tilde{\sigma}$ that contain p , they have the same germ at p .

The property that each leaf of a s.r.f.s is locally equifocal allow us to extend the holonomy map to the whole local section and hence to define the notion of singular holonomy map (see Proposition 2.3). If we fix a local section σ , we can define a Weyl pseudogroup of σ as the pseudogroup generated by the singular holonomy maps defined on σ (see Definition 2.3). The Weyl pseudogroup of a local section centered at q and the s.r.f.s on a slice S_q describe completely the s.r.f.s. on a neighborhood of q .

We are finally ready to recall the conjecture that motivate our next result.

In [9] Molino proved that, if M is compact, the closure of the leaves of a (regular) riemannian foliation form a partition of M which is a singular riemannian foliation. He also proved that the leaf closure are orbits of a locally constant sheaf of germs of (transversal) Killing fields.

If the foliation is singular riemannian foliation and M is compact, then Molino was able to prove (see [9] Theorem 6.2 page 214) that the closure of the leaves should be a transnormal system, but as he remarked, it remains to prove that the closure of the leaves is in fact a singular foliation. In this paper we prove the Molino's conjecture, when \mathcal{F} is a s.r.f.s. In addition we study the singular holonomy of \mathcal{F} and in particular the tranverse orbits of the closure of a leaf. We will not assume that M is compact.

Theorem 1.7. *Let \mathcal{F} be a s.r.f.s. on a complete riemannian manifold M . Then*

- a) *the closure of the leaves of \mathcal{F} form a partition of M which is a singular riemannian foliation, i.e. $\{\bar{L}\}_{L \in \mathcal{F}}$ is a singular riemannian foliation.*
- b) *Each point q is contained in an homogenous submanifold \mathcal{O}_q (possible with dimension 0). If we fix a local section σ that contains q , then \mathcal{O}_q is a connected component of an orbit of the closure of the Weyl pseudogroup of σ .*
- c) *If q is a point of the submanifold \bar{L} , then a neighborhood of q in \bar{L} is the product of the homogenous submanifold \mathcal{O}_q with plaques with the same dimension of the plaque P_q .*
- d) *Let q be a singular point and T the intersection of the slice S_q with the singular stratum that contains q . Then the normal connection of T in S_q is flat.*
- e) *Let q be a singular point and T defined as in Item d). Let v be a parallel normal vector field along T , $x \in T$ and $y = \exp_x(v)$. Then $\mathcal{O}_y = \eta_v(\mathcal{O}_x)$.*

One can construct examples that illustrate the above theorem by means of suspension of homomorphisms (see Appendix for details). In fact, the suspension technique is very useful to construct examples of s.r.f.s. with nonembedded leaves, with exceptional leaves and also inhomogeneous examples. Other techniques to construct examples of s.r.f.s on nonsymmetric spaces are suitable changes of metric and surgery (see [4] for details).

Corollary 1.8. *Let \mathcal{F} be a s.r.f.s. on a complete manifold M and q a singular point. Let T denote the intersection of the slice S_q with the stratum that contains q . Suppose that $T = \{q\}$. Then all the leaves of \mathcal{F} are closed.*

Remark 1.9. According to the slice theorem (see Theorem 2.4) the restriction of the foliation \mathcal{F} to the slice S_q is diffeomorphic to an isoparametric foliation $\tilde{\mathcal{F}}$ on an open set of an euclidean space. Therefore the condition that T is a point is equivalent to saying that a regular leaf of $\tilde{\mathcal{F}}$ is a full isoparametric submanifold.

Remark 1.10. In the Appendix we construct an example of s.r.f.s. such that, for a local section σ , the isometric action of \overline{W}_σ (the closure of the Weyl pseudogroup of σ) is not polar. In particular, this implies that there exists

a s.r.f.s \mathcal{F} such that the partition formed by the closure of the leaves of \mathcal{F} is not a s.r.f.s. Although the isometric action of \overline{W}_σ need not to be polar in a neighborhood of a singular point q , if σ is flat, there always exists a totally geodesic submanifold N_q that is orthogonal to the orbits of \overline{W}_σ . This submanifold N_q is a slice of T in σ (see Equation 4.3). It fails to be a section of \overline{W}_σ , if its dimension is lower than the codimension of the regular orbits of \overline{W}_σ . To prove that N_q is orthogonal to the orbits of \overline{W}_σ which it meets, one can use the above theorem and a result of Heintze, Olmos and Liu[6].

This paper is organized as follows. In Section 2 we review some facts about s.r.f.s. and fix the notation. In Section 3 and 4 we prove Theorem 1.4 and Theorem 1.7 respectively. In Section 5 we prove Corollary 1.8. In Section 6, the Appendix, we recall the suspension technique and present an example which illustrates Theorem 1.7 and Remark 1.10.

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2. FACTS ABOUT S.R.F.S.

In this section we recall some results about s.r.f.s. that will be used in this work. Details can be found in [2]. Throughout this section we assume that \mathcal{F} is a s.r.f.s. on a complete riemannian manifold M .

Let us start with a result that relates s.r.f.s. to equifocal submanifolds (see Definition 1.5 to recall the definitions of equifocal submanifolds and the endpoint map η_ξ).

Theorem 2.1. *The regular leaves of \mathcal{F} are equifocal. In particular, the union of regular leaves that have trivial normal holonomy is an open and dense set in M , provided that all the leaves are compact.*

A consequence of the first statement of the theorem is that given a regular leaf L with trivial holonomy then we can reconstruct \mathcal{F} by taking all parallel submanifolds of L . More precisely we have

Corollary 2.2. *Let L be a regular leaf of \mathcal{F} .*

- a) *Let β be a smooth curve of L and ξ a parallel normal field along β . Then the curve $\eta_\xi \circ \beta$ belongs to a leaf of \mathcal{F} .*
- b) *Let L be a regular leaf with trivial holonomy and Ξ denote the set of all parallel normal fields along L . Then $\mathcal{F} = \{\eta_\xi(L)\}_{\xi \in \Xi}$. In particular, if ξ is a parallel normal field along L then the endpoint map $\eta_\xi : L \rightarrow L_q$ is surjective, where $q = \eta_\xi(x)$ for $x \in L$.*

Corollary 2.2 allows us to define singular holonomy map, which will be very useful to study \mathcal{F} .

