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CONDENSED MATTER

Real Space Renormalization Group of the Chiral Potts Model

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Abstract
The chiral Potts model is studied by means of the real space renormalization group. We use the renormalization group scheme proposed by Niemeyer and Van Leeuwen with an approximation that retains only the first term in the cumulant expansion. The recurrence relations are obtained for any number of states, renormalization factor and lattice dimension. By using a renormalization factor $b = 4$, the phase diagram for the three-state model on a square lattice is obtained in terms of the chiral field and temperature. It shows two regions: a disordered one, which corresponds to the paramagnetic phase, and a modulated one, consisting of structures described by rational wave numbers.

Keywords Critical phenomena · Renormalization group · Modulated phases · Potts chiral

1 Introduction

Condensed matter physics has many examples of systems that present modulated structures, with periods that are commensurate or incommensurate with the underlying lattice [1]. Structures such as a periodic lattice distortion, magnetic systems with harmonic or helical ordering and a harmonic charge density waves are examples of modulated systems.

In the case of magnetic systems, the modulated variable is the magnetization, described by a propagation vector. *CeBi* and *CeSb* [2] are examples of magnetic systems that show modulated phases. These crystals have been studied experimentally by means of neutron diffraction experiments; in the case of *CeSb*, the magnetization is represented by a 3d order parameter that varies harmonically showing many commensurate structures. In order to explain the behavior of such systems, some theoretical models have been proposed. A common characteristic of these models is that their Hamiltonian contains competitive interactions: for instance, in the axial nearest neighbor Ising model (ANNNI), there is a competition in the interactions between nearest and next nearest neighbors, because they favor opposite alignments; in the chiral Potts model, the competition concerns two opposite tendencies in the interactions just between nearest neighbors. The chiral Potts model was first studied by Ostlund [3] and Huse [4] independently; Ostlund performed a low temperature analysis of the system by means of a transfer matrix technique while Huse solved the model using the Migdal-Kadanoff renormalization group scheme.

Our study is on the chiral Potts model with symmetric interactions on a square lattice and much of our attention is addressed to the three-state Potts model, which is studied using the Niemeyer and van Leeuwen renormalization group with an approximation that retains only the first term in the cumulant expansion. Iterating the recurrence relations of the renormalization group scheme, we verified that the nature of the interactions is preserved, which is a requirement for the soundness of the method. A phase diagram is obtained by means of an analysis of the flows as proposed by Huse [4]. On the phase diagram we find a paramagnetic (or disordered) phase and a set of modulated phases corresponding to rational wave numbers q (in the first Brillouin zone).

In Sect. 2 the model is defined and the wave number q is characterized. In Sect. 3 we describe the method and show how the recurrence relations are obtained. The results are

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presented in Sect. 4: we show how the flows are obtained from the recurrence relations and the phase diagram is constructed, we also analyze the critical region corresponding to the usual Potts model. The results are discussed in Sect. 5 while the conclusion is found in Sect. 6.

2 The Chiral Potts Model

2.1 The Model

The original p state chiral Potts model was proposed by Ostlund [3]: in every lattice site there is a spin varibable $\vec{s}_i$ that may assume one out of p states:

$\vec{e}_0, \vec{e}_1, \vec{e}_2, ..\vec{e}_{p-1}$, where $\vec{e}_i$ are planar unitary vectors (see Fig. 1), that can be written as

$$\vec{e}_i = (\cos i\theta, \sin i\theta) \qquad 0 \leq i \leq p-1. \tag{1}$$

The Hamiltonian for this model is expressed as:

$$\mathcal{H} = -\sum_{i,j} J(\theta_{ij}), \tag{2}$$

where the sum is over nearest neighbor spins; the interaction $J(\theta_{ij})$ depends only on the relative angle between them,

$$J(\theta_{ij}) \equiv J_{ij} = J_1 \vec{s}_i \cdot \vec{s}_j + J_2 \left(\vec{s}_i \times \vec{s}_j\right) \cdot \hat{z}. \tag{3}$$

J_1 and J_2 are constants that favor parallel and orthogonal alignment of neighboring spins respectively. In general, $J_{ij} \neq J_{ji}$, in other words, the interaction must be specified by an ordered pair (s_i, s_j). Equation 3 can be written in a more convenient form, in terms of new variables J and Δ:

$$J_{ij} = J \cos\left[\frac{2\pi}{p}\left(n_i - n_j + \Delta\right)\right], \tag{4}$$

where $J^2 = J_1^2 + J_2^2$ and $\Delta = (1/2\pi)\arccos(J_1/\sqrt{J_1^2 + J_2^2})$, with $J \geq 0$ and $-p/2 \leq \Delta < p/2$; the variables n_i are defined in terms of θ_i as $n_i = (p/2\pi)\theta_i$.

