RT-MAE-8005

A CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

by

PEDRO A. MORETTIN

PALAVRAS CHAVES: Finite Walsh-Fourier transform
(Key Words) Walsh-Fourier spectrum
Central Limit Theorem

CLASSIFICAÇÃO AMS : 60G10
(Classification AMS) 62M10

NOVEMBRO DE 1980

A CENTRAL LIMIT THEOREM FOR STATIONARY PROCESSES

Pedro A.Morettin
University of São Paulo, Brasil.

1. INTRODUCTION

Let $\{X(n),\ n=0,\pm1,\pm2,\dots\}$ be a strictly stationary sequence with zero mean and covariance function $R(k)=E\{X(n)X(n+k)\}$.

For $N=2^t$, $t>0$ integer, let

$$d_N(\lambda)=N^{-1/2}\sum_{n=0}^{N-1} X(n)W(n,\lambda),\quad 0\leq\lambda<1 \tag{1}$$

be the finite Walsh - Fourier transform of the values $X(0),\dots,X(N-1)$, where $\{W(n,\lambda),\ n=0,1,2,\dots,\ 0\leq\lambda<1\}$ is the orthonormal system of Walsh functions. For the difinitions, notations and properties concerned with this system we refer to the papers by Fine (1949, 1950, 1957), Morettin (1974b, 1980) and Kohn (1980a, 1980b).

Our purpose is to prove a central limit theorem for $d_N(\lambda)$. Some previous results were derived by Morettin (1973, 1974a) and Kohn (1980a).

Let

$$\tau(j)=N^{-1}\sum_{k=0}^{N-1} R(j\oplus k-k) \tag{2}$$

be the logical covariance (Robinson (1972), Kohn (1980a)).

It follows that $E\{d_N(\lambda)\}=0$ and

$$\mathrm{Cov}\{d_N(\lambda),\ d_N(\mu)\}=\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1} W(j,\lambda)W(k,\mu)\ \mathrm{Cov}\{X(j),X(k)\}$$

$$= N^{-1} \sum_{j,k=0}^{N-1} W(j,\lambda)\ W(k,\mu)\ R(j-k) \qquad (3)$$

It follows that the variance of $d_N(\lambda)$ is

$$\mathrm{Var}\{d_N(\lambda)\} = N^{-1} \sum_{j,k=0}^{N-1} W(j \oplus k,\lambda)\ R(j-k)$$

$$= N^{-1} \sum_{j,k=0}^{N-1} R(j \oplus k-k)\ W(j,\lambda)$$

and using (2) we have that

$$\mathrm{Var}\{d_N(\lambda)\} = \sum_{j=0}^{N-1} \tau(j)\ W(j,\lambda). \qquad (4)$$

If we further assume that

$$\sum_{j=0}^{\infty} |\tau(j)| < \infty, \qquad (5)$$

then $\mathrm{Var}\{d_N(\lambda)\}$ converges to

$$f(\lambda) = \sum_{j=0}^{\infty} \tau(j)\ W(j,\lambda) \qquad (6)$$

which we call the Walsh - Fourier spectrun of $\{X(n), n=0,\pm 1,\ldots\}$.

We now state

Assumption 1 [Kohn (1980a)] : $\lim_{n\to\infty} \sum_{|j|<2^n} (1-\frac{|j|}{2^n})\ |R(j)| < \infty$.

It is easy to see that if Assumption 1 holds, then (5) follows and the spectrum $f(\lambda)$ is well defined.

Let M_n be the σ-algebra generated by $\{X(j);\ j\leq n\}$ and put

$$\alpha_j = [E\{E[X(n)|M_{n-j}] - E[X(n)|M_{n-j-1}]\}^2]^{1/2}, \quad j\geq 0.$$

Assuming that $M_{-\infty}$ is trivial and that $\sum_{j=0}^{\infty} \alpha_j < \infty$, Kohn (1980a) proves a central limit theorem for $d_N(\lambda)$. The proof consists in adapting the proof of a central limit theorem for time series regression due to Hannan (1973), which is complex.

As mentioned by Hannan (1973), it is possible to relax the conditions $\Sigma|\alpha(j)|<\infty$, if we require $\{X(n)\}$ to have higher moments. Following this line, we impose a further assumption and derive the same result of Kohn (1980a), the proof being considerably simpler.

2. THE CENTRAL LIMIT THEOREM

Let the cumulant of order r of X(n) be denoted by

$$C_r(j_1,\ldots,j_r)=\mathrm{cum}\{X(j_1),\ldots,\ X(j_r)\}, \qquad (7)$$

for $j_1,\ldots,j_r=0,\pm 1,\ldots$, assuming $E\{|X(n)|^r\}<\infty$. By stationarity,

$$C_r(j_1,\ldots,j_r)=C_r(j_1+j,\ldots,\ j_r+j),$$

for $j_1,\ldots,j_r$, $j=0,\pm 1,\ldots$, and in asymmetric notation,

$$C_r(j_1,\ldots,\ j_{r-1})=C_r(j_1,\ldots,\ j_{r-1},0). \qquad (8)$$

Assumption 2. $$\sum_{j_1=0}^{\infty}\cdots\sum_{j_{r-1}=0}^{\infty}|C_r(j_1,\ldots,\ j_{r-1})|<\infty. \qquad (9)$$

Under this assumption, values of the sequence well separated in time are weakly dependent; it is a form of "mixing" condition.

