Physica Scripta

PAPER

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IOP Publishing Physica Scripta

Phys. Scr. **88** (2013) 045005 (9pp) doi:10.1088/0031-8949/88/04/045005

Semiclassical description of quantum rotator in terms of $SU(2)$ coherent states

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Received 12 August 2013
Published 17 September 2013
Online at stacks.iop.org/PhysScr/88/045005

Abstract
We introduce coordinates of the rigid body (rotator) using mutual positions between body-fixed and space-fixed reference frames. Wave functions that depend on such coordinates can be treated as scalar functions of the group $SU(2)$. Irreducible representations of the group $SU(2) \times SU(2)$ in the space of such functions describe their possible transformations under independent rotations of the both reference frames. We construct sets of the corresponding group $SU(2) \times SU(2)$ Perelomov coherent states (CS) with a fixed angular momentum j of the rotator as special orbits of the latter group. Minimization of different uncertainty relations is discussed. The classical limit corresponds to the limit $j \to \infty$. Considering Hamiltonians of rotators with different characteristics, we study the time evolution of the constructed CS. In some cases, the CS time evolution is completely or partially reduced to their parameter time evolution. If these parameters are chosen as Euler angles, then they obey the Euler equations in the classical limit. Quantum corrections to the motion of the quantum rotator can be found from exact equations on the CS parameters.

PACS numbers: 03.65.Sq, 03.65.Fd

1. Introduction

Coherent states (CS) play an important role in modern quantum mechanics due to their fundamental theoretical importance and a wide range of applications, e.g. in semiclassical description of quantum systems, in quantization theory, in condensed matter physics, in radiation theory, in quantum computations and so on (see e.g. [1]). Due to Glauber and co-workers [2–4] (see also Klauder and Sudarshan [1]) there exists a well-developed scheme of constructing CS for systems with quadratic Hamiltonians. Some non-trivial generalizations of the Glauber approach are developed by Klauder and Gazeau (see [6]). According to Perelomov [5] (see also Perelomov [1]), one can construct some kind of CS for systems with a given Lie group of symmetry. An important example of Perelomov CS are CS of $SU(N)$ groups; see [7–14] for $SU(2)$ and [15, 16] for symmetrical representations of $SU(N)$ with arbitrary N. However, physical applications of the latter CS are not as well known as those of the Glauber-type CS.

In this paper, we consider an important application of CS of the $SU(2)$ group in quantum theory of a rigid body (rotator in what follows) and on this basis we study semiclassical description of this system.

Basis elements of quantum theory of the rotator can be found, e.g. in [18–20]. In its simple version, the theory describes only the rotational motion of a many-particle system that is tightly bound. The Hamiltonian of such a system reads

$$\hat{H} = \frac{1}{2A_1}\hat{\mathcal{I}}_1^2 + \frac{1}{2A_2}\hat{\mathcal{I}}_2^2 + \frac{1}{2A_3}\hat{\mathcal{I}}_3^2, \qquad (1)$$

where A_a, $a = 1, 2, 3$ are the principal moments of inertia, and $\hat{\mathcal{I}}_a$ is projection of the angular momentum operator with respect to the axes of the body-fixed reference frame (brf). The most complete description of the quantum rotator in terms of the discrete basis can be found in [21, 22].

Attempts to construct CS of the quantum rotator should be mentioned; see [23–25]. These CS are parameterized by three complex numbers. The total angular momentum j of such CS was not fixed. Janssen's CS [23] have the property of mixing half-integer and integer quantum numbers. Morales *et al* [24] had introduced a different set of CS following closely the definition of Janssen; this set includes integer quantum numbers only. Similar sets of CS were considered in [25]. However, the introduced CS did not solve the problem of complete semiclassical description of the quantum rotator.

In this paper, we argue that the quantum rotator as any extended object has to be described in terms of two reference frames: brf and space-fixed reference frame (srf). As a consequence, two types of CS have to be introduced. According to the Perelomov scheme, they are orbits of the highest weight of irrep of the $SU(2)\times SU(2)$ group. In addition, it turns out that for semiclassical calculations, it is convenient to construct CS with a fixed angular momentum j. Then we study the time evolution of the constructed CS for systems with a quadratic in generators Hamiltonian. Finally, we consider the semiclassical limit. It is demonstrated that exact quantum equations of the CS parameters are reduced to the classical ones for big j. In the special case of the fundamental representation ($j=1/2$), CS always conserve their shape. However, in this case, equations for CS parameters differ essentially from the corresponding classical equations. In the appendix, we have placed some basic formulae to avoid doubts in parameterization definitions (unfortunately, in the literature different definitions are used [21, 26, 28]) and usual confusion with the signs.

2. Quantum description of the rotator

2.1. Space-fixed and body-fixed reference frames

Positions of a rotator can be described by the rotation matrix $V=\| v_a^i \|$ that belongs to the group $SO(3)\sim SU(2)$. This matrix relates two reference frames, one the brf associated with the rotator and defined by the orthobasis $\boldsymbol{\xi}_a$ and another one the srf associated with the laboratory and defined by the orthobasis $\boldsymbol{e}_i$,

$$\boldsymbol{\xi}_a = v^i_{\ a}\boldsymbol{e}_i. \tag{2}$$

The matrix V is orthogonal, $V^{\mathrm{T}}=V^{-1}$.

