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Abstract

We classify contact manifolds (M, \mathcal{D}) which are homogeneous under a connected semisimple Lie group G , and symmetric in the sense that there exists a contactomorphism of (M, \mathcal{D}) normalizing G , fixing a point o in M and restricting to minus identity along \mathcal{D}_o .

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1. Introduction

In a short note published in 1990, the first named author described a simple construction of all contact homogeneous spaces $M = G/H$ of a given Lie group G in terms of the orbits of the coadjoint representation of the group G [1] analogous to the Kirillov–Kostant–Souriau construction of symplectic homogeneous spaces.

A *contact structure* on a smooth manifold M of dimension $2n + 1$ is a maximally non-integrable distribution \mathcal{D} of hyperplanes in the tangent spaces of M ; the pair (M, \mathcal{D}) is then called a *contact manifold*. Locally \mathcal{D} can be given as the field of kernels of a locally defined 1-form θ , called a local *contact form*, and the maximal non-integrability condition refers to the

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fact that $\theta \wedge (d\theta)^n$ is never zero. In case M is transversally orientable, a contact form can be chosen globally and moreover uniquely up to a conformal factor.

The automorphism group of a contact structure is infinite-dimensional; its elements are sometimes called *contactomorphisms*. A contact manifold (M, \mathcal{D}) is called *homogeneous* if it admits a transitive Lie group of contactomorphisms; in this case the contact structure is called *invariant*. The problem of describing invariant contact structures on a homogeneous space of a Lie group can be formulated in terms of Lie algebras. Fix a Lie group G . According to main theorem in [1], the invariant contact structures on a simply-connected homogeneous space M of G fall into one of the two following disjoint classes:

1. M is the universal covering of the projectivization of a conical coadjoint orbit.
2. M is the total space of a 1-dimensional bundle over a covering of a non-conical coadjoint orbit.

Recall that a nonzero coadjoint orbit of a Lie group G is called *conical* if together with any point in the orbit also the positive ray through the point is contained in the orbit. In Sections 2.1 and 2.2 we review in detail and in our context the two constructions above.

Following [8,15,19,22,27] we say that a contact manifold (M, \mathcal{D}) is *symmetric* if each point p of M is fixed under an involutive contactomorphism that restricts to minus identity along \mathcal{D}_p ; such a contactomorphism is called a *symmetry* at p (it does not have to be unique). In the quoted references, the contact manifold carries additional geometric structure related to \mathcal{D} and the symmetries are required to preserve it (e.g. sub-Riemannian, sub-conformal, sub-Hermitian, Cauchy–Riemann or parabolic structure). Contrary to the case of Riemannian symmetric spaces, a contact symmetric space needs not be homogeneous, namely, the group generated by symmetries may act non-transitively on the space. The first known example (an odd-dimensional projective space with two deleted points) was constructed by Lenka Zalabová [27]. For this reason, in this paper we further restrict to homogeneous spaces and we say that a homogeneous contact manifold (M, \mathcal{D}) of a Lie group G is *symmetric* if it admits a symmetry at the basepoint $o = eH \in M$ that normalizes G . Our main result is a complete classification of simply-connected symmetric homogeneous contact manifolds of a semisimple Lie group. For simplicity, we call such objects (*semisimple*) *symmetric contact spaces*.

Some remarks are in order. Although [8] assumes there is a compatible Riemannian structure on the contact distribution, their classification result includes many homogeneous spaces of non-semisimple Lie groups; the existence of a Hermitian Cauchy–Riemann structure on the distribution follows from the compactness of the isotropy group; their examples associated to simple Lie groups are listed in Table 5. More generally [19] considers symmetric Hermitian Cauchy–Riemann structures on distributions more general than of contact type (of arbitrary codimension), albeit with no general classification results. The examples in Tables 6–8 admit an invariant (para-) Cauchy–Riemann structure that is the pullback of an invariant (para-) complex structure on the base coadjoint orbit. The paper [27] considers symmetric contact structures associated to parabolic geometries, which endow the Lie algebra of G with a so-called contact grading; the flat models of such parabolic geometries correspond to our examples of conical type listed in Table 1. Further in [17,18] the authors study parabolic contact manifolds carrying a smooth system of symmetries and give conditions as to when such manifolds are fibered over a reflexion space in the sense of Loos; such spaces are related to our examples of non-conical type listed in Tables 5–8. Finally, we believe that the examples in Tables 2 and 4 are too simple and/or already known, but those in Table 3 are possibly new.

Table 1

Symmetric contact spaces of an absolutely simple group G which are projectivized orbits of a highest root vector (real adjoint varieties).

Cartan's type	\mathfrak{g}	\mathfrak{z}	$W^{\mathbb{C}}$	Depth
AI	$\mathfrak{sl}_n\mathbb{R}$	$\mathfrak{gl}_{n-2}\mathbb{R}$	$\pi_1 + \pi_{n-1}$	2
AIII	$\mathfrak{su}_{p,q}$	$\mathfrak{u}_{p-1,q-1}$	$\pi_1 + \pi_{n-1}$	2
BDI	$\mathfrak{so}_{p,q}$	$\mathfrak{so}_{p-2,q-2} + \mathfrak{sl}_2\mathbb{R}$	$\pi_1 \otimes \pi'_1$	2
CI	$\mathfrak{sp}_n\mathbb{R}$	$\mathfrak{sp}_{n-1}\mathbb{R}$	π_1	2
DIII	\mathfrak{so}_{2n}^*	$\mathfrak{so}_{2n-4}^* + \mathfrak{so}_3$	$\pi_1 \otimes \pi'_1$	2
EI	$\mathfrak{e}_{6(6)}$	$\mathfrak{sl}_6\mathbb{R}$	π_3	2
EII	$\mathfrak{e}_{6(2)}$	$\mathfrak{su}_{3,3}$	π_3	2
EIII	$\mathfrak{e}_{6(-14)}$	$\mathfrak{su}_{1,5}$	π_3	2
EV	$\mathfrak{e}_{7(7)}$	$\mathfrak{so}_{6,6}$	π_6	2
EVI	$\mathfrak{e}_{7(-5)}$	\mathfrak{so}_{12}^*	π_6	2
EVII	$\mathfrak{e}_{7(-25)}$	$\mathfrak{so}_{2,10}$	π_6	2
EVIII	$\mathfrak{e}_{8(8)}$	$\mathfrak{e}_{7(7)}$	π_7	2
EIX	$\mathfrak{e}_{8(-24)}$	$\mathfrak{e}_{7(-25)}$	π_7	2
FI	$\mathfrak{f}_{4(4)}$	$\mathfrak{sp}_3\mathbb{R}$	π_3	2
G	$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_2\mathbb{R}$	$3\pi_1$	2

Table 2

Symmetric contact space of group $G_{2(2)}$ which is the projectivized orbit of a short root vector.

\mathfrak{g}	\mathfrak{z}	$W^{\mathbb{C}}$	$V^{\mathbb{C}}$	Depth
$\mathfrak{g}_{2(2)}$	$\mathfrak{sl}_2\mathbb{R}$	$\pi_1 + \pi_1 + \pi_1$	π_1	3

Table 3

Symmetric contact spaces of an absolutely simple G which are projectivized orbits of an even nilpotent element.

$(\mathfrak{g}, \mathfrak{s} + \mathfrak{z})$	$(V + W)^{\mathbb{C}}$	Depth	Conditions
$(\mathfrak{sl}_3\mathbb{R}, \mathfrak{so}_{1,2})$	$4\pi_1$	4	—
$(\mathfrak{su}_{1,2}, \mathfrak{so}_{1,2})$	$4\pi_1$	4	—
$(\mathfrak{so}_{p,q}, \mathfrak{so}_{1,2} \oplus \mathfrak{so}_{p-1,q-2})$	$\pi_1 \otimes \pi'_1$	2	$p \geq 1, q \geq 2$

Table 4

Symmetric contact spaces $M = G/H$ with non-absolutely simple Lie groups G .

\mathfrak{g}	\mathfrak{s}	\mathfrak{z}	$(V + W)^{\mathbb{C}}$	Depth
$\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{sl}_2\mathbb{R}$	Diagonal	0	$2\pi_1$	2
$\mathfrak{sl}_2\mathbb{C}$	Normal real form	0	$2\pi_1$	2

1.1. Summary of arguments and results

Since we are assuming the group G semisimple, we may identify the dual space \mathfrak{g}^* of the Lie algebra with the Lie algebra \mathfrak{g} via the Killing form and identify coadjoint orbits with adjoint orbits. Then the class (1) is identified with the projectivization $P\text{Ad}_G e = \text{Ad}_G(\mathbb{R}e) \subset P\mathfrak{g}$ of the adjoint orbit of a nilpotent element $e \in \mathfrak{g}$, and contact manifolds in class (2) can then be described in terms of one-dimensional bundles over non-nilpotent adjoint orbits.

Table 5
Hermitian CR symmetric contact spaces G/H .

\mathfrak{g}	$\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}$
\mathfrak{su}_n	$\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathfrak{u}_1$
$\mathfrak{su}_{p,n-p}$	$\mathfrak{su}_p \oplus \mathfrak{su}_{n-p} \oplus \mathfrak{u}_1$
\mathfrak{so}_{2n}^*	$\mathfrak{su}_n \oplus \mathfrak{so}_2$
\mathfrak{so}_{2n}	$\mathfrak{su}_n \oplus \mathfrak{so}_2$
\mathfrak{so}_n	$\mathfrak{so}_{n-2} \oplus \mathfrak{so}_2$
$\mathfrak{so}_{n-2,2}$	$\mathfrak{so}_{n-2} \oplus \mathfrak{so}_2$
$\mathfrak{sp}_n(\mathbb{R})$	$\mathfrak{su}_n \oplus \mathfrak{so}_2$
\mathfrak{sp}_n	$\mathfrak{su}_n \oplus \mathfrak{so}_2$
\mathfrak{e}_6	$\mathfrak{so}_{10} \oplus \mathfrak{so}_2$
$\mathfrak{e}_6(-14)$	$\mathfrak{so}_{10} \oplus \mathfrak{so}_2$
\mathfrak{e}_7	$\mathfrak{e}_6 \oplus \mathfrak{so}_2$
$\mathfrak{e}_7(-25)$	$\mathfrak{e}_6 \oplus \mathfrak{so}_2$
$\mathfrak{so}_{p,1}$	$\mathfrak{so}_{p-1} \oplus \mathbb{R}$

Table 6
Pseudo-Hermitian CR symmetric contact spaces G/H .

\mathfrak{g}	$\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}$
$\mathfrak{su}_{p,q}$	$\mathfrak{su}_{r,s} \oplus \mathfrak{su}_{p-r,q-s} \oplus \mathfrak{so}_2$ ($r > 0$ and $s > 0$)
$\mathfrak{sl}_{2n}(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{so}_2$
\mathfrak{su}_{2n}^*	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathfrak{so}_2$
\mathfrak{so}_{2n}^*	$\mathfrak{su}_{p,n-p} \oplus \mathfrak{so}_2$ ($0 < p < n$)
\mathfrak{so}_{2n}	$\mathfrak{so}_{2n-2}^* \oplus \mathfrak{so}_2$
$\mathfrak{so}_{p,q}$	$\mathfrak{so}_{p-2,q} \oplus \mathfrak{so}_2$ ($p > 2$ and $q > 0$)
$\mathfrak{so}_{2p,2q}$	$\mathfrak{su}_{p,q} \oplus \mathfrak{so}_2$ ($p > 0$ and $q > 0$)
$\mathfrak{sp}_n(\mathbb{R})$	$\mathfrak{su}_{p,n-p} \oplus \mathfrak{so}_2$ ($0 < p < n$)
$\mathfrak{sp}_{p,q}$	$\mathfrak{su}_{p,q} \oplus \mathfrak{so}_2$ ($p > 0$ and $q > 0$)
$\mathfrak{e}_6(-14)$	$\mathfrak{so}_{2,8} \oplus \mathfrak{so}_2$
$\mathfrak{e}_6(-14)$	$\mathfrak{so}_{10}^* \oplus \mathfrak{so}_2$
$\mathfrak{e}_6(2)$	$\mathfrak{so}_{10}^* \oplus \mathfrak{so}_2$
$\mathfrak{e}_6(2)$	$\mathfrak{so}_{4,6} \oplus \mathfrak{so}_2$
$\mathfrak{e}_7(7)$	$\mathfrak{e}_6(2) \oplus \mathfrak{so}_2$
$\mathfrak{e}_7(-5)$	$\mathfrak{e}_6(2) \oplus \mathfrak{so}_2$
$\mathfrak{e}_7(-5)$	$\mathfrak{e}_6(-14) \oplus \mathfrak{so}_2$
$\mathfrak{e}_7(-25)$	$\mathfrak{e}_6(-14) \oplus \mathfrak{so}_2$