The partition function for this model is given by

$$Z = \sum_{n_i} \exp \bar{\mathcal{H}}, \tag{5}$$

where the reduced Hamiltonian $\bar{\mathcal{H}}$ is expressed as

$$\bar{\mathcal{H}} = \beta\mathcal{H} = \sum_{i,j} \frac{1}{T}\left[\frac{2\pi}{p}\left(n_i - n_j + \Delta\right)\right]; \tag{6}$$

the variables reduced temperature, $T \equiv 1/\beta J$ and chiral field, Δ are the parameters in terms of which the p-state chiral Potts model will be described thereafter.

2.2 Symmetries

Here we consider the reduced Hamiltonian (Eq. 6) defined on a square lattice, where the lattice sites are labeled by the indices (i, j) and the layers, by ℓ, such that the lattices sites in layer ℓ_0 satisfy $j = -i + \ell_0$ (see Fig. 2). The partition function (Eq. 5), associated to the reduced Hamiltonian, has the following properties:

- invariance by inversion of Δ: $Z(T, \Delta) = Z(T, -\Delta)$;
- invariance by rotation: $Z(T, \Delta) = Z(T, \Delta - m)$, where m integer, $1 \leq m < p$.

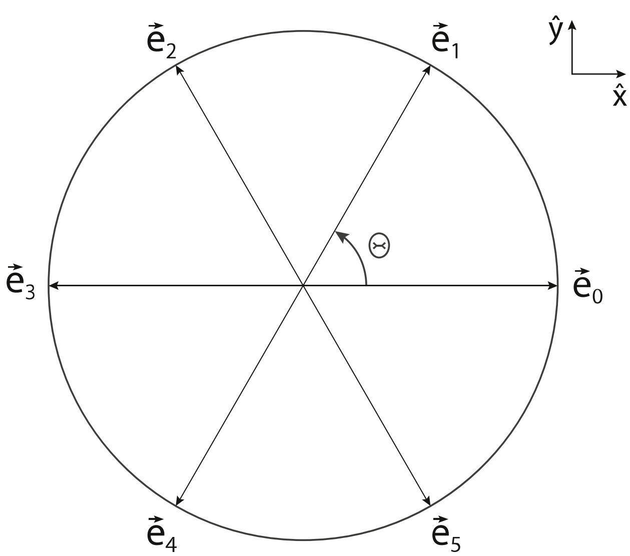

Fig. 1 In this example the dynamic variable $\vec{s}_i$ is in one out of six states: $\vec{e}_1, \vec{e}_2, \vec{e}_3, ..\vec{e}_6$

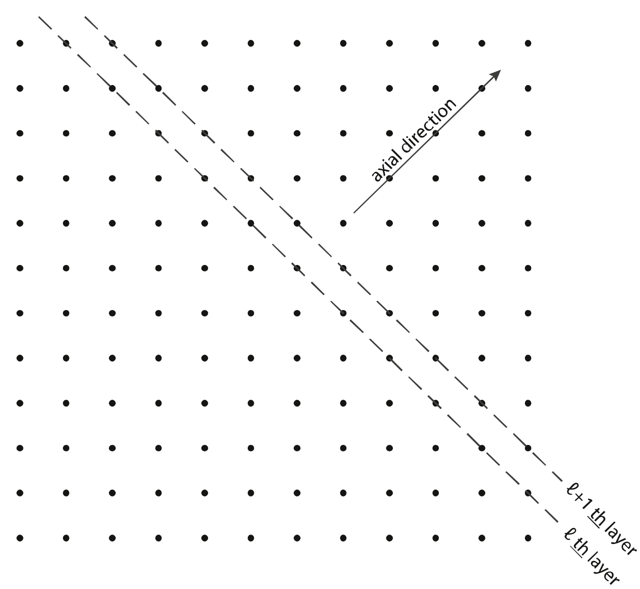

Fig. 2 Configuration of a 2d lattice showing how the layers are labeled

The competing interactions described above produce a macroscopic magnetization with modulated character. One defines the magnetization of the ℓ_0^{th} layer $\vec{m}_{\ell_0}$, as the mean over the spins belonging to this layer, so that the $\vec{m}_{\ell_0} \equiv m_{\ell_0}(\cos\alpha_{\ell_0}, \sin\alpha_{\ell_0})$ is also a $2d$ variable. The modulated behavior in a given direction can be described by a wave number q, restricted to the first Brillouin zone ($0 \le q < 1$). This wave number is defined as follows: suppose the magnetization has period equal to N (layers), that is, $\vec{m}_{\ell+N} = \vec{m}_{\ell}$, for any ℓ, then