We consider v_a^i as coordinate set of the rotator. These coordinates can be expressed via the Euler angles; see equation (A.1) in the appendix. We also consider the left index, which labels the lines of the matrix v_a^i, as 'external', the right, which labels the columns, as 'internal'.

Introducing 2×2 matrices $\Xi=\sigma^a\boldsymbol{\xi}_a$ and $E=\sigma^i\boldsymbol{e}_i$, we represent relation (2) in terms of the complex Cayley–Klein parameters z_i,

$$\Xi = Z^{\dagger}EZ, \quad Z\in SU(2),$$
$$Z=\begin{pmatrix} z_1 & z_2\\ -\bar z_2 & \bar z_1\end{pmatrix}, \quad |z_1|^2+|z_2|^2=1. \tag{3}$$

The Cayley–Klein parameters can be expressed via the Euler angles ψ,θ,ϕ,

$$z_1=\cos(\theta/2)\,\mathrm{e}^{-\mathrm{i}(\phi+\psi)/2}, \quad z_2=-\sin(\theta/2)\,\mathrm{e}^{\mathrm{i}(-\phi+\psi)/2}. \tag{4}$$

Thus, there exists the correspondence $V\Leftrightarrow z$, such that one can describe the rotator orientation by two complex parameters z_i, $i=1,2$.

One can consider two types of transformations: rotations of the srf, which we call external transformations; and rotations of the brf, which we call internal transformations. It is obvious that the quantities $\{\boldsymbol{e}_i\}$ transform as vectors under the external transformations, whereas they remain unchanged with respect to internal transformations (rotations of the body). In contrast, the quantities $\{\boldsymbol{\xi}_a\}$ remain unchanged with respect to the external transformations, whereas they are vectors under the internal transformations.

To describe transformations of the coordinates v_a^i under the two types of the above rotations, we represent equation (2) as $\xi=eV$, considering ξ and e as columns composed from the vectors $\boldsymbol{\xi}_a$ and $\boldsymbol{e}_i$. An external transformation $e'=e\Lambda$ changes the matrix V as follows:

$$\xi=eV=e'\Lambda^{-1}V=e'V'\Longrightarrow V'=\Lambda^{-1}V. \tag{5}$$

An internal transformation $\xi'=\xi\underline{\Lambda}$ changes the same matrix as

$$\xi=\xi'\underline{\Lambda}^{-1}=eV\Longrightarrow V'=V\underline{\Lambda}. \tag{6}$$

Thus, under external transformation Λ results in the left multiplication of the matrix V by the matrix Λ^{-1}, whereas an internal transformation $\underline{\Lambda}$ results in the right multiplication of the matrix V by the matrix $\underline{\Lambda}$.

If both transformations are made at the same time, we call such a transformation the general transformation in what follows, then the following change of the matrix V takes place:

$$V'=\Lambda^{-1}V\underline{\Lambda}. \tag{7}$$

Both the rotation matrices Λ and $\underline{\Lambda}$ can be parameterized by the Euler angles. In the representation (7), generators are given by the standard 3×3 matrices. In addition, the matrices of generators of transformations (5) and (6) have the same form (however, their action is different, being related to left- and right-multiplication).

Let us consider general transformation (7) in terms of complex matrices (3). It affects both the matrices Ξ and E. The change $\Xi\to\Xi'$ induced by internal transformations is given by the matrices $g_{\mathrm{r}}\in SU(2)_{\mathrm{int}}$ as follows:

$$\Xi'=(g_{\mathrm{r}})^{-1}\Xi g_{\mathrm{r}},$$

whereas a change $E\to E'$ induced by external transformations is given by the matrices $g_{\mathrm{l}}\in SU(2)_{\mathrm{ext}}$:

$$E'=(g_{\mathrm{l}})^{-1}Eg_{\mathrm{l}}.$$

As it follows from equation (3), an internal transformation g_{r} modifies the matrix Z as $Z'=Zg_{\mathrm{r}}$, whereas an external transformation g_{l} modifies the matrix Z as $Z'=g_{\mathrm{l}}^{-1}Z$. The general transformation is a combination of both transformations and changes the matrix Z according to the rule

$$Z'=g_{\mathrm{l}}^{-1}Zg_{\mathrm{r}}, \tag{8}$$

where

$$g_{\mathrm{l}}=\begin{pmatrix} u_1 & u_2\\ -\bar u_2 & \bar u_1\end{pmatrix}, \quad g_{\mathrm{r}}=\begin{pmatrix} v_1 & v_2\\ -\bar v_2 & \bar v_1\end{pmatrix}. \tag{9}$$

Obviously the general transformation

$$\Pi(g_{\mathrm{l}},g_{\mathrm{r}})\in SU(2)_{\mathrm{ext}}\times SU(2)_{\mathrm{int}}. \tag{10}$$

2.2. Wave functions of the rotator as functions on $SU(2)$ group

In quantum theory, wave functions depend on positions of the system under consideration. As was demonstrated, positions of the rotator can be described either by elements of the matrix V or by two complex parameters $z_{1,2}$, $|z_1|^2+|z_2|^2=1$, that constitute the matrix $Z\in SU(2)$ according to equation (3). In what follows, we consider the rotator state vectors as functions of such matrices, $\Psi=\Psi(Z), Z\in SU(2)$.