Our starting point for the classification of semisimple symmetric contact spaces in class (1), namely, those of projective type, is the Morozov–Jacobson theorem, which allows to include a nilpotent element $e \in \mathfrak{g}$ into a standard basis (h, e, f) , called a \mathfrak{sl}_2 -triple, of a 3-dimensional subalgebra \mathfrak{s} . Denote by $\mathfrak{z} = Z_{\mathfrak{g}}(\mathfrak{s})$ the centralizer of \mathfrak{s} , by $N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s} + \mathfrak{z}$ the normalizer of \mathfrak{s} , and by \mathfrak{q} a reductive complement of $N_{\mathfrak{g}}(\mathfrak{s})$ in \mathfrak{g} . The stability subalgebra of the projectivized orbit $M = \text{Ad}_G(\mathbb{R}e) \subset P\mathfrak{g}$ can be written as

$$\mathfrak{h} = N_{\mathfrak{g}}(\mathbb{R}e) = \mathbb{R}h + \mathbb{R}e + \mathfrak{z} + V$$

where $V = Z_{\mathfrak{q}}(e)$ is the span of highest weight vectors in the \mathfrak{s} -module \mathfrak{q} . Denote by W the ad_h -invariant complementary subspace to V such that $\mathfrak{q} = V + W$. Then $\mathfrak{g} = \mathfrak{h} + \mathfrak{m}$ where $\mathfrak{m} = \mathbb{R}f + W$ is identified with the tangent space T_oM and W is identified with the contact hyperplane \mathcal{D}_o . The semisimple element h defines a gradation $\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i$ where \mathfrak{g}^i denotes

the i -eigenspace; the largest m such that $\mathfrak{g}^m \neq 0$ is called the *depth* of the gradation. We also say that e and the corresponding \mathfrak{sl}_2 -subalgebra \mathfrak{s} are *odd* (resp. *even*) if this gradation is odd (resp. even), which means that there exists (resp. does not exist) an odd number j with $\mathfrak{g}^j \neq 0$.

In the case in which the Lie algebra is absolutely simple, we prove the following theorem which describes all symmetric contact spaces $M = G/H$ which are projectivized orbits of odd nilpotent elements. A depth 2 gradation

$$\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2 \quad (1)$$

is called a *contact gradation* if $\dim \mathfrak{g}^{\pm 2} = 1$ and the bilinear form $\mathfrak{g}^{-1} \times \mathfrak{g}^{-1} \rightarrow \mathfrak{g}^{-2}$ induced by the Lie bracket is non-degenerate. It is a real form of the canonical contact gradation of the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ constructed from the regular 3-dimensional subalgebra $\mathfrak{s}^{\mathbb{C}}(\mu) = \mathbb{C}h_{\mu} + \mathbb{C}e_{\mu} + \mathbb{C}f_{\mu}$ associated with a long root μ ; this gradation is the eigenspace decomposition of $\text{ad}_{h_{\mu}}$ and $(\mathfrak{g}^{\mathbb{C}})^2 = \mathbb{C}e_{\mu}$, $(\mathfrak{g}^{\mathbb{C}})^0 = \mathbb{C}h_{\mu} + Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}}(\mu))$. All contact gradations on real simple Lie algebras are known and listed, for instance, in [17, Table 1].

Theorem 1.1. *For an absolutely simple Lie algebra \mathfrak{g} and a contact gradation (1), the projectivized orbit $M_e = \text{Ad}_G(\mathbb{R}e)$ of the nilpositive element $e \in \mathfrak{g}^2$ is a symmetric contact space, and these manifolds exhaust all symmetric contact spaces which are projectivized orbits of odd nilpotent elements if \mathfrak{g} is not of \mathbf{G}_2 -type. For the normal real form of \mathbf{G}_2 -type, the projectivized orbit of the nilpositive element of the regular 3-subalgebra $\mathfrak{s}(\beta)$ associated with a short root β is also a symmetric contact space; the associated gradation has depth 3. The complete list is given in Tables 1 and 2.*

The projectivized orbits M_e are called *real adjoint varieties* and have many remarkable properties. The associated symmetric decomposition has the form

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- = (\mathfrak{s} + \mathfrak{z}) + (V + W).$$

In particular, the corresponding manifold G/G_+ is a *symmetric para-quaternionic Kähler space* [2,12].

In case of the projectivized orbit of a long root vector, the \mathfrak{z} -modules V and W are isomorphic; in Table 1 we indicate the subalgebra \mathfrak{z} and the complexification of its representation on $\mathfrak{g}^{-1} = W$, which is always of quaternionic type.

Throughout this paper, we indicate the (complex) representations by their highest weights and denote the i th fundamental weight of a complex simple Lie algebra by π_i (cf. [16, Table 1, p. 224]).

The case of the projectivized orbit of a short root occurs for $\mathfrak{g} = \mathfrak{g}_{2(2)}$ only; the situation is summarized in Table 2.

The following theorem describes symmetric contact spaces of which are projectivized orbits of even nilpotent elements in an absolutely simple Lie algebra \mathfrak{g} . For an even nilpotent element $e \in \mathfrak{g}$ and the corresponding $\mathfrak{sl}_2\mathbb{R}$ -subalgebra \mathfrak{s} , the complexification $\mathfrak{s}^{\mathbb{C}}$ is an $\mathfrak{sl}_2\mathbb{C}$ -subalgebra of $\mathfrak{g}^{\mathbb{C}}$ whose adjoint action on $\mathfrak{g}^{\mathbb{C}}$ admits irreducible components of odd dimension only, hence it defines an SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$ in the sense of Vinberg [25]. A symmetric contact space which is the projectivized orbit of an even nilpotent element in \mathfrak{g} gives rise to an SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$ which is of *symmetric type*, in the sense that the normalizer $N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$ is a symmetric subalgebra of $\mathfrak{g}^{\mathbb{C}}$. In this case we will prove that the dimensions of the irreducible components of the adjoint action of $\mathfrak{s}^{\mathbb{C}}$ on $\mathfrak{g}^{\mathbb{C}}$ do not exceed 5 and hence the SO_3 -structure is in addition *short*. In Section 5.2 we will refer to the classification of short SO_3 -structures on complex simple Lie algebras in [25] and check which ones are of symmetric type to prove the following theorem.

Theorem 1.2. *For an absolutely simple Lie algebra \mathfrak{g} , the symmetric contact spaces that are projectivized orbits of even nilpotent elements of \mathfrak{g} are in bijective correspondence with the isomorphism classes of short SO_3 -structures of symmetric type on $\mathfrak{g}^{\mathbb{C}}$. The complete list is given in Table 3.*

Some remarks about the table are in order. Recall that $\mathfrak{s} + \mathfrak{z}$ embeds into \mathfrak{g} as a symmetric subalgebra. The case $p = q = 3$ in Table 3 gives

$$(\mathfrak{so}_{3,3}, \mathfrak{so}_{1,2} \oplus \mathfrak{so}_{2,1}) = (\mathfrak{sl}_4\mathbb{R}, \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{sl}_2\mathbb{R}).$$

We have $\mathfrak{su}_4^* = \mathfrak{sl}_2\mathbb{H} = \mathfrak{so}_{5,1}$, $\mathfrak{su}_{2,2} = \mathfrak{so}_{2,4}$, $\mathfrak{so}_{2,2} = \mathfrak{so}_{1,2} \oplus \mathfrak{so}_{2,1}$, and $\mathfrak{so}_4^* = \mathfrak{sl}_2\mathbb{H} = \mathfrak{so}_{1,2} \oplus \mathfrak{so}_3$.

The following theorem explains the case of a non-absolutely simple Lie algebra, in which the possibilities are very limited.

Theorem 1.3. *For a non-absolutely simple Lie algebra \mathfrak{g} , the only possible symmetric contact spaces that are projectivized orbits of a nilpotent element in \mathfrak{g} are described as follows:*

- (i) $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{sl}_2\mathbb{R}$, $\mathfrak{z} = 0$, $\mathfrak{g}_+ = \mathfrak{s}$ is the diagonal subalgebra and $V + W$ is the skew-diagonal.
- (ii) $\mathfrak{g} = \mathfrak{sl}_2\mathbb{C}$, $\mathfrak{z} = 0$, $\mathfrak{g}_+ = \mathfrak{s}$ is the normal real form of \mathfrak{g} and $V + W = i\mathfrak{s}$.

(cf. Table 4).

The last theorem describes all symmetric contact spaces $M = G/H$ of a semisimple Lie group G which are associated with non-nilpotent orbits (non-conical type). We prove that M is a canonical contactization of a symplectic symmetric space [7]. More precisely, let (N, ω) be a symplectic manifold. It is called *quantizable* if there exists a principal bundle $\pi : P \rightarrow M$ with one-dimensional structure group $A = \mathbb{R}$ or S^1 and connection θ such that $d\theta = \pi^*\omega$. The contact manifold $(M, \ker\theta)$ is called a *contactization* of N , see [3].

Let $N = \mathrm{Ad}_G\xi = G/K \subset \mathfrak{g}$ be a non-nilpotent adjoint orbit which is a symmetric symplectic manifold with symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and symplectic form defined by the $\mathrm{ad}_{\mathfrak{k}}$ -invariant closed 2-form $\omega(x, y) = d(B \circ \xi)(x, y) := -B(\xi, [x, y])$ for $x, y \in \mathfrak{p}$, where B is the Killing form of \mathfrak{g} . Then ξ is semisimple and the centralizer $K = Z_G(\xi)$ is connected. Let \mathfrak{h} be the B -orthogonal complement to ξ in \mathfrak{k} . Then \mathfrak{h} is a codimension one ideal of \mathfrak{k} . Assume that the connected subgroup H generated by \mathfrak{h} is a closed subgroup of K . Then the principal $A = K/H$ -bundle $M = G/H \rightarrow N = G/K$ is a contactization of N . In fact, the 1-form $\theta := B \circ \xi$ is Ad_H -invariant and defines a contact form θ on $M = G/H$ with associated contact distribution \mathcal{D} which is the invariant extension of the hyperplane $\mathfrak{p} \subset T_oM$. We prove that every symmetric contact space M is of this form.

Theorem 1.4. *Let G be a connected semisimple Lie group with Lie algebra \mathfrak{g} and let $N = \mathrm{Ad}_G\xi = G/K$ be a (semisimple) adjoint orbit which is a symplectic symmetric space. Assume that the B -orthogonal complement to ξ in $\mathfrak{k} = Z_{\mathfrak{g}}(\xi)$ generates a closed subgroup of the centralizer $K = Z_G(\xi)$. Then N admits a contactization $M = G/H$ which is a symmetric contact space of G . Conversely, every (semisimple) symmetric contact space of non-conical type arises in this way.*

The semisimple symplectic symmetric spaces were introduced and classified by P. Bieliavsky [7]. Every simply-connected symplectic symmetric space of a connected semisimple Lie group is a direct product of symplectic symmetric spaces of simple Lie groups. The lists of all possible pairs $(\mathfrak{g}, \mathfrak{k})$ for \mathfrak{g} simple are given in Tables 5–8. The tables are organized and labeled by the existing type of induced CR structure on the contact distribution of G/H .

Table 7

Para-pseudo-Hermitian CR symmetric contact spaces G/H .

\mathfrak{g}	$\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}$
$\mathfrak{sl}_n(\mathbb{R})$	$\mathfrak{sl}_p(\mathbb{R}) \oplus \mathfrak{sl}_{n-p}(\mathbb{R}) \oplus \mathbb{R}$
\mathfrak{su}_{2n}^*	$\mathfrak{su}_{2p}^* \oplus \mathfrak{su}_{2n-2p}^* \oplus \mathbb{R}$
$\mathfrak{su}_{n,n}$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{R}$
\mathfrak{so}_{4n}^*	$\mathfrak{su}_{2n}^* \oplus \mathbb{R}$
$\mathfrak{so}_{p,q}$	$\mathfrak{so}_{p-2,q} \oplus \mathbb{R}$
$\mathfrak{so}_{n,n}$	$\mathfrak{sl}_n(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{so}_{p,q}$	$\mathfrak{so}_{p-1,q-1} \oplus \mathbb{R}$
$\mathfrak{sp}_n(\mathbb{R})$	$\mathfrak{sl}_n(\mathbb{R}) \oplus \mathbb{R}$
$\mathfrak{sp}_{n,n}$	$\mathfrak{su}_{2n}^* \oplus \mathbb{R}$
$\mathfrak{e}_{6(6)}$	$\mathfrak{so}_{5,5} \oplus \mathbb{R}$
$\mathfrak{e}_{6(-26)}$	$\mathfrak{so}_{1,9} \oplus \mathbb{R}$
$\mathfrak{e}_{7(7)}$	$\mathfrak{e}_{6(6)} \oplus \mathbb{R}$
$\mathfrak{e}_{7(-25)}$	$\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$

Table 8

CR symmetric contact spaces G/H with pseudo-Hermitian and para-pseudo-Hermitian structures.

