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MOUFANG UNIT LOOPS TORSION OVER THE CENTRE

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ABSTRACT. Let L be an RA loop, that is, a loop whose loop ring in any characteristic is an alternative, but not associative, ring. We find necessary and sufficient conditions for the (Moufang) unit loop of RL to be torsion over its centre when R is the ring of rational integers or an arbitrary field. Over a field, torsion over the centre turns out to be equivalent to torsion of bounded exponent.

1. INTRODUCTION

Let L be a *Moufang loop*, that is, a loop satisfying the identity

$$((xy)z)y = x((yz)y). \tag{1.1}$$

Any group is a Moufang loop and, given a commutative and associative ring R with unity, one can form the loop ring RL just as if L were a group. In general, the Moufang identity does not "lift" from L to RL (the repeated variable is the problem), but sometimes it does. If L is a Moufang loop such that RL also satisfies (1.1), then L is called an *RA loop*. If L is an RA loop, the set $\mathcal{U}(RL)$ of *units* (invertible elements) in RL is also a Moufang loop (containing L) and it is interesting to compare the properties of L and $\mathcal{U}(RL)$. An RA loop L is very special. The torsion units of L form a subloop. The loop L is solvable, in fact, nilpotent. Each element of L has at most two conjugates and L is torsion over the centre; indeed, ℓ^2 is central for any $\ell \in L$. Is it possible for $\mathcal{U}(RL)$ to share with L any of these properties? Conditions under which the torsion units of RL form a subloop when R is either the ring of rational integers or a field were given in two papers [GM95b, GM96c]. Nilpotence and the "finite conjugacy property" of $\mathcal{U}(RL)$, for various R, were explored in a series of papers, [GM95b, GM97, GM96b, GM95a] and, more recently, solvability has been investigated [GM]. In this paper, we consider the possibility that the unit loop $\mathcal{U}(RL)$ might be *torsion over its centre*—that is, for any $\mu \in \mathcal{U}(RL)$, μ^n is central for some $n = n(\mu)$—answering the question when $R = \mathbb{Z}$ and when R is a field. In each case, we consider also whether or not there is a uniform bound on the exponents n. For group rings, these questions have been investigated by Sehgal [Seh78, Section II.2], Cliff and Sehgal [CS80], Coelho [Coe82] and Bist [Bis94].

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A ring with 1 which satisfies the Moufang identity (1.1) is necessarily an *alternative ring*; that is, a ring which also satisfies the right and left alternative laws

$$(yx)x = yx^2 \text{ and } x(xy) = x^2y$$

for all x and y. Our basic reference for Moufang loops, alternative rings and loop rings is the monograph [GJM96], though we endeavour to cite original references throughout this paper too. The most useful property of a Moufang loop (or an alternative ring) is its *diassociativity*: the subloop (or subring) generated by any pair of elements is associative; moreover, any three elements which associate generate an associative subloop (or subring). We use these facts throughout this work.

2. The unit loop of $\mathbb{Z}L$

A *torsion element* in a loop is an element of finite order. In an RA loop, the set of torsion elements forms a normal subloop which is locally finite and, if L is finitely generated, actually finite [GJM96, Lemma VIII.4.1], [GM95b, Lemma 2.1]. A loop M is *torsion* if every element of M is torsion, *torsion over its centre* if, for each $x \in M$, there exists a positive integer $n = n(x)$ so that x^n is central, and *torsion over its centre of bounded exponent* if there exists a positive integer n such that x^n is central for all $x \in M$.

The unit loop of an integral loop ring $\mathbb{Z}L$ always contains the elements $\pm\ell$, $\ell \in L$. Such units are called *trivial* and it is often that case that assumptions about $\mathcal{U}(\mathbb{Z}L)$ imply that all units are trivial. For a torsion Moufang loop L, $\mathcal{U}(\mathbb{Z}L) = \pm L$ if and only if L is an abelian group of exponent 1, 2, 3, 4 or 6, or a hamiltonian 2-loop. This is a theorem of Higman for group rings [Hig40], [Seh78, Theorem II.4.1, p. 57] and, for alternative loop rings which are not associative, of Parmenter and Goodaire [GJM96, Theorem VIII.3.2], [GP86, Theorem 7]. A loop is *hamiltonian* if it is not commutative and every subloop is normal. A group is hamiltonian if and only if it is the direct product $Q_8 \times E \times A$, where Q_8 is the quaternion group of order 8, E is elementary abelian of exponent 2 and A is a group all of whose elements are of odd order [Hal59, Theorem 12.5.4]. A Moufang loop which is not associative is hamiltonian if and only if it is the direct product $C \times E \times A$ where $C = M_{16}(Q_8)$ is the Cayley loop [GJM96, §4.1], [Che74, Case 2(d)] and E and A are as before [GJM96, Theorem II.4.8], [Nor52].