According to (8), general transformations (10) define a representation $T(g_l, g_r)$ of the group $\Pi(g_l, g_r)$ in the space of scalar functions $\Psi(Z)$ (rotator wave functions)

$$T(g_l, g_r)\Psi(Z)=\Psi'(Z)=\Psi(g_l^{-1}Zg_r). \qquad (11)$$

It is obvious that the generators of the representation $T(g_l, g_r)$ consist of the left generators $\hat{J}_i$ (A.2) of $T_L(g_l)$ and the right generators $\hat{J}_a^R$ (A.3) of $T_R(g_r)$ regular representations of $SU(2)$.

The straightforward calculation of the left and the right generators (see the appendix) gives expressions (A.5), (A.6) and the following standard commutation relations (below we use the angular momentum measured in units of $\hbar$, $J_i=\mathcal{J}_i/\hbar$, $I_i=\mathcal{I}_i/\hbar$):

$$[\hat{J}_i, \hat{J}_a^R]=0, \quad [\hat{J}_i, \hat{J}_k]=\mathrm{i}\epsilon^{ikl}\hat{J}_l, \quad [\hat{J}_a^R, \hat{J}_b^R]=\mathrm{i}\epsilon^{abc}\hat{J}_c^R. \qquad (12)$$

Let us turn to the physical interpretation of the left and right generators. The operators $\hat{J}_k$ are projections of the angular momentum in the srf. The projection $\hat{I}_a$ of the vector $\hat{\mathbf{J}}=(\hat{J}_1, \hat{J}_2, \hat{J}_3)$ on the unit vector $\boldsymbol{\xi}_a$ of brf is given by the scalar product

$$\hat{I}_a=(\hat{\mathbf{J}}, \boldsymbol{\xi}_a)=\hat{J}_k v_a^k. \qquad (13)$$

It should be noted that $\hat{J}_k$ and v_a^k commute (rotation does not affect components v_a^k of vectors $\boldsymbol{\xi}_a$ that are parallel to rotation axis $\boldsymbol{e}_k$); see (A.7). The straightforward calculations give the following result:

$$\hat{I}_a=-\hat{J}_a^R, \qquad (14)$$

which means that the operators $\hat{I}_a$ coincide with the right generators up to a sign. In turn, this means that the commutation relations for the operators $\hat{I}_a$ differ by a sign from the usual (12) commutation relations for the angular momentum operators

$$[\hat{I}_a, \hat{I}_b]=-\mathrm{i}\epsilon^{abc}\hat{I}_c. \qquad (15)$$

The operators $\hat{I}_a$ can be considered as generators that correspond to changed-by-a-sign rotation parameters.

The reason for such a difference in the commutation relations can be easily seen in the example of the two-dimensional rotator with the symmetry group $SO(2)\sim U(1)$. The wave function ψ of such a rotator depends only on the angle ϕ. The transformation of the function $z=\mathrm{e}^{\mathrm{i}\phi}$ is given by the multiplication on $g=\mathrm{e}^{\mathrm{i}\alpha}\in U(1)$ from the left, $z'=g^{-1}z$, or from the right, $z'=zg$ and $g^{-1}zg=z$. Thus, in the two-dimensional case, an external rotation is equivalent to an inverse internal rotation if we speak about the wave function transformation. Respectively, the corresponding generators differ by the sign only, $\hat{J}=-\mathrm{i}\mathrm{d}/\mathrm{d}\phi$, $\hat{J}^R=\mathrm{i}\mathrm{d}/\mathrm{d}\phi$ and $\hat{J}^R=-\hat{J}$. This is a consequence of the fact that the group $U(1)$ is commutative.

However, in the three-dimensional case, where we deal with the $SO(3)\sim SU(2)$ group, such an interpretation (see [18]) is not correct; the latter group is non-commutative and a general external rotation is not equivalent to its inverse internal rotation. If we suppose that such equivalence take place, then $\Lambda^{-1}V\Lambda=V$ or $[\Lambda, V]=0$. The latter holds only in the case when Λ and V correspond to rotations about one and the same axis. The difference between the generators of the external and internal rotations is not reduced to the difference in signs. However, angular momentum operators in brf are $\hat{I}_a=-\hat{J}_a^R$, and the difference in the commutation relations is reduced to the sign change.

Let $\hat{J}_i$ be generators that correspond to the subgroup $SU(2)_{\mathrm{ext}}$ and $\hat{I}_a$ are generators that correspond to the subgroup $SU(2)_{\mathrm{int}}$. Operators $\hat{J}_k$ are transformed as vectors under $SU(2)_{\mathrm{ext}}$ transformations and are invariants under $SU(2)_{\mathrm{int}}$ transformations, whereas the operators $\hat{I}_a$ are transformed as vectors under the latter transformations and are invariants under the $SU(2)_{\mathrm{ext}}$ transformations.