\mathfrak{g}	$\mathfrak{k} = \mathfrak{h} \oplus \mathbb{R}$
$\mathfrak{sl}_n(\mathbb{C})$	$\mathfrak{sl}_p(\mathbb{C}) \oplus \mathfrak{sl}_{n-p}(\mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{so}_{2n}(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{so}_n(\mathbb{C})$	$\mathfrak{so}_{n-2}(\mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{sp}_n(\mathbb{C})$	$\mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{e}_6^{\mathbb{C}}$	$\mathfrak{so}_{10}(\mathbb{C}) \oplus \mathbb{C}$
$\mathfrak{e}_7^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}} \oplus \mathbb{C}$

2. Homogeneous contact manifolds of a semisimple Lie group

The main result of this work is a classification of symmetric contact spaces of semisimple Lie groups according to the following definition.

Definition 2.1. A homogeneous contact manifold $(M = G/H, \mathcal{D})$ is called a *symmetric contact space* if there is an involutive contactomorphism of (M, \mathcal{D}) that fixes the point $o = eH$, acts on the contact subspace \mathcal{D}_0 as minus identity, and normalizes G .

We first give a description of homogeneous contact manifolds $(M = G/H, \mathcal{D})$ of a semisimple Lie group G , which follows from the general construction of homogeneous contact manifolds of a Lie group given in terms of coadjoint orbits [1]. In the semisimple case, we may use the Killing form to identify coadjoint orbits and adjoint orbits.

There are two types of homogeneous contact manifolds: manifolds of conical type (constructed as the projectivization of a nilpotent orbit), and manifolds of non-conical type (constructed as homogeneous line bundles over a non-nilpotent adjoint orbit). We next describe the structure of such manifolds.

2.1. Manifolds of conical type

We describe the invariant contact structure on the projectivization $M = P\text{Ad}_G e = G/N_G(\mathbb{R}e) \subset P\mathfrak{g}$ of a nilpotent orbit $N = \text{Ad}_G e$ (that is, the orbit of a nilpotent element $e \in \mathfrak{g}$). They are characterized as contact manifolds that have no invariant contact form. There are only finitely many such manifolds for any semisimple Lie group G .

By the real version of the Morozov–Jacobson theorem [11], we can find elements $h, f \in \mathfrak{g}$ such that (h, e, f) is an \mathfrak{sl}_2 -triple in \mathfrak{g} , namely, the following bracket relations are satisfied:

$$[h, e] = 2e, [h, f] = -2f, [e, f] = h.$$

Denote by \mathfrak{s} the corresponding $\mathfrak{sl}_2\mathbb{R}$ -subalgebra of \mathfrak{g} . Recall that the finite-dimensional real irreducible representations of $\mathfrak{sl}_2\mathbb{R}$ are precisely the real forms of complex irreducible representations of $\mathfrak{sl}_2\mathbb{C}$ [20].

To describe the infinitesimal structure of such homogeneous manifolds, we introduce the following notation:

$$\begin{aligned} \mathfrak{z} = Z_{\mathfrak{g}}(\mathfrak{s}) &: \text{the centralizer of } \mathfrak{s} \text{ in } \mathfrak{g} \\ V &: \text{the span of all highest weight vectors of} \\ &\quad \text{irreducible } \mathfrak{s}\text{-submodules of } \mathfrak{g} \text{ other than } \mathfrak{s} \\ W = \sum_{i>0} \text{ad}_f^i V &: \text{the remaining weight spaces not contained in } \mathfrak{s} \\ \mathfrak{k} = Z_{\mathfrak{g}}(e) = \mathbb{R}e + \mathfrak{z} + V &: \text{the centralizer of } e \text{ in } \mathfrak{g} \\ \mathfrak{h} = N_{\mathfrak{g}}(\mathbb{R}e) = \mathbb{R}h + \mathbb{R}e + \mathfrak{z} + V &: \text{the normalizer of the line } \mathbb{R}e \text{ in } \mathfrak{g} \\ B &: \text{the ad-invariant symmetric bilinear form on} \\ &\quad \mathfrak{g}, \text{ normalized so that } B(e, f) = 1. \end{aligned}$$

Note that $N = G/K$ and $M = G/H$ as homogeneous spaces, where $H = N_G(\mathbb{R}e)$ is a closed subgroup of G with Lie algebra \mathfrak{h} and $K = Z_G(e)$ is a closed codimension one subgroup of H with Lie algebra \mathfrak{k} .

The subspace $\mathfrak{m} = \mathbb{R}f + W$ is a complementary subspace to \mathfrak{h} in \mathfrak{g} which is identified with the tangent space $T_o M = \mathfrak{g}/\mathfrak{h}$ at the point $o = eH$, so that we have a (non-reductive) decomposition

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathbb{R}h + \mathbb{R}e + \mathfrak{z} + V) + (\mathbb{R}f + W). \quad (2)$$

Definition 2.2. The decomposition (2) is called the *canonical decomposition* of \mathfrak{g} associated to the \mathfrak{sl}_2 -triple (h, e, f) .

We denote by $\theta = B \circ e$ the linear form on \mathfrak{g} dual to the vector e . Since H preserves θ up to a multiple, the kernel $\ker \theta = \mathfrak{h} + W$ defines an H -invariant subspace $\mathcal{D}_0 = (\mathfrak{h} + W)/\mathfrak{h} \cong W$ of $T_o M$. We extend it to an invariant codimension one distribution \mathcal{D} in M .

Proposition 2.1. *The distribution \mathcal{D} is an invariant contact distribution on the manifold $M = G/H = P\text{Ad}_G e$.*

Proof. It is sufficient to check that $d\theta$ is not degenerate on W . We show that $\ker d\theta = \mathfrak{k}$. Indeed, if $x \in \ker d\theta$, then

$$0 = d\theta(x, \mathfrak{g}) = -\theta(\text{ad}_x \mathfrak{g}) = (\text{ad}_x^* \theta)(\mathfrak{g})$$

which means that $x \in Z_{\mathfrak{g}}(\theta) = Z_{\mathfrak{g}}(e) = \mathfrak{k}$. \square

2.2. Manifolds of non-conical type

We start with a characterization of a conical orbit.

Lemma 2.2 ([1]). *Let G be a connected Lie group and let $\theta \in \mathfrak{g}^*$ be a 1-form. Denote by \mathfrak{k} the centralizer $Z_{\mathfrak{g}}(\theta)$. Then the coadjoint orbit $N = \text{Ad}_G^*(\theta)$ is conical if and only if $\theta(\mathfrak{k}) = 0$.*

Proof. Let $G_{\mathbb{R}\theta}$ (resp. $\mathfrak{g}_{\mathbb{R}\theta}$) denote the normalizer of the line $\mathbb{R}\theta$ in G (resp. \mathfrak{g}). The action of $G_{\mathbb{R}\theta}$ (resp. $\mathfrak{g}_{\mathbb{R}\theta}$) on $\mathbb{R}\theta$ defines a homomorphism $\ell : G_{\mathbb{R}\theta} \rightarrow \mathbb{R}^+$ (resp. $d\ell : \mathfrak{g}_{\mathbb{R}\theta} \rightarrow \mathbb{R}$). The surjectivity of ℓ is equivalent to that of $d\ell$, so N is conical if and only if there exists $z \in \mathfrak{g}$ such that $\text{ad}_z^*\theta = \theta$.

On the other hand, the linear map $L_\theta : X \in \mathfrak{g} \mapsto \text{ad}_X^*\theta = -\theta \circ \text{ad}_X \in \mathfrak{g}^*$ is skew-symmetric in the sense that $(L_\theta X)Y = (L_\theta Y)X$ for all $X, Y \in \mathfrak{g}$. It follows that $\text{im}(L_\theta) = \text{ann}(\ker L_\theta) = \text{ann}(\mathfrak{k})$. This shows that conicity of the coadjoint orbit through θ is equivalent to $\theta \in \text{ann}(\mathfrak{k})$, so we are done. \square

Let G be a connected semisimple Lie group and fix $\theta \in \mathfrak{g}^*$ such that $N = \text{Ad}_G^*(\theta) \cong G/K$ is a non-conical orbit, where $K = Z_G(\theta)$. Then θ does not vanish identically on \mathfrak{k} , where $\mathfrak{k} = Z_{\mathfrak{g}}(\theta)$ is the Lie algebra of K . Let

$$\mathfrak{h} = \ker \theta \cap \mathfrak{k}.$$

As in Sub Section 2.1, one sees that $\mathfrak{k} = \ker d\theta$. It follows that

$$\theta([\mathfrak{k}, \mathfrak{g}]) = d\theta(\mathfrak{k}, \mathfrak{g}) = 0$$

proving that \mathfrak{h} is a codimension one ideal of \mathfrak{k} . Let H be the associated connected subgroup of K . If H is closed, then $M := G/H$ is a homogeneous space which is the total space of a 1-dimensional bundle over N and which carries a natural contact structure:

Proposition 2.3. *The 1-form θ defines an invariant contact form on the manifold $M = G/H$.*

Proof. Since $\theta|_{\mathfrak{h}} \equiv 0$, θ induces a Ad_H -invariant element of $\mathfrak{g}^*/\mathfrak{h}^* \cong (\mathfrak{g}/\mathfrak{h})^* = T_o^*M$, so a globally defined invariant 1-form on M . Note that $d\theta$ is the pull-back of the Kirillov–Kostant–Souriau form on $N \cong G/K$ under $G/H \rightarrow G/K$, so θ is a contact form on M , whose associated contact distribution is denoted by \mathcal{D} . \square

Denote by ξ the element of \mathfrak{g} dual to θ under the Killing form B . Note that ξ can be B -isotropic, but we can choose $\eta \in \mathfrak{k} \setminus \mathfrak{h}$ such that $\theta(\eta) = B(\xi, \eta) = 1$, η generates a closed subgroup C of K and $K = C \rtimes H$ (the semi-direct product; compare [16, Thm. 3.1, p. 51]). We may now write

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{h} + \mathbb{R}\eta, \tag{3}$$

where \mathfrak{p} is a subspace complementary to \mathfrak{h} in $\ker \theta$. Note that $d\theta$ is non-degenerate on \mathfrak{p} .

3. Symmetric contact manifolds of conical type

The main result of this section is Proposition 3.1 which reduces the problem of classification of symmetric contact spaces of conical type of a semisimple Lie group G to the description of \mathfrak{sl}_2 -subalgebras of symmetric type of the Lie algebra \mathfrak{g} . We will use the notation from Section 2.1 and refer to the canonical decomposition (2).

3.1. Gradation associated with an \mathfrak{sl}_2 -triple

The semisimple element h of an \mathfrak{sl}_2 -triple (h, e, f) defines a gradation of the Lie algebra:

$$\mathfrak{g} = \sum_{i \in \mathbb{Z}} \mathfrak{g}^i,$$

where \mathfrak{g}^i is the i -eigenspace of ad_h . The largest index m such that $\mathfrak{g}^m \neq 0$ is called the *depth* of the gradation. The gradation is called *odd* if $\mathfrak{g}^i \neq 0$ for some odd i and *even* otherwise.

In our setting

$$\mathfrak{g}^0 = \mathfrak{z} + \mathbb{R}h + W^0 \quad \text{where} \quad W^i = W \cap \mathfrak{g}^i$$

and

$$V = \sum_{i>0} V^i \quad \text{where} \quad V^i = V \cap \mathfrak{g}^i.$$

Definition 3.1. Let \mathfrak{g} be a semisimple Lie algebra. Consider an \mathfrak{sl}_2 -triple (h, e, f) of \mathfrak{g} and the associated 3-dimensional subalgebra \mathfrak{s} .

- (i) We say that (h, e, f) and \mathfrak{s} are *odd* (resp. *even*) if the associated ad_h -gradation of \mathfrak{g} is odd (resp. even);
- (ii) We say that (h, e, f) and \mathfrak{s} are *of symmetric type* if the normalizer $N_{\mathfrak{g}}(\mathfrak{s})$ is a symmetric subalgebra, that is, the reductive decomposition

$$\mathfrak{g} = N_{\mathfrak{g}}(\mathfrak{s}) + \mathfrak{q} = (\mathfrak{s} + \mathfrak{z}) + (V + W)$$

is a symmetric decomposition. (Note that in this case the homogeneous manifold $G/N_G(\mathfrak{s})$, which is the space of all 3-dimensional subalgebras conjugate to \mathfrak{s} , is a para-quaternionic Kähler symmetric space [2].)

The following proposition reduces the classification of contact symmetric spaces of conical type of a semisimple Lie group G to the description of $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras of symmetric type of the Lie algebra \mathfrak{g} .