Proposition 2.3 (Singular Holonomy). *Let L_p be a regular leaf, β a smooth curve in L_p and let $[\beta]$ denote the homotopy class of β . Let U be a local section centered at $p = \beta(0)$. Then there exists a local section V centered at $\beta(1)$ and an isometry $\varphi_{[\beta]} : U \rightarrow V$ that has the following properties:*

- 1) $\varphi|_{\beta}(x) \in L_x$ for each $x \in U$,
- 2) $d\varphi|_{\beta}\xi(0) = \xi(1)$, where ξ is a parallel normal field along β .

An isometry as above is called *singular holonomy map along β* .

We remark that, in the definition of the singular holonomy map, singular points can be contained in the domain U . If the domain U and the range V are sufficiently small, then the singular holonomy map coincides with the holonomy map along β .

Now we recall some results about the local structure of \mathcal{F} , in particular about the structure of the set of singular points in a local section.

Note that, if q is a singular point, the restriction $\mathcal{F}|_{S_q}$ of \mathcal{F} to the slice S_q is also a singular foliation. In fact, since $\mathcal{X}_{\mathcal{F}}$ acts transitively on the leaves, the plaques (the connected components of the leaves intersected with $\text{Tub}(P)$) are transversal to S_q .

The relation between a slice S_q , a local section and $\mathcal{F}|_{S_q}$ are given by the next result.

Theorem 2.4 (Slice Theorem). *Let q be a singular point of M and S_q a slice at q . Then*

- a) Let ϵ be the radius of the slice S_q . Denote $\Lambda(q)$ the set of local sections σ containing q , such that $\text{dist}(p, q) < \epsilon$ for each $p \in \sigma$. Then $S_q = \bigcup_{\sigma \in \Lambda(q)} \sigma$.
- b) $S_x \subset S_q$ for all $x \in S_q$.
- c) $\mathcal{F}|_{S_q}$ is a s.r.f.s. on S_q with the induced metric of M .
- d) $\mathcal{F}|_{S_q}$ is diffeomorphic to an isoparametric foliation on an open set of \mathbb{R}^n , where n is the dimension of S_q .

From d) it is not difficult to derive the following corollary.

Corollary 2.5. *Let σ be a local section. Then the set of singular points of \mathcal{F} contained in σ is a finite union of totally geodesic hypersurfaces. These hypersurfaces are sent by a diffeomorphism to focal hyperplanes contained in a section of an isoparametric foliation on an open set of a euclidean space.*

We will call the set of singular points of \mathcal{F} contained in σ the *singular stratum of the local section σ* . Let M_r denote the set of regular points in M . A *Weyl Chamber* of a local section σ is the closure in σ of a connected component of $M_r \cap \sigma$.

Corollary 2.6. *A Weyl Chamber of a local section is a convex set.*

Corollary 2.7. *Let γ be a geodesic orthogonal to a regular leaf. Then the singular points are isolated on γ .*

Remark 2.8. The above fact can be proved without the Slice Theorem. In fact it was used in [2] to prove the Slice Theorem.

The Slice Theorem establishes a relation between s.r.f.s. and isoparametric foliations. By analogy with the classical theory of isoparametric submanifolds, it is natural to ask if we can define a (generalized) Weyl group action on σ . The next definitions and results answer this question.

Definition 2.9 (Weyl Pseudogroup W). The pseudosubgroup generated by all singular holonomy maps $\varphi_{|\beta|}$ such that $\beta(0)$ and $\beta(1)$ belong to the same local section σ is called *generalized Weyl pseudogroup* of σ . Let W_σ denote this pseudogroup. In a similar way we define W_Σ for a section Σ . Given a slice S we will define W_S as the set of all singular holonomy maps $\varphi_{|\beta|}$ such that β is contained in the slice S .

Remark 2.10. To recall the definition of pseudogroups and orbifolds see E. Salem [9, Appendix D].

Proposition 2.11. *Let σ be a local section. Then the reflections in the hypersurfaces of the singular stratum of the local section σ leave $\mathcal{F}|\sigma$ invariant. Moreover these reflections are elements of W_σ .*

One can construct an example of a s.r.f.s. by suspension such that W_σ is larger than the pseudogroup generated by the reflections in the hypersurfaces of the singular stratum of σ . The next result gives a sufficient condition to guarantee that both pseudogroups coincide.

Proposition 2.12. *Suppose that each leaf of \mathcal{F} is compact and has trivial normal holonomy. Let σ be a local section. Then W_σ is generated by the reflections in the hypersurfaces of the singular stratum of the local section.*

Remark 2.13. In [4] Töben and I proved that if M is simply connected and the leaves of \mathcal{F} are compact, then each regular leaf has trivial holonomy.

Finally let us recall a result related to Theorem 1.7.

Proposition 2.14. *Let σ be a local section. Consider a point $p \in \sigma$, then*

$$\overline{W_\sigma} \cdot p = \overline{L_p \cap \sigma}.$$

In other words, the closure of $L_p \cap \sigma$ is an orbit of complete closed pseudogroup of local isometries. In particular $\overline{L_p \cap \sigma}$ is a closed submanifold.

3. PROOF OF THEOREM 1.4

In this section we will prove Theorem 1.4.