As follows from equation (13), the square of the total momentum is the same in both reference frames

$$\hat{\mathbf{I}}^2=\sum_a \hat{J}_i v_a^i \hat{J}_k v_a^k=\hat{\mathbf{J}}^2. \qquad (16)$$

In the representation $T(g_l, g_r)$, both subgroups act in the same space of functions that depend on three real parameters (Euler angles). One can see that there are three mutually commuting operators

$$\hat{J}_3, \quad \hat{I}_3, \quad \hat{\mathbf{J}}^2=\hat{\mathbf{I}}^2. \qquad (17)$$

We denote the common eigenfunctions of the operator set (17) as $|j\,mk\rangle$,

$$\begin{aligned}
&\hat{J}_3|jmk\rangle=m|jmk\rangle, \quad -j\leqslant m\leqslant j,\\
&\hat{I}_3|j\,m\,k\rangle=k|j\,m\,k\rangle, \quad -j\leqslant k\leqslant j, \qquad (18)\\
&\hat{\mathbf{J}}^2|j\,m\,k\rangle=j(j+1)|j\,m\,k\rangle, \quad 2j=0,1,2,\ldots.
\end{aligned}$$

They correspond to rotator states with a given angular momentum j and its z-projection m with respect to the srf and z-projection k with respect to the brf.

As mentioned above, general transformations (8) belong to the direct product $SU(2)\times SU(2)$. In the general case, the irreps of $SU(2)\times SU(2)$ are characterized by eigenvalues of two different Casimir operators $\hat{\mathbf{J}}^2$ and $\hat{\mathbf{I}}^2$. However, in the case under consideration, $\hat{\mathbf{J}}^2=\hat{\mathbf{I}}^2$, and rotator states are labeled only by three numbers: the total momentum j and the two projections m, k. This is a consequence of the fact that in the case under consideration, the operators of both subgroups act in the same space of functions depending only on three parameters. In this space, only a part of representations of the direct product $SU(2)\times SU(2)$ is realized; see the appendix.

The algebra of operators $\hat{J}_a^R=-\hat{I}_a$ has the same commutation relations as the algebra of operators $\hat{J}_i$, and therefore, standard results of the angular momentum theory hold true here. We obtain multiplets of dimension $2j+1$, where j is the integer or half-integer maximal value of projection $k=I_3$ to the fixed axis $\boldsymbol{\xi}_3$.

Thus, there are $(2j+1)^2$ states with the same j. As is known, explicit form of the states $|j\,m\,k\rangle$ is given by the Wigner D-functions that are matrix elements of the irreps $T_j(g)$ of the group $SU(2)$ [20, 21]. Here we write the Wigner D-functions via the Cayley–Klein parameters

$$\langle z|j\,m\,k\rangle = D^j_{m,k}(z) = \sqrt{(j+m)!(j-m)!(j+k)!(j-k)!} \times \sum_{n_\alpha}{}' \frac{z_1^{n_1} z_2^{n_2} \bar{z}_1^{n_3} (-\bar{z}_2)^{n_4}}{n_1! n_2! n_3! n_4!}, \; n_\alpha \in \mathbb{Z}_+, \tag{19}$$

where summation over n_α is restricted by the constraints

$$n_1+n_2+n_3+n_4 = 2j, \quad -n_1-n_2+n_3+n_4 = 2m, \quad -n_1+n_2+n_3-n_4 = 2k, \tag{20}$$

see [26]. In fact, only one of all n_α is independent.

For the highest weights of the irreps of $SU(2)_{\text{ext}}$ or $SU(2)_{\text{int}}$, i.e. for $m = \pm j$ (functions of only one column of the matrix (3) z_1, z_2 or $\bar{z}_1$, $\bar{z}_2$) or $k = \pm j$ (functions of only one line of the matrix (3)) the summation is absent:

$$\langle z|j\,m\,j\rangle = \left(\frac{(2j)!}{(j+m)!(j-m)!}\right)^{1/2} \bar{z}_1^{j+m} (z_2)^{j-m}, \tag{21}$$

$$\langle z|j\,j\,k\rangle = \left(\frac{(2j)!}{(j+k)!(j-k)!}\right)^{1/2} \bar{z}_1^{j+k} (-\bar{z}_2)^{j-k}.$$

The scalar product in the space of the functions $\Psi(Z)$, $Z \in SU(2)$ is given by the integration over the Cayley–Klein parameters with the invariant measure $\mathrm{d}\mu(z)$,

$$\int \bar{\Psi}_1(z)\Psi_2(z)\,\mathrm{d}\mu(z),$$

$$\mathrm{d}\mu(z) = \frac{1}{8\pi^2}\delta(|z_1|^2+|z_1|^2-1)\,\mathrm{d}^2 z_1 \mathrm{d}^2 z_2 = \frac{1}{8\pi^2}\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\psi. \tag{22}$$

The wave functions in z-representation (19) are normalized with respect to such a scalar product. In fact, these functions represent the scalar product (22) of states $|z\rangle$ with a given orientation and states $|jmk\rangle$ with a given angular momentum and its projections.

Wave functions that do not depend on the angle ψ are eigenfunctions of $\hat{I}_3$ with the eigenvalue $k=0$. In addition, in this case the operators $\hat{J}_k$ (A.5) acquire the form of the 'usual' operators of angular momentum for a non-orientable point particle, which depend only on the two angles θ and ϕ. Such states are $|j\,m\rangle = |j\,m\,0\rangle$.

