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Simple circuits are not that simple

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Abstract

The teaching of basic circuits in early undergraduate physics courses is widespread and, to some extent, quite uniform. In most textbooks, the introduction of resistors, capacitors and inductors is followed by their assemblage into larger systems, together with mathematical calculations of charges and currents as functions of time. In this process, spatial features of circuits tend to be omitted. Here, we argue that this kind of omission is not 'natural' but, rather, should be justified in terms of internal time-scales operating in the system, in the framework of the continuity equation. When space is brought back to the discussion of circuits, a number of important physical features, absent in most textbooks, emerge. Among them, one has a spatial uniformity of currents, the existence of charges coating metallic surfaces, and the presence of charge distributions inside wires at the endings of resistors and coils. We believe that the discussion of these qualitative issues with students could foster a more mature and unified relationship with electromagnetism.

Keywords: circuits, Maxwell's equations, time-scales, spatial features

(Some figures may appear in colour only in the online journal)

1. Introduction

Circuits are important in the teaching of basic electromagnetism. Although they may prove to be useful in practice and even in laboratory situations, the subject is present in introductory courses mostly for conceptual reasons. Indeed, circuits provide an early instance in which Maxwell's equations and Ohm's law, discussed previously in isolation, can be applied together to a single class of problems. As such, they represent an important step towards maturity. In undergraduate courses, circuits are usually discussed after an introduction to resistors R, capacitors C, and inductors L, which are afterwards assembled using either their characteristic voltages or energies and powers.

The assembling of these elements is a moment of synthesis, for their joint action gives rise to time-dependent effects, such as electric oscillations, which were absent when they were taken in isolation. In most syllabuses, oscillations are discussed earlier in mechanics courses, by means of mass-spring systems, and circuits allow one to revisit mathematical techniques used

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to solve simple second order differential equations. So, in principle, circuits could be a nice subject for a balanced blending of physics and mathematics. However, inspection of textbooks [1–4] indicates a more formal bias.

The mathematical analogy between mechanical and electric oscillations is well known: a mass m corresponds to L, a spring constant k corresponds to $1/C$, a displacement x corresponds to a charge Q, and so on. In spite of this compelling logic, electric oscillations tend to be more difficult to students than mechanical ones, suggesting reasons that go beyond mathematics. We wonder whether this could derive from the fact that descriptions of circuits are somehow more abstract than those of mass-spring systems. In particular, the variable x is rather tangible, whereas Q is more abstract. In the framework of Maxwell's equations, charges are internal quantities, unrelated to space an time [5] and, as a consequence, mechanical oscillations are formulated in familiar terms of both space and time, whereas just time is present in electric oscillations.

The elimination of space from the discourse on circuits relies on a number of approximations and simplifications which most textbooks fail to mention. An important exception is *The Feynman Lectures on Physics* [6], where the discussion of circuits at chapter 22 begins with the warning: 'even such a mundane subject, when looked at in sufficient detail, can contain great complications'. The physics of simple circuits is indeed conceptually involved and simplified mathematical treatments that skip spatial coordinates are not 'natural'. Although reasonable, they need justification in order to provide a coherent picture of electromagnetism.

Here, we discuss implications of bringing space back to the teaching of circuits. A first question to be tackled concerns currents, which run through wires and other components located at different places. In usual approaches, after solving the appropriate differential equation with suitable initial conditions, one finds an electric current $I(t)$ which depends just on time. Its does not have spatial features and those that notice this are pushed into the striking conclusion that the current is the same across any section of either wire or battery, with the obvious exception of gaps in capacitors. This also holds for the time derivative of $I(t)$, indicating that changes within the circuit, which is an extended object, are identical at points separated by a distance, suggesting a kind of non-locality between causes and effects. The assumption that currents in simple systems are spatially uniform corresponds to an approximation and can be justified. This requires the use of the continuity equation describing charge conservation, complemented by a discussion of time-scales which determine various effects in metallic conductors. And, as we argue in the sequence, simple circuits correspond to quasi-electrostatic systems.

According to Ohm's law, a current in a wire derives from an internal resultant electric field $\vec{E}_\mathrm{w}$ and, if the former is spatially uniform, the same happens with the intensity of the latter. This raises a forgotten issue. As the intensity of the electric field from a battery decreases with the distance to its poles, the uniformity of $|\vec{E}_\mathrm{w}|$ along the wire indicates the presence of another electric field in the system. This other field is due to very thin layers of charges which coat external metallic surfaces of the system and its most conspicuous action is to contribute to keeping the flow of free electrons confined within metallic wires. In textbooks, the issue of surface charges was discussed more than 50 years ago in the books by Sommerfeld [7] and Jefimenko [8] and, more recently, hinted in reference [9] and extensively developed by Chabay and Sherwood [10]. There is also an established literature on the subject dealing with models and qualitative features and a relevant summary can be found in reference [11]. Theoretical studies devoted to rather schematic situations also exist [12] and of special interest are the very instructive pictures produced long ago by Jefimenko [8, 13].

A necessary feature of quasi-electrostatic systems is that the influence of a given element spreads quasi-simultaneously to all parts of a circuit. In the case of batteries and capacitors, it is well known that this is due to electrostatic fields associated with explicit charge distributions.

The behaviour of inductors, on the other hand, is determined by Faraday's law and is apparently non-electrostatic. Nevertheless, they also give rise to quasi-electrostatic interactions, owing to charges distributed inside the wires close to their endings. This justifies the procedure found in many textbooks of using potential differences associated with isolated components to write down equations describing circuits.

The discussion presented here regards qualitative aspects of the physics of simple circuits [14] and complements usual approaches emphasizing calculations of charges and currents. We feel that the presentation of these features to students may contribute to turn the subject less abstract and more attractive. Also, by setting problems into a wider qualitative framework of electromagnetism, one allows students to feel and enjoy the high coherence of the theory and, hopefully, to incorporate it as part of their own scientific culture.