Proposition 3.1. *The universal covering of the homogeneous contact manifold $M = G/H = \text{Ad}_G(\mathbb{R}e) \subset P\mathfrak{g}$ of conical type is a symmetric contact space if and only if e can be included in an \mathfrak{sl}_2 -triple of symmetric type.*

Proof. Let $M = G/H = \text{Ad}_G(\mathbb{R}e)$ be a symmetric contact space which is the projectivization of a nilpotent orbit and consider the canonical decomposition (2)

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{m} = (\mathbb{R}h + \mathbb{R}e + \mathfrak{z} + V) + (\mathbb{R}f + W).$$

If (h, e, f) is an \mathfrak{sl}_2 -triple of symmetric type, then

$$\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_- = (\mathfrak{s} + \mathfrak{z}) + (V + W)$$

under an involution s . Let \tilde{G} be the simply-connected Lie group with Lie algebra \mathfrak{g} , and \tilde{H} the connected subgroup for \mathfrak{h} ; it is known that $\tilde{H} = Z_{\tilde{G}}(e)^0$ [11, § 6.1]. Now $\tilde{M} = \tilde{G}/\tilde{H}$ (almost effective presentation) is a simply-connected homogeneous contact manifold of conical type covering M . The involutive automorphism s of \mathfrak{g} integrates to an involutive automorphism of \tilde{G} which preserves the stability subgroup \tilde{H} and hence induces an involutive contactomorphism of \tilde{M} fixing the point o and acting on \mathcal{D}_o as -1 . Hence \tilde{M} is a symmetric contact space.

Next we prove the converse statement. Assume that the universal covering \tilde{M} of M is a symmetric contact space. The symmetry at the basepoint of \tilde{M} induces an involutive automorphism s of \mathfrak{g} with symmetric decomposition $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ such that $s[\mathfrak{h}] = \mathfrak{h}$ and $W \subset \mathfrak{g}_-$.

Note that $\mathfrak{g}^{-2} = \mathbb{R}f + W^{-2}$. Since $d\theta$ is non-degenerate on W , we can find $x \in W^i$, $y \in W^j$ with $i + j = -2$ and $-\theta([x, y]) = d\theta(x, y) = -1$. Now

$$s[x, y] = s(f + [x, y]_W) = sf - [x, y]_W$$

and

$$[sx, sy] = [-x, -y] = f + [x, y]_W$$

implying that $sf = f$.

Since the ± 1 -eigendecomposition $\mathfrak{m} = \mathbb{R}f + W$ under s is ad_h -invariant, we see that ad_h and s commute on \mathfrak{m} , so

$$\text{ad}_h \circ s = s \circ \text{ad}_h = \text{ad}_{sh} \circ s$$

as operators on \mathfrak{m} . Put $z = h - sh \in Z_{\mathfrak{h}}(f) = \mathfrak{z}$. Since z centralizes W and e , it also centralizes V . Consider the adjoint action of \mathfrak{z} on $\mathfrak{s} + V + W$. The kernel \mathfrak{n}_1 of this action is an ideal of \mathfrak{g} contained in \mathfrak{h} , thus $\mathfrak{n}_1 = 0$. We have proved above that $z \in \mathfrak{n}_1$, hence $sh = h$.

Now it follows from the theory of \mathfrak{sl}_2 -triples that $se = e$ [16, Prop. 2.1, ch. 6, p. 194]. We deduce that

$$s|_{\mathfrak{s}} = \text{id}, \text{ and thus } s(\mathfrak{z}) = \mathfrak{z} \text{ and } s(V) = V.$$

For $0 \neq x \in V$, we have $0 \neq [f, x] \in W$ so

$$-[f, x] = s[f, x] = [sf, sx] = [f, sx]$$

implying that $sx + x$ is an element of V that centralizes f , hence zero. This proves $s|_V = -1$.

We have already shown that

$$\mathfrak{g}_+ = \mathfrak{s} + \mathfrak{z}_+ \quad \text{and} \quad \mathfrak{g}_- = \mathfrak{z}_- + V + W$$

where $\mathfrak{z} = \mathfrak{z}_+ + \mathfrak{z}_-$ under s . It only remains to check that $\mathfrak{z}_- = 0$.

We first claim that $\mathfrak{z}_+ + \mathfrak{s} + V + W$ is a subalgebra of \mathfrak{g} and

$$\mathfrak{g} = (\mathfrak{z}_+ + \mathfrak{s} + V + W) + \mathfrak{z}_-$$

is a reductive decomposition; indeed this follows from

$$[V + W, V + W] \subset [\mathfrak{g}_-, \mathfrak{g}_-] \subset \mathfrak{g}_+ = \mathfrak{s} + \mathfrak{z}_+$$

and

$$[\mathfrak{z}_-, V + W] \subset (V + W) \cap \mathfrak{g}_+ = 0.$$

Next we can consider the kernel \mathfrak{n}_2 of the adjoint representation of $\mathfrak{z}_+ + \mathfrak{s} + V + W$ on \mathfrak{z}_- . Of course \mathfrak{n}_2 is an ideal of \mathfrak{g} , and we have seen that it contains $\mathfrak{s} + V + W$. Since

$$\begin{aligned} B(\mathfrak{z}, \mathfrak{s} + V + W) &= B(\mathfrak{z}, \mathbb{R}h + W^0) \\ &= B(\mathfrak{z}, \text{ad}_e(\mathbb{R}f + W^{-2})) \\ &= B(\text{ad}_e \mathfrak{z}, \mathbb{R}f + W^{-2}) \\ &= 0, \end{aligned}$$

the Killing orthogonal \mathfrak{n}_2^\perp is contained in $\mathfrak{z} \subset \mathfrak{h}$ and thus $\mathfrak{n}_2^\perp = 0$. This implies $\mathfrak{z}_- = 0$, as desired. \square

4. Contact symmetric space associated with contact gradation of a semisimple Lie algebra

Definition 4.1 (*Čap-Slovak*). A depth 2 gradation $\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$ of a real (resp. complex) semisimple Lie algebra \mathfrak{g} is called a *contact gradation* (resp. complex contact gradation) if

- (a) $\dim \mathfrak{g}^{-2} = 1$; and
- (b) the skew-symmetric bilinear form $\mathfrak{g}^{-1} \times \mathfrak{g}^{-1} \rightarrow \mathfrak{g}^{-2}$ induced by the Lie bracket is nondegenerate.

It turns out contact gradations can exist only on simple Lie algebras [10, Proposition 3.2.4].

A contact gradation defines a homogeneous manifold $M = G/G^{\geq 0}$, where G is the simply connected Lie group with the Lie algebra \mathfrak{g} and $G^{\geq 0}$ the subgroup generated by the non-negative subalgebra $\mathfrak{g}^{\geq 0} = \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2$.

Proposition 4.1. *The manifold $M = G/G^{\geq 0}$ associated with a contact gradation is a symmetric contact space.*

Proof. The contact distribution \mathcal{D} is defined as a natural extension of the isotropy invariant subspace \mathfrak{g}^{-1} of the space $\mathfrak{g}^{<0} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1}$ which is identified with the tangent space $T_o M = \mathfrak{g}/\mathfrak{g}^{\geq 0}$. Denote by s the involutive automorphism of \mathfrak{g} associated with symmetric decomposition $\mathfrak{g} = \mathfrak{g}^{ev} + \mathfrak{g}^{odd} = (\mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2) + (\mathfrak{g}^{-1} + \mathfrak{g}^1)$. It defines an involutive automorphism σ of the Lie group G which preserves the subgroup $G^{\geq 0}$. Then the transformation $gG^{\geq 0} \mapsto \sigma(g)G^{\geq 0}$ preserves the contact distribution \mathcal{D} , fixes the point o and acts as -1 on \mathcal{D}_o , hence defines a symmetry. \square

4.1. Canonical contact gradation of a complex simple Lie algebra and adjoint variety

A complex simple Lie algebra \mathfrak{g} admits a canonical complex contact gradation associated with a highest root (cf. [26, Theorem 4.2] or [10, Proposition 3.2.4]). Let \mathfrak{g} be a complex simple Lie algebra with a Cartan subalgebra \mathfrak{a} and corresponding root space decomposition

$$\mathfrak{g} = \mathfrak{a} + \sum_{\alpha \in R} \mathfrak{g}_{\alpha}.$$

Let $\Pi \subset R$ be a simple root system and R_+ the associated system of positive roots. The highest root $\mu \in R_+$ defines a contact gradation of \mathfrak{g} as follows. Denote by $R^0 = \mu^{\perp} \cap R$ (resp., $R_+^0 = R^0 \cap R_+$) the roots (resp., positive roots) orthogonal to μ with respect to the Killing form, and set $R^1 = R_+ \setminus (\{\mu\} \cup R_+^0)$. For a set of roots $P \subset R$ we denote by $\mathfrak{g}(P) = \sum_{\alpha \in P} \mathfrak{g}_{\alpha}$ the space which is the span of the root spaces associated with roots from P . Then

$$\mathfrak{g} = \mathfrak{g}_{-\mu} + \mathfrak{g}(-R^1) + \mathfrak{g}(R^0) + \mathfrak{g}(R^1) + \mathfrak{g}_{\mu}$$

is a complex contact gradation which is called the *canonical complex contact gradation* associated with the highest root. The associated (compact) complex homogeneous manifold $G/G^{\geq 0} = \text{Ad}_G[\mathfrak{g}_{\mu}]$ (here G is the simply-connected complex semisimple Lie group with Lie algebra \mathfrak{g}) is the orbit of the highest weight line \mathfrak{g}_{μ} in the projectivization $P\mathfrak{g}$ of the Lie algebra and is called the *adjoint variety* (it is the only closed orbit of G in $P\mathfrak{g}$.)

4.2. Contact gradations of a real absolutely simple Lie algebra

Let $\mathfrak{g} = \sum_j \mathfrak{g}^j$ be a gradation of a complex semisimple Lie algebra. It is the eigenspace decomposition of the adjoint operator ad_h , where $h \in \mathfrak{g}$ is the uniquely defined element of \mathfrak{g} such that $\text{ad}_h|_{\mathfrak{g}^j} = j \cdot \text{id}$, called the *grading element*.

A real form \mathfrak{g}^σ of \mathfrak{g} defined by an anti-involution (conjugate-linear involutive automorphism) σ is called *consistent* with the gradation of \mathfrak{g} if it inherits a gradation $\mathfrak{g}^\sigma = \sum_j (\mathfrak{g}^\sigma)^j$ or, equivalently, $\sigma(h) = h$. Any gradation of \mathfrak{g}^σ is induced by a gradation of \mathfrak{g} .

4.2.1. The contact gradations of classical Lie algebras

Herein we describe the canonical gradations of the classical Lie algebras. At first, we consider the real and complex Lie algebras $\mathfrak{sl}_n(\mathbb{K}) = \mathfrak{sl}(V)$, $\mathfrak{sp}_m(\mathbb{K}) = \mathfrak{sp}(V)$, where $\mathbb{K} = \mathbb{R}$ or \mathbb{C} and $V = \mathbb{K}^n$, with $n = 2m$ in the symplectic case. Consider the gradation

$$V = V^{-1} + V^0 + V^1 = \mathbb{K}p + V^0 + \mathbb{K}q$$

where, in the symplectic case, ω is a symplectic form, p, q are isotropic vectors with $\omega(p, q) = 1$ which span a nondegenerate subspace, and V^0 is its orthogonal complement. It induces a contact gradation of the Lie algebras $\mathfrak{g} = \mathfrak{sl}(V)$, $\mathfrak{sp}(V)$, where $\mathfrak{g}^j = \{A \in \mathfrak{g}, AV^i \subset V^{i+j}\}$. In matrix notation with respect to the decomposition $V = V^{-1} + V^0 + V^1$, the gradation is described as

$$\begin{pmatrix} \mathfrak{g}^0 & \mathfrak{g}^{-1} & \mathfrak{g}^{-2} \\ \mathfrak{g}^1 & \mathfrak{g}^0 & \mathfrak{g}^{-1} \\ \mathfrak{g}^2 & \mathfrak{g}^1 & \mathfrak{g}^0 \end{pmatrix}$$

More precisely, it is given by

$$\begin{aligned} \mathfrak{sl}(V) = & \mathbb{K}p \otimes q^* + (p \otimes (V^0)^* + V^0 \otimes q^*) \\ & + \mathfrak{sl}(\mathbb{K}p \otimes p^* + \mathfrak{gl}(V^0) + \mathbb{K}q \otimes q^*) + (q \otimes (V^0)^* + V^0 \otimes p^*) + \mathbb{K}q \otimes p^* \end{aligned} \quad (4)$$

and

$$\mathfrak{sp}(V) = S^2V = \mathbb{K}p^2 + pV^0 + (\mathbb{K}pq + \mathfrak{sp}(V^0)) + qV^0 + \mathbb{K}q^2.$$

We have identified $\mathfrak{sl}(V)$ with a codimension one subspace of $V \otimes V^*$ (the traceless endomorphisms of V), and $\mathfrak{sp}(V)$ with the space S^2V of symmetric bilinear forms on V^* using ω . We denote by ab the symmetric product $\frac{1}{2}(a \otimes b + b \otimes a)$, for $a, b \in V$. In the first case, the grading element is $h = -p \otimes p^* + q \otimes q^*$ which is included into the \mathfrak{sl}_2 -triple

$$h, \quad e = q \otimes p^*, \quad f = p \otimes q^*.$$

In the second case the triple is

$$h = 2pq, \quad e = -q^2, \quad f = p^2.$$

Next we fix in the space $V = \mathbb{C}^n$, where $n > 2$, a Hermitian metric γ of signature $(k+1, \ell+1)$ for $k+\ell > 0$, such that the subspace V^0 is nondegenerate and orthogonal to the space spanned by p, q with $\gamma(p, q) = 1$. Then the corresponding real form $\mathfrak{su}_{k+1, \ell+1}$ is consistent with the gradation of $\mathfrak{sl}(V)$. Relative to the decomposition $V = \mathbb{C}p + V^0 + \mathbb{C}q$, matrices from $\mathfrak{su}_{k+1, \ell+1}$ have the form

$$\begin{pmatrix} \lambda + i\mu & -X_-^* & i\alpha_- \\ X_+ & A - \frac{2i\mu}{k+\ell} & X_- \\ i\alpha_+ & -X_+^* & -\lambda + i\mu \end{pmatrix}$$

where $A \in \mathfrak{su}_{k,\ell}$, $\lambda, \mu, \alpha_{\pm} \in \mathbb{R}$, $X_{\pm} \in \mathbb{C}^{k,\ell}$ and $X_{\pm}^* = \gamma(X_{\pm}, \cdot)$ is the complex conjugate covector.