3. Coherent states (CS) of the rotator

3.1. Instantaneous CS

We construct CS as orbits in spaces of group irreps; see [5] (see also Perelomov [1]). To construct CS with semiclassical properties, we consider the variance that is invariant under the general transformations (8) (i.e. changes of reference frames). In states that corresponding to the discrete basis $|j\,m\,k\rangle$, we have

$$\Delta J^2_{\text{ext}} = \langle J^2\rangle - \langle J\rangle^2 = j(j+1)-m^2,$$
$$\Delta J^2_{\text{int}} = \langle J^2\rangle - \langle I\rangle^2 = j(j+1)-k^2 \tag{23}$$

such that the complete variance reads

$$\Delta J^2_\Sigma = \Delta J^2_{\text{ext}} + \Delta J^2_{\text{int}} = 2j(j+1)-m^2-k^2. \tag{24}$$

We chose the ratio

$$\frac{\Delta J^2_\Sigma}{2J^2} = 1 - \frac{m^2+k^2}{2j(j+1)} \tag{25}$$

as the measure of the classicality (how close a quantum state is to the corresponding classical state). The quantity (25) is minimal and equal to $1/(1+j)$ for states of discrete basis $|jmk\rangle$ (19) with $|m|=|k|=j$ and tends to zero as $j\to\infty$.

Thus, for a given angular momentum j, the states $|jmk\rangle$ with $|m|=|k|=j$ can be considered as semi-classical states. The same holds true for states that can be obtained from the states $|jmk\rangle$ after transformations (10). They have the same value $1/(1+j)$ for quantity (25).

We first consider separately the CS of groups $SU(2)_{\text{ext}}$ and $SU(2)_{\text{int}}$.

Applying the external transformations to the state $|j\,j\,j\rangle = \bar{z}_1^{2j}$, we obtain 'left' CS $|j\,u\,j\rangle$,

$$|j\,u\,j\rangle = (u_1\bar{z}_1 + \bar{u}_2 z_2)^{2j} = \sum_{m=-j}^{j}\left(\frac{(2j)!}{(j+m)!(j-m)!}\right)^{1/2} u_1^{j+m}\bar{u}_2^{j-m}|j\,m\,j\rangle, \tag{26}$$

where $u_1 = \cos(\gamma/2)\,\mathrm{e}^{\mathrm{i}\delta/2}$ and $u_2 = \sin(\gamma/2)\,\mathrm{e}^{-\mathrm{i}\delta/2}$. We stress that these states are exactly the CS of the angular momentum [8] (see also Perelomov [1]). However, in the rotator case, the variables z_1, z_2 depend on three Euler angles ϕ, θ, ψ, which are coordinates on the group $SU(2)$, whereas in the angular momentum CS they depend only on two variables ϕ, θ, which are coordinates on the homogeneous space $SU(2)/U(1)$.

One can see that the state $|j\,u\,j\rangle$ is an eigenvector for the projector $\hat{J}_{\boldsymbol{n}}$ on the direction given by a unit vector $\boldsymbol{n}$:

$$\boldsymbol{n} = (\sin\gamma\cos\delta, \sin\gamma\sin\delta, \cos\gamma),$$
$$n_i = \sigma_i^{\alpha\beta}\bar{u}_\alpha u_\beta, \tag{27}$$

i.e. $\hat{J}_{\boldsymbol{n}}|j\,u\,j\rangle = j|j\,u\,j\rangle$.

The overlapping of two 'left' CS (26) one with the vector $\mathbf{n}$ and another one with the vector $\boldsymbol{n}'$ can be easily calculated

$$\langle j\,u\,j|j\,u'\,j\rangle = (\cos(\beta'/2))^{2j}. \tag{28}$$

Here β' is the angle between the vectors $\boldsymbol{n}$ and $\boldsymbol{n}'$.

In the limit $j\to\infty$, the 'left' CS states $|j\,u\,j\rangle$ and $|j\,u'\,j\rangle$ are orthogonal if $u \neq u'$,

$$\lim_{j\to\infty}\langle j\,u\,j|j\,u'\,j\rangle = 0, \;\; u \neq u'.$$

The corresponding relative uncertainties (25) tend to zero and we obtain semi-classical states with the angular momentum j and rotation axis given by the angles α and β.

Applying the internal transformations to the state $|j\ j\ j\rangle = \bar{z}_1^{2j}$, we obtain 'right' CS $|j\ j\ v\rangle$ of the rotator

$$|j\ j\ v\rangle = (\bar{v}_1\bar{z}_1 + v_2(-\bar{z}_2))^{2j} = \sum_{k=-j}^{j}\left(\frac{(2j)!}{(j+k)!(j-k)!}\right)^{1/2} \bar{v}_1^{j+k} v_2^{j-k}|j\ j\ k\rangle, \quad (29)$$

where $v_1 = \cos(\gamma/2)\,e^{i\delta/2}$ and $v_2 = \sin(\gamma/2)\,e^{-i\delta/2}$.