A singular riemannian foliation is called *integrable* if the distribution of the normal spaces of the regular leaves is integrable. A neighborhood of a leaf of this distribution is called *regular section*.

In what follows we will need the following result due to Boualem (see Lemma 1.2.1, Proposition 1.2.3 and Lemma 1.2.4 of [5]).

Proposition 3.1 (Boualem [5]). *Let \mathcal{F} be an integrable singular riemannian foliation on a complete riemannian manifold M . Let S_q denote the slice at q of radius ϵ . Let σ be a regular section contained in the normal ball $B_\epsilon(q)$. Suppose also that σ intersects the slice S_q . Then*

- a) *There exists a subspace $V \subset T_q S_q$ such that $\sigma \subset D_\epsilon(q) := \exp_q(V \cap U)$, where U is the ball of radius ϵ in $T_q S_q$ centered at 0.*

- b) the map \exp_q^{-1} is a diffeomorphism between the foliation $\mathcal{F}|_{S_q}$ and an integrable singular riemannian foliation $\tilde{\mathcal{F}}$ on the open set U of the inner product space $(T_q S_q, \langle, \rangle_q)$, where \langle, \rangle_q denote the metric of $T_q M$. Furthermore, the submanifold $\tilde{\sigma} := \exp_q^{-1}(\sigma)$ is a regular section of $\tilde{\mathcal{F}}$.

Remark 3.2. Let \langle, \rangle_0 denote the canonical euclidean product. Then there exists a positive definite symmetric matrix A such that $\langle X, Y \rangle_q = \langle AX, Y \rangle_0$. The isometry $\sqrt{A} : (T_q S_q, \langle, \rangle_q) \rightarrow (\mathbb{R}^n, \langle, \rangle_0)$ is a diffeomorphism between the foliation $\tilde{\mathcal{F}}$ and an integrable singular riemannian foliation $\hat{\mathcal{F}}$ on an open set of the inner product space $(\mathbb{R}^n, \langle, \rangle_0)$.

Proposition 3.3. *Let \mathcal{F} be an integrable singular riemannian foliation on a complete riemannian manifold M and γ be a geodesic orthogonal to a regular leaf. Then the set of singular points is isolated on γ .*

Proof. Since the set of regular points on γ is open, we can suppose that $q = \gamma(0)$ is a singular point and that $\gamma(t)$ is a regular point for $-\delta < t < 0$. We will show that there exists $\epsilon > 0$ so that the point $\gamma(t)$ is a regular point if $0 < t < \epsilon$.

It follows from Proposition 3.1 that there exist an open set $U \subset T_q S_q$ and an integrable singular riemannian foliation $\tilde{\mathcal{F}}$ on U such that $\exp_q^{-1}(\mathcal{F}) = \tilde{\mathcal{F}}$. Set $\tilde{\gamma} := \exp_q^{-1}(\gamma)$.

Note that, for $t_0 < 0$, the point $0 = \tilde{\gamma}(0)$ is a focal point of the regular leaf $\tilde{L}_{\tilde{\gamma}(t_0)}$. Indeed, since 0 is a leaf of $\tilde{\mathcal{F}}$ with dimension zero and $\tilde{\mathcal{F}}$ is a singular riemannian foliation, each geodesic that joins 0 to $\tilde{L}_{\tilde{\gamma}(t_0)}$ must be orthogonal to $\tilde{L}_{\tilde{\gamma}(t_0)}$.

Since focal points are isolated along $\tilde{\gamma}$, we can choose $\epsilon > 0$, such that $\tilde{\gamma}(t)$ is not a focal point of the plaque $\tilde{P}_{\tilde{\gamma}(t_0)}$ along $\tilde{\gamma}$ for $0 < t < \epsilon$.

Suppose that there exists $0 < t_1 < \epsilon$ such that $x = \tilde{\gamma}(t_1)$ is a singular point.

Let σ be a regular section of $\tilde{\gamma}(t_0)$. It follows from Proposition 3.1 that σ is contained in a subspace V of dimension k . Note that $x \in V$. Now we choose straight lines $l_1, \dots, l_k : [0, 1] \rightarrow V$ such that

- (1) $l_i(0) \in \sigma$,
- (2) $l_i(1) = x$,
- (3) $\{l'_i(1)\}$ is a basis to $T_x V$.

Since σ is a regular section and $\tilde{\mathcal{F}}$ is a singular riemannian foliation, then the straight lines l_i are orthogonal to the plaque \tilde{P}_x . Therefore we have:

$$\sigma \subset \nu_x := x + \nu_x \tilde{P}_x.$$

Since x is a singular point, it follows that:

$$\dim \sigma < \dim \nu_x.$$

The equations above imply that $\dim \tilde{P}_{\tilde{\gamma}(t_0)} \cap \nu_x > 0$. Thus we can find geodesics in the subspace ν_x that join x to the submanifold $\tilde{P}_{\tilde{\gamma}(t_0)} \cap \nu_x$. Since the foliation is riemannian, these geodesics are also orthogonal to $\tilde{P}_{\tilde{\gamma}(t_0)} \cap \nu_x$. This implies that x is a focal point of the submanifold $\tilde{P}_{\tilde{\gamma}(t_0)}$. This contradicts our choice of ϵ and completes the proof. \square

Lemma 3.4. *Let S_q be the slice at $q \in M$. Let $\tilde{\mathcal{F}}$ be the integrable singular riemannian foliation on an open set U of $T_q S_q$ defined in Proposition 3.1. Consider a subspace V orthogonal to a regular leaf of $\tilde{\mathcal{F}}$. Then*