Now we describe the canonical gradation of complex orthogonal and real pseudo-orthogonal Lie algebras $\mathfrak{so}(V)$ where $V = \mathbb{C}^n$ or $\mathbb{R}^{k+2,\ell+2}$ with $k + \ell > 0$. Using the metric g in V , we identify $\mathfrak{so}(V)$ with the space $\Lambda^2 V$ of bivectors. The gradation of $\mathfrak{so}(V)$ is induced by the gradation $V = V^{-1} + V^0 + V^1$ of V where $V^{\pm 1}$ are isotropic 2-dimensional subspaces such that $V^{-1} + V^1$ is non degenerate and V^0 is its orthogonal complement. Then

$$\mathfrak{so}(V) = \Lambda^2 V = \Lambda^2 V^{-1} + V^{-1} \otimes V^0 + (V^{-1} \otimes V^1 + \mathfrak{so}(V^0)) + V^0 \otimes V^1 + \Lambda^2 V^1$$

is a contact gradation. We denote by p, p' a basis of $P := V^{-1}$ and by q, q' the dual basis of $Q = V^1 \simeq (V^{-1})^*$. Note that $P \otimes Q \simeq \mathfrak{gl}(P) \simeq \mathfrak{gl}(Q)$ and $\mathfrak{g}^0 \simeq \mathfrak{gl}_2(\mathbb{K}) \oplus \mathfrak{so}(V^0)$. The grading element is $h = 2(q \wedge p + q' \wedge p')$. It is included into \mathfrak{sl}_2 triple

$$h, f = 2(p \wedge p'), e = -2(q \wedge q')$$

which corresponds to the graded subalgebra $\mathbb{K}p \wedge p' + \mathbb{K}h + \mathbb{K}q \wedge q'$ of the graded algebra $\mathfrak{so}(P + Q)$, isomorphic to $\mathfrak{so}_4(\mathbb{C})$ for $\mathbb{K} = \mathbb{C}$ and $\mathfrak{so}_{2,2}$ for $\mathbb{K} = \mathbb{R}$.

Finally, in case $V = \mathbb{C}^n$ with $n = 2m$ there is a further real form \mathfrak{so}_{2m}^* of the Lie algebra $\mathfrak{so}(V)$ consisting of the elements preserving a nondegenerate skew-Hermitian form. The gradation of \mathfrak{so}_{2m}^* is induced by the gradation of V as above, and $\mathfrak{so}(P + Q)$ is isomorphic to \mathfrak{so}_4^* .

4.3. Classification of real simple Lie algebras which admit a contact gradation

Here we describe all real simple Lie algebras which admits a contact gradation. Note that a contact gradation in a real or complex semisimple Lie algebra \mathfrak{g} is a *fundamental gradation*, i.e. the negative subalgebra $\mathfrak{g}^{<0}$ is generated by \mathfrak{g}^{-1} . Any fundamental gradation of a complex semisimple Lie algebra \mathfrak{g} is associated with a subset Π^1 of the system of simple roots Π and defined by the condition that $\deg \mathfrak{g}_{\alpha} = 1$ for $\alpha \in \Pi^1$ and $\deg \mathfrak{g}_{\alpha} = 0$ for $\alpha \in \Pi \setminus \Pi^1$ [5].

The following result by Djoković (see [13] or [4, Prop. 3.8] or [5, §6.2]) gives a description of all gradations of a real form \mathfrak{g}^{σ} of a complex semisimple Lie algebra \mathfrak{g} in terms of Satake diagrams.

Proposition 4.2. *Let \mathfrak{g} be a complex semisimple Lie algebra with a gradation defined by Π^1 and let \mathfrak{g}^{σ} be a noncompact real form defined by a Satake diagram. Then \mathfrak{g}^{σ} is consistent with the gradation if and only if all nodes in the Satake diagram associated with roots from Π^1 are white and there is no curved arrow which connects a root from Π^1 with a root which is not in Π^1 .*

The contact gradation of \mathfrak{g} is defined by the set Π^1 which consists of the simple roots α_i associated with fundamental weights π_i which appear in the decomposition of the highest root μ in terms of fundamental weights. For example for A_n , $\mu = \pi_1 + \pi_n$ and $\Pi^1 = \{\alpha_1, \alpha_n\}$. Following [21], we write down the decomposition of the highest root in terms of fundamental weights for all complex simple Lie algebras in Table 9.

Using Djoković's criterion (Proposition 4.2) and analyzing the list of Satake diagrams of real absolutely simple Lie algebras, we deduce:

Proposition 4.3. *The contact gradations of the classical noncompact real absolutely simple Lie algebras are exhausted by the contact gradations of $\mathfrak{sl}_n \mathbb{R}$, $n > 2$, $\mathfrak{su}_{k+1,\ell+1}$, $k + \ell > 0$,*

Table 9

Highest roots for complex simple Lie algebras.

A_1	$2\pi_1$
$A_n \ (n \geq 2)$	$\pi_1 + \pi_n$
$B_n \ (n \geq 2)$	π_2
$C_n \ (n \geq 3)$	$2\pi_1$
$D_n \ (n \geq 4)$	π_2
E_6	π_6
E_7	π_6
E_8	π_1
F_4	π_4
G_2	π_2

$\mathfrak{sp}_n\mathbb{R}$, $n > 1$, $\mathfrak{so}_{k+2,\ell+2}$, $k + \ell > 0$, and \mathfrak{so}_{2m}^* , $m > 2$, described in Section 4.2.1. Each exceptional noncompact real absolutely simple Lie algebra admits a unique contact gradation, up to conjugation, with the exception of the Lie algebras *EIV* and *FII*, which admit no contact gradations.

5. Classification of $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras of symmetric type in an absolutely simple Lie algebra \mathfrak{g}

Definition 5.1. An $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra \mathfrak{s} of a real semisimple Lie algebra \mathfrak{g} is called *regular* if its complexification $\mathfrak{s}^{\mathbb{C}}$ is a regular 3-dimensional subalgebra $\mathfrak{s}(\mu)$ associated with some root μ (with respect to a Cartan subalgebra of $\mathfrak{g}^{\mathbb{C}}$). Here $\mathfrak{s}(\mu)$ is spanned over \mathbb{C} by an \mathfrak{sl}_2 -triple (h_μ, e_μ, f_μ) , where e_μ, f_μ are root vectors. In this case may assume that \mathfrak{s} is spanned over \mathbb{R} by (h_μ, e_μ, f_μ) .

Assume that \mathfrak{s} is a regular $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of a real absolutely simple Lie algebra \mathfrak{g} , spanned by the \mathfrak{sl}_2 -triple (h, e, f) , such that $\mathfrak{s}^{\mathbb{C}} = \mathfrak{s}(\mu)$, where μ is a long root of $\mathfrak{g}^{\mathbb{C}}$ with respect to some Cartan subalgebra. Then the associated gradation of \mathfrak{g} has the form

$$\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2,$$

where $\mathfrak{g}^2 = \mathbb{R}e$, $\mathfrak{g}^{-2} = \mathbb{R}f$ and $\mathfrak{g}^0 = \mathbb{R}h + \mathfrak{z}$, where \mathfrak{z} is the centralizer of \mathfrak{s} . This is a contact gradation and \mathfrak{s} is an $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of \mathfrak{g} of symmetric type, with symmetric decomposition

$$\mathfrak{g} = \mathfrak{g}^{even} + \mathfrak{g}^{odd} = (\mathfrak{s} + \mathfrak{z}) + (V + W),$$

where $V = \mathfrak{g}^1$, $W = \mathfrak{g}^{-1}$.

Any two roots of the same length in a complex simple Lie algebra are conjugate by an inner automorphism, so the classification of regular $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras of an absolutely simple Lie algebra \mathfrak{g} amounts to the description of anti-linear involutions (real forms) of the complex simple Lie algebra $\mathfrak{g}^{\mathbb{C}}$ that preserve a regular 3-dimensional subalgebra $\mathfrak{s}(\mu)$; here μ is a fixed root of $\mathfrak{g}^{\mathbb{C}}$ in the simply-laced case, but it can be either a fixed long root or a fixed short root in the multiply-laced case.

5.1. Case of odd $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras

Let \mathfrak{s} be an odd $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of symmetric type of an absolutely simple Lie algebra \mathfrak{g} . We prove that \mathfrak{s} must be a regular subalgebra. By assumption, the Killing orthogonal

decomposition $\mathfrak{g} = N_{\mathfrak{g}}(\mathfrak{s}) + \mathfrak{q}$ is a symmetric decomposition. Moreover, the ad_h -gradation $\mathfrak{g} = \sum_i \mathfrak{g}^i$ is odd, that is, $\mathfrak{g}^j \neq 0$ for some odd j .

Lemma 5.1. $N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s} + \mathfrak{z} = \mathfrak{g}^{\text{even}} = \mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2$, with $\mathfrak{g}^{-2} = \mathbb{R}f$, $\mathfrak{g}^2 = \mathbb{R}e$, and $\mathfrak{q} = \mathfrak{g}^{\text{odd}}$.

Proof. Owing to the fact that \mathfrak{g}^i is the i -eigenspace of ad_h , we have

$$N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{s} + \mathfrak{z} \subset \mathbb{R}f + \mathfrak{g}^0 + \mathbb{R}e \subset \mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2 \subset \mathfrak{g}^{\text{even}}$$

and then $0 \neq \mathfrak{g}^{\text{odd}} \subset \mathfrak{q}$. Since $[\mathfrak{g}^{\text{odd}}, \mathfrak{g}^{\text{odd}}] + \mathfrak{g}^{\text{odd}}$ is a non-trivial ideal of the simple Lie algebra \mathfrak{g} , it coincides with \mathfrak{q} . This shows that $N_{\mathfrak{g}}(\mathfrak{s}) = \mathfrak{g}^{\text{even}} = [\mathfrak{g}^{\text{odd}}, \mathfrak{g}^{\text{odd}}]$ and $\mathfrak{g}^{\text{odd}} = \mathfrak{q}$. \square

Proposition 5.2. An odd $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra \mathfrak{s} of symmetric type must be regular. Moreover:

- if \mathfrak{g} not of type G_2 , then the complexification $\mathfrak{s}^{\mathbb{C}}$ is of the form $\mathfrak{s}(\mu)$, where μ is a long root of $\mathfrak{g}^{\mathbb{C}}$ with respect to some Cartan subalgebra, and hence the associated gradation of \mathfrak{g} is a contact gradation (of depth 2);
- if \mathfrak{g} is of type G_2 and $\mathfrak{s}^{\mathbb{C}} = \mathfrak{s}(\mu)$ for a short root μ , then the associated gradation of \mathfrak{g} has depth 3.

Proof. Let (h, e, f) be an \mathfrak{sl}_2 -triple spanning \mathfrak{s} . Then h is a semisimple element of \mathfrak{g} and belongs to a Cartan subalgebra \mathfrak{a} , which is necessarily contained in $Z_{\mathfrak{g}}(h) = \mathfrak{g}^0$. Owing to Lemma 5.1, $\mathfrak{g}^0 = \mathbb{R}h + \mathfrak{z}$. Now e is a root vector of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$, say associated to the root μ . Then f is a root vector associated to $-\mu$. This already proves that \mathfrak{s} is a regular subalgebra.