The state $|jjk\rangle$ has definite projection k on the axis $\boldsymbol{\xi}^3$ of brf. Respectively, CS $|jj\,v\rangle$ is the state with the maximal projection $k = j$ of the angular momentum on the direction

$$\boldsymbol{v} = (\sin\gamma\cos\delta, \sin\gamma\sin\delta, \cos\gamma), \qquad v_i = \sigma_i^{\alpha\beta}\bar{v}_\alpha v_\beta,$$

i.e. CS $|jj\,v\rangle$ is an eigenvector of the projector $\hat{I}_{\boldsymbol{v}}$ on the direction $\boldsymbol{v}$ defined by the angles γ, δ in the brf

$$\hat{I}_{\boldsymbol{v}}|j\ j\ v\rangle = j|j\ j\ v\rangle.$$

In the classical theory, if brf coincides with the principal axis frame of the rotator, then $\mathcal{I}_a = \hbar I_a = A_a\omega_a$, where A_a are principal inertia momenta and ω_a are components of angular velocity. The quasiclassical rotation vector or angular velocity is then defined with components

$$\omega_a = A_a^{-1}\hbar\langle\hat{I}_a\rangle \quad (30)$$

and for CS (29) we have $\omega_a = A_a^{-1} j\hbar v_a$.

In the general case, when we apply the general transformations (10) to the state $|j\ j\ j\rangle$, we obtain CS $|j\ u\ v\rangle$ of the rotator that can be expressed in terms of the Wigner D-functions (19)

$$|j\ u\ v\rangle = (u_1\bar{v}_1\bar{z}_1 + \bar{u}_2\bar{v}_1 z_2 + \bar{u}_2 v_2 z_1 + u_1 v_2(-\bar{z}_2))^{2j} = \sum_{m,k=-j}^{j}\frac{(2j)!}{\sqrt{(j+m)!(j-m)!(j+k)!(j-k)!}} \times u_1^{j+m}\bar{u}_2^{j-m}\bar{v}_1^{j+k}v_2^{j-k}|j\ m\ k\rangle \quad (31)$$

The states $|j\ u\ v\rangle$ are characterized by projections j of angular momentum on the axis $\boldsymbol{n}$ in srf and on the axis $\boldsymbol{v}$ in brf. At fixed j, these states possess minimal invariant variance (24) $\Delta J_\Sigma^2 = 2j$.

3.2. Time evolution of rotator CS euler equations

Due to the spatial isotropy, the Hamiltonian of a free rotator cannot depend explicitly on the rotator orientation and is, therefore, an 'external' invariant. Such a Hamiltonian can depend on the combination of the left generators $\hat{J}_k$, which is the Casimir operator $\hat{\mathbf{J}}^2$. In addition, it can be a function of the operators $\hat{I}_a$, which are 'external' invariants.

For a completely symmetric rotator, not only the external transformations, but also each of the internal transformations are symmetry transformations of the Hamiltonian; in this, case the symmetry group is $SO(3)\times SO(3)$. In the case of the axial symmetry, there exists internal symmetry with respect to the right rotations about the $\boldsymbol{\xi}_3$-axis (with the generator $\hat{I}_3$); in this case the symmetry group is $SO(3)\times SO(2)$. This symmetry corresponds to an additive quantum number k. Finally, in the case when all three inertia momenta are different (there is no internal symmetry), the internal transformations with generators $\hat{I}_a$ are not Hamiltonian symmetries and the symmetry group is $SO(3)$.

Thus, a symmetry with respect to external transformations (or an external symmetry) is interpreted as a symmetry of the embedding space, in which the rotator is placed, and a symmetry with respect to internal transformations (or an internal symmetry) is interpreted as a symmetry of the rotator itself.

Let us consider the time evolution of the rotator CS. We chose the rotator Hamiltonian in the following form:

$$\hat{H} = \hbar^2\left(\frac{1}{2A_1}\hat{I}_1^2 + \frac{1}{2A_2}\hat{I}_2^2 + \frac{1}{2A_3}\hat{I}_3^2\right) + U, \quad (32)$$

where A_b are principal inertia momenta, brf coincides with the principal axis frame and U is the rotator potential energy, which in the general case depends on the rotator position. In classical theory, the rotator motion is described by the Euler equations

$$A_a\dot{\omega}_a = \epsilon^{abc}A_b\omega_b\omega_c + K_a, \quad (33)$$

where $K_a = i\hat{I}_a U$ is the torque.

Our aim here is to obtain equations for CS evolution in term of angular velocity ω_a (30) and to compare them with Euler equations.

Consider first the Hamiltonian of the axial-symmetric $(A_1 = A_2 = A)$ free rotator

$$\hat{H} = \frac{1}{2}\hbar^2\left(\frac{1}{A}\hat{J}^2 + \Omega\hat{I}_3^2\right), \quad \Omega = \left(\frac{1}{A_3} - \frac{1}{A}\right). \quad (34)$$

States with a given energy are eigenvectors of the operators $\hat{J}^2$ and $\hat{I}_3$, in particular, these are both states of the discrete basis $|j\,m\,k\rangle$ and the 'left' CS $|j\ u\ j\rangle$ (26). If initial states are the states $|j\ m\ k\rangle$ or $|j\ u\ j\rangle$, the time evolution modifies them only by phase factors. The time evolution of initial states that are not eigenvectors for the operator $\hat{I}_3$ is more complicated. As an example, one can mention the 'right' CS $|j\ j\ v\rangle$ (29).