2. Battery and wire—steady currents

We begin by considering a very simple circuit, made just by a battery of electromotive force (emf) $\mathcal{E}_\mathrm{b}$ and internal resistance R_b, attached to a wire of uniform cross section S_w, length ℓ_w and resistivity ρ_w, which also contains a small gap acting as a switch. The same battery and wire are considered in all situations discussed and the resistivity, as a free parameter, can be adjusted to produce the resistance R_w required by specific cases.

Switches, although usually unnoticed, convey interesting physics [15]. Indeed, in a short article, Leon [16] acknowledges the contribution from the Italian physicist Gilberto Bernardini to his education and recalls: '*Over time, Bernardini and I learnt how to communicate and I began to watch Gilberto. There was his habit of entering a dark room, pushing the light switch: light. Pushing it again, off. On, off five or six times. Each time there would be a loud "fantastico!" Why? He seemed to have this remarkable sense of wonder about simple things.*' A nice description of the physics underlying a similar phenomenon can be found in reference [17].

A battery is a device which is always active, irrespective of being connected to a circuit or not and its emf $\mathcal{E}_\mathrm{b}$ gives rise to a potential difference V_b between its poles, associated with an electrostatic field $\vec{E}_\mathrm{b}$ extended allover space, as suggested in figure 1. As the value of V_b calculated over a field line is constant, the average electric field decreases as the size of that line increases. Except at the poles, the field $\vec{E}_\mathrm{b}$ is close to that produced by two point-like charges of opposite signs, kept apart by the length of the battery.

In our simple circuit, when the switch is open, the battery field $\vec{E}_\mathrm{b}$ displaces free electrons inside the wire and induces charge distributions on its surface. These, in turn, give rise to another electrostatic field $\vec{E}_\mathrm{q}$, which is also present everywhere, but is especially intense around the tips of the open gap. The fields $\vec{E}_\mathrm{b}$ and $\vec{E}_\mathrm{q}$ are superimposed and, in all points inside the metallic components, give rise to a typical electrostatic screening effect associated with the condition $\vec{E}_\mathrm{w} = \vec{E}_\mathrm{b} + \vec{E}_\mathrm{q} = 0$. As a consequence, there is no current in the circuit.

When the gap is closed, the charges induced in its edges disappear, superficial densities rearrange, the field $\vec{E}_\mathrm{q}$ changes and a current I begins to flow very quickly. The fact that the field $\vec{E}_\mathrm{b}$ is independent of the switch stresses the rather important contribution of $\vec{E}_\mathrm{q}$ to this problem. Very shortly after the closing of the switch, the behaviour of the system becomes steady and remains the same when observed at different times. This kind of time independence has a number of related consequences. One of them is that the superficial charge distributions on the metallic surfaces do not change and their field $\vec{E}_\mathrm{q}$ is electrostatic. Consistently, the magnetic field $\vec{B}$ produced by the current is constant and Faraday's law does not yield non-conservative contributions.

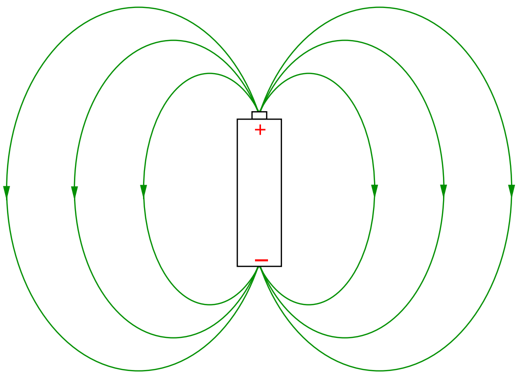

Figure 1. Schematic battery surrounded by the electrostatic field $\vec{E}_b$ created by charges at its poles.

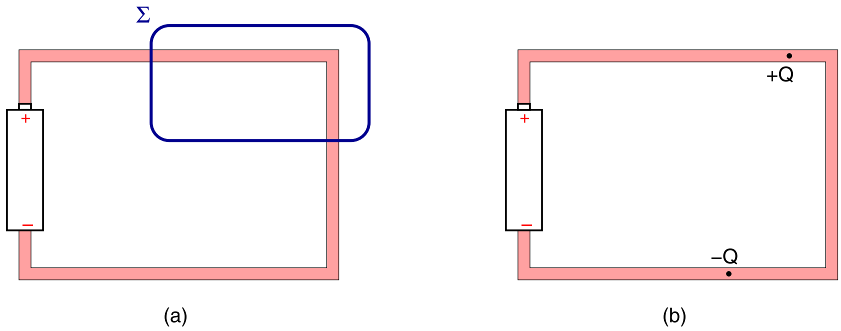

Figure 2. A battery attached to a wire: (a) surface Σ, (b) hypothetical charge concentrations.

Another consequence is the spatial uniformity of the current. To see this, we resort to the law of charge conservation, described by the continuity equation

$$\oiint_{\Sigma} \vec{j} \cdot \hat{n}\, d\Sigma = -\frac{dQ_{int}}{dt}, \tag{1}$$

where Σ is a closed oriented surface, $\hat{n}$ its outward normal, $\vec{j}$ is the current density over Σ and Q_{int} the charge contained in it. Applying this law with a surface cutting the wire of the circuit at two different points, as in figure 2(a), there is a current I_{in} entering it at a given point and another one I_{out} leaving it somewhere else. If one assumes for a while that the current is not spatially uniform, the continuity equation would yield $I_{out} - I_{in} = -dQ_{int}/dt \neq 0$, indicating that Q_{int} changes with time, in contradiction with the fact that one is dealing with a steady situation. Therefore one must have the condition $I_{out} = I_{in}$ for any surface Σ, which corresponds to a spatially uniform current.