Consider the root decomposition of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$. Note that each $(\mathfrak{g}^i)^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^i$ for $i \neq 0$ is a sum of root spaces. Fix an ordering of the roots that puts h into the Weyl chamber. Then μ is a positive root. Denote the depth of the ad_h -gradation of \mathfrak{g} by m . If $\tilde{\alpha}$ denotes the highest root, then the highest root space $(\mathfrak{g}^{\mathbb{C}})_{\tilde{\alpha}} \subset (\mathfrak{g}^m)^{\mathbb{C}}$. Recall that $\text{ad}_f^k[(\mathfrak{g}^{\mathbb{C}})_{\tilde{\alpha}}] = (\mathfrak{g}^{\mathbb{C}})_{\tilde{\alpha} - k\mu}$ if $\tilde{\alpha} - k\mu$ is a root. The main observation now is that the length of the μ -chain of roots through $\tilde{\alpha}$ can be at most 4, and it equals 4 if and only if $\mathfrak{g}^{\mathbb{C}}$ is of G_2 -type and μ is a short root [9, ch. VI, §1, no.3].

If the depth $m = 2$, from Lemma 5.1 we see that

$$\mathfrak{g} = \mathfrak{g}^{-2} + \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 + \mathfrak{g}^2,$$

where

$$\mathfrak{g}^{-2} = \mathbb{R}f, \quad \mathfrak{g}^0 = \mathbb{R}h + \mathfrak{z}, \quad \mathfrak{g}^2 = \mathbb{R}e,$$

so μ is the highest root.

If $m \geq 3$, then $V = \mathfrak{g}^m$ and $\text{ad}_f^k : (\mathfrak{g}^m)^{\mathbb{C}} \rightarrow (\mathfrak{g}^{m-2k})^{\mathbb{C}}$ is injective for $k : 1, \dots, m$. By the main observation above, this implies that $m = 3$, \mathfrak{g} is of G_2 -type and μ is a short root. \square

Proof of Theorem 1.1. The projectivized adjoint orbit of a nilpositive element of a contact gradation is a symmetric contact space due to Proposition 4.1.

Conversely, in view of Proposition 3.1 we need to classify odd $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras of \mathfrak{g} of symmetric type.

Owing to Proposition 5.2, the contact gradations of real absolutely simple Lie algebras described in Proposition 4.3 exhaust all the possibilities, unless we are in case G_2 .

In the case of the normal real form $\mathfrak{g} = \mathfrak{g}_{2(2)}$ we check that the $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra associated to a short root is of symmetric type and odd. Let $\tilde{\alpha}$ and β be the highest root and highest short root of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{a}^{\mathbb{C}}$, respectively, where \mathfrak{a} is a Cartan subalgebra of \mathfrak{g} . Then the

normalizer $N_{\mathfrak{g}}(\mathfrak{s}(\beta)) = \mathfrak{s}(\beta) + \mathfrak{s}(\tilde{\alpha})$ coincides with $N_{\mathfrak{g}}(\mathfrak{s}(\tilde{\alpha}))$, which is already known to be a symmetric subalgebra of \mathfrak{g} . Hence $\mathfrak{s}(\beta)$ is of symmetric type. Moreover $V + W = 2\text{Sym}^3(\mathbb{R}^2)$ as an $\mathfrak{s}(\beta)$ -module, which says that the eigenvalues of ad_h are $\pm 3, \pm 1$, that is, $\mathfrak{s}(\beta)$ is odd. We obtain Tables 1 and 2. \square

5.2. Case of even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras

Let \mathfrak{s} be an even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of symmetric type of an absolutely simple Lie algebra \mathfrak{g} . We prove here that the complexification $\mathfrak{s}^{\mathbb{C}}$ defines a short SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$ in the sense of E. Vinberg [25].

By assumption, the Killing orthogonal decomposition $\mathfrak{g} = N_{\mathfrak{g}}(\mathfrak{s}) + \mathfrak{q}$ is a symmetric decomposition. Moreover, the ad_h -gradation $\mathfrak{g} = \sum_i \mathfrak{g}^i$ is even, that is, $\mathfrak{g}^j = 0$ for all odd j . Although \mathfrak{s} need not be regular, we will see that it is not far from being regular by means of the following concept.

Definition 5.2. An even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra \mathfrak{s} of a real semisimple Lie algebra \mathfrak{g} , spanned by an \mathfrak{sl}_2 -triple (h, e, f) , will be called *short* if the eigenvalues of the endomorphism ad_h belong to the set $\{0, \pm 2, \pm 4\}$ or, equivalently, irreducible submodules of the $\text{ad}_{\mathfrak{s}}$ -module \mathfrak{g} have dimensions 1, 3 or 5.

If \mathfrak{s} is a short even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of a real semisimple Lie algebra \mathfrak{g} , then the complexification $\mathfrak{s}^{\mathbb{C}}$ clearly is a short SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$ in the sense of Vinberg [25].

Proposition 5.3. *An even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra \mathfrak{s} of symmetric type of an absolutely simple Lie algebra \mathfrak{g} is short.*

Proof. Let (h, e, f) be a \mathfrak{sl}_2 -triple spanning \mathfrak{s} . We shall prove that the ad_h -gradation of \mathfrak{g} has depth at most 4.

In fact, the simplicity of \mathfrak{g} implies $[\mathfrak{g}_-, \mathfrak{g}_+] = \mathfrak{g}_+$. Moreover we may assume that the adjoint action of \mathfrak{g}_+ on \mathfrak{g}_- is irreducible. Indeed, otherwise the simplicity of \mathfrak{g} implies that there is an $\text{ad}_{\mathfrak{g}_+}$ -irreducible decomposition $\mathfrak{g}_- = \mathfrak{g}^{(1)} + \mathfrak{g}^{(-1)}$ such that $\mathfrak{g} = \mathfrak{g}^{(-1)} + \mathfrak{g}^{(0)} + \mathfrak{g}^{(1)}$ defines a gradation of depth 1, where $\mathfrak{g}^{(0)} = \mathfrak{g}_+$ [23, App., Lem. 2 and comments thereafter]. Since $\mathfrak{sl}_2\mathbb{C}$ is a factor of $\mathfrak{g}^{(0)} \otimes \mathbb{C}$, the classification of gradations of complex simple Lie algebras (e.g. [10, p. 297]) gives that $(\mathfrak{g} \otimes \mathbb{C}, \mathfrak{g}_+ \otimes \mathbb{C}) = (\mathfrak{sl}_{n+1}\mathbb{C}, \mathfrak{s}(\mathfrak{gl}_2\mathbb{C} \oplus \mathfrak{gl}_{n-1}))$, but in this case the gradation induced by the semisimple element of $\mathfrak{sl}_2\mathbb{C}$ is not even.

Denote by \mathfrak{s} the span of (h, e, f) and consider the canonical decomposition (2). The adjoint action of $\mathfrak{g}_+ = \mathfrak{s} \oplus \mathfrak{z}$ on \mathfrak{g}_- is irreducible. Since \mathfrak{z} must preserve the \mathfrak{s} -isotypical decomposition of \mathfrak{g}_- , there must be only one isotypical component, that is $\mathfrak{g}_- = V + W = P_m + \dots + P_m$ as an $\text{ad}_{\mathfrak{s}}$ -module, where $P_m = \text{Sym}^m(\mathbb{R}^2)$ is the real irreducible representation of $\mathfrak{sl}_2(\mathbb{R})$ of dimension $m + 1$.

Owing to the contact condition, for every $0 \neq w_0 \in W^0$ there exists $w_{-2} \in W^{-2}$ such that $[w_0, w_{-2}] = f \in \mathfrak{s} \subset \mathfrak{g}^+$. If $m \geq 6$, then we reach a contradiction to the Jacobi identity in \mathfrak{g} as follows. Let $0 \neq w_{-4} \in W^{-4}$. The Lie bracket $[w_{-4}, [w_0, w_{-2}]] = [w_{-4}, f] \in W^{-6}$ is nonzero. However, $[w_{-4}, w_0]$ and $[w_{-4}, w_{-2}]$ are weight vectors of weights -4 and -6 , respectively, lying in $\mathfrak{g}^+ = \mathfrak{s} + \mathfrak{z}$; hence they are zero. This contradicts the Jacobi identity applied to w_{-4}, w_0, w_{-2} . Therefore $m = 2$ or $m = 4$, as desired. \square

Proposition 5.4. *There are no even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebras of symmetric type in an absolutely simple Lie algebra \mathfrak{g} of exceptional type.*

Table 10

Short structures on exceptional Lie algebras and their centralizers.

$\mathfrak{g}^{\mathbb{C}}$	Index of $\mathfrak{s}^{\mathbb{C}}$	$\dim(\mathfrak{s}^{\mathbb{C}} \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}}))$
$\mathfrak{g}_2^{\mathbb{C}}$	4	3
$\mathfrak{f}_4^{\mathbb{C}}$	4	11
	8	17
$\mathfrak{e}_6^{\mathbb{C}}$	4	19
	8	17
$\mathfrak{e}_7^{\mathbb{C}}$	4'	38
	8	20
	3''	55
	7	17
$\mathfrak{e}_8^{\mathbb{C}}$	4''	81
	8	31

Proof. Let \mathfrak{s} be an even $\mathfrak{sl}_2(\mathbb{R})$ -subalgebra of symmetric type of \mathfrak{g} . In view of [Proposition 5.3](#), there is an induced short SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$. We shall see however $N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$ can never be a symmetric subalgebra of $\mathfrak{g}^{\mathbb{C}}$, reaching a contradiction.

Recall that $N_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}}) = \mathfrak{s}^{\mathbb{C}} \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$. We run through the classification of short structures on exceptional complex simple Lie algebras given in [\[25, §2.2\]](#) and, in each case, use [\[14, Table 21\]](#) to determine $\dim Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$. We obtain [Table 10](#); the *index* refers to the Dynkin index of the 3-dimensional Lie subalgebra as listed in [\[14\]](#).

From the classification of Berger [\[6\]](#), we immediately see that $\dim \mathfrak{s}^{\mathbb{C}} \oplus Z_{\mathfrak{g}^{\mathbb{C}}}(\mathfrak{s}^{\mathbb{C}})$ is not the dimension of a symmetric subalgebra. \square

Proof of Theorem 1.2. Owing to [Propositions 5.3](#) and [5.4](#), it suffices to run through the cases of complex simple Lie algebras $\mathfrak{g}^{\mathbb{C}}$ of classical type; from the list of symmetric subalgebras, select those that contain an ideal isomorphic to $\mathfrak{sl}_2\mathbb{C}$; and check which of those induce a short SO_3 -structure on $\mathfrak{g}^{\mathbb{C}}$.

Recall the classification of short SO_3 -structures on a classical Lie algebra $\mathfrak{g}^{\mathbb{C}}$ [\[25, p. 257\]](#). In case $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sl}_n\mathbb{C}$, an SO_3 -structure is determined by an n -dimensional representation $\rho : \mathfrak{sl}_2\mathbb{C} \rightarrow \mathfrak{sl}_n\mathbb{C}$, which in turn is characterized by the dimensions n_1, n_2, \dots of its irreducible components, which must have all the same parity. The SO_3 -structure is short if and only if all the n_i 's do not exceed 3. The same holds for $\mathfrak{g}^{\mathbb{C}} = \mathfrak{so}_n\mathbb{C}$ (resp. $\mathfrak{g}^{\mathbb{C}} = \mathfrak{sp}_n\mathbb{C}$), with the addendum that the number of n_i 's equal to 2 (resp. 3) must be even. In [Table 11](#), for each classical complex simple Lie algebra, and for each symmetric subalgebra containing $\mathfrak{sl}_2(\mathbb{C})$ as an ideal, we list ρ and check Vinberg's criterion for a short SO_3 -structure.

[Table 11](#) contains the list of symmetric subalgebras of classical complex simple Lie algebras of the form $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{z}$ (cf. [\[6\]](#)) and an indication of whether the $\mathfrak{sl}_2(\mathbb{C})$ -factor defines a short SO_3 -structure.

Finally, we collect real forms of $(\mathfrak{sl}_3\mathbb{C}, \mathfrak{so}_3\mathbb{C})$, $(\mathfrak{sl}_4\mathbb{C}, \mathfrak{so}_4\mathbb{C})$ and $(\mathfrak{so}_n\mathbb{C}, \mathfrak{so}_3\mathbb{C} \oplus \mathfrak{so}_{n-3}\mathbb{C})$ that give examples and obtain [Table 3](#). \square

6. Short \mathfrak{sl}_2 -subalgebras of symmetric type in non-absolutely simple, semisimple Lie algebras

Assume that the Lie algebra \mathfrak{g} is not absolutely simple, i.e. the complex Lie algebra $\mathfrak{g}^{\mathbb{C}}$ is not simple.