Let L be an RA loop. The main theorem of this section gives necessary and sufficient conditions for the unit loop of the integral loop ring $\mathbb{Z}L$ to be torsion over its centre. First, however, we settle a perhaps more obvious question.

Proposition 2.1. *$\mathcal{U}(\mathbb{Z}L)$ is torsion if and only if L is a hamiltonian Moufang 2-loop.*

Proof. If $\mathcal{U}(\mathbb{Z}L)$ is torsion, then certainly the torsion units of $\mathbb{Z}L$ form a subloop, so the torsion units are trivial by Corollary 3.2 of [GM95b]. As a subloop of a torsion loop, L is torsion, so the alternative analogue of Higman's Theorem described above tells us that L is a hamiltonian Moufang 2-loop. Conversely, if L is a hamiltonian Moufang 2-loop, then L is a torsion loop and $\mathcal{U}(\mathbb{Z}L) = \pm L$ and Higman's Theorem again gives the desired result and completes the proof. (See also [GJM96, Proposition 3.1].) □

The proof of the main result of this section, Theorem 2.2, requires the concept of the "support" of an element in a loop ring. If $\alpha = \sum_{\ell \in L} \alpha_\ell \ell$, $\alpha_\ell \in R$, is an element of the loop ring RL, the *support* of α is the set

$$\operatorname{supp}(\alpha) = \{\ell \in L \mid \alpha_\ell \neq 0\}.$$

We also require the fact that if L is an RA loop, then ℓ^2 is central for all $\ell \in L$ [GJM96, Theorem IV.1.8], [Goo83, Theorem 3].

Theorem 2.2. *Let L be an RA loop with torsion subloop T. Then the following conditions are equivalent:*

(i) *$\mathcal{U}(\mathbb{Z}L)$ is torsion over its centre, $\mathcal{Z}(\mathcal{U}(\mathbb{Z}L))$.*
(ii) *T is an abelian group or a hamiltonian Moufang 2-loop and, if $x \in L$ does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$.*
(iii) *$\mathcal{U}(\mathbb{Z}L)$ is torsion of bounded exponent 2 over $\mathcal{Z}(\mathcal{U}(\mathbb{Z}L))$.*
(iv) *$\mathcal{U}(\mathbb{Z}L)$ is nilpotent.*

Proof. The equivalence of conditions (ii), (iv) and several others is given in [GJM96, Corollary XII.2.14] and [GM95b]. Since (iii) implies (i), it suffices here to show that (i) implies (ii) and (ii) implies (iii).

Assume (i) and suppose that $\alpha \in \mathbb{Z}L$ satisfies $\alpha^2 = 0$. Then $1 + \alpha$ is a unit (with inverse $1 - \alpha$). Also, for any positive integer n, we have $(1 + \alpha)^n = 1 + n\alpha$. Our hypothesis thus implies that $1 + n\alpha$ is central for some n. Thus α is central and $(\mathbb{Z}L)\alpha$ is a nilpotent ideal in $\mathbb{Z}L$. It is known that an alternative loop ring RL is semiprime if and only if $R\mathcal{Z}$ is semiprime, where $\mathcal{Z} = \mathcal{Z}(L)$ denotes the centre of L [GJM96, Corollary VI.3.6], [GP87]. Since $\mathbb{Z}\mathcal{Z}$ is semiprime [GJM96, Theorem VI.2.7], [May76, Proposition 2], $\alpha = 0$. All this shows that $\mathbb{Z}L$ contains no nilpotent elements.