The continuity equation also points to another problem associated with the assumption $I_{out} \neq I_{int}$. If the current were not spatially uniform, there would be an amount of charge of a given sign accumulating in some region. As the circuit is neutral, there would also be an amount of charge of opposite sign accumulating somewhere else, as suggested in figure 2(b). One would then have a mechanism that produces charge concentrations at different points, which would increase the potential energy of the system. However, in actual systems, there is a spontaneous trend towards the minimization of potential energy which, in a metal, is characterized by a dispersion time-scale. If at a given instant t there is a free charge density $\rho(t)$ inside a metal, its change rate is given by the continuity equation as

$$\frac{\partial \rho(t)}{\partial t} = -\vec{\nabla} \cdot \vec{j}. \tag{2}$$

Using Ohm's law $\vec{j} = \vec{E}/\rho_{\mathrm{w}}$ and Gauss's law $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, one gets the differential equation

$$\frac{\partial \rho(t)}{\partial t} = -\frac{\rho}{\rho_{\mathrm{w}}\, \epsilon_0}, \tag{3}$$

whose solution is

$$\rho(t) = \rho_0\, \mathrm{e}^{-t/T_{\mathrm{d}}}\,, \quad \text{with } T_{\mathrm{d}} = \epsilon_0\, \rho_{\mathrm{w}}, \tag{4}$$

where ρ_0 is the value of $\rho(t)$ at $t = 0$. For typical metals, $\rho_{\mathrm{w}} \sim 10^{-8}\ \Omega\,\mathrm{m}$ and $T_{\mathrm{d}} \sim 10^{-19}$ s. This time-scale, which is rather short, determines the dispersion of free charges in excess through the metal and, again, the continuity equation rules that the current entering any surface Σ must be the same as that leaving it.

The spatial uniformity of the current is important because it organizes the conceptual understanding of simple circuits. When the switch is on, the potential difference between the poles of the battery is $V_{\mathrm{b}} = R_{\mathrm{w}}\, \mathcal{E}_{\mathrm{b}}/R$, where

$$R_{\mathrm{w}} = \frac{\rho_{\mathrm{w}}\, \ell_{\mathrm{w}}}{S_{\mathrm{w}}} \tag{5}$$

is the wire's resistance, $R = R_{\mathrm{b}} + R_{\mathrm{w}}$, and the time-independent current is

$$I = \frac{\mathcal{E}_{\mathrm{b}}}{R} = \frac{V_{\mathrm{b}}}{R_{\mathrm{w}}}. \tag{6}$$

This current is driven by an electric field $\vec{E}_{\mathrm{w}}$ existing inside the wire, which is related to the current density $\vec{j}_{\mathrm{w}}$ by Ohm's law

$$\vec{j}_{\mathrm{w}} = \frac{\vec{E}_{\mathrm{w}}}{\rho_{\mathrm{w}}}. \tag{7}$$

As $\vec{j}_{\mathrm{w}}$ is everywhere parallel to the wire, the same happens with $\vec{E}_{\mathrm{w}}$. The local orientation of the wire is described by the unit vector $\hat{u}_j$, associated with $\vec{j}_{\mathrm{w}}$ and defined as

$$\hat{u}_j = \frac{\vec{j}_{\mathrm{w}}}{|\vec{j}_{\mathrm{w}}|}. \tag{8}$$

Recalling that $I = |\vec{j}_{\mathrm{w}}|\, S_{\mathrm{w}}$ and using equations (5)–(7), one has

$$\vec{E}_{\mathrm{w}} = \frac{V_{\mathrm{b}}}{\ell_{\mathrm{w}}}\, \hat{u}_j. \tag{9}$$

The intensity of $\vec{E}_{\mathrm{w}}$ is the same allover the wire, both close to and far from the battery. This is a remarkable conclusion, for it indicates that $\vec{E}_{\mathrm{w}}$ cannot be identified with just $\vec{E}_{\mathrm{b}}$, the field the battery produces and, again, stresses the important role played by the static charges distributed

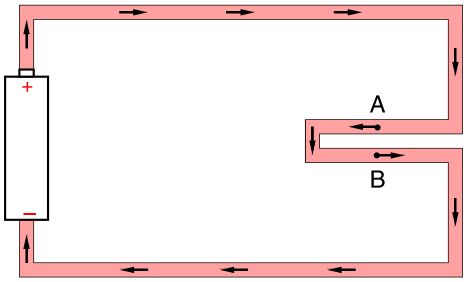

Figure 3. A battery attached to a bent wire; the arrows indicate the field $\vec{E}_{\rm w} = \vec{E}_{\rm b} + \vec{E}_{\rm q}$.

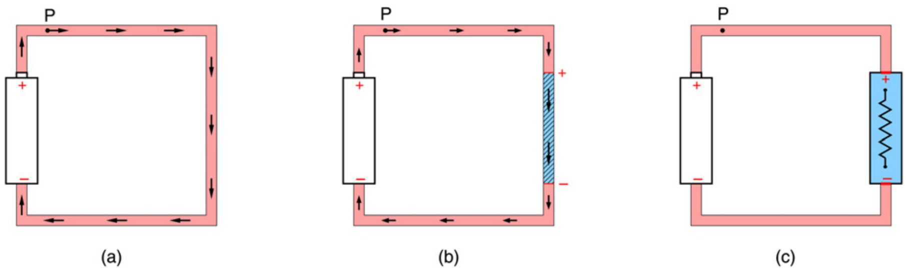

Figure 4. Circuits containing a battery attached to (a) circuit x, with a single wire; (b) circuit y, with two kinds of wires; (c) circuit y, with a perfect wire 1 and wire 2 taken as a resistor.

over the external surfaces of the wire, which give rise to the field $\vec{E}_{\rm q}$. The details of $\vec{E}_{\rm q}$ depend on the geometry of the wire and if this geometry changes, local values of $\vec{E}_{\rm q}$ also change. Nevertheless, if $\ell_{\rm w}$ is kept fixed, independently of the geometry chosen, the resultant field $\vec{E}_{\rm w} = \vec{E}_{\rm b} + \vec{E}_{\rm q}$ always fulfills equation (9).