Table 11

Classical complex simple Lie algebras and their symmetric subalgebras.

$\mathfrak{g}^{\mathbb{C}}$	Symm subalg	$\mathfrak{sl}_2\mathbb{C}$ is an ideal	ρ	Short SO_3 -struct
$\mathfrak{sl}_n\mathbb{C}$	$\mathfrak{so}_n\mathbb{C}$	Only if $n = 3$ or $n = 4$	$n_1 = 3$ $n_1 = n_2 = 2$	Yes Yes
	$\mathfrak{s}(\mathfrak{gl}_k \oplus \mathfrak{gl}_{n-k}\mathbb{C})$	Only if $n \geq 3, k = 2$	$n_1 = 2, n_2 = \dots = n_{n-2} = 1$	No
	$\mathfrak{sp}_n\mathbb{C}$ ($n \geq 4$ even)	No	–	–
$\mathfrak{so}_n\mathbb{C}$	$\mathfrak{so}_k\mathbb{C} \oplus \mathfrak{so}_{n-k}\mathbb{C}$	Only if $k = 3$ or $k = 4$	$n_1 = 3, n_2 = \dots = n_{n-3} = 1$ $n_1 = n_2 = 2, n_3 = \dots = n_{n-4} = 1$	Yes No
	$\mathfrak{gl}_{n/2}\mathbb{C}$ ($n \geq 6$ even)	No	–	–
$\mathfrak{sp}_n\mathbb{C}$	$\mathfrak{sp}_k\mathbb{C} \oplus \mathfrak{sp}_{n-k}\mathbb{C}$	Only if $k = 1$	$n_1 = 2, n_2 = \dots = n_{2n-2} = 1$	No
	$\mathfrak{gl}_n\mathbb{C}$ ($n \geq 3$)	No	–	–

Proposition 6.1. *If \mathfrak{g} is semisimple but not simple, then the only \mathfrak{sl}_2 -triple of symmetric type is the triple $(h + h', e + e', f + f')$ associated with the diagonal subalgebra $(\mathfrak{sl}_2\mathbb{R})^d$ of the Lie algebra $\mathfrak{g} = \mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{sl}'_2\mathbb{R}$. Here $\mathfrak{z} = 0$, $\mathfrak{g}_+ = \mathfrak{s} = (\mathfrak{sl}_2\mathbb{R})^d$ and the associated canonical decomposition (2) is*

$$\mathfrak{g} = (\mathbb{R}(h + h') + \mathbb{R}(e + e') + 0 + \mathbb{R}(e - e')) + (\mathbb{R}(f + f') + (\mathbb{R}(h - h') + \mathbb{R}(f - f'))).$$

Proof. Let \mathfrak{s} be an $\mathfrak{sl}_2\mathbb{R}$ -subalgebra of symmetric type of \mathfrak{g} . Then $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ under an involution s of \mathfrak{g} , where $\mathfrak{g}_+ = \mathfrak{s} + \mathfrak{z}$ and \mathfrak{z} is the centralizer of \mathfrak{s} in \mathfrak{g} . There is an s -invariant decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \dots \oplus \mathfrak{g}_r$ into a direct sum of ideals, where for each $i = 1, \dots, r$, either the Lie algebra \mathfrak{g}_i is simple, or $\mathfrak{g}_i = \mathfrak{d} \oplus \mathfrak{d}$ is a sum of two copies of a simple Lie algebra \mathfrak{d} and $s(x, y) = (y, x)$ for $(x, y) \in \mathfrak{d} \oplus \mathfrak{d}$. For each i , there is an involutive decomposition $\mathfrak{g}_i = (\mathfrak{g}_i)_+ + (\mathfrak{g}_i)_-$ under the restriction of s .

Consider the projection $\pi_i : \mathfrak{g} \rightarrow \mathfrak{g}_i$ and put $\mathfrak{s}_i := \pi_i(\mathfrak{s})$. Since \mathfrak{s} does not centralize nonzero elements in \mathfrak{g}_- , $\mathfrak{s}_i \neq 0$ for all i and hence, by simplicity of \mathfrak{s} , π_i defines an isomorphism $\mathfrak{s} \cong \mathfrak{s}_i$. Upon this identification, we can now write

$$\mathfrak{s} = \{(x, \dots, x) \in \mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_r \mid x \in \mathfrak{s}\}.$$

The centralizer

$$\mathfrak{z} = Z_{\mathfrak{g}}(\mathfrak{s}) = Z_{\mathfrak{g}_1}(\mathfrak{s}_1) \oplus \dots \oplus Z_{\mathfrak{g}_r}(\mathfrak{s}_r)$$

and $Z_{\mathfrak{g}_i}(\mathfrak{s}_i) \cap \mathfrak{s}_i = \{0\}$ for all i . Since \mathfrak{s} is a proper subset of $\mathfrak{s}_1 \oplus \dots \oplus \mathfrak{s}_r$ in case $r \geq 2$, the condition $\mathfrak{g}_+ = \mathfrak{s} \oplus \mathfrak{z}$ forces $r = 1$.

Since $\mathfrak{g} = \mathfrak{g}_1$ is assumed non-simple, $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{d}$ where $\mathfrak{d} \cong \mathfrak{g}_+$ is simple. Since $\mathfrak{g}_+ = \mathfrak{s} \oplus \mathfrak{z}$ is a sum of ideals, this implies $\mathfrak{z} = 0$ and $\mathfrak{d} = \mathfrak{s} \cong \mathfrak{sl}_2\mathbb{R}$, as we wished. \square

Proposition 6.2. *If \mathfrak{g} is a complex simple Lie algebra viewed as real, then the only \mathfrak{sl}_2 -triple of symmetric type is the triple (h, e, f) associated with the real subalgebra $\mathfrak{sl}_2\mathbb{R}$ of the Lie algebra $\mathfrak{sl}_2\mathbb{C}$. The associated canonical decomposition (2) is given by*

$$\mathfrak{sl}_2(\mathbb{C}) = (\mathbb{R}h + \mathbb{R}e + 0 + \mathbb{R}(ie)) + (\mathbb{R}f + (\mathbb{R}ih + \mathbb{R}if))$$

Proof. Denote by J the ad-invariant complex structure on \mathfrak{g} , and denote by s the involutive automorphism that defines $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$. The complex structure sJs is also ad-invariant, so

simplicity of \mathfrak{g} implies that $sJs = \pm J$. This means J either commutes or anti-commutes with s . In the former case, $J\mathfrak{g}_+ = \mathfrak{g}_+$ so $J\mathfrak{s}$ would be a simple ideal of $\mathfrak{g}_+ = \mathfrak{s} + \mathfrak{z}$. Since $\mathfrak{s} \cap J\mathfrak{s}$ is a complex subalgebra of \mathfrak{s} and \mathfrak{s} admits no non-trivial complex subalgebras, we would have $J\mathfrak{s} \subset \mathfrak{z}$. However, this is a contradiction because \mathfrak{z} centralizes $J\mathfrak{s}$ and $J\mathfrak{s}$ is not Abelian.

We have seen that J anti-commutes with s , so $J\mathfrak{s} \subset \mathfrak{g}_-$ and $\mathfrak{s} \subset J\mathfrak{g}_- \subset \mathfrak{g}_+ = \mathfrak{s} + \mathfrak{z}$. Recall that the adjoint action of \mathfrak{s} admits no trivial components in \mathfrak{g}_- , and hence none on $J\mathfrak{g}_-$. It follows that $\mathfrak{s} = J\mathfrak{g}_-$.

The ad-invariance of J also implies that $J\mathfrak{z} \subset \mathfrak{z}$, so $\mathfrak{z} = J\mathfrak{z} \subset \mathfrak{g}_+ \cap \mathfrak{g}_- = 0$. We have proved that $\mathfrak{g} = \mathfrak{s} + J\mathfrak{s}$ is the complexification of $\mathfrak{s} \cong \mathfrak{sl}_2\mathbb{R}$. \square

Note that the symmetric pairs $(\mathfrak{sl}_2\mathbb{C}, \mathfrak{sl}_2\mathbb{R})$ and $(\mathfrak{sl}_2\mathbb{R} \oplus \mathfrak{sl}_2\mathbb{R}, \mathfrak{sl}_2\mathbb{R})$ given in [Propositions 6.1](#) and [6.2](#) have the same complexification and in a certain sense are dual one to the other; the results of these propositions are collected in [Table 4](#).

7. Symmetric contact manifolds of non-conical type

In this section, we prove [Theorem 1.4](#). We first recall some facts about symplectic symmetric spaces and refer to [\[7\]](#) for more details on them.

7.1. Symplectic symmetric spaces

A *symplectic symmetric space* is a connected affine symmetric manifold endowed with a symplectic structure which is invariant under the geodesic symmetries. The transvection group G of a symplectic symmetric space N (i.e. the connected group generated by the geodesic symmetries) acts transitively on N . The symmetry at the basepoint normalizes G and induces on its Lie algebra \mathfrak{g} an involution s and hence a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ into the ± 1 -eigenspaces. The symplectic structure at the basepoint yields an $\text{ad}_{\mathfrak{k}}$ -invariant non-degenerate 2-form ω on \mathfrak{p} , and the triple $(\mathfrak{g}, s, \omega)$ is called a *symplectic symmetric Lie algebra*. Conversely, every symplectic symmetric Lie algebra gives rise to a unique, up to isomorphism, simply-connected symplectic symmetric space.

7.1.1. Symplectic symmetric spaces of a semisimple Lie group

Let $N = G/K$ be a simply-connected symplectic symmetric space of a connected semisimple Lie group G , where K is connected. The Whitehead lemmas imply that the invariant symplectic form can be written as $d\theta$ for a unique $\theta \in \mathfrak{g}^*$, where d is the Chevalley coboundary operator. This element is dual under the Killing form to an element ξ in the center of \mathfrak{k} . It follows that N decomposes as a product $G_1/K_1 \times \cdots \times G_k/K_k$, where the G_i are simple Lie groups, $K_i = K'_i \cdot Z(K_i)$, K'_i is a semisimple Lie group, the center $Z(K_i) \cong T^1, \mathbb{R}$ or \mathbb{C}^\times , and $\xi = \xi_1 + \cdots + \xi_k$, where $\xi_i \in Z(\mathfrak{k}_i)$ is non-trivial. Moreover, N is an equivariant symplectic covering of the adjoint orbit of ξ in \mathfrak{g} .

7.1.2. Symplectic symmetric spaces of a complex simple Lie group and graded Lie algebras of depth one

Let G be a complex simple Lie group with Lie algebra \mathfrak{g} . Fix a Cartan subalgebra and an ordering in the set of associated roots R . Denote the system of simple roots by Π . Recall that the *Dynkin mark* of a simple root $\alpha \in \Pi$ is its (necessarily positive, integral) coefficient in the expression of the highest root as a linear combination of simple roots. Each $\alpha \in \Pi$ with Dynkin mark 1 defines a gradation of depth one

$$\mathfrak{g} = \mathfrak{g}^{-1} + \mathfrak{g}^0 + \mathfrak{g}^1 = \mathfrak{g}(-R_1) + \mathfrak{g}(R_0) + \mathfrak{g}(R_1)$$

where $\xi = \frac{1}{2}h_\alpha$ is the grading element, h_α is the coroot, $R_0 = \{\beta \in R \mid B(\beta, \alpha) = 0\}$ and $R_1 = R^+ \setminus R^0$. It turns out that every gradation of depth one of \mathfrak{g} arises in this way, up to conjugation ([10, §3.2.3]; see also [26]). The associated symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p} = \mathfrak{g}^{\text{even}} + \mathfrak{g}^{\text{odd}} = \mathfrak{g}^0 + (\mathfrak{g}^{-1} + \mathfrak{g}^1)$$

defines a complex symmetric space G/K with complex symplectic structure defined by the $\text{ad}_{\mathfrak{k}}$ -invariant 2-form $\omega = d(B \circ \xi)$, whose kernel is \mathfrak{k} .

7.1.3. Realifications of complex symplectic symmetric spaces

Let $N = G/K$ be a complex symplectic symmetric space with symplectic form ω as in Section 7.1.2. Then N can be considered as a real manifold with real invariant symplectic structure $\omega^r = a\Re\omega + b\Im\omega$, where $a, b \in \mathbb{R}$, $a^2 + b^2 > 0$, and it becomes a (real) symplectic symmetric space.