- a) *The set of regular points of $\tilde{\mathcal{F}}$ is open and dense in $V \cap U$, which is orthogonal to the leaves of $\tilde{\mathcal{F}}$. In particular the foliation $\hat{\mathcal{F}}$ defined in Remark 3.2 is an isoparametric foliation.*
- b) *the set of regular points of \mathcal{F} is open and dense in $D_\epsilon(q) := \exp_q(V \cap U)$.*
- c) *The submanifold $D_\epsilon(q)$ is orthogonal to the leaves of \mathcal{F} .*
- d) *$D_\epsilon(q)$ is a totally geodesic submanifold.*

Proof. a) By assumption there is a regular section $\tilde{\sigma} \subset V$. The fact that the set of regular points of $\tilde{\mathcal{F}}$ is open and dense in $V \cap U$ follows direct from Proposition 3.3. To prove that $V \cap U$ is orthogonal to the plaque \tilde{P}_x at $x \in V \cap U$, define straight lines $l_1, \dots, l_k : [0, 1] \rightarrow V$ such that

- (1) $l_i(0) \in \tilde{\sigma}$,
- (2) $l_i(1) = x$,
- (3) $\{l'_i(1)\}$ is a basis to $T_x V$.

Since $\tilde{\sigma}$ is a regular section and $\tilde{\mathcal{F}}$ is a singular riemannian foliation, the straight lines l_i are orthogonal to the plaque \tilde{P}_x . To conclude the proof of item a) we note that the properties of V imply that $\tilde{\mathcal{F}}$ is a s.r.f.s. Now the slice theorem (Theorem 2.4) implies that $\tilde{\mathcal{F}}$ is an isoparametric foliation.

b) The item b) follows direct from item a).

c) Let $y \in D_\epsilon(q)$ be a regular point and σ_y a regular section that contains y . We want to prove that $\sigma_y \subset D_\epsilon(q)$. Define $\tilde{\sigma}_y := \exp_q^{-1}(\sigma_y)$. It follows from Proposition 3.1 that $\tilde{\sigma}_y$ is a regular section of $\tilde{\mathcal{F}}$. Now item a) implies that $\tilde{\sigma}_y \subset V \cap U$. Therefore $\sigma_y \subset D_\epsilon(q)$, i.e., the regular leaf L_y is orthogonal to $D_\epsilon(q)$ at y . Since regular points are dense in $D_\epsilon(q)$, the singular leaves are also orthogonal to $D_\epsilon(q)$.

d) It follows from item c) and from the fact that regular points are dense in $D_\epsilon(q)$. \square

Proposition 3.5. *Let q be a point of M and $D_\epsilon(q)$ the submanifold defined in Proposition 3.1. Let $B_\delta(q)$ be the normal ball of radius δ in M centered at q . Define the disc $D_\delta(q) := \{\exp_q(v) \text{ such that } v \in T_q D_\epsilon(q) \text{ and } \|v\| < \delta\}$. Then*

- a) *the set of regular points of \mathcal{F} is open and dense in $D_\delta(q)$.*
- b) *The submanifold $D_\delta(q)$ is orthogonal to the leaves of \mathcal{F} .*

c) $D_\delta(q)$ is a totally geodesic submanifold.

Remark 3.6. Note that δ (the radius of the normal neighborhood of q) is often greater than ϵ (the radius of the slice S_q) which depends on the leaf L_q .

Proof. Let $\xi \in T_q D_\epsilon(q)$ be a vector such that the geodesic $\gamma(t) := \exp_q(t\xi)$ has regular points ($0 \leq t \leq 1$). It follows from Proposition 3.3 that the singular points are isolated on γ . Since γ is compact, we can cover this geodesic with a finite number of slices with centers $\gamma(t_i)$ so that $\gamma(t_i)$ is the unique possible singular point in $\gamma \cap S_{\gamma(t_i)}$. Now we can construct a submanifold T by gluing the totally geodesic submanifolds $D_\epsilon(\gamma(t_i))$ defined in Lemma 3.4. Note that T satisfies the conditions of item a) b) and c).

Since \exp_q is a diffeomorphism and T is totally geodesic, we can find a neighborhood W of ξ in $T_q D_\epsilon(q)$ and define a neighborhood of γ in T as $N := \exp_q(W)$. Therefore $N \subset D_\delta(q)$. Hence we have constructed a neighborhood N of γ in $D_\delta(q)$ that satisfies the conditions of item a), b) and c).

To conclude the proof, note that we can choose the vector ξ arbitrarily close to any vector of $T_q D_\epsilon(q)$ for the set of regular points of \mathcal{F} is an open and dense set in $D_\epsilon(q)$. □

Proof of the Theorem

Let $q \in M$ be a point. Define $D_{r_1}(q) := \{\exp_q(v) / v \in T_q D_\epsilon(q) \text{ and } \|v\| \leq r_1\}$, where r_1 is a positive number. Since $D_{r_1}(q)$ is compact, we can find a compact neighborhood of $D_{r_1}(q)$ and a number $\delta > 0$ such that, for each point x in this neighborhood, $B_\delta(x)$ is a normal ball. Let γ be a geodesic with length r_1 such that $\gamma'(0) \in T_p D_\epsilon(q)$.

At first we note that we can extend $D_\epsilon(q)$ along the geodesic γ to a noncomplete submanifold which satisfies the items of Proposition 3.5, gluing totally geodesic discs of radius $\delta/2$. To see this let the first disc be $D_{\delta/2}(q)$. It follows from Proposition 3.5 that $D_{\delta/2}(q)$ satisfies the items a), b) and c). Now we define the second disc as $D_{\delta/2}(\gamma(\delta/4)) := \{\exp_{\gamma(\delta/4)}(v) \text{ for } v \in T_{\gamma(\delta/4)} D_{\delta/2}(q) \text{ and } \|v\| \leq \delta/2\}$. Again, it follows from Proposition 3.5 that $D_{\delta/2}(\gamma(\delta/4))$ satisfies the items a), b) and c). We continue in this way, defining discs D_i 's by induction and extending $D_\epsilon(q)$ along γ .