$$q = \frac{1}{2\pi}\frac{\phi_1 + \phi_2 + \ldots + \phi_N}{N}; \tag{7}$$

where ϕ_ℓ is the relative angle between $\vec{m}_{\ell+1}$ and $\vec{m}_\ell$. q has the following properties:

$$\begin{aligned} q(T, -\Delta) &= -q(T, \Delta) \qquad (mod\ 1),\ \text{m integer} \\ q(T, \Delta + m) &= q(T, \Delta) + m/p \qquad (mod\ 1),\ \text{where } 1 \le m < p \end{aligned} \tag{8}$$

2.3 The System Ground State

We consider a $2d$ lattice assembled as a succession of parallel layers (see Fig. 2). The interactions between nearest neighbor spins belonging to neighboring layers are described by Eq. 3, with the additional convention that the second spin variable in the cross product term refers to a layer that succeeds the layer which contains the spin variable that appears first.

One supposes that at $T = 0$, all spins on a given layer are aligned. Moreover, the relative angle between all spins in neighboring layers is constant: for the $p = 3$ model, the constant is $n(2\pi/3)$, where n is an integer in the interval $0 \le n < 3$. In this case, the free energy per spin ϵ is given by

$$\epsilon = \begin{cases} -J\cos\frac{2\pi}{3}\Delta & \text{ferro phase } 0,0,0,0\ldots; \\ -J\cos\frac{2\pi}{3}(-1+\Delta) & 0,1,2,0,1,2,\ldots \text{ phase} \\ -J\cos\frac{2\pi}{3}(-2+\Delta) & 0,2,1,0,2,1,\ldots \text{ phase.} \end{cases} \tag{9}$$

Therefore we conclude that, for $T = 0$,

$$q = \begin{cases} \frac{2}{3} & \text{for } -1.5 < \Delta < -0.5; \\ \text{ferro} & \text{for } -0.5 < \Delta < 0.5 \text{ and} \\ \frac{1}{3} & \text{for } 0.5 < \Delta < 1.5. \end{cases} \tag{10}$$

Figure 3 shows the stable phases at $T = 0$.

In this section we presented the model, and defined the variable q that describes the ordered phases; we also studied the system ground state. In the next section, we describe the approach used to solve the problem.

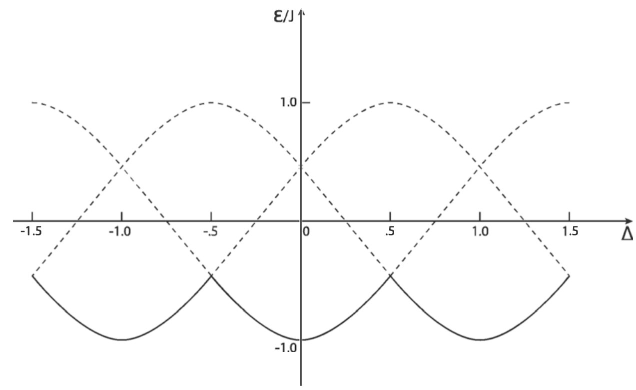

Fig. 3 Normalized free energy per spin as a function of the chiral field at $T = 0$

3 The Renormalization Group and the Recurrence Relations

In order to study this problem we have used the real space renormalization group [5, 6] which we outline next.

3.1 Criticality and Fixed Point: the Real-State Renormalization Group Approach

The underlying idea of the renormalization group approach is that, near criticality, the dynamic variables are strongly correlated: there is a quite strong collective behavior. In this region, it is expected that the correlation length ξ is large with respect to the lattice parameter. Based on these facts, the strategy is to propose a partition of the system into blocks of size b ($b \ll \xi$) and assign a new representative dynamic variable to each block. This procedure corresponds to a scaling transformation where every length is rescaled by a factor b, while the original spin variable is rescaled with respect to the original one.

Consider the reduced Hamiltonian defined as in Eq. 6. Let $\mathcal{R}$ be a transformation that takes the Hamiltonian $\bar{\mathcal{H}}$ into a new Hamiltonian $\bar{\mathcal{H}}'$. It must be understood as a transformation of the parameters that characterize the system by means of a set of non linear analytic recurrence relations.