For $t \in T$ of order n, write $\widehat{t} = 1 + t + t^2 + \cdots + t^{n-1}$. Let ℓ be any element of L. Since $t\widehat{t} = \widehat{t}$, $\alpha = (1 - t)\ell\widehat{t}$ satisfies $\alpha^2 = 0$, so $\alpha = 0$ and $\ell\widehat{t} = t\ell\widehat{t}$. In this last equation, ℓ is in the support of the left hand side, so it is in the support of the right. This means that $\ell = t\ell t^i$ for some $i \geq 0$ and hence $\ell^{-1}t\ell = t^{-i} \in \langle t\rangle$. In an RA loop, this is sufficient to ensure normality of $\langle t\rangle$ [GJM96, Corollary IV.1.11], [CG90, Corollaries 2.4 and 2.11]. It follows that every subloop of T is normal in L. In particular, every subloop of T is normal in T, so T is either an abelian group or a hamiltonian Moufang loop. In the latter case, if T contains a nonidentity element of odd order, then $\mathbb{Z}T$ contains nontrivial units by Higman's Theorem (for alternative loop rings) and hence a free group of rank 2 by [HP80] in the case T is a group and [GJM96, Theorem VIII.5.8] otherwise. Since a free group cannot be torsion over its centre, T cannot have elements of odd order, so T is a 2-loop in the hamiltonian case. Moreover, if T is a hamiltonian Moufang 2-loop, the known structure of such loops makes it easy to see that the rest of condition (ii) is satisfied. On the other hand, suppose T is an abelian group and $x \in L$ does not centralize T. The subloop $G = \langle T, x\rangle$ generated by T and x is a group [GJM96, Corollary IV.2.4], [GM96b, Lemma 1.3] with $\mathcal{U}(\mathbb{Z}G)$ torsion over its centre. That $x^{-1}tx = t^{-1}$ for all $t \in T$ now follows from a result of Bist [Bis94, Corollary 6]. So we have (ii) in this case as well.

Assume condition (ii), which is condition (10) in Corollary XII.2.14 of [GJM96]. Then $\mathcal{U}(\mathbb{Z}T) = \mathcal{H}L$ is the product of L and an abelian group $\mathcal{H}$, central in $\mathcal{U}(\mathbb{Z}L)$ (see also the proof of Theorem 3.3 in [GM95b]). Let $\mu = \nu\ell \in \mathcal{U}(\mathbb{Z}L)$, $\nu \in \mathcal{H}$. Then $\mu^2 = \nu^2\ell^2 \in \mathcal{Z}(\mathcal{U}(\mathbb{Z}L))$ because $\nu^2 \in \mathcal{H}$ is central and ℓ^2 is central in L. This is (iii). □

3. The Unit Loop of a Loop Algebra over a field

Now we turn our attention to alternative loop algebras over fields and completely determine when the unit loop of an alternative algebra is torsion (possibly of bounded exponent) over its centre. Our results depend on the characteristic of the field and are summarized in Theorems 3.8, 3.9, 3.13 and 3.14. As in the previous section, we first dismiss the case of torsion unit loops which, in the present discussion, are not very interesting.

Proposition 3.1. *Let L be an RA loop and K a field. Then $\mathcal{U}(KL)$ is torsion if and only if L and $K^* = K \setminus \{0\}$ are both torsion.*

Proof. Necessity is clear. Conversely, if K^* and L are torsion, then $\operatorname{char} K = p > 0$, so K has a finite prime field P. Given $\mu = \sum_{i=1}^{n} k_i \ell_i \in \mathcal{U}(KL)$, $k_i \in K$, $\ell_i \in L$, let $E = \mathsf{P}(k_1, k_2, \ldots, k_n)$ be the subfield of K generated by P and the coefficients of μ. Let L_0 be the finite loop generated by $\ell_1, \ell_2, \ldots, \ell_n$. Then each element of the finite Moufang loop $\mathcal{U}(EL_0)$ (for instance, μ) has finite order. □

Corollary 3.2. *If $\mathcal{U}(KL)$ is not torsion, there exist central units in K or in L of infinite order.*

Proof. If $\mathcal{U}(KL)$ is not torsion, Proposition 3.1 says that either L contains an element ℓ of infinite order, in which case $\ell^2 \in \mathcal{Z}(L)$ also has infinite order, or else K^* contains an element of infinite order. Since elements of K are central in KL, the result follows. □

Lemma 3.3. *Let K be a field and let L be an RA loop with torsion subloop T. Suppose $\mathcal{U}(KL)$ is torsion over its centre, but not torsion. Then every idempotent of KL is central.*

Proof. By Corollary 3.2, KL contains a central unit z of infinite order which is either in K^* or in L. Let $e \in KL$ be an idempotent. Following [CS80, Lemma 2.4], we observe that $\mu = ze + (1-e)$ is a unit, with inverse $z^{-1}e + (1-e)$, so $\mu^n = z^n e + (1-e) = 1 + (z^n - 1)e$ is central for some positive integer n. Thus $(z^n-1)e$ is central and $x^{-1}(z^n-1)ex = (z^n-1)e$ for any $x \in L$. (Note that x, z and e associate because z is central.) Writing $e^x = x^{-1}ex$, this gives $z^n(e^x - e) = e^x - e$ for any $x \in L$. Replacing z by z^k in the preceding arguments, we have in fact that $z^{kn}(e^x - e) = e^x - e$ for any x and for any $k = 1, 2, 3, \ldots$. If $z \in K^*$, $z^{kn}\alpha = \alpha$ for any $\alpha \in KL$ implies $z^{kn} = 1$, contradicting the fact that z has infinite order. Thus $z \in L$, but then, if ℓ is in the support of $e^x - e$, so is $z^{kn}\ell$. Since this happens for infinitely many k and the support of $e^x - e$ is finite, it must be that $e^x - e = 0$, hence $e^x = e$ (for any x) and e is central. □