A pedagogical example showing the importance of $\vec{E}_{\rm q}$ is given in figure 3. As points A and B are close together, one may expect the battery field $\vec{E}_{\rm b}$ to be approximately the same at both. However, the total field $\vec{E}_{\rm w}$ points at opposite directions at A and B, an effect which highlights the contribution of $\vec{E}_{\rm q}$.

Let us now consider the two circuits shown in figures 4(a) and (b), labelled x and y. In circuit x, the battery is attached to wire 1, of length $\ell_{\rm w}$, uniform section $S_{\rm w}$ and resistivity ρ_1. In circuit y, a part of the wire is replaced by another one, of length ℓ_2, cross section $S_{\rm w}$ and resistivity $\rho_2 > \rho_1$.

In both circuits, currents are uniform across any of their transverse sections and one has

$$I_x = \frac{V_{\rm b}\, S_{\rm w}}{\rho_1\, \ell_{\rm w}}, \tag{10}$$

$$I_y = \frac{V_{\rm b}\, S_{\rm w}}{\rho_1\, (\ell_{\rm w} - \ell_2) + \rho_2\, \ell_2} \quad < I_x. \tag{11}$$

This result is interesting, for it shows that the insertion of wire 2 produces a change of the current in wire 1, even at places distant from it, such as the point P shown in the figures. According to Ohm's law, this is due to local changes in the electric fields. In circuit x, $|\vec{E}_{\rm w}|$ is

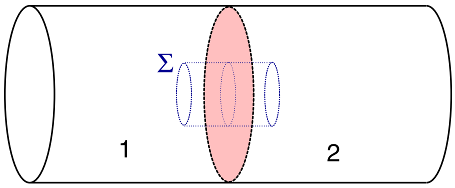

Figure 5. Wire and Gaussian surface Σ around interface 1–2.

uniform all over the wire and given by

$$|\vec{E}_{wx}| = \frac{V_b}{\ell_w}, \tag{12}$$

whereas in circuit y it is uniform over each section of the wire and reads

$$|\vec{E}_{wy}|_1 = \frac{\rho_1 V_b}{\rho_1 (\ell_w - \ell_2) + \rho_2 \ell_2} \quad < |\vec{E}_{wx}|, \tag{13}$$

$$|\vec{E}_{wy}|_2 = \frac{\rho_2 V_b}{\rho_1 (\ell_w - \ell_2) + \rho_2 \ell_2} \quad > |\vec{E}_{wx}|. \tag{14}$$

These results mean that the electric field $\vec{E}_{wy}$ inside the wire is discontinuous along the interfaces between the two different metals and indicate the presence of charge distributions there.

Assuming densities to be superficial, they can be evaluated using Gauss's law. At interface 1–2, we adopt a Gaussian surface Σ as in figure 5, with outward normal $\hat{n}$. Using $\hat{n} \cdot \hat{u}_j = -1$ in wire 1 and $\hat{n} \cdot \hat{u}_j = +1$ in wire 2, we have

$$\oiint_{\Sigma} \vec{E} \cdot \hat{n} \, d\Sigma = \left[-|\vec{E}_{wy1}| + |\vec{E}_{wy2}| \right] \Sigma_{\parallel} = \frac{\sigma \Sigma_{\parallel}}{\epsilon_0}, \tag{15}$$

where $\Sigma_{\parallel}$ is the area parallel to the interface. Thus, Gauss's law yields

$$|\sigma| = \epsilon_0 \, |\vec{E}_{wy2} - \vec{E}_{wy1}| = \epsilon_0 \, \frac{(\rho_2 - \rho_1) \, V_b}{\rho_1 (\ell_w - \ell_2) + \rho_2 \ell_2}, \tag{16}$$

with signs shown in figure 4(b). The corresponding charges $|Q| = |\sigma| S_w$ give rise to an electrostatic field similar to that of a dipole, which exists both outside and inside the wire. It is the action of this field at point P in circuit y that makes $I_y < I_x$ there. As expected, this dipole disappears when $\rho_2 \to \rho_1$.

If wire 1 is a very good conductor, we may consider the limit $\rho_1 \to 0$ and equation (14) yields $V_b = |\vec{E}_{wy2}| \, \ell_2$ for the potential difference between the endings of wire 2. One then recovers the classical textbook situation, in which the whole resistance is encompassed into a resistor. This result is instructive, because it shows that a resistor spreads its influence over a circuit by means of electrostatic fields due to charges accumulated on its extremities, with densities given by $|\sigma| = \epsilon_0 V_b / \ell_2$. In order to produce a feeling for the size of $|\sigma|$, we assume $V_b = 1.5$ V and $\ell_2 = 2.0$ cm, typical of a laboratory resistor. This amounts to $|\sigma| = 6.6 \times 10^{-10}$ C m^{-2} or, alternatively, to $|\sigma/e| = 4.1 \times 10^9$ electrons/m^2. If the good conductor is made of copper, whose density of free electrons is around $n = 8.5 \times 10^{28}$ electrons/m^3, the deviation from the average two-dimensional density is $|\sigma/e|/n^{2/3} \sim 2 \times 10^{-10} = 2$/(ten billion). As stressed by

Rosser [18], tiny amounts of charge in excess can give rise to large macroscopic effects. The same holds for the case of forces between wires [19].