Let $\bar{\mathcal{H}}$ be represented by an array in the parameter space $\mathbf{K} = \{K_1, K_2, K_3, \ldots\}$ (in our case, $\mathbf{K} = \{T, \Delta\}$); in this representation, the corresponding transformed Hamiltonian is expressed in terms of a new set $\mathbf{K}' = \{K_1', K_2', K_3', \ldots\}$. One can express the transformation $\bar{\mathcal{H}} \to \bar{\mathcal{H}}'$ as

$$\mathbf{K}' = \mathcal{R}(\mathbf{K}), \tag{11}$$

or, in terms of the i^{th} component of $\mathbf{K}$:

$$K_i' = \mathcal{R}_i\{K_1, K_2, K_3, \ldots\}. \tag{12}$$

Applying this procedure successively we obtain a trajectory in the parameter space until a fixed point $\mathbf{K}^*$ is reached such that the relation $\mathbf{K}^* = \mathcal{R}(\mathbf{K}^*)$ is satisfied. In the vicinity of criticality, the Hamiltonian and the thermodynamic functions are analyzed by linearizing the Hamiltonian about the fixed point:

$$\mathbf{K}' = \mathcal{R}(\mathbf{K}^* + \Delta\mathbf{K}) = \mathbf{K}^* + \mathbf{L}\Delta\mathbf{K}, \tag{13}$$

where $\mathbf{L}$ is a matrix such that

$$L_{ij} = \left(\frac{\partial \mathcal{R}_i}{\partial K_j}\right)_{\mathbf{K}=\mathbf{K}^*}, \tag{14}$$

where $\mathcal{R}_i$ is defined according to Eq. 12.

It is useful to study criticality in terms of the eigenvalues of a diagonal matrix obtained from $\mathbf{L}$ through a similarity transformation. Indeed, the iterations that define the flux lines take place in the space of the normal components associated to a fixed point.

3.2 The Renormalization Group by Niemeyer and van Leeuwen

We suppose the system is near criticality and use renormalization group scheme in the version proposed by Niemeyer and van Leeuwen [7], taking into account terms up to first order in the cumulant expansion.

In this case the square lattice is partitioned into N' blocks of b^d spins where $d = 2$. The reduced Hamiltonian that describes the interactions in this lattice is given by Eq. 4, which can be written as

$$\bar{\mathcal{H}}(s) = K \sum_{(ij)} \mathfrak{R}\left(s_i e^{i\delta} s_j^*\right), \tag{15}$$

where $K = 1/T$ and $\delta = (2\pi/p)\Delta$; the dynamical variables represented by unit vectors $\vec{s}_i$ are replaced by complex numbers s_i that may assume one out of p values $s_i = e_n = \exp(j\, n2\pi/p)$, with $0 \leq n \leq p-1$. The sum is again over spins in nearest neighbors layers. A dynamical variable s'_k is assigned to each block and it assumes the same values e_n. One introduces a projection operator $\mathcal{P}\{s'_i, s_i\}$, which is a function of the variables $\{s'_i\}$ and $\{s_i\}$, such that $\exp(\bar{\mathcal{H}}'(s')) = \sum_{s_i} \exp(\bar{\mathcal{H}}(s))\mathcal{P}(s', s)$. $\mathcal{P}(s', s)$ defines a rule to assign a value for the variable s'_k of the k^{th} block, from all the spins $s_{l,k}$ of a given block, according to a rule to be established. It can be shown that the partition function is invariant with respect to the transformation $\exp(\bar{\mathcal{H}}(s)) \to \exp(\bar{\mathcal{H}}'(s'))$.

In order to obtain the recurrence relations, one subdivide $\mathcal{H}(s)$ as

$$\mathcal{H}(s) = \mathcal{H}_0(s) + V(s), \tag{16}$$

where $\mathcal{H}_0(s)$ represents the nearest neighbor interactions within the same block and $V(s)$ refers to the interactions between nearest neighbors belonging to adjacent blocks. Therefore

$$\exp(\bar{\mathcal{H}}'(s')) = \mathcal{Z}_0 \langle \exp V \rangle_0, \tag{17}$$

where

$$\mathcal{Z}_0(s') = \sum_s \mathcal{P}(s', s) \exp(\mathcal{H}_0(s)) \tag{18}$$

and

$$\langle \mathcal{F}_0(s') \rangle = \frac{1}{\mathcal{Z}_0(s')} \left[\sum_s \mathcal{F}\mathcal{P}(s', s) \exp(\mathcal{H}_0(s)) \right]. \tag{19}$$