The Schrödinger equation for states with a definite angular momentum j has the form

$$i\hbar\frac{\partial\Psi(z,t)}{\partial t} = \frac{1}{2}\hbar^2\left[A^{-1}j(j+1) + \Omega\hat{I}_3^2\right]\Psi(z,t). \quad (35)$$

Here, the case $j = 1/2$ is special. In such a case $\hat{I}_1^2 = \hat{I}_2^2 = \hat{I}_3^2 = 1/4$ (in the spinor representation $j = 1/2$ generators satisfy both the commutation relations (15) and the anticommutation relations

$$\{\hat{I}_i, \hat{I}_k\} = \frac{1}{4}\delta_{ik},$$

see details in [27]). That is why the rhs of equation (35) takes the form

$$\frac{1}{2}\left[A^{-1}j(j+1) + \Omega\hat{I}_3^2\right]\Psi_{1/2}(z,t) = \frac{1}{8}(3A^{-1} + \Omega)\Psi_{1/2}(z,t) = \frac{1}{4}\left(\frac{1}{A} + \frac{1}{2A_3}\right)\Psi_{1/2}(z,t).$$

Thus, in this special case the time evolution of any initial state changes it only by a phase factor

$$\Psi_{1/2}(z,t)=\exp\left[-\mathrm{i}\hbar\left(\frac{1}{A}+\frac{1}{2A_3}\right)\frac{t}{4}\right]\Psi_{1/2}(z,0).$$

Let us choose the rotator CS $|j\ j\ v\rangle$ (29) as the initial state. In the general case, its evolution with time changes the form of the wave packet (spreading of the wave packet) and parameters of the CS. Consider first the spreading problem. To this end, we make the change of the wave function $\Psi(z,t)$ by the one $\tilde{\Psi}(z,t)$:

$$\tilde{\Psi}(z,t)=\Psi(z,t)\exp\left[-\frac{\mathrm{i}}{\hbar}\frac{j(j+1)}{2A}t\right]. \tag{36}$$

The new function $\tilde{\Psi}(z,t)$ satisfies the following equation:

$$\mathrm{i}\hbar\frac{\partial\tilde{\Psi}(z,t)}{\partial t}=\frac{\Omega}{2}\hat{I}_3^2\tilde{\Psi}(z,t). \tag{37}$$

Let us suppose that the CS does not change its form such that the time evolution changes only the parameter v, i.e. there exist solutions of equation (37) of the form $|j\ j\ v(t)\rangle$

$$\tilde{\Psi}(z,t)=|j\ j\ v(t)\rangle=[\bar{v}_1(t)\bar{z}_1+v_2(t)(-\bar{z}_2)]^{2j}\,.$$

Then equation (37) takes the form

$$2ij\left[\bar{v}_1'\bar{z}_1+v_2'(-\bar{z}_2)\right][\bar{v}_1\bar{z}_1+v_2(-\bar{z}_2)]$$
$$=\frac{\Omega}{2}\left\{j(j-1/2)\,[\bar{v}_1\bar{z}_1-v_2(-\bar{z}_2)]^2+j\,[\bar{v}_1\bar{z}_1+v_2(-\bar{z}_2)]^2\right\}. \tag{38}$$

Decomposing the right and the left sides of equation (38) in powers of z_k, we obtain three equations (for the case $j=1/2$ two equations, and for states with $v_1(0)=0$ or $v_2(0)=0$ only one equation). In the general case these three equations are inconsistent, which means that in the general case solutions of equation (35) cannot be written as $|j\ j\ v(t)\rangle$. (For semisimple groups, the only Hamiltonians that are linear in group generators or are functions of Casimir operators preserve CS form; for solvable groups such Hamiltonians may contain some bilinear combinations of the generators that form a simple algebra, see [17].)

However, one can construct solutions that are close in a sense to the CS. To this end, we represent the wave function $\tilde{\Psi}(z,t)$ in the following form:

$$\tilde{\Psi}(z,t)=\sum_k c_k(t)|j\ j\ k\rangle,$$
$$c_k(0)=\left(\frac{(2j)!}{(j+k)!(j-k)!}\right)^{1/2}\bar{v}_1^{j+k}v_2^{j-k}. \tag{39}$$

At $t=0$, the corresponding state $\Psi(z,t)$ from (36) is CS, $\Psi(z,0)=|j\ j\ v\rangle$. Substituting the function (39) into equation (37), we obtain

$$c_k(t)=c_k(0)\exp\left(-\frac{\mathrm{i}}{2\hbar}k^2\Omega t\right)$$

such that

$$\Psi(t)=\exp\left[\frac{\mathrm{i}}{\hbar}\frac{j(j+1)}{2A}t\right]\sum_k c_k(0)$$
$$\times\exp\left(-\frac{\mathrm{i}}{2\hbar}k^2\Omega t\right)|j\ j\ k\rangle. \tag{40}$$

The function $\tilde{\Psi}(z,t)$ is periodic with the period $T_0=4\pi\hbar\Omega^{-1}$. Thus, in the time instants nT_0 the wave function $\Psi(z,t)$, which differs from $\tilde{\Psi}(z,t)$ by a phase factor, again takes the form of a CS. This means that the variance ΔI^2 is not growing and the wave packet is not spreading with time.