As another interesting limit, we make $\rho_2 \to \infty$ in equations (13) and (14), for this amounts to transforming wire 2 into a kind of insulating gap. In this case, $|\vec{E}_{wy}|_1 \to 0$, corresponding to electrostatic equilibrium and the current in wire 1 vanishes, whereas charges of opposite signs remain present at its tips. In case ℓ_2 is small, this simulates a switch.

3. Time dependent currents

In circuits containing capacitors and inductors, currents become time-dependent and one needs to consider the various different time-scales of the problem. One of them may be said to be 'natural' to a circuit because, in the SI (https://stacks.iop.org/EJP/41/055202/mmedia), one has the dimensions: $[R] = \text{m}^2\ \text{kg}\ \text{s}^{-3}\ \text{A}^{-2}$, $[L] = \text{m}^2\ \text{kg}\ \text{s}^{-2}\ \text{A}^{-2}$ and $[C] = \text{m}^{-2}\ \text{kg}^{-1}\ \text{s}^4\ \text{A}^2$. The combinations L/R, RC and $\sqrt{LC}$ have dimensions of time and, therefore, it is not surprising that these time-scales T_n occur naturally in calculations. For ordinary circuits, typical values of R, L and C yield $T_n \sim 10^{-6}$ s. Another relevant time-scale is set by f_p, the plasma frequency of the metal making the system. This frequency represents the threshold beyond which the masses of free electrons in a metal begin to dominate the behaviour of the current and the usual Ohm's law no longer holds. For most metals, the scale of this frequency is above the visible spectrum, around $\sim 10^{15}\ \text{s}^{-1}$. It corresponds to times $T_p \sim 10^{-15}$ s, much smaller than T_n and can be ignored, allowing Ohm's law to be used safely. The last important time-scale is determined by the speed of light c. According to classical electrodynamics, every time an electron is accelerated in the circuit, it emits electromagnetic waves which carry this information to other parts of the system. Assuming the circuit to have a typical size $d \sim 10$ cm, this gives rise to a time-scale $T_c = d/c \sim 10^{-9}$ s, which rules the communication among its various parts by means of electromagnetic waves.

As this discussion suggests, T_p is just too small and the relevant ratio is $T_n/T_c \sim 10^3$, a factor comparable to 1 h/1 s. It indicates that the various parts of the circuit have plenty of time to influence themselves during the relatively slow time evolution associated with T_n. In this scenario, the circuit can be considered as being *quasi-electrostatic*. Strictly speaking, the concept of potential applies to conservative fields only and should not be used to describe circuits containing capacitors and inductors, where there are time-varying fields. Nevertheless, the fact that T_n is large when compared to T_c opens room for a looser language and explains the common practice of describing circuits in terms of potential differences.

4. Capacitor

We consider a circuit containing a pedagogical capacitor of capacitance C, made by two parallel plates of area A in the vacuum, kept apart by a distance $d \ll \sqrt{A}$, attached to the wire with resistivity ρ_w, cross section S_w, length ℓ_w, with a switch in the form of a small gap. The resistance of the wire is given by equation (5) whereas the capacitance reads

$$C = \frac{\epsilon_0 A}{d}. \tag{17}$$

When the switch is open, as in figure 6(a), the charges on the plates, assumed to be $+Q_0$ and $-Q_0$, produce a field $\vec{E}_C$, which exists everywhere. It induces charge distributions over all metallic surfaces, giving rise to a field $\vec{E}_q$. This situation is fully electrostatic and, for any point inside the wire, the resultant field is $\vec{E}_w = \vec{E}_C + \vec{E}_q = 0$, whereas for points outside them

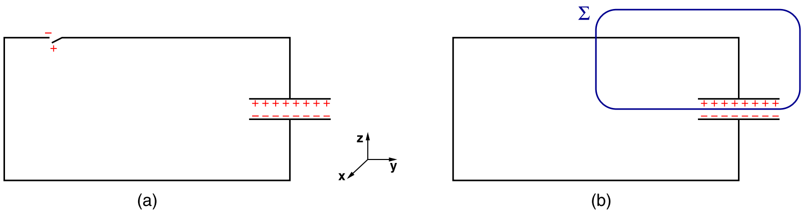

Figure 6. Circuit containing a capacitor and a wire: (a) open switch, (b) surface Σ.

$\vec{E} = \vec{E}_{\rm C} + \vec{E}_{\rm q} \neq 0$, in general. As the density of electrostatic energy is $\mathrm{d}U/\mathrm{d}V = \epsilon_0\,|\vec{E}|^2/2$, this energy is spread everywhere outside the metallic parts and the usual notion that the energy $Q^2/2C$ of the system is stored in the region between the plates, although fair, is an approximate one.

The closing of the switch allows a flux of electrons from the negative to the positive plate, the system releases heat and eventually becomes neutral. At a given instant t, the amount of charge $Q(t)$ on a plate of the capacitor is

$$Q(t) = Q_0\,\mathrm{e}^{-\,t/RC}. \tag{18}$$

The charges $+Q(t)$ and $-Q(t)$ are located mostly in the inner sides of plates of the capacitor and, according to Gauss's law, give rise to a field $\vec{E}_{\rm Cv}(t)$ in the vacuum region between them, given by

$$\vec{E}_{\rm Cv}(t) = -\,\frac{Q_0}{\epsilon_0\,A}\,\mathrm{e}^{-\,t/RC}\;\hat{e}_z. \tag{19}$$

As the wire has a spatial extension, one must resort to the continuity equation to relate equation (18) to the current running at any given point. Using equation (1) with the surface Σ indicated in figure 6(b) and denoting by $I(t)$ the current leaving it, one has

$$I(t) = -\frac{\mathrm{d}Q(t)}{\mathrm{d}t} = \frac{Q_0}{RC}\,\mathrm{e}^{-\,t/RC}. \tag{20}$$

In this result, $I(t)$ represents the current at the point where Σ crosses the wire, whereas $Q(t)$ is the charge inside Σ. As it holds for any surface Σ, $I(t)$ is the same at all points of the wire, irrespective of whether they are close or far from the capacitor. So, even time-dependent currents are spatially uniform in quasi-electrostatic circuits, since they follow the scale $T_{\rm n}$ and vary slowly.