Taking up to the first term in the cumulant expression, the renormalized Hamiltonian is

$$\bar{\mathcal{H}}'(s') \approx \ln(\mathcal{Z}_0(s')) + \langle V \rangle_0, \tag{20}$$

or

$$\bar{\mathcal{H}}'(s') \approx N' \ln \zeta + \sum_{k,k'} \tilde{V}(s'_k, s'_{k'}), \tag{21}$$

where $\zeta = \left(\mathcal{Z}_0(s')\right)^{1/N'}$, the sum is over spins representing nearest neighbor blocks and $\tilde{V}(s'_k, s'_{k'})$ can be expressed as

$$\begin{aligned} \tilde{V}(s'_k, s'_{k'}) &= K \sum_{ll'} \mathfrak{R}(s'^*_k (v_l{}^* \exp(-i\delta) v_{l'}) s'_{k'}) \\ &= K \sum_{ll'} \mathfrak{R}(s'^*_k \exp(-i\delta') s'_{k'}); \end{aligned} \tag{22}$$

expressions for v_l and $v_{l'}$ are derived in the Appendix. Comparing Eqs. 15 and 21, we see that $\bar{\mathcal{H}}'(s')$ has the same functional form as $\bar{\mathcal{H}}(s)$. The relation between the original parameters (K, δ) and the renormalized ones (K', δ') are:

$$K \sum_{ll'} v_l{}^* \exp(-i\delta) v_{l'} = K' \exp(-i\delta'). \tag{23}$$

We decompose v_l and v'_l into two linearly independent components (that depend on K and δ) and the recurrence relations in the space defined by $(K_1, K_2) = (K\cos\delta, K\sin\delta)$ are obtained either analytically or by an exact numerical counting of the configurations. We recall that K_1 and K_2 are related to the original T and Δ through

$$\begin{cases} K_1 = \frac{1}{T}\cos\left(\frac{2\pi}{p}\Delta\right) \quad \text{and} \\ K_2 = \frac{1}{T}\sin\left(\frac{2\pi}{p}\Delta\right). \end{cases} \tag{24}$$

In this section we described how the recurrence relations are obtained within the renormalization group scheme adopted. The recurrence relations are valid for any number of states p ($p \geq 3$), renormalization factor b and dimensionality d. Next we present the results obtained by running the recurrence relations with the corresponding flux lines.

4 Results

4.1 Flow Analysis

We look for the phase diagram for the $p = 3$ model in terms of the chiral field Δ and reduced temperature T. Every point on the phase diagram corresponds to a single wave number q associated to the spatial modulation of the magnetic structure. We use the renormalization factor $b = 4$ and the recurrence relations obtained in the last section are iterated. The determination of q associated to each point in the phase is obtained by the analysis of the fluxes. We found an analytical limitation to obtain the recurrence relations: the number of configurations within a block is too large; therefore, we used an exact numerical method that takes into account all the configurations and their respective Boltzmann factors. In this case, the projection operator considered $\mathcal{P}(s', s)$ takes into account only the four terms in the square diagonal according to the following rules:

$$\mathcal{P}(s'_k, s_k) = \begin{cases} 1 & \text{if } s'_k = s_{kl}, \text{ for the majority of the diagonal spins } s_{kl}, \\ 0 & \text{if } s'_k \neq s_{kl}, \text{ for the majority of the diagonal spins } s_{kl} \text{ and} \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad (25)$$

4.2 Constructing the Phase Diagram

Every point (T, Δ) on the phase diagram is associated to a wave number $q(T, \Delta)$. Starting from a point $q(T, \Delta)$ and applying the recurrence relations n times, one obtains $q^n(T, \Delta)$. Next we describe how $q(T, \Delta)$ is obtained.

The symmetry relations 8 show that to obtain the full phase diagram it is sufficient to find $q(T, \Delta)$ in the interval $0 \leq \Delta < 0.5$; therefore we restrict our attention to this interval.

Let $q \equiv q(T, \Delta)$ and $q' \equiv q(T', \Delta')$. After one iteration, all the lengths are rescaled by a factor b, therefore the wave number q will be rescaled by a factor $1/b$; nevertheless, q is restricted to the first Brillouin zone, then, after one iteration,

$$q \to q' = bq - [bq], \quad (26)$$

where $[x]$ denotes the integer part of x. The inverse of the transformation (Eq. 26) is

$$q = \frac{q'}{b} + \frac{m}{b}, \quad (27)$$

therefore, if q' is known, Eq. 27 gives the values of q with multiplicity b. This ambiguity can be raised by imposing that q is a weak monotonic function of Δ.