7.1.4. Real forms of symplectic complex symmetric spaces

Let $N = G/K$ be a real symplectic symmetric space of an absolutely simple Lie group G with symmetric decomposition

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}.$$

The symplectic form corresponds to an $\text{ad}_{\mathfrak{k}}$ -invariant 2-form ω on \mathfrak{g} with $\ker \omega = \mathfrak{k}$. Moreover, ω is the differential of an $\text{ad}_{\mathfrak{k}}$ -invariant 1-form θ which can be written as $\theta = B \circ \xi$ for some semisimple element ξ in the center of \mathfrak{k} such that $\text{ad}_{\xi}\mathfrak{p} = \mathfrak{p}$. Let $x, y \in \mathfrak{p}^{\mathbb{C}}$ be eigenvectors with respective eigenvalues λ and μ . Then $[x, y] \in \mathfrak{k}^{\mathbb{C}}$ is an eigenvector with eigenvalue $\lambda + \mu = 0$. This shows that ad_{ξ} has only two eigenvalues $\pm\lambda$, where $\lambda \neq 0$ is either real or purely imaginary. In either case we get a gradation of depth 1

$$\mathfrak{g}^{\mathbb{C}} = \mathfrak{k}^{\mathbb{C}} + \mathfrak{p}^{\mathbb{C}} = (\mathfrak{g}^{\mathbb{C}})^0 + ((\mathfrak{g}^{\mathbb{C}})^{-1} + (\mathfrak{g}^{\mathbb{C}})^1),$$

whose grading element d is $\frac{1}{\lambda}\xi$ or $\frac{i}{\lambda}\xi$. Moreover, the Lie algebra is the real form of $\mathfrak{g}^{\mathbb{C}}$ defined by an anti-involution σ such that $\sigma(d) = \pm d$. We say that \mathfrak{g} is *consistent* with the depth 1 gradation of $\mathfrak{g}^{\mathbb{C}}$.

Conversely, if we are given a real form \mathfrak{g}^{σ} of a complex simple Lie algebra \mathfrak{g} , defined by an anti-involution σ which is consistent with a depth one gradation of \mathfrak{g} in the sense it preserves the grading element d up to sign, then the symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ restricts to a symmetric decomposition $\mathfrak{g}^{\sigma} = \mathfrak{k}^{\sigma} + \mathfrak{p}^{\sigma}$ and the element $\xi \in \mathfrak{g}^{\sigma}$ equal to d or $i \cdot d$ defines an $\text{ad}_{\mathfrak{k}^{\sigma}}$ -invariant 2-form $\omega = d(B \circ \xi)$ on \mathfrak{g}^{σ} , with kernel \mathfrak{k}^{σ} . The corresponding symmetric manifold $N = G^{\sigma}/K^{\sigma}$ is a symplectic symmetric space with symplectic form induced by ω .

7.2. Symmetric contact spaces of non-conical type as 1-dimensional bundles over symplectic symmetric spaces

Let $(M = G/H, \mathcal{D})$ be a homogeneous contact manifold of a connected semisimple Lie group G which is the total space of a 1-dimensional bundle over $N = G/K$ as in Section 2.2. Assume (M, \mathcal{D}) is a symmetric contact space, so there exists an involutive automorphism s of \mathfrak{g} that preserves \mathfrak{h} and induces -1 on \mathcal{D} .

Proposition 7.1. *There exists a symmetric decomposition*

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$$

under s , where

$$\mathfrak{k} = \mathfrak{h} + \mathbb{R}\eta$$

and η is a nonzero element in the center of \mathfrak{k} . Moreover, \mathcal{D} is the G -invariant extension of \mathfrak{p} and there exists an $\text{ad}_{\mathfrak{k}}$ -invariant symplectic structure on \mathfrak{p} given by $d\theta$, where $\theta \in \mathfrak{g}^*$, so that $(\mathfrak{g}, s, d\theta)$ is a symplectic symmetric Lie algebra. Finally, η generates a central closed subgroup C of K and $K = C \times H$ (direct product).

Proof. Recall that

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p}, \quad \mathfrak{k} = \mathfrak{h} + \mathbb{R}\eta,$$

where \mathfrak{k} is the centralizer of $\theta \in \mathfrak{g}^*$, $\mathfrak{h} = \ker \theta \cap \mathfrak{k}$ is a codimension one ideal of \mathfrak{k} and $\theta(\eta) = 1$ (cf. (3)). Since s must preserve the contact structure, $s^*\theta = c\theta$ for some $c \neq 0$, and the involutivity forces $c = \pm 1$. Now s is a semisimple automorphism of \mathfrak{g} that preserves $\ker \theta$ and $\mathfrak{k} = \ker d\theta$, so we can choose the subspace \mathfrak{p} to be s -invariant.

We have $\ker \theta = \mathfrak{h} + \mathfrak{p}$ so \mathfrak{p} induces the distribution \mathcal{D} and $s|_{\mathfrak{p}} = -1$. Let $\xi \in \mathfrak{g}$ be the element dual to θ under the Killing form B . Then

$$0 = \theta(\mathfrak{h} + \mathfrak{p}) = B(\xi, \mathfrak{h} + \mathfrak{p}). \quad (5)$$

Plainly, $\xi \in Z_{\mathfrak{g}}(\xi) = Z_{\mathfrak{g}}(\theta) = \mathfrak{k}$, so ξ lies in the center of \mathfrak{k} .

We already know that $s\xi = \pm\xi$. Non-degeneracy of $d\theta$ on \mathfrak{p} gives $x, y \in \mathfrak{p}$ such that $d\theta(x, y) = -1$ and then

$$s[x, y] = s(\eta + [x, y]_{\mathfrak{p}} + [x, y]_{\mathfrak{h}}) = s\eta - [x, y]_{\mathfrak{p}} + (s[x, y])_{\mathfrak{h}}$$

and

$$[sx, sy] = [-x, -y] = \eta + [x, y]_{\mathfrak{p}} + [x, y]_{\mathfrak{h}}$$

implying that $s\eta = \eta$. Hence $s^*\theta = \theta$ and $s\xi = \xi$. It also follows that

$$B(\eta, \mathfrak{p}) = 0. \quad (6)$$

Write $\mathfrak{g} = \mathfrak{g}_+ + \mathfrak{g}_-$ and $\mathfrak{h} = \mathfrak{h}_+ + \mathfrak{h}_-$ under s . Note that $\mathfrak{g}_+ = \mathfrak{h}_+ + \mathbb{R}\eta$ is a reductive subalgebra of \mathfrak{g} (this follows for instance from [24, Lemma 20.5.12]) and $\mathfrak{g}_- = \mathfrak{h}_- + \mathfrak{p}$, $\ker \theta = \mathfrak{h}_+ + \mathfrak{g}_-$. Since $[\mathfrak{k}, \mathfrak{g}] \subset \mathfrak{h}$, we have $[\mathfrak{h}_-, \mathfrak{g}_+] \subset \mathfrak{h}_-$. Therefore we can change \mathfrak{p} , if necessary, so that \mathfrak{p} is in addition $\text{ad}_{\mathfrak{g}_+}$ -invariant.

We next claim $\text{ad}_{\xi} : \mathfrak{p} \rightarrow \mathfrak{p}$ is an isomorphism. In fact, if $\text{ad}_{\xi}x = 0$ for some $x \in \mathfrak{p}$, then $d\theta(x, \mathfrak{p}) = B(\text{ad}_{\xi}x, \mathfrak{p}) = 0$ implying $x = 0$ by nondegeneracy of $d\theta$ on \mathfrak{p} .

It is clear that B is nondegenerate on $\mathfrak{g}_+ \subset \mathfrak{k}$. Next, note that $[\mathfrak{h}_-, \mathfrak{p}] \subset \mathfrak{g}_+$ and $B([\mathfrak{h}_-, \mathfrak{p}], \mathfrak{g}_+) = B(\mathfrak{p}, [\mathfrak{h}_-, \mathfrak{g}_+]) \subset B(\mathfrak{p}, \mathfrak{h}_-) = B(\text{ad}_{\xi}\mathfrak{p}, \mathfrak{h}_-) = B(\mathfrak{p}, \text{ad}_{\xi}\mathfrak{h}_-) = 0$, since ξ centralizes \mathfrak{h} . We have proved that $[\mathfrak{h}_-, \mathfrak{p}] = 0$. It follows that there is a symmetric decomposition

$$\mathfrak{g} = (\mathfrak{g}_+ + \mathfrak{p}) + \mathfrak{h}_-.$$

This argument also shows that

$$B(\mathfrak{h}, \mathfrak{p}) = B(\mathfrak{h}, \text{ad}_{\xi}\mathfrak{p}) = B(\text{ad}_{\xi}\mathfrak{h}, \mathfrak{p}) = 0. \quad (7)$$

Denote by \mathfrak{n}_3 the kernel of the adjoint action of $\mathfrak{g}_+ + \mathfrak{p}$ on \mathfrak{h}_- ; this is an ideal of \mathfrak{g} that contains $\mathbb{R}\xi + \mathfrak{p}$. Using (5)–(7), we deduce that

$$\mathfrak{n}_3^\perp \subset \mathfrak{p}^\perp \cap \xi^\perp = (\mathfrak{h} + \mathbb{R}\eta) \cap (\mathfrak{h} + \mathfrak{p}) = \mathfrak{h},$$

and hence $\mathfrak{n}_3^\perp = 0$, owing to the effectiveness of the presentation $M = G/H$. Finally $\mathfrak{n}_3 = \mathfrak{g}$ and thus $\mathfrak{h}_- = 0$, so that we arrive at

$$\mathfrak{g} = \mathfrak{k} + \mathfrak{p},$$

symplectic involutive Lie algebra, where $\mathfrak{k} = \mathfrak{h} + \mathbb{R}\eta = \mathfrak{g}_+$ and the symplectic structure on $\mathfrak{p} = \mathfrak{g}_-$ is induced by $d\theta(x, y) = -B(\text{ad}_\xi x, y)$ for $x, y \in \mathfrak{p}$. Since $\mathfrak{k} = \mathfrak{g}^+$ is a reductive subalgebra of \mathfrak{g} , $\mathfrak{k} = [\mathfrak{k}, \mathfrak{k}] \oplus Z(\mathfrak{k})$, where $Z(\mathfrak{k})$ denotes the center of \mathfrak{k} , $[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{h}$, and η can be chosen in the center of \mathfrak{k} and to generate a closed subgroup of G . \square

The next example shows that ξ and η in Proposition 7.1 do not have to coincide.

Example 7.1. Consider the complex Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ with its standard basis (h, e, f) . Its Killing form B^c satisfies $B^c(h, h) = 8$ and $B^c(e, f) = 4$ with other values zero. Let \mathfrak{g} denote the realification of $\mathfrak{sl}_2(\mathbb{C})$. The Killing form of \mathfrak{g} is $B = 2\Re B^c$. We obtain a symmetric contact space of non-conical type by choosing $\xi = \lambda h \in \mathfrak{g}$, where $0 \neq \lambda \in \mathbb{C}$. In fact $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ is a symmetric decomposition, where $\mathfrak{k} = \mathbb{C}h$, $\mathfrak{p} = \mathbb{C}e + \mathbb{C}f$ and $\mathfrak{h} = \mathbb{R}(\mu h)$, where $\Re(\lambda\mu) = 0$. Note that ξ is B -isotropic if and only $\Re(\lambda^2) = 0$; in that case, $\eta \neq \xi$.

Proof of Theorem 1.4. Let $N = \text{Ad}_G \xi$ be a symplectic symmetric space as in the statement. Then $N = G/K$, where K is connected, and there exists a symmetric decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ under an involution s . Moreover the symplectic structure on N is induced from the Ad_K -invariant form given by the restriction to \mathfrak{p} of $\omega = d\theta$, where $\theta \in \mathfrak{g}^*$ is dual to ξ under the Killing form B of \mathfrak{g} . Let $\mathfrak{h} = \ker \theta \cap \mathfrak{k}$ (\mathfrak{h} is also the B -orthogonal of ξ in \mathfrak{k}) and consider the associated connected subgroup H of K . It turns out H is a codimension one normal subgroup of K . By assumption, H is closed in K , so $M = G/H$ is the total space of a principal K/H -bundle over N . The 1-form θ defines an invariant contact structure on M which is the invariant extension of the hyperplane $\mathfrak{p} \subset T_o M$. The involution s preserves \mathfrak{h} and θ , thus it induces a contactomorphism of M . Moreover, it induces -1 on $\mathcal{D} = (\ker \theta)/\mathfrak{h}$ and hence M is a symmetric contact space.

The converse follows from Proposition 7.1. \square

Remark 7.2. In cases $\dim Z(\mathfrak{k}) = 1$ or $\dim_{\mathbb{C}} Z(\mathfrak{k}) = 1$ (i.e. \mathfrak{g} is absolutely simple or complex simple; such cases are exactly those listed in Tables 5–8), H is closed for any choice of $\xi \in Z(\mathfrak{k})$; in the first case, in addition, η can be taken to be a multiple of ξ . In general, H is closed if and only if $H \cap Z(K)^0$ is closed; a sufficient condition is that there exists a non-compact (closed) one-parameter subgroup of K not contained in H .

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