Let Σ_{r_1} denote the union of the extensions constructed above along every geodesics γ with length r_1 such that $\gamma(0)' \in T_q D_\epsilon(q)$.

Lemma 3.7. Σ_{r_1} can be covered by a finite number of discs D_i 's (of radius δ) which satisfy the items a), b) and c) of Proposition 3.5.

Proof. We start the proof with the following claim.

Claim 3.8. Let $x \in D_{r_1}(q)$. Then $D_{r_1}(q) \cap B_{\delta/4}(x)$ belongs to a finite union of discs D_i 's (of radius $\delta/2$) which satisfy the items a), b) and c) of Proposition 3.5.

To check this claim, let K denote the set $\{v \in T_p D_\epsilon(q) \text{ such that } \exp_p(v) \in \overline{B_{\delta/4}(x)} \text{ and } \|v\| \leq r_1\}$ and choose a vector $v_0 \in K$. As explained above, we can extend $D_\epsilon(q)$ along the geodesic $\exp_p(tv_0)$ to a totally geodesic noncomplete submanifold T . Since \exp is continuous, we can find a neighborhood W of the segment v_0 , such that $\exp_q(tu) \subset T$ for all $u \in W \cap K$ and $0 < t \leq 1$. In other words, we can find a neighborhood W of v_0 such that for all $u \in W \cap K$, the extensions along $\exp_q(tu)$ are the same in $\overline{B_{\delta/4}(x)}$. The claim now follows from the fact that K is compact.

To conclude the proof of the lemma, we cover $D_{r_1}(q)$ with a finite number of balls $B_{\delta/4}^i$. Then, using Claim 3.8, we deduce that $D_{r_1}(q)$ is contained in a finite union of discs $D_{\delta/2}(p_i)$ which satisfy the items a), b) and c) of Proposition 3.5. Thus, by the definition of Σ_{r_1} , we conclude that, for each $x \in \Sigma_{r_1}$, there exists i_0 such that $x \in D_\delta(p_{i_0})$. \square

We are ready to show that Σ_{r_1} is an immersed submanifold in M . Let $\mathbf{G}^k(M)$ be the k -Grassmann bundle of TM and $\Pi : \mathbf{G}^k(M) \rightarrow M$ the natural projection. Let $D_i^* \subset \mathbf{G}^k(M)$ be the set $\{T_x D_i / x \in D_i\}$ and $\Sigma_{r_1}^* \subset \mathbf{G}^k(M)$ be the set $\{T_x D_i / \text{for all } x \in \tilde{\Sigma}_{r_1}\}$. Hence we have defined the blow-up's of D_i^* and Σ_{r_1} . It is easy to see that D_i^* is embedded in $\mathbf{G}^q(M)$. In addition, if D_i^* and D_j^* have a common point then they coincide in a neighborhood of this point. Since $\Sigma_{r_1}^*$ can be covered by a finite numbers of D_i^* , we see that $\Sigma_{r_1}^*$ is embedded in $\mathbf{G}^q(M)$. We conclude that Σ_{r_1} is immersed for $\Sigma_{r_1} = \Pi \Sigma_{r_1}^*$.

To finish the proof, we choose a sequence $r_1 < \dots < r_n \rightarrow \infty$ and construct Σ_{r_i} and $\Sigma_{r_i}^*$ in a similar way as we have done before. As above we can also show that $\Sigma_{r_i}^*$ is embedded and Σ_{r_i} is immersed. Finally we define $\Sigma^* := \cup_{r_i} \Sigma_{r_i}^*$. We can see that Σ^* is a submanifold and that $\Sigma := \cup_{r_i} \Sigma_{r_i}$ is immersed for $\Sigma = \Pi \Sigma^*$. The Hopf and Rinow's Theorem implies that Σ is complete.

4. PROOF OF THEOREM 1.7

In this section we will prove some propositions and claims, which will imply direct the Theorem 1.7. In fact, item a) of the theorem will follow from Corollary 4.9 and Proposition 4.11. Item b) will follow from Proposition 4.1 and Definition 4.3. The items c) and d) will be proved in Proposition 4.4 and Proposition 4.10 respectively. Finally, item e) will follow from Claim 4.13.

Proposition 4.1. *Let S_q denote a slice at a point $q \in M$ and consider two local sections σ_0 and σ_1 that contain q . Then*

- a) *There exists $h \in WS_q$ such that $W_{\sigma_1} = hW_{\sigma_0}h^{-1}$.*
- b) *$\overline{W_{\sigma_1}} \cdot q = \overline{W_{\sigma_0}} \cdot q \subset \sigma_1 \cap \sigma_0$.*

Proof. a) Let $w \in W_{\sigma_1}$. Then there exists a curve β_1 contained in a regular leaf such that $\beta_1(0) \in \sigma_1$, $\beta_1(1) \in \sigma_1$ and $w = \varphi_{[\beta_1]}$.

Let α be a curve contained in a regular leaf of $\mathcal{F} \cap S_q$ such that $\alpha(0) \in \sigma_0$ and $\alpha(1) \in \sigma_1$. Set $h := \varphi_{[\alpha]}$.

Now we define two curve that join the local section σ_0 to σ_1 as $\rho(t) := \varphi_{[\alpha|_{[0,t]}}(h^{-1}\beta_1(0))$ and $\varrho(t) := \varphi_{[\alpha|_{[0,t]}}(h^{-1}\beta_1(1))$. Finally we define the concatenation $\beta_0 := \varrho^{-1} * \beta_1 * \rho$. Since $h = \varphi_{[\alpha]} = \varphi_{[\rho]} = \varphi_{[\varrho]}$, we have that $\varphi_{[\beta_0]} = h^{-1}\varphi_{[\beta_1]}h$. From the last equation, we conclude that $W_{\sigma_1} \subset hW_{\sigma_0}h^{-1}$. By a similar argument one can prove that $W_{\sigma_1} \supset hW_{\sigma_0}h^{-1}$ and conclude the proof of item a).

b) If q is a regular point, then $\sigma_0 = \sigma_1$ and the result is trivial. Hence we can assume that q is a singular point.