Lemma P4.5 of Brillinger (1975, p.403), plus the fact that all cumulants of order greater than two are zero for a normal distribution, will be used to prove the central limit theorem that follows.

Theorem 1. Suppose Assumptions 1 and 2 hold. Then $d_N(\lambda)$ converges in distribution to a variable with distribution $\mathcal{N}(0,f(\lambda))$, where $f(\lambda)$ is given by (6).

Proof. We begin by noting that $E\{d_N(\lambda)\}=0$ and $\mathrm{Var}\{d_N(\lambda)\}\to f(\lambda)$, when $N\to\infty$, hence the first and second-order cumulants behave in the manner required by the theorem.

Now,

$$\mathrm{cum}\{d_N(\lambda_1),\ldots,d_N(\lambda_r)\} =$$

$$= N^{-r/2} \sum_{j_1=0}^{N-1} \cdots \sum_{j_r=0}^{N-1} W(j_1,\lambda_1)\ldots W(j_r,\lambda_r)\,\mathrm{cum}\{X(j_1),\ldots,X(j_r)\}$$

$$= N^{-r/2} \sum_{j_1=0}^{N-1} \cdots \sum_{j_r=0}^{N-1} C_r(j_1,\ldots,j_{r-1}) W(j_1,\lambda_1)\ldots W(j_r,\lambda_r).$$

Hence,

$$|\mathrm{cum}\{d_N(\lambda_1),\ldots,d_N(\lambda_r)\}| \le N^{-r/2+1} \sum_{j_1=0}^{N-1} \cdots \sum_{j_{r-1}=0}^{N-1} |C_r(j_1,\ldots,j_{r-1})|$$

and by Assumption 2 the cumulant in question is of order $O(N^{-r/2+1})$ and therefore tends to zero, as $N\to\infty$, if $r>2$. By the above mentioned lemma, the theorem is proved.

Remarks. a) For $\lambda=0$ we have a central limit theorem for the the variable $Y_N = N^{-1/2} \sum_{n=0}^{N-1} X(n)$: $Y_N \to \mathcal{N}(0,f(0))$, as $N\to\infty$, with $f(0)=\sum_{j=0}^{\infty} r(j)$.

b) In contrast to the trigonometric case, the finite Walsh-Fourier transforms are not necessarily asymptotic independent. For details, see Kohn (1980a).

c) The theorem holds for "frequencies" $\lambda_j(N)$ which depend on N, for example, of the form $\lambda_j(N)=j/N$, in such a way that $\lambda_j(N)\oplus\lambda\to 0$.

REFERENCES

Brillinger, D.R. (1975): Time Series - Data Analysis and Theory. New York, Holt, Rinehart and Winston, Inc.

Fine, N.J. (1950): The generalized Walsh functions. Trans. Amer. Math.

Soc., 69, 66-77.

Fine,N.J. (1949): On the Walsh functions. Trans. Amer. Math. Soc., 65, 372-414.

Fine,N.J. (1957): Fourier-Stieltjes series of Walsh functions.Trans. Amer. Math. Soc., 86, 246-255.

Hannan,E.J. (1973): Central Limit theorems for time series regression. Z.Warscheinlichkeits., 26, 157-170.

Kohn,R. (1980a): On the spectral decomposition of stationary time series using Walsh functions. I. Adv. Appl. Prob., 12, 183-199.

Kohn,R. (1980b): On the spectral decomposition of stationary time series using Walsh functions.II. Adv. Appl. Prob., 12, 462-474.

Morettin,P.A. (1973): A note on a central limit theorem for dependent randon variables. Bol. Soc. Brasil. Mat., 4. 47-49.

Morettin,P.A. (1974a): Limit theorems for stationary and diadic-stationary processes. Bol. Soc. Brasil. Mat., 5, 97-104.

Morettin,P.A. (1974b): Walsh-function analysis of a certain class of time series. Stoch. Proc. and Their Appl., 2. 183-193.

Morettin,P.A. (1980): Walsh espectral analysis. SIAM REVIEW. forthcoming.

Robinson,G.S. (1972): Discrete Walsh and Fourier power spectra.Proc Symp. Appl. of Walsh Functions, Washington,D.C., 298-309.

RELATÓRIO TÉCNICO DO DEPARTAMENTO DE ESTATÍSTICA

Títulos Publicados

RT-MAE-7901 - Wagner de Souza Borges, On the Limiting Distribution of the Failure Time of Composite Material, outubro de 1979.

RT-MAE-7902 - A.Galves, J.G.Leite & M.Roussignol, The Invariance Principle For the One-Dimensional Symmetric Simple Exclusion Process, novembro de 1979.

RT-MAE-8001 - Raúl P.Mentz, Juan C.Abril, Zulema N.Cardoso & Maria B.Ceballos, Exploratory Fitting of Autoregressive and Moving Models to Well-behaved Time Series Data. fevereiro de 1980.

RT-MAE-8002 - Pedro A.Morettin, Walsh Spectral Analysis, março de 1980.

RT-MAE-8003 - Josemar Rodrigues, Robust Estimation and Finite Population, março de 1980.

RT-MAE-8004 - Wagner de Souze Borges & Flávio Wagner Rodrigues. On the Axiomatic Theory of Maultistate Coherent Structures, julho de 1980.