As figure 6 suggests, the electric field $\vec{E}_{\rm C}(t)$ is highly concentrated between the plates of the capacitor, but its influence extends over the whole circuit. Indeed, during the time the switch is closed, it is the joint action of $\vec{E}_{\rm C}$ and $\vec{E}_{\rm q}$, due to charges coating metallic surfaces, that moves charges from one plate to the other. The resultant field $\vec{E}_{\rm w}(t) = \vec{E}_{\rm C}(t) + \vec{E}_{\rm q}(t)$ is quasi-electrostatic and satisfies Ohm's law, equation (7). Using equations (8) and (20), we find

$$\vec{j}_{\rm w}(t) = \frac{\mathrm{d}\,Q_0}{\rho_{\rm w}\,\ell_{\rm w}\,\epsilon_0\,A}\,\mathrm{e}^{-\,t/RC}\,\hat{u}_j, \tag{21}$$

$$\vec{E}_{\rm w}(t) = \frac{\mathrm{d}\,Q_0}{\ell_{\rm w}\,\epsilon_0\,A}\,\mathrm{e}^{-\,t/RC}\,\hat{u}_j. \tag{22}$$

Both $|\vec{j}_{\rm w}(t)|$ and $|\vec{E}_{\rm w}(t)|$ are spatially uniform inside the wire at each instant t. The quasi-electrostatic nature of the problem becomes clear when equations (19) and (22) are used to write

$$|\vec{E}_{\rm w}(t)| = \frac{d}{\ell_{\rm w}}\,|\vec{E}_{\rm Cv}(t)| \tag{23}$$

since, for a closed path Γ passing inside the wire and in the region between the plates of the capacitor, this amounts to the condition

$$\oint_\Gamma \vec{E}\cdot {\rm d}\vec{\gamma} = -\ell_{\rm w}\,|\vec{E}_{\rm w}(t)| + d\,|\vec{E}_{\rm Cv}(t)| = 0, \tag{24}$$

even for time-dependent fields.

5. Inductor

We conclude by considering a schematic circuit with an inductor. A battery of emf $\mathcal{E}_{\rm b}$ is attached to a wire of length $\ell_{\rm w}$, cross section $S_{\rm w}$, resistivity $\rho_{\rm w}$, containing a switch in the form of a small gap, which is closed at $t = 0$. A part of length $\ell_{\rm c}$ of this wire is winded as a cylindrical coil of radius a and total height $h \gg a$, containing N loops, so that $\ell_{\rm c} = N2\pi a$. As the solenoid is especially important in this discussion, it is convenient to divide the circuit into two parts by a mathematical plane passing through points A and B and to call region 1 its left sector, which includes the battery, the gap and the straight parts of the wire, whereas the coil is kept isolated in region 2, as shown in figure 7.

This is a simple RL circuit, in which $R = R_{\rm b} + R_{\rm w}$, with $R_{\rm w}$ given by equation (5) and

$$L = \frac{\mu_0\, N^2\, \pi\, a^2}{h}, \tag{25}$$

assuming self-induction to be dominated by the coil. The current in the circuit is given by

$$I(t) = 0 \qquad \text{for } t \leqslant 0, \tag{26}$$

$$I(t) = \frac{\mathcal{E}_{\rm b}}{R}\left[1 - {\rm e}^{-R\,t/L}\right] \qquad \text{for } t \geqslant 0. \tag{27}$$

When the gap is open, just the fields $\vec{E}_{\rm b}$, produced by the battery and $\vec{E}_{\rm q}$, by the static charges located over the metallic surfaces are present, which yield $\vec{E}_{\rm w} = \vec{E}_{\rm b} + \vec{E}_{\rm q} = 0$ inside the wire and there is no current.

When the gap is closed, surface charges are redistributed, both the current $I(t)$ and its magnetic field $\vec{B}(t)$ begin to vary and an electric field $\vec{E}_{\rm ind}$ due to self-induction becomes important. The field $\vec{B}(t)$ extends to all regions encompassing the circuit, but it is much more intense inside the solenoid. Therefore, a good point of departure for a simple description of the system consists in neglecting $\vec{B}(t)$ outside the coil.

In the situation shown in figure 7, the current $I(t)$ in the coil is clockwise and, at a generic instant $t > 0$, the magnetic field given by Ampère's law reads

$$\vec{B}(t) = -\,\frac{\mu_0\,N}{h}\,I(t)\,\hat{e}_z. \tag{28}$$

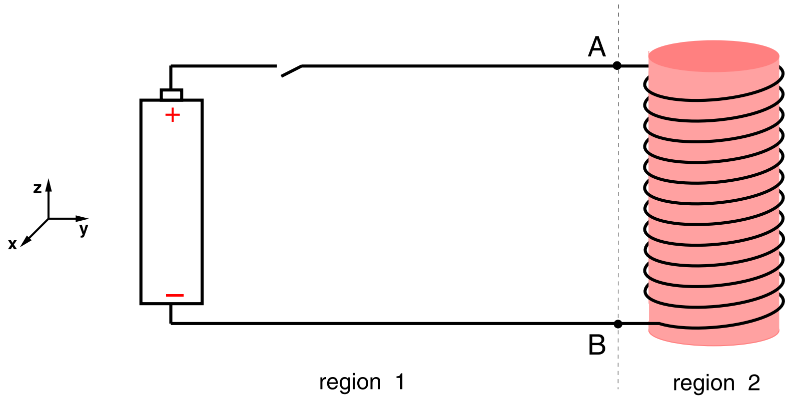

Figure 7. A cylindrical solenoid coupled to a battery; a plane passing through points A and B divides the circuit into two sectors.