The symmetry relations, Eq. 8, show that, for all T, $q(T, 0) = 0$ and $q(T, 0.5) = 1/6$; these lines represent the first part of the phase diagram. In general, we can write:

$$\frac{m}{6} \leq q(T, \Delta) \leq \frac{m+1}{6}, \text{ where } m = \begin{cases} 0 & \text{for } 0 < \Delta < 0.5 \\ 1 & \text{for } 0.5 < \Delta < 1.0 \\ 2 & \text{for } 1.0 < \Delta < 1.5 \\ 3 & \text{for } -1.5 < \Delta < -1.0 \\ 4 & \text{for } -1.0 < \Delta < -0.5 \\ 5 & \text{for } -0.5 < \Delta < 0. \end{cases} \quad (28)$$

The determination of a given wave number $q(T, \Delta)$ is made in two steps: iteration of the recurrence relations n times until a (T^n, Δ^n) point on the phase diagram is reached where $q(T^n, \Delta^n)$ is already known; after that, using Eq. 27, q^{n-1} is obtained from q^n, q^{n-2} from q^{n-1},... and finally q from q^1, as shown in Fig. 4. Recalling that about a trivial fixed point, the conditions $\xi = \xi/b$ and $q = bq - [bq]$, then $\xi = 0$; two families of solutions satisfy the constraint: (i) a paramagnetic phase, corresponding to a high temperature fixed point and (ii) a set of low temperature phases, located on the $T = 0$ line: three different values of q are found: $q = m/3$, with $m = 0, 1$ or 2, as described in Sects. 2.2 and 2.3.

With this analysis, it was possible to determine q^n after a given number of iterations. The phase diagram obtained, shown in Figure contains all the symmetries predicted in Sect. 2.2; in fact Fig. 4 shows just the broader phases, nevertheless, between two phases, there are an infinite number of intervening phases, separated by the paramagnetic phase that extends down to the $T = 0$ line.

4.3 Determination of the Critical Temperature and Critical Exponents

We analyzed the critical region corresponding to the non-trivial fixed point $(T^*, \Delta^*) = (T_c, 0)$. Our analysis gives $T^* = 1.2703$ that compares well with the exact value obtained by Wu [8], $T^*_{exact} = 1.4925$. A critical exponent ϕ describes the dependence of T on Δ through $(T - T_c) \simeq \Delta^{1/\phi}$. We obtained $\phi = 0.35 > 0$, which indicates that T_c is a repulsive fixed point; similar studies in the literature give $\phi = 0.506$ and $\phi = 0.2$ for the chiral Potts model on a hierarchical lattice [4] and asymmetric interactions on a square lattice respectively [9].

4.4 Constructing the Devil's Staircases

Given a fixed temperature, a devil staircase is a curve that displays the phases that emerge in the plane wave number versus chiral field; it is obtained from an analysis of the phase diagram. We fixed $T = 0.45$ and obtained a set of three plots by successive amplifications of a given Δ-interval (Fig. 5).