We have noted and, indeed, made frequent use of the fact that the set of torsion elements of an RA loop L is a subloop. In fact, for any prime p, the sets T_p of p-elements (order a power of p) and $T_{p'}$ of p'-*elements* (order relatively prime to p) are normal subloops of L and $T = T_p \times T_{p'}$ [GM96a, Lemma 1.2]. (In the case $p = 2$, see also [GJM96, Proposition V.1.1].)

Corollary 3.4. *Let K be a field of characteristic $p > 0$ and let L be an RA loop with torsion subloop T. Suppose $\mathcal{U}(KL)$ is torsion over its centre but not torsion. Let $T_{p'}$ denote the subloop of p'-elements in T, that is, those of order relatively prime to p. Then*

(i) *every subloop of $T_{p'}$ is normal in L;*

(ii) *$T_{p'}$ is an abelian group.*

Proof. (i) Let $t \in T$ have order n relatively prime to p and let $\widehat{t} = 1 + t + t^2 + \cdots + t^{n-1}$. Then $e = \frac{1}{n}\widehat{t}$ is an idempotent, so $x^{-1}ex = e$ for any $x \in L$ by the lemma. It follows that $x^{-1}tx = t^i$ for some i. As previously noted, the test for normality in an RA loop is the same as it is in a group, hence $\langle t \rangle$ is normal in L, a condition which implies that every subloop of $T_{p'}$ is normal in L.

(ii) Since elements in L of odd order are central [GJM96, Proposition V.1.1], [CG86, Theorem 6], the result is clear if $p = 2$ while, if $p \neq 2$, it suffices to show that the set T_2 of elements of L whose order is a power of 2 forms an abelian group. Since $T_2 \subseteq T_{p'}$, every subloop of T_2 is normal by part (i) so, if T_2 is not an abelian group, it is hamiltonian and hence contains Q_8, the quaternion group of order 8. Denoting by P the prime field of K, we have $PQ_8 \subseteq KL$. Now the semi-simple group algebra $\mathsf{P}Q_8$ contains a quaternion algebra [GJM96, Corollary VII.2.4], [Seh78, Proposition 1.14] which is necessarily a ring of 2×2 matrices since it is finite. Thus KL contains noncentral idempotents, contradicting the lemma. □

The first theorem of this section makes use of the fact that if G is an associative subloop of an RA loop, then either G is abelian or else the quotient $G/\mathcal{Z}(G)$ of G by its centre is $C_2 \times C_2$, the Klein 4-group [GJM96, Proposition III.3.6 and Corollary IV.2.2], [CG86, Theorem 5]. We also require the following lemma the proof of which can be deduced from Lemma XII.1.1 of [GJM96] and its proof. See also [GM95b, Lemma 2.3].

Lemma 3.5. *Let K be a field and let L be an RA loop with torsion subloop T. Suppose $KT = \bigoplus D_i$ is the direct sum of division rings and that every idempotent of KT is central in KL. Then the following are true.*

(i) *Any sum S of a subset of the division rings D_i is* normal *in KT in the sense that $S\alpha = \alpha S$, $(S\alpha)\beta = S(\alpha\beta)$, $(\alpha S)\beta = \alpha(S\beta)$ and $\alpha(\beta S) = (\alpha\beta)S$ for any $\alpha, \beta \in KL$.*

(ii) *Each unit $\mu \in KL$ can be written in the form $\mu = \sum \mu_q q$, where $q \in L$, the μ_q are in KT and, if $q_1 \neq q_2$, the set of division rings required to write μ_{q_1} as a sum in $\bigoplus D_i$ has empty intersection with the set required to write μ_{q_2} as a sum; in particular, $\mu_{q_1}\mu_{q_2} = 0$ if $q_1 \neq q_2$.*

Remark 3.6. When $KT = \bigoplus D_i$ is the direct sum of division rings, it will be convenient in certain parts of what follows to write $\mu \sim \nu$ if μ and ν are sums of elements in the same subset of division rings. Clearly this defines an equivalence relation.