According to Faraday's law, this magnetic field induces an electric field $\vec{E}_{\rm ind}$, with circular field lines concentric with the coil, oriented counterclockwise and with intensity

$$E_{\rm ind} = \frac{\mu_0\, N\, r}{2\, h}\, \frac{{\rm d}I(t)}{{\rm d}t} \quad \text{for } r \leqslant a, \tag{29}$$

$$E_{\rm ind} = \frac{\mu_0\, N\, a^2}{2\, h\, r}\, \frac{{\rm d}I(t)}{{\rm d}t} \quad \text{for } r \geqslant a, \tag{30}$$

where r is the distance to the symmetry axis and $I(t)$ is given by equation (27). The direct influence of $\vec{E}_{\rm ind}$ over the currents in regions 1 and 2 is quite different. In region 1 its field lines cross metallic conductors at various oblique angles, affecting little the flow of the current and contribute just to accumulate charges over the external surfaces of the wire. Recalling that $\hat{u}_j$ is given by equation (8), in region 1 $\vec{E}_{\rm ind} \cdot \hat{u}_j$ is in average negligible and its effects can be incorporated into $\vec{E}_{\rm q}$.

Inside the wire of the coil, on the other hand, the field $\vec{E}_{\rm ind}$ is always along it and opposite to the current, being given by

$$\vec{E}_{\rm ind} = -\frac{\mu_0\, N\, a}{2\, h}\, \frac{{\rm d}I(t)}{{\rm d}t}\, \hat{u}_j. \tag{31}$$

Using equations (27) and (25) and recalling that $\ell_{\rm c} = N2\pi a$, this result is rewritten as

$$\vec{E}_{\rm ind} = -\frac{\mathcal{E}_{\rm b}}{\ell_{\rm c}}\, {\rm e}^{-R\, t/L}\, \hat{u}_j. \tag{32}$$

Thus, the field $\vec{E}_{\rm ind}$ is effective only in region 2 and influences the behaviour of the current $I(t)$ just there. This is a general feature, since magnetic fields concentrated within solenoids tend to restrict the influence of induced electric fields to its metallic loops. This may seem to contradict the fact that one is dealing with a low time-scale $T_{\rm n}$ and, consequently, with a current which is spatially uniform. Indeed, in the present instance, $I(t)$ must be the same in regions 1 and 2, even if $\vec{E}_{\rm ind}$ is effective only in the latter.

In the case of an inductor, the spatial uniformity of the current on the circuit derives from an interesting mechanism. When a coil is present in a circuit, changes of $\vec{B}(t)$ in its interior are accompanied by a small but important accumulation of charges across the section of the wire in the vicinity of points A and B. In the present case, these charges are $Q_A(t) > 0$ around A and $Q_B(t) = -Q_A(t) < 0$ around B, amounting to an ephemeral electric dipole. The electric field $\vec{E}_{dip}(t)$ of this dipole exists allover space, both inside and outside the conductors and tends to decrease the current in region 1 and increase it in region 2. It is this mechanism that ensures the uniformity of the current over the circuit.

Inside metallic conductors, the resultant electric field $\vec{E}_w$ is related to the current density $\vec{j}_w$ by Ohm's law, equation (7) and, using equations (5) and (27), for both regions, we get

$$|\vec{E}_{w1}(t)| = |\vec{E}_{w2}(t)| = \frac{\mathcal{E}_b R_w}{R \ell_w} \left[1 - e^{-Rt/L}\right]. \tag{33}$$

Inside the wire in region 1, the field is written as

$$\vec{E}_{w1}(t) = \left[\vec{E}_b + \vec{E}_q(t) + \vec{E}_{dip}(t)\right]_1 \cdot \hat{u}_j \, \hat{u}_j, \tag{34}$$

whereas in region 2, inside the wire of the coil, we have

$$\begin{aligned}\vec{E}_{w2}(t) &= \left[\vec{E}_b + \vec{E}_q(t) + \vec{E}_{dip}(t) + \vec{E}_{ind}(t)\right]_2 \cdot \hat{u}_j \, \hat{u}_j \\ &= \left[\vec{E}_b + \vec{E}_q(t) + \vec{E}_{dip}(t)\right]_2 \cdot \hat{u}_j \, \hat{u}_j + \vec{E}_{ind}(t).\end{aligned} \tag{35}$$

Comparing equations (34) and (35), we learn that $[\vec{E}_b + \vec{E}_q + \vec{E}_{dip}] \cdot \hat{u}_j$ is discontinuous at the transition between regions 1 and 2, indicating the existence of a charge distribution inside the wire around point A. Assuming it to be a two-dimensional interface, its charge density σ_A can be derived from Gauss's law, using the same Gaussian surface Σ shown in figure 5. Using $\hat{n} \cdot \hat{u}_j = -1$ in region 1 and $\hat{n} \cdot \hat{u}_j = +1$ in region 2, we write

$$\begin{aligned}\oiint_\Sigma \vec{E} \cdot \hat{n} \, d\Sigma &= \left[-|\vec{E}_b + \vec{E}_q + \vec{E}_{dip}|_1 + |\vec{E}_b + \vec{E}_q + \vec{E}_{dip}|_2\right] \Sigma_\parallel \\ &= \left[-\vec{E}_{ind} \cdot \hat{u}_j\right] \Sigma_\parallel = \frac{\sigma_A \Sigma_\parallel}{\epsilon_0},\end{aligned} \tag{36}$$