We can deduce the next claim from the Slice Theorem.

Claim 4.2. *Let T be the intersection of the local section σ_0 with the singular stratum that contains q . Then $T \subset \sigma_0 \cap \sigma_1$ and $h|_T$ is the identity.*

On the other hand, it follows direct from the definition of W_{σ_0} that

$$(4.1) \quad W_{\sigma_0} \cdot q \subset T.$$

Now Equation 4.1 and Claim 4.2 imply that

$$(4.2) \quad hW_{\sigma_0} \cdot q = W_{\sigma_0} \cdot q.$$

Finally, using Proposition 2.14, Item a), Claim 4.2 and Equation 4.2, we can prove Item b) as follows:

$$\begin{aligned} \overline{W_{\sigma_1}} \cdot q &\stackrel{\text{Prop. 2.14}}{=} \overline{W_{\sigma_1} \cdot q} \\ &\stackrel{\text{Item a)}}{=} \overline{hW_{\sigma_0}h^{-1} \cdot q} \\ &\stackrel{\text{Claim 4.2}}{=} \overline{hW_{\sigma_0} \cdot q} \\ &\stackrel{\text{Eq. 4.2}}{=} \overline{W_{\sigma_0} \cdot q} \\ &\stackrel{\text{Prop. 2.14}}{=} \overline{W_{\sigma_0}} \cdot q \end{aligned}$$

□

Definition 4.3 (Transverse Orbits). Consider a point q of M and σ a local section that contains q . Let \mathcal{O}_q denote the submanifold $\overline{W_{\sigma}^0} \cdot q$, where $\overline{W_{\sigma}^0}$ is the connected component of the identity element. It follows from Proposition 4.1 that this definition is independent of the choice of the local section σ .

Proposition 4.4. *Let L be a leaf of \mathcal{F} and \overline{L} the closure of L . Then \overline{L} is a submanifold of M . In addition if q is a point of \overline{L} , then a neighborhood of q in \overline{L} is the product of \mathcal{O}_q with plaques with the same dimension of P_q .*

Proof. Let S_q be a slice at q .

Claim 4.5. *Let T denote the intersection of S_q with the stratum that contains L_q . Then $L \cap S_q \subset T$.*

To check the above claim, we can assume that q is a singular point, because the claim is trivial when q is a regular point. Suppose that $L \cap S_q$ is not contained in T . Then Proposition 2.3 implies that $L \cap S_q$ must be equidistant from T . Thus q does not belong to \bar{L} what contradicts our assumption.

From Claim 4.5 we can deduce the next claim.

Claim 4.6. *$L \cap S_q$ are points of S_q .*

Now Claim 4.6 and the item b) of the slice theorem (Theorem 2.4) imply the following claim.

Claim 4.7. *If $x \in \bar{L} \cap S_q$, then*

- (1) *the plaque $P_x \subset \bar{L}$,*
- (2) *the plaque P_x intersect S_q in one point,*
- (3) *the plaque P_x is orthogonal to S_q .*

According to Claim 4.7, to prove the proposition it suffices to show that the connected component of $\bar{L} \cap S_q$ which contains q is the submanifold \mathcal{O}_q .

At first we note that if q is a regular point, then S_q is a local section and it follows from Proposition 2.14 that the connected component of $\bar{L} \cap S_q$ is the submanifold \mathcal{O}_q .

Now we suppose that q is a singular point and choose a local section σ that contains T . Since $q \in \bar{L}$, we have that $q \in \bar{W}_\sigma \cdot z$, for $z \in L$. This implies that $\bar{W}_\sigma \cdot q = \bar{W}_\sigma \cdot z$. Therefore \mathcal{O}_q is the connected component of $\bar{W}_\sigma \cdot z$ that contains q . To conclude the proof we note that Proposition 2.14 implies that the connected component of $\bar{W}_\sigma \cdot z$ that contains q is the connected component of $\bar{L} \cap S_q$ that contains q . □

Corollary 4.8. *Let L be a leaf of \mathcal{F} , \bar{L} the closure of L and $q \in \bar{L}$. Then $L_q \subset \bar{L}$ and $\bar{L}_q = \bar{L}$.*

Corollary 4.9. *The singular partition $\{\bar{L}\}_{L \in \mathcal{F}}$ is a transnormal system.*

Proof. Let γ be a geodesic orthogonal to \bar{L} so that $\gamma(0) \in \bar{L}$. Since $L_{\gamma(0)} \subset \bar{L}$ (see Corollary 4.8), the geodesic γ is orthogonal to the leaf $L_{\gamma(0)}$. Therefore, by the slice theorem, the segment of geodesic $\gamma|_{(-\epsilon, \epsilon)}$ is contained in a local section σ , for a small ϵ .

Since γ is orthogonal to \bar{L} , it follows from Proposition 4.4 that γ is orthogonal to $\bar{W}_\sigma \cdot \gamma(0)$. Since \bar{W}_σ is a pseudogroup of isometries, γ is orthogonal to the orbits of \bar{W}_σ . This fact and Proposition 4.4 imply that γ is orthogonal to $\bar{L}_{\gamma(t)}$ for every $t \in (-\epsilon, \epsilon)$.