Corollary 3.7. *With notation as in the lemma, suppose that μ and ν are sums of elements in certain division rings D_i and that α and β are any elements of KL. Then there exist μ', ν' with $\mu' \sim \mu$ and $\nu' \sim \nu$ such that $(\mu\alpha)(\nu\beta) = (\mu'\nu')(\alpha\beta)$.*

Proof. Part (i) of the lemma gives the existence of $\mu_1, \mu_2 \sim \mu$ and $\nu_1, \nu_2, \nu_3 \sim \nu$ such that

$$(\mu\alpha)(\nu\beta) = \mu_1(\alpha \cdot \nu\beta) = \mu_1(\alpha\nu_1 \cdot \beta) = \mu_1(\nu_2\alpha \cdot \beta) = \mu_1(\nu_3 \cdot \alpha\beta) = (\mu_2\nu_3)(\alpha\beta)$$

from which the result follows. □

In the theory of group rings, one of the best known theorems is surely that of H. Maschke: if G is a finite group and K is a field of characteristic 0 or $p > 0$ not dividing $|G|$, then KG

is semisimple artinian [Row88b, §8.1] and hence the direct sum of matrices over division rings. This result will prove useful to us here.

Theorem 3.8. *Let L be an RA loop with torsion subloop T and let K be a field of characteristic 0. Suppose $\mathcal{U}(KL)$ is not torsion. Then the following conditions are equivalent.*

(i) *$\mathcal{U}(KL)$ is torsion over its centre.*
(ii) *T is central.*
(iii) *$\mathcal{U}(KL)$ is torsion of exponent 2 over its centre; that is, $[\mathcal{U}(KL)]^2 \subseteq \mathcal{Z}(\mathcal{U}(KL))$.*
(iv) *$\mathcal{U}(KL)$ is torsion of bounded exponent over its centre.*

Proof. Assume (i). By Lemma 3.3, every idempotent of KT is central in KL. Let $t \in T$, $x \in L$ and let $G = \langle t, x \rangle$ be the group generated by t and x. Certainly $\mathcal{U}(KG)$ is torsion over its centre but, since $\operatorname{char} K = 0$, $\mathcal{U}(KG)$ is not torsion by Proposition 3.1. As noted above, either G is abelian or $G/\mathcal{Z}(G) \cong C_2 \times C_2$. In any event, G is solvable, so the torsion element t is central in G by Theorem 1 of [CS80]. In particular $tx = xt$, giving (ii).

Assuming (ii), it is clear that every idempotent of KT is central in KL. Let $\mu \in \mathcal{U}(KL)$. Replacing L by the support of μ (and, if necessary, at most three additional elements from L which do not associate), we may assume that L is finitely generated and, hence, that T is finite. By Maschke's theorem, the group algebra KT is a direct sum of fields. Now we apply Lemma 3.5 and write $\mu = \sum \mu_q q$ as in that lemma. Since T is central, so is each μ_q, so $\mu^2 = \sum \mu_q^2 q^2$. This gives (iii) because each $q^2 \in \mathcal{Z}(L)$.

Since it is trivial that (iii) implies (iv) and (iv) implies (i), the proof is complete. □

Some of our arguments in the sequel make use of the concept of *augmentation ideal*, one reference for which is [GJM96, §VI.1]. Let R be any commutative and associative ring with 1. Let N be a normal subloop of an RA loop (or a group) L and let $\epsilon_N \colon RL \to R[L/N]$ denote the linear extension to RL of the natural homomorphism $L \to L/N$. This map is a ring homomorphism whose kernel is the ideal $\Delta(L,N) = \sum_{n \in N} RL(n-1)$. In the special case $N = L$, we write $\epsilon = \epsilon_L$, calling this the *augmentation map* on RL, note that $\epsilon(\sum \alpha_\ell \ell) = \sum \alpha_\ell$ and set $\Delta(L) = \Delta(L, L)$. We call $\Delta(L)$ the *augmentation ideal* of RL. The identity $\ell_1(\ell_2 - 1) = (\ell_1\ell_2 - 1) - (\ell_1 - 1)$ shows that

$$\Delta(L) = \{\sum_{\ell \in L} \alpha_\ell(\ell - 1) \mid \alpha_\ell \in R\}.$$

For any normal subloop N of L, note that $\Delta(L,N) = (RL)\Delta(N)$.

Since the number 2 is so important in the theory of RA loops (for example, finite RA loops which are not direct products are 2-loops), it is not surprising that theorems about RA loop algebras over fields often vary depending on whether or not the characteristic is 2.