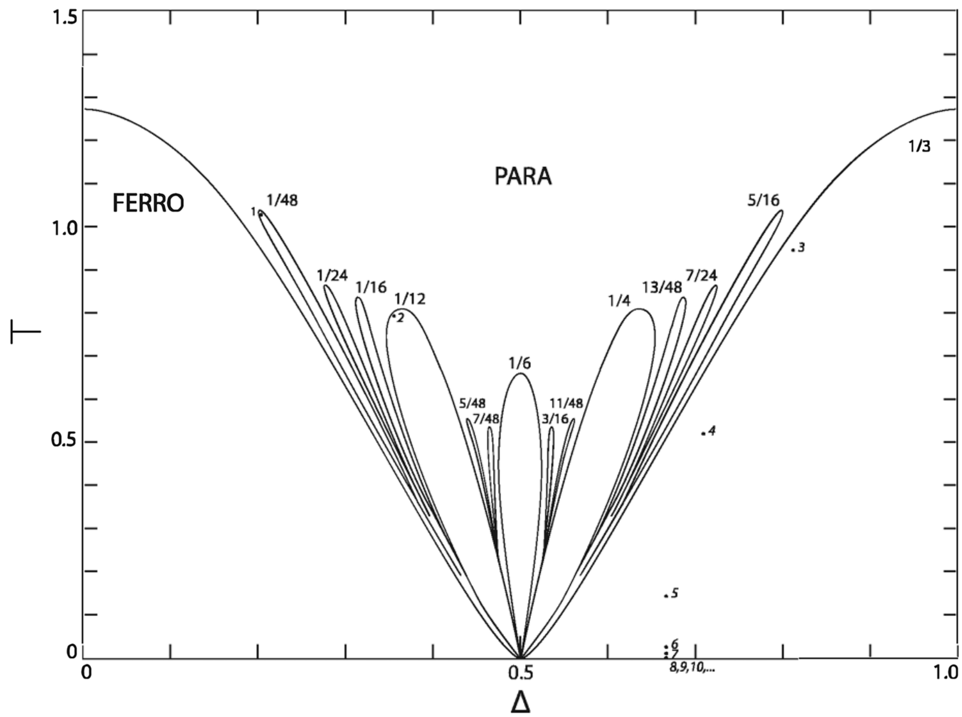

Fig. 4 Phase diagram in terms of the variables chiral field versus reduced temperature; the plot represents one-third of the full phase diagram. The wave number values corresponding to each phase are shown near the boundaries. Iterations produce fluxes that are also illustrated on this figure

These curves seem to be self-similar, presenting three intermediate plateaus: a central one wider than the side ones.

4.5 Fractal Dimension

The fractal dimension associated to the points belonging to the paramagnetic phase can be obtained by an analysis of the devil staircase. We proceed as follows: (i) consider a Δ region at a temperature T where we believe the stair is complete, in this region we have an infinity of commensurate phases; (ii) a length scale ϵ is fixed and we search for the phases that have a width larger than this length, summing them up; (iii) this sum is subtracted from the total Δ-length considered, which gives the measure of the complementary interval $L(\epsilon)$ where the paramagnetic sets in at this temperature. The fractal dimension is defined as

$$d_F = \lim_{\epsilon \to 0} \frac{N(\epsilon)}{\epsilon^{-1}}, \quad (29)$$

where $N(\epsilon) = L(\epsilon)/\epsilon$. On a given Δ range, d_F is calculated from the plot $\ln \epsilon^{-1}$ versus $\ln N(\epsilon)$ considering a set of decreasing values of ϵ from 10^{-5} down to 10^{-9}. For instance, at $T = 0.2$, $d_F = 0.33$ and at $T = 0.45$, $d_F = 0.35$.

5 Discussion

We use renormalization group, which is an approach based on the hypothesis of criticality. Therefore, in general, our results should be more reliable in the vicinity of the

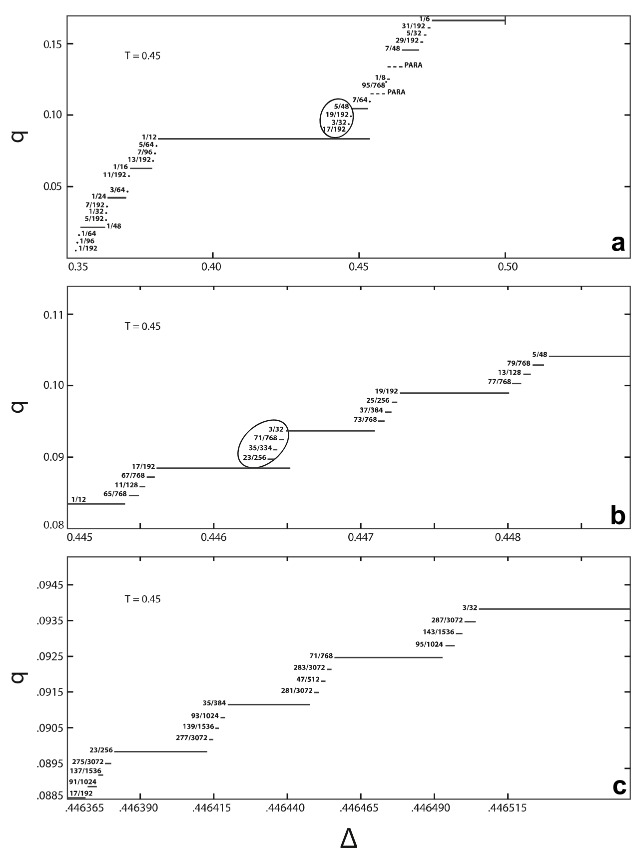

Fig. 5 Devil staircases obtained for $T = 0.45$

non-trivial fixed points. There are examples where the real space renormalization group gives reliable results also far from criticality, as in the work by Oliveira et al. [10] where they have considered two- and three-dimensional Ising systems and obtained the surface tension between coexisting phases. In general, it is always possible to apply renormalization group and then obtain thermodynamic properties, even far from criticality; nevertheless, the approach may lead to misleading results.

The method used here is not exact because it considers finite blocks to obtain the thermodynamic properties; this approximate character brings some consequences that we discuss next.