With the above argument one can prove that $\gamma|_{(-r, r)}$ is orthogonal to $\{\bar{L}\}_{L \in \mathcal{F}}$. To prove that γ is also orthogonal to $\bar{L}_{\gamma(r)}$, we note that γ is orthogonal to the leaf $L_{\gamma(r)}$ and hence is contained in a local section σ . By

Proposition 4.4, γ is orthogonal to the orbits of \overline{W}_σ and in particular to the orbit $\overline{W}_\sigma \cdot \gamma(r)$. Thus γ is orthogonal to $\overline{L}_{\gamma(r)}$. The result follows then from the connectedness of γ . \square

Proposition 4.10. *Let q be a singular point and T the intersection of the slice S_q with the stratum that contains q . Then the normal connection of T in S_q is flat.*

Proof. Let x_0 and x_1 be points of T and v be a vector of the normal space $\nu_{x_0}T \cap T_{x_0}S_q$. Let $||^\nu(\alpha)_0^1 v$ denote the parallel normal transport of v along a curve $\alpha \subset T$ so that $\alpha(0) = x_0$ and $\alpha(1) = x_1$. We want to prove that $||^\nu(\alpha)_0^1 v$ independ of the choise of the curve α . Let $||(\alpha)_0^1 v$ denote the parallel transport of v along α . Since T is totally geodesic, $||^\nu(\alpha)_0^1 v = ||(\alpha)_0^1 v$. Thus, it suffices to prove that $||(\alpha)_0^1 v$ independ of the choise of the curve α that joins x_0 to x_1 .

We choose a local section σ that is tangent to v . It follows from the slice theorem that $T \subset \sigma$. Since σ is totally geodesic, $||(\alpha)_0^1 v \subset \sigma$ for all t .

From now on we consider σ as the ambient space of T .

Let N_{x_0} be the slice at x_0 of T i.e.,

$$(4.3) \quad N_{x_0} := \{\exp_{x_0} v, \text{ for } v \in (\nu_{x_0}T \cap T_{x_0}\sigma) \text{ and } \exp_{x_0} v \subset S_q\}.$$

It follows from the slice theorem that we can choose a basis $\{\xi_i\}_{i=1 \dots k}$ of $T_{x_0}N_{x_0}$ such that $x_0, \exp_{x_0}(\xi_1), \dots, \exp_{x_0}(\xi_k)$ are vertices of a simplicial cone contained in the singular stratification of N_{x_0} . Let T_i denote the singular stratum that contains the curve $s \rightarrow \exp_{x_0}(s\xi_i)$, for small s .

Since T_i is totally geodesic, the parallel transport $||(\alpha)_0^1 \xi_i$ is always tangent to T_i . On the other hand, T is a totally geodesic hypersurface of T_i . These two facts implies that $||(\alpha)_0^1 \xi_i$ independ of the choice of α . Therefore $||(\alpha)_0^1 v$ independ of the choice of α because $\{\xi_i\}$ is a basis of N_{x_0} . \square

Proposition 4.11. *The singular partition $\overline{\mathcal{F}} := \{\overline{L}\}_{L \in \mathcal{F}}$ is a singular foliation.*

Proof. Let S_q denote the slice at a point q . It suffices to prove that $\overline{\mathcal{F}} \cap S_q$ is a singular foliation. Since $\mathcal{F} \cap S_q$ is already a singular foliation, Proposition 4.4 implies that it is enough to prove that $\{\mathcal{O}_x\}_{x \in S_q}$ is a singular foliation.

If q is a regular point, then the slice S_q is a local section σ and the result follows from the fact that $\{\mathcal{O}_x\}_{x \in \sigma}$ are orbits of the pseudogroup of isometries \overline{W}_σ . Thus we can suppose that q is a singular point.

Let T denote the intersection of the slice S_q with the singular stratum that contains q . Since the normal connection of T in S_q is flat (see Proposition 4.10), we can define the end point map $\eta_\xi : T \rightarrow S_q$ where ξ is a parallel normal vector field along T . One can easily check that:

Claim 4.12. *The partition $\{\eta_\nu(T)\}_{\nu \in \Upsilon}$ is a regular foliation, where Υ is the set of all parallel normal vector field along T .*

Using Claims 4.12 above and Claim 4.13 below one can show that $\{\mathcal{O}_x\}_{x \in S_q}$ is a singular foliation and conclude the proof.

Claim 4.13. *Let $x \in T$ and $y = \exp_x(v)$ for $v \in \nu_x T$. Then $\mathcal{O}_y = \eta_v(\mathcal{O}_x)$*

We start the proof of Claim 4.13 choosing a local section σ so that w is tangent to σ . We also consider a curve $w : (-\epsilon, \epsilon) \rightarrow \overline{W}_\sigma^0$ in the connected component of the identity of \overline{W}_σ and define a curve $\alpha(t) := w(t) \cdot x$. Next we review the definitions used in the proof of Proposition 4.10. Let $N_{\alpha(t)}$ be the slice at $\alpha(t)$ of T (see (4.3)) and $\{\xi_i\}$ a basis of $T_x N_x$ so that $x, \exp_x(\xi_1), \dots, \exp_x(\xi_k)$ are vertices of a simplicial cone contained in the singular stratification of N_x . Let T_i denote the singular stratum that contains the curve $s \rightarrow \exp_x(s \xi_i)$, for small s .