Theorem 3.9. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Suppose $\mathcal{U}(KL)$ is not torsion. Then $\mathcal{U}(KL)$ is torsion over its centre if and only if T is an abelian group, every idempotent of KT is central in KL and, if T is not central, then K is algebraic over its prime field.*

Proof. Suppose $\mathcal{U}(KL)$ is torsion over its centre. Then every idempotent of KL (and hence certainly of KT) is central by Lemma 3.3. Since T is a subloop and since commutative RA loops are associative [GJM96, Corollary IV.1.3], [CG90, Corollary 2.5], to show that T is an abelian group, it suffices just to show that the elements of T commute. So let $a, b \in T$. Since elements of odd order in L are central, we may assume that a and b each have order a power of 2. The group $G = \langle a, b\rangle$ generated by a and b is a (finite) 2-group because L has a unique central commutator of order 2 [GJM96, Theorem IV.1.8], [Goo83, Theorem 3]. Let P denote the prime field of K. Since $p \neq 2$, the finite group algebra $\mathsf{P}G$ is a direct sum of fields and matrix rings. Since idempotents in KL are central, the same is true for $\mathsf{P}G \subseteq KL$, so there are no matrix rings. Thus G is commutative, so $ab = ba$ and T is an abelian group. Suppose T is not central and let G be any associative subloop of L which is not torsion and which contains noncentral elements of order a power of 2 (for example, the group generated by a noncentral element of 2-power order and another element which is not torsion, such element existing because T is an abelian group and L is not). Then $\mathcal{U}(KG)$ is torsion over its centre, $\mathcal{U}(KG)$ is not torsion and not every p' element of G is central (since p is odd). It follows from Theorem 1 of [CS80] that K is algebraic over its prime field.

Conversely, suppose that T is an abelian group, that every idempotent of KT is central in KL and, if T is not central, that K is algebraic over its prime field. Let $\mu \in KL$ be a unit. Replacing L by a finitely generated subloop which is not associative and contains the support of L, we may assume that T is finite. Write the abelian group $T = P \times A$, where P is the p-primary component of T and $A = T_{p'}$ is the set of p'-elements. Since $p \neq 2$, P is central and hence normal in L. Also, the torsion subloop of L/P is $T/P \cong A$ which has no elements of order $p = \operatorname{char} K$. By Maschke's Theorem, $K[T/P]$ is a direct sum of fields. Since P is a p-group and $\operatorname{char} K = p$, $\Delta(P)$ is nilpotent [Jen41]. Centrality of P shows that $\Delta(L,P)^n \subseteq (KL)\Delta(P)^n$ for any $n \geq 1$; thus $\Delta(L,P)$ is also nilpotent. (See also [MZ, Theorem 3.4].) Let $\overline{e} \in K[T/P]$ be an idempotent, where $\overline{\alpha} = \epsilon_P(\alpha)$ denotes the image of $\alpha \in KL$ under the homomorphism $\epsilon_P \colon KL \to K[L/P] \cong KL/\Delta(L,P)$. Then $\overline{e}$ "lifts" to an idempotent in KT; that is, $\overline{e} = \overline{f}$ for some idempotent $f \in KT$ [Row88a, Corollary 1.1.28]. Since idempotents of KT are central in KL, $\overline{e}$ is central in $K[L/P]$.

Now apply Lemma 3.5 and write the unit $\overline{\mu}$ in the form $\overline{\mu} = \sum \overline{\mu_q}\,\overline{q}$, $\overline{\mu_q} \in K[T/P]$, $\overline{q} \in L/P$, and $\overline{\mu_{q_1}}\,\overline{\mu_{q_2}} = \overline{0}$ if $\overline{q_1} \neq \overline{q_2}$, hence $\overline{\mu'_{q_1}}\,\overline{\mu'_{q_2}} = \overline{0}$ if $\overline{q_1} \neq \overline{q_2}$ whenever $\overline{\mu'_{q_1}} \sim \overline{\mu_{q_1}}$ and $\overline{\mu'_{q_2}} \sim \overline{\mu_{q_2}}$. Thus $\mu = \sum \mu_q q + \delta$ for some $\delta \in \Delta(L,P)$, $\mu_q \in KA$, $q \in L$, and $\mu_{q_1}\mu_{q_2} \in \Delta(L,P)$ if $q_1 q_2^{-1} \notin P$.