$\Sigma_\parallel$ being the area parallel to the interface. Recalling equation (32), we find

$$\sigma_A = \epsilon_0 E_{ind} = \epsilon_0 \frac{\mathcal{E}_b}{\ell_c} e^{-Rt/L}. \tag{37}$$

Very close to the interface, the field $\vec{E}_{A\sigma}$ created by this charge distribution is given by

$$\vec{E}_{A\sigma 1} = -\frac{E_{ind}}{2} \hat{u}_j \quad \text{in region 1}, \tag{38}$$

$$\vec{E}_{A\sigma 2} = +\frac{E_{ind}}{2} \hat{u}_j, \quad \text{in region 2}. \tag{39}$$

This field is opposite to the current in region 1. In region 2, as the field $\vec{E}_{ind} = -E_{ind} \hat{u}_j$ is also present, the effect of the charge distribution at the interface A is to make $\vec{E}_{A\sigma 1} = \vec{E}_{A\sigma 2} + \vec{E}_{ind}$, allowing the current to be continuous. Concerning point B, the same type of reasoning yields $\sigma_B = -\sigma_A$, as expected in a neutral wire. As the charges at points A and B are equal and opposite, their effects inside the wire in region 1 tend to boycott the effects produced by the battery, weakening the current in that sector while it lasts.

The maximum charge density occurs at $t = 0$ and, for $\mathcal{E}_b = 1.5$ V and a cylindrical coil with radius $a = 2$ cm and 1000 loops, having $\ell_c \sim 125$ m, equation (37) yields $|\sigma| = 1.1 \times 10^{-13}$ C m^{-2}, which corresponds to $|\sigma/e| = 0.7 \times 10^6$ electrons/m^2. In the case of copper, the relative deviation from the two-dimensional density is $|\sigma/e|/n^{2/3} \sim 3.4 \times 10^{-14}$.

Although this discussion involves a number of simplifying assumptions, such as a long cylindrical coil with the associated neglecting of magnetic fields outside it, a sharp division of the circuit into two regions and two-dimensional charge distributions at points A and B, its main qualitative features are general and remain valid for other systems containing inductors. The most remarkable of them is, as pointed out by Feynman [6], that an inductor spreads its influence over a circuit by means quasi-electrostatic interactions and one is entitled to talk about potential differences between it tips.

6. Overview

Simple circuits are important in early physics education because they provide concrete applications of basic electromagnetism and, also, for allowing training on simple differential equations. Treatments of the subject in the literature tend to be rather uniform and, once Ohm's law is used to define the quantity R and Maxwell's equations are used to derive the quantities C and L, fields are usually suppressed from the discussion. The discourse is then narrowed to manipulations of charges Q and currents I, in a framework provided by R, L, C, and an external time t. Realistic drawings of circuits are replaced by symbolic representations and space is no longer mentioned.

We suggest here that the elimination of space from the presentation of circuits is not natural and can be justified by the various time-scales inherent to systems with metallic components. There are at least four such scales, associated with their characteristic times: $T_d \sim 10^{-19}$ s, for the dispersion of excess charge concentrations inside metals, $T_p \sim 10^{-15}$ s, for plasma oscillations limiting the validity of Ohm's law, $T_c \sim 10^{-9}$ s, for radiation interactions within the circuit and $T_n \sim 10^{-6}$ s, for effects directly associated with R, L, and C. As T_n is much larger than the other ones, in some situations it becomes the only relevant scale.

As stressed by Feynman [6], when just T_n is relevant one is dealing with an 'almost static approximations of Maxwell's equations'. However, this information deserves little room in the literature. Together with the continuity equation, the quasi-electrostatic picture supports the assumption that currents in circuits are spatially uniform. Moreover, it allows one to understand why specific paths are followed by free electrons running within metallic components. Wires do not have rigid impermeable external walls and are not similar to water pipes. Instead, they convey the flow of free electrons with the help of charge densities lying on their external surfaces and, accordingly, a proper description of a circuit should acknowledge the field $\vec{E}_q$ they produce. When a uniform current is flowing, Ohm's law in the form $\vec{j}_w = \vec{E}_w/\rho_w$ ensures that the intensity $|\vec{E}_w|$ of the resultant field driving it is also spatially uniform. This feature is independent of the shape of the wire considered and requires the field $\vec{E}_q$ to be explained.

In the quasi-electrostatic picture, all elements of a circuit contribute quasi-simultaneously to the behaviour of the current at all its points. Batteries accumulate charges at their poles, wires are covered by thins layers of charges, capacitors contain charges on their plates, whereas both resistors and inductors have charges distributed inside the wires close to their endings. All these charges give rise to quasi-electrostatic fields, which spread out over the circuit, producing quasi-simultaneous nets of mutual influences. This justifies another tacit procedure found in textbooks, that of describing circuits in terms of potential differences associated with isolated components.

7. Conclusion

Introductory presentations of simple circuits found in university textbooks rely tacitly on the assumption that their behaviour is quasi-electrostatic and tend to concentrate narrowly on calculations of charges and currents as functions of time, using R, L, and C as free parameters. The discussion presented here affects very little these quantitative approaches. However, even if the mathematical determination of quasi-electrostatic fields present in a circuit may be expected to be very difficult and beyond the scope of basic physics courses, their qualitative aspects are well within reach of students. The qualitative restoration of both space and fields into pedagogical discussions of circuits can make them less abstract, more realistic, and therefore, more instructive. Most important, the ensuing images of nature may motivate students and invite them to derive joy from the 'sense of wonder about simple things' allowed by physics.

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