Since the local isometries of \overline{W}_σ leave T invariant, we can conclude that

$$(4.4) \quad dw(t)_x \xi_i \in T_{\alpha(t)} N_{\alpha(t)}$$

We can also deduce that $w(t) : T_i \rightarrow T_i$, using the fact that \overline{W}_σ leave singular stratum invariant. Therefore we have:

$$(4.5) \quad dw(t)_x \xi_i \in T_{\alpha(t)} T_i.$$

On the other hand, the fact that T_i is totally geodesic and T is a totally geodesic hypersurface of T_i imply that:

$$(4.6) \quad \|(\alpha)_0^t \xi_i \in T_{\alpha(t)} T_i \cap T_{\alpha(t)} N_{\alpha(t)},$$

where $\|(\alpha)_0^t \xi_i$ is the parallel transport of ξ_i along the curve α

Finally we can deduce from Equations 4.4, 4.5 and 4.6 that:

$$(4.7) \quad dw(t)_x \xi_i = \|(\alpha)_0^t \xi_i.$$

Since $\{\xi_i\}$ is a basis of N_x , it follows from Equation 4.7 that

$$(4.8) \quad dw(t)_x v = \|(\alpha)_0^t v,$$

which implies Claim 4.13. □

5. PROOF OF COROLLARY 1.8

Let σ be a local section that contains q . The fact that $T = \{q\}$ implies that W_σ has discrete orbits. Therefore the leaves in a tubular neighborhood of L_q are closed. Now suppose that there exists a nonclosed leaf L . Let γ be a geodesic orthogonal to \overline{L} such that $\gamma(0) \in \overline{L}$ and $\gamma(1)$ is a regular point of σ . Since L is not closed, the submanifold $\mathcal{O}_{\gamma(0)}$ has dimension greater than zero. It follows from item c) of Theorem 1.7 that γ is orthogonal to $\mathcal{O}_{\gamma(0)}$. Since the leaves near to a neighborhood of $\gamma(1)$ in σ intersect σ only one time, we can use Corollary 2.2 to conclude that all points of γ near to $\gamma(1)$ are focal points of $\mathcal{O}_{\gamma(0)}$. This contradicts the fact that focal points are isolated.

6. APPENDIX

In this appendix, we use the method of suspension of homomorphism to construct a simple example of s.r.f.s. that illustrates Theorem 1.7 and Remark 1.10.

We start by recalling the method of suspension. For further details see, for example, the book of Molino [9][page 28,29; 96,97].

Let Q and V be riemannian manifolds with dimension p and n respectively and let $\rho : \pi_1(Q, q_0) \rightarrow Iso(V)$ be a homomorphism from the fundamental group of Q to the group of isometries of V . Let $\hat{P} : \hat{Q} \rightarrow Q$ be the projection of the universal cover of Q into Q . Then we can define an action of $\pi_1(Q, q_0)$ on $\widetilde{M} := \hat{Q} \times V$ by

$$[\alpha] \cdot (\hat{q}, v) := ([\alpha] \cdot \hat{q}, \rho(\alpha^{-1}) \cdot v),$$

where $[\alpha] \cdot \hat{q}$ denotes the deck transformation associated to $[\alpha]$ applied to a point $\hat{q} \in \hat{Q}$.

We denote the set of orbits of this action by M and the canonical projection by $\Pi : \widetilde{M} \rightarrow M$. It is possible to see that M is a manifold. Indeed, given a simple open neighborhood $U_j \subset Q$, we can construct the following bijection:

$$\begin{aligned} \Psi_j : \Pi(\hat{P}^{-1}(U_j) \times V) &\rightarrow U_j \times V \\ \Pi(\hat{q}, v) &\rightarrow (\hat{P}(\hat{q}) \times v). \end{aligned}$$

If $U_i \cap U_j \neq \emptyset$ and connected, we can see that

$$\Psi_i \cap \Psi_j^{-1}(q, v) = (q, \rho([\alpha]^{-1})v)$$

for a fixed $[\alpha]$. So there exists a unique manifold structure on M for which Ψ_j are local diffeomorphisms. We define a map P by

$$\begin{aligned} P : M &\rightarrow Q \\ \Pi(\hat{q}, v) &\rightarrow \hat{P}(\hat{q}) \end{aligned}$$

It follows that M is a total space of a fiber bundle, which has P as the projection over the basis Q . Besides the fiber of this bundle is V and the structural group is given by the image of ρ .

At last we define $\mathcal{F} := \{\Pi(\hat{Q}, v)\}$, i.e., the projection of the trivial foliation defined as the product of \hat{Q} with each v . It is possible to see that this is a foliation transverse to the fibers of the fiber bundle. In addition this foliation is a riemannian foliation whose tranverse metric coincides with the metric of V .

Example 6.1. In what follows we construct a s.r.f.s such that the intersection of a local section with the closure of a regular leaf is an orbit of an action of a subgroup of isometries of the local section. This isometric action is not a polar action.

Let V denote the product $\mathbf{R}^2 \times \mathbf{C} \times \mathbf{C}$ and $\hat{\mathcal{F}}_0$ the singular foliation of codimension 5 on V whose leaves are the product of points in $\mathbf{C} \times \mathbf{C}$ with circles in \mathbf{R}^2 centered at $(0,0)$. It is easy to see that the foliation $\hat{\mathcal{F}}_0$ is a singular riemannian foliation with sections. Let Q be the circle \mathbf{S}^1 and k be an irrational number. Then we define the homomorphism ρ as

$$\begin{aligned} \rho: \pi_1(Q, q_0) &\rightarrow \text{Iso}(V) \\ n &\rightarrow ((x, z_1, z_2) \rightarrow (x, \exp(ink) \cdot z_1, \exp(ink) \cdot z_2)). \end{aligned}$$

Finally we set $\mathcal{F} := \Pi(\hat{Q} \times \hat{\mathcal{F}}_0)$. It turns out that \mathcal{F} is s.r.f.s. such that the intersection of the section $\Pi(0 \times \mathbf{R} \times \mathbf{C} \times \mathbf{C})$ with the closure of a regular leaf is an orbit of an isometric action on the section. This isometric action is not a polar action, since the isometric action

$$\begin{aligned} \mathbf{S}^1 \times \mathbf{C} \times \mathbf{C} &\rightarrow \mathbf{C} \times \mathbf{C} \\ (s, z_1, z_2) &\rightarrow (s \cdot z_1, s \cdot z_2) \end{aligned}$$

is not a polar action, as one can easily check.

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