If T is central in L, each μ_q is central in KL and hence associates with all pairs of elements in KL. Thus, for any q_1, q_2, $(\mu_{q_1} q_1)(\mu_{q_2} q_2) = \mu_{q_1}\mu_{q_2} q_1 q_2$. It follows that the product $\mu_{q_1} q_1 \mu_{q_2} q_2$ is $\mu_{q_1}^2 p q_1{}^2$ for some $p \in P$ if $q_1 q_2^{-1} \in P$ and an element of $\Delta(L,P)$ otherwise. Note that $\mu_{q_1}^2 p q_1{}^2$ is central. Thus $\mu^2 = \gamma + \delta_1$, with γ central and $\delta_1 \in \Delta(L,P)$. There exists a positive integer n such that $[\Delta(L,P)]^{p^n} = 0$. Since $(a+b)^{p^n} = a^{p^n} + b^{p^n}$ for commuting elements a and b in characteristic p, we see that $\mu^{2p^n} = \gamma^{p^n}$ is central. Thus $\mathcal{U}(KL)$ is torsion over its centre in this case.

Suppose that T (and hence A) are not central. By hypothesis, K is algebraic over its prime field. Since each idempotent of KT is central in KL, the algebraic closure of the prime field of K in K is finite [GJM96, Theorem XIII.1.6], [GM96a, Theorem 2.1], so K and hence KT are finite. Thus KT is a direct sum of (finite) fields K_i. Choose an integer m such that $\alpha^m = 1$ for any nonzero α in any K_i. By Corollary 3.7, for any $\overline{q_1}, \overline{q_2}$, there exist

$\overline{\mu'_{q_1}} \sim \overline{\mu_{q_1}}$ and $\overline{\mu'_{q_2}} \sim \overline{\mu_{q_2}}$ such that $(\overline{\mu_{q_1}}\,\overline{q_1})(\overline{\mu_{q_2}}\,\overline{q_2}) = (\overline{\mu'_{q_1}}\,\overline{\mu'_{q_2}})(\overline{q_1}\,\overline{q_2})$ for some $\overline{\mu'_{q_1}} \sim \overline{\mu q_1}$ and $\overline{\mu'_{q_2}} \sim \overline{\mu_{q_2}}$. Thus

$$(\mu_{q_1}q_1)(\mu_{q_2}q_2) \in \Delta(L,P) \quad \text{if } q_1q_2^{-1} \notin P,$$

while

$$(\mu_{q_1}q_1)(\mu_{q_2}q_2) = (\mu'_{q_1}\mu'_{q_2})pq_1^2 \text{ with } p \in P \text{ and } pq_1^2 \text{ central, if } q_1q_2^{-1} \notin P.$$

It follows that

$$\mu^2 = \sum \alpha_q z_q + \delta_1 \tag{3.1}$$

with each $\alpha_q \in KT$, $z_q \in L$ central and $\delta_1 \in \Delta(L,P)$. Recalling that KT is the direct sum of fields K_i, we may assume that each α_q appearing in (3.1) is an element of some K_i. In particular, ${\alpha_q}^m = 1$ for all q. Centrality of the z_q now gives

$$\mu^{2m} = \sum z_q^{2m} + \delta_2, \tag{3.2}$$

with $\sum z_q^{2m}$ central and $\delta_2 \in \Delta(L,P)$. There exists p^n such that $[\Delta(L,P)]^{p^n} = \{0\}$. Thus μ^{2mp^n} is central. □

Remark 3.10. It is known, in terms of L and K, precisely when all the idempotents of KT are central in KL [GJM96, §XIII.1], [GM96a].

With reference to Theorem 3.9 and its proof, suppose K is a field of positive characteristic $p \neq 2$ and L is an RA loop with no elements of order p. Then $P = \{1\}$, so $\Delta(L,P) = \{0\}$. If L has central torsion, we showed that that the square of any unit is central. So we have the following result for semiprime alternative loop algebras.

Corollary 3.11. *Let K be a field of characteristic $p \neq 2$ and let L be an RA loop which has central torsion and which contains no elements of order p. Then $\mathcal{U}(KL)$ is torsion over its centre if and only if it is torsion of exponent 2 over its centre.*

We wish now to explore the case of bounded exponent. In so going, we shall make use of the following lemma of S. Coelho [Coe82].

Lemma 3.12. *Let $\mathcal{U}(KL)$ denote the unit loop of an alternative loop algebra over a field K of characteristic $p > 0$ and suppose $[\mathcal{U}(ZL)]^n \subseteq \mathcal{Z}(\mathcal{U}(ZL))$. Write $n = p^a n'$ with $p \nmid n'$. Then x^{p^a} is central for any nilpotent $x \in KL$.*

Proof. Let $x \in KL$ be a nilpotent element. Then $1 + x$ is a unit, so $(1+x)^n$ central; that is, $(1 + x^{p^a})^{n'}$ is central. Now $x^{p^r} = 0$ for some p^r and there exist integers i and j with $ip^r + jn' = 1$. Thus $1 + x^{p^a} = (1 + x^{p^a})^{ip^r+jn'} = (1 + x^{p^r})^{ip^a}(1 + x^{p^a})^{jn'} = (1 + x^{p^a})^{jn'}$ is central, so x^{p^a} is central. □

