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SEMI-PARALLEL SURFACES IN SPACE FORMS

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1. Introduction

Symmetric spaces are locally characterized by the condition $\nabla R = 0$, where R is the riemannian connection (extended to act on tensors) and R is the curvature tensor of the space. The integrability condition of $\nabla R = 0$ is $R \cdot R = 0$, where R is extended to act as a derivation on tensors. Spaces which satisfy the latter condition are called semi-symmetric and were classified by Szabó [10,11].

In submamifold theory, the condition analogue to $\nabla R = 0$ is $\nabla \alpha = 0$ (see (2) below), where α is the second fundamental form of the submanifold, and the condition analogue to $R \cdot R = 0$ is $R \cdot \alpha = 0$. Submanifolds - or isometric immersions - satisfying the first condition are called parallel and have been studied by Ferus [6], Backes and Reckziegel [2], and Takeuchi [12]. Submanifolds satisfying the condition $R \cdot \alpha = 0$ are called semi-parallel and have been studied in the past years by several authors, especially Deprez [3.4] and Lumiste [8,9].

In this communication I will briefly describe some results - Theorems 1, 2 and 3 below - on semi-parallel surfaces which are better detailed in a joint work with Francesco Mercuri, see [1]. Here, as well in the above mentioned articles, the ambient manifold is a space form.

2. Notation

Let M^n be a connected n-dimensional riemannian manifold and let $Q^N(c)$ be a complete, simply connected N-dimensional riemannian manifold with constant sectional curvature c. Given an isometric immersion $f : M^n \to Q^N(c)$, we denote by ∇ and R the riemannian connection of M and respective curvature, by

α the second fundamental form of f, and by $\nabla^\perp$ and $R^\perp$ the normal connection and respective curvature. The immersion is said to be *semi-parallel* if for every tangent vectors X, Y, Z we have:

$$R(X,Y)\alpha := \nabla_X\nabla_Y\alpha - \nabla_Y\nabla_X\alpha - \nabla_{[X,Y]}\alpha = 0 \tag{1}$$

where

$$(\nabla_X\alpha)(Y,Z) := \nabla_X^\perp(\alpha(Y,Z)) - \alpha(\nabla_XY,Z) - \alpha(Y,\nabla_XZ). \tag{2}$$

It follows from the classical equations of Gauss, Codazzi-Mainard ad Ricci that condition (1) can be rewritten as:

$$R^\perp(X,Y)(\alpha(Z,W)) = \alpha(R(X,Y)Z,W) + \alpha(R(X,Y)W,Z). \tag{3}$$

It is not difficult to see that if the immersion is semi-parallel then M is semi-symmetric.

3. Semi-parallel surfaces

From now on we restrict to the case $n = 2$. Let $\{e_1, e_2\}$ be a local orthonormal frame tangent to a semi-parallel immersion $f : M^2 \to Q^N(c)$ and define $\alpha_{i,j} := \alpha(e_i, e_j)$. Then (3) becomes:

$$R^\perp\alpha_{11} = -R^\perp\alpha_{22} = K\alpha_{12}, \ R^\perp\alpha_{12} = K(\alpha_{11} - \alpha_{22}), \tag{4}$$

where K is the Gaussian curvature of M and $R^\perp := R^\perp(e_1, e_2)$ is the normal curvature operator. If $R^\perp \neq 0$ then α_{12} and $\alpha_{22} - \alpha_{11}$ are linearly independent and, in this case, the equations of Ricci and Gauss applied to (4) give:

$$\begin{array}{lll} \|\alpha_{12}\|^2 = K, & \|\alpha_{ii}\|^2 = 4K - c. & \langle\alpha_{ii}, \alpha_{12}\rangle = 0, \ i = 1, 2, \\ \|H\|^2 = 3K - c, & \|\alpha_{11} - \alpha_{22}\|^2 = 4K, & \langle\alpha_{11}, \alpha_{22}\rangle = 2K - c, \end{array} \tag{5}$$

where $H = (\alpha_{11} + \alpha_{22})/2$ is the mean curvature vector of the immersion. Then by considerations on whether $R^\perp \equiv 0$ or $R^\perp \neq 0$ somewhere, we can deduce:

Theorem 1 *([3]) Let $f : M^2 \to Q^N(c)$ be a semi-parallel immersion. Then there exists an open and dense set $U \subset M$ such that the connected components of U are of the following types:*

- *(i) Open parts of a umbilical $Q^2(K)$ in $Q^N(c)$, $K \geq c$;*
- *(ii) Flat surfaces with $R^\perp \equiv 0$;*
- *(iii) Isotropic immersions with $R^\perp \neq 0$ and $\|H\|^2 = 3K - c$.*

For the case $N = 5$, we have the following result which was proven by Lumiste [9] in the case $c = 0$.

Theorem 2 *Let $f : M^2 \to Q^5(c)$ be a semi-parallel immersion of a connected surface. If $R^\perp \neq 0$ somewhere, then $f(M^2)$ is an open piece of a Veronese surface in some $Q^4(\bar{c})$ totally umbilical in $Q^5(c)$, $\bar{c} > 0$.*

The main point in the proof of Theorem 2 is to show that the Gaussian curvature has to be constant. For this we choose an orthonormal frame $(e_1, \ldots, e_5)$ in $Q^5(c)$ adapted to the immersion and in a way that:

$$e_3 = H/\sqrt{3K - c},\ e_4 = (\alpha_{11} - \alpha_{22})/2\sqrt{K},\ e_5 = \alpha_{12}/\sqrt{K}.$$

This is always possible by (5) except when $H = 0$ or, equivalently, when $3K = c > 0$, but this case can be studied separately (see Proposition 3.6 of [1]). Next, manipulating the structure equations for the dual and connection forms of the choosen frame, we arrive to the desired result when $c \geq 0$. For the case $c < 0$ we have also to use a classical result of Beltrami (see [5] p. 161) on the differential parameters on a surface.

A beautiful theorem of Kuiper and Pohl [7] has as a consequence that a tight immersion of the real projective space into $\mathbf{R}^N$ is projectively equivalent to a Veronese surface and its image is contained in some $\mathbf{R}^5 \subset \mathbf{R}^N$. We use this result and the similarity between (5) and the normal data of the classical Veronese surface in Euclidean space to obtain:

Theorem 3 *Let $f : M^2 \to \mathbf{R}^N$ be a semi-parallel immersion of a compact connected surface with $K \equiv 0$. Then $\chi(M) \leq \tau(f) \leq 3\chi(M)$. Moreover:*

- *(i) If M is non orientable then f embeds M as a Veronese surface in some 4-sphere of* $\mathbf{R}^N$;

- *(ii) If M is orientable and f is tight, then f is totally umbilical.*

Here $\chi(M)$ is the Euler characteristic of M, which is positive since $K \not\equiv 0$, and $\tau(f)$ is the total absolute curvature of f.

It would be interesting to extend (i) of Theorem 3 to the orientable case as well. Also one can ask in what extent the restriction $N = 5$ is necessary in Theorem 2.

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