Having b finite, Eq. 26 tells us that incommensurate phases will never be achieved. We find phases labeled by wave numbers of the form $q = m/3(4^n)$; the 4^n factor in the denominator is a result of the $b = 4$ renormalization factor in the iterations.

In the region $0 \leq \Delta < 0.5$ of the phase diagram, the 1/12 is larger than the others, the same occurs for the corresponding regions on the phase diagram; this feature may be also a consequence of the renormalization group scheme. The $\Delta = 0.5$ line (and the corresponding lines) is analogous to the usual Potts model with antiferromagnetic interactions. It is conjectured [1, 5–7, 9] that, for the later model, a disordered phase extends down to $T = 0$; therefore one believes that the modulated phase with $q = 1/6$ at $\Delta = 0.5$ should not appear; the same way, the phases 1/2 at $\Delta = 1.5$ and 5/6 at $\Delta = -0.5$ shouldn't be present. We believe that, by increasing b, the area corresponding to these phases will shrink.

We have tried variations of the projection operator $\mathcal{P}(s'_k, s_k)$, but they did not produce any significant modifications in the phase diagram.

Although the phase diagram presents an envelope curve that tends to separate the paramagnetic from the ordered regions, the disordered region occurs down to temperatures close to zero. In Sect. 4.5 we calculated the fractal dimension d_F of the paramagnetic phase and found $d_F < 1$ showing that the paramagnetic character is irrelevant and the devil staircase is complete.

6 Conclusion

We have studied the three-state chiral Potts model in a two-dimensional lattice. The Hamiltonian has competing interactions: the first term favors alignment of neighboring spins, while the second, produces a chiral twist. The phase diagram, in terms of temperature versus chiral field, shows modulated phases labeled by wave numbers that describe the modulation pitch. In order to study this problem, we used a renormalization group scheme in the version proposed by Niemeyer and van Leeuwen [7], with a renormalization factor $b = 4$. We obtained the recurrence relations, and, by iterating them, obtained fluxes; the phase diagram was constructed by the analysis of the fluxes. The critical temperature and the critical exponent corresponding to the equivalent usual Potts model were found: they compare well with results from the literature. A devil staircase was constructed and the fractal dimension for the disordered phase in the low temperature region was calculated: it indicates that the paramegnetic phase is irrelevant in the low temperature region. Some imperfections of our method were discussed which stem mainly from the finite size of the renormalization blocks.

Appendix. Explicit Recurrence Relations

We recall Eq. 23:

$$K \sum_{ll'} v_l^{\,*} \exp(-i\delta) v_{l'} = K' \exp(-i\delta'); \tag{30}$$

and decompose v_l and $v_{l'}$ into two linearly independent components e_0 and e_1:

$$\begin{cases} v_l = u_l e_0 + w_l e_1 \quad \text{and} \\ v_{l'} = u_{l'} e_0 + w_{l'} e_1, \end{cases} \tag{31}$$

with

$$\begin{cases} u_{l,(l')} = 2\, z_{l(l'),0} + z_{l(l'),1} - 1 \quad \text{and} \\ w_{l,(l')} = z_{l(l'),0} + 2\, z_{l(l'),1} - 1, \end{cases} \tag{32}$$

where $z_{l(l'),n}$ is the probability of finding site $l(l')$ in the state n. Defining $(K_1, K_2) = (K\cos\delta, K\sin\delta)$, we rewrite Eq. 30 as

$$\begin{cases} K_1' = K_1(\Omega_{11} + \Omega_{22}) + K_1(\Omega_{12} + \Omega_{21})\cos\frac{2\pi}{p} + K_2(\Omega_{12} - \Omega_{21})\sin\frac{2\pi}{p} \quad \text{and} \\ K_2' = K_2(\Omega_{11} + \Omega_{22}) + K_2(\Omega_{12} + \Omega_{21}\cos\frac{2\pi}{p} - K_1(\Omega_{12} - \Omega_{21})\sin\frac{2\pi}{p}, \end{cases} \tag{33}$$

with

$$\Omega_{11} = \sum_{l,l'} = u_l u_{l'}, \tag{34}$$

$$\Omega_{12} = \sum_{l,l'} = u_l w_{l'},$$

$$\Omega_{21} = \sum_{l,l'} = w_l u_{l'}$$

$$\Omega_{21} = \sum_{l,l'} = w_l u_{l'}.$$

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Declarations

Ethical Approval The authors declare that the present work does not involve any human or animal so no ethical permission is required.

Ethics Conduct This article does not contain any studies involving animals performed by any of the authors.

Conflict of Interest The authors declare no competing interests.

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