Theorem 3.13. *Let K be a field of positive characteristic $p \neq 2$ and let L be an RA loop with torsion subloop T. Let P and $A = T_{p'}$ denote, respectively, the sets of p- and p'-elements in T. Suppose $\mathcal{U}(KL)$ is not torsion. Then the following statements are equivalent.*

(i) *$\mathcal{U}(KL)$ is torsion of bounded exponent over its centre.*
(ii) *T is abelian, there exists a such that x^{p^a} is central for all $x \in \Delta(L,P)$ and either*

- *T is central and $\mathcal{U}(KL)$ has exponent $2p^a$ over its centre, or*
- *K is finite, $A^m = \{1\}$ for some m and $\mathcal{U}(KL)$ has exponent $2rp^a$ over its centre, where $r = |K(\zeta)| - 1$, ζ a primitive mth root of unity.*

Proof. Clearly we have only to prove that (i) implies (ii), so assume (i) is the case. Since $\mathcal{U}(\mathbb{Z}L)$ is torsion over its centre (but not torsion), T is abelian by Theorem 3.9. There exists n such that $[\mathcal{U}(KL)]^n \subseteq \mathcal{Z}(\mathcal{U}(KL))$. Write $n = p^a n'$ with $p \not| n'$. Given $\delta \in \Delta(L, P)$, by restricting to a subloop which is not associative and which contains the support of δ, we may assume that L is finitely generated, hence that P is finite. As shown in the proof of Theorem 3.9 (the second part), $\Delta(L, P)$ is nilpotent, so δ^{p^a} is central by Lemma 3.12.

If T is central and $\mu \in KL$ is a unit, exactly as in the proof of Theorem 3.9, we have $\mu^2 = \gamma + \delta_1$, with γ central and $\delta_1 \in= \Delta(L, P)$. Since ${\delta_1}^{p^a}$ is central, so is $\mu^{2p^a} = \gamma^{p^a} + \delta_1^{p^a}$.

Suppose T is not central. Then there exist $t \in T$ and $\ell \in L$ such that t and ℓ do not commute. Since P is central, we may assume that $t \in A$. Since T is abelian, ℓ has infinite order, so the subloop $\langle A, \ell\rangle$ generated by A and ℓ is not torsion, not commutative and a group by [GJM96, Corollary IV.2.4], [GM96b, Lemma 1.3]. By [Coe82, Theorem B], $A^m = \{1\}$ for some m and K is finite. Let $\mu \in \mathcal{U}(KL)$. Once again, without loss of generality, we may assume that L is finitely generated and hence that T is finite. Thus $T = P \times A$ and, by Maschke's Theorem, $K[T/P] \cong KA = \bigoplus K_i$ is the direct sum of fields, necessarily finite and all contained in $K(\zeta)$, ζ a primitive mth root of unity. For $r = |K(\zeta)| - 1$ and nonzero α in any K_i, we have $\alpha^r = 1$. As in the proof of Theorem 3.9, we can write $\mu^2 = \sum \alpha_q z_q + \delta$ with z_q central, $\alpha_q \in K_i$ and $\delta \in \Delta(L, P)$. Thus $\mu^{2r} = \gamma + \delta_1$, with γ central and $\delta_1 \in \Delta(L, P)$. So $(\mu^{2r})^{p^a}$ is central. □

Theorems 3.9 and 3.13 have assumed that the characteristic of K is different from 2. The remaining case is answered by the following very easy result.

Theorem 3.14. *Let L be an RA loop and let K be a field of characteristic 2. Then $\mathcal{U}(KL)$ is of bounded exponent 2 over its centre.*

Proof. Since L has a unique nonidentity commutator-associator s (necessarily central and of order 2) [GJM96, Theorem IV.1.8], [Goo83, Theorem 3], then $\ell_1\ell_2 - \ell_2\ell_1$ is either 0 or $\ell_1\ell_2(1-s)$ for any $\ell_1, \ell_2 \in L$. Thus for any $\alpha, \beta \in KL$, the ring commutator $\alpha\beta - \beta\alpha$, which we denote $[\alpha, \beta]$, is of the form $\gamma(1+s)$ for some $\gamma \in KL$. Thus $[\alpha^2, \beta] = \alpha[\alpha, \beta] + [\alpha, \beta]\alpha = (1+s)(\alpha\gamma + \gamma\alpha) \in (1+s)^2(KL) = 0$. So α^2 is central. □

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