Let us consider mean values $\langle\hat{I}_i\rangle$ in the time-dependent CS $|j\ j\ v(t)\rangle$ (29). Suppose that the time evolution is due to the Hamiltonian (32). Then we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle\hat{I}_a\rangle=\frac{\mathrm{i}}{\hbar}\langle[\hat{H},\hat{I}_a]\rangle$$
$$=\hbar\sum_j\epsilon^{abc}\frac{1}{2A_b}\langle\hat{I}_b\hat{I}_c+\hat{I}_c\hat{I}_b\rangle+\frac{1}{\hbar}K_a. \tag{41}$$

To find explicit expressions for the mean values entering equation (41), we use the fact that the operators $\hat{I}_b$ can be expressed via the operators $\hat{T}_\alpha^\beta=a_\alpha\partial/\partial a_\beta$, $a_1=\bar{z}_1$, $a_2=-\bar{z}_2$,

$$\hat{I}_1=\hat{T}_2^1+\hat{T}_1^2,\quad \hat{I}_2=\mathrm{i}(\hat{T}_2^1-\hat{T}_1^2),\quad \hat{I}_3=\hat{T}_2^2-\hat{T}_1^1$$

and Q-symbols of the operators $T_\beta^\alpha(\bar{v},v)=\langle\hat{T}_\beta^\alpha\rangle$ and $(T_\beta^\alpha T_\delta^\gamma)(\bar{v},v)=\langle\hat{T}_\beta^\alpha\hat{T}_\delta^\gamma\rangle$ that were calculated in [10, 16]

$$T_\beta^\alpha(\bar{v},v)=2j\bar{v}_\alpha v_\beta,$$
$$(T_\beta^\alpha T_l^\gamma)(\bar{v},v)=2j(2j-1)\bar{v}_\alpha\bar{v}_\gamma v_\beta v_\delta+2j\bar{v}_\alpha v_\delta\delta_\beta^\gamma.$$

Taking into account the expression (30) for angular velocity ω_a of the rotator in terms of CS parameters, we obtain the following equation:

$$A_a\dot{\omega}_a=\frac{2j-1}{2j}\epsilon^{abc}A_b\omega_b\omega_c+K_a. \tag{42}$$

For $j=1/2$ the CS are stabile and their evolution is reduced to the change of CS parameters according to the equation

$$A_a\dot{\omega}_a=K_a$$

and the precession is absent; at $K_i=0$ the only phase of the wave function is changing with time.

Equation (42) can be interpreted as quantum version of the Euler equations for the classical rotator (33). They differ from the classical equations by the factor $(2j-1)/2j$ in the rhs. Note that for small j, this factor differs from its classical value 1 essentially. This difference results in slowdown of the rotator precession. This factor tends to its classical value 1 as $j\to\infty$. Thus, $(j)^{-1}$ can be considered as a small dimensionless parameter that provides the classical limit. Remembering that $(j)^{-1}=\hbar\mathcal{J}^{-1}$, we see that in the problem under consideration, as in many other quantum mechanical problems, formal decompositions in the Planck constant can be interpreted as semiclassical decompositions.

Let $\omega_a^{\rm cl}$ obey classical Euler equations (33) and we present solution of equation (42) in the form $\omega_a = \omega_a^{\rm cl} + \Delta\omega_a + O((j)^{-2})$, where $\Delta\omega_a$ are quantum corrections of the order $(j)^{-1}$. Then these corrections satisfy the set of linear first-order differential equations

$$A_a \Delta\dot{\omega}_a = -(2j)^{-1} A_a \dot{\omega}_a^{\rm cl} + \epsilon^{abc} A_b (\Delta\omega_b \omega_c^{\rm cl} + \omega_b^{\rm cl} \Delta\omega_c).$$

Acknowledgments

DMG thanks FAPESP and CNPq for permanent support and for the support from the project 2.3684.2011 of Tomsk State University and FTP, contract no. 14.B37.21.0911.

Appendix

The orientation of a three-dimensional rotator is determined by a 3×3 orthogonal matrix, $V \in O(3)$, composed of the coefficients of a re-decomposition of the bases (srf and brf), see section 2 . If both systems $\{\boldsymbol{e}_i\}$ and $\{\boldsymbol{\xi}_a\}$ are right or left, the matrix $V \in SO(3)$ depends on three real-valued parameters, which can be chosen as the Euler angles:

$$V = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \times \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} \begin{bmatrix} \cos\phi\cos\psi\cos\theta \\ -\sin\phi\sin\psi \end{bmatrix} & \begin{bmatrix} -\sin\phi\cos\psi - \cos\phi \\ \times\sin\psi\cos\theta \end{bmatrix} & \cos\phi\sin\theta \\ \begin{bmatrix} \sin\phi\cos\psi\cos\theta \\ +\cos\phi\sin\psi \end{bmatrix} & \begin{bmatrix} \cos\phi\cos\psi \\ -\sin\phi\sin\psi\cos\theta \end{bmatrix} & \sin\phi\sin\theta \\ -\cos\psi\sin\theta & \sin\psi\sin\theta & \cos\theta \end{pmatrix}. \tag{A.1}$$

To find generators of an arbitrary irrep of $SO(3)$, one has to examine representations in the space of functions on the group, i.e. functions $f(\phi,\psi,\theta)$ of the rotator orientation.

The left regular representation $T_{\rm L}(g)$ acts in the space of functions $f(q)$, $q = q(\phi,\psi,\theta) \in SO(3)$, on the group as follows:

$$T_{\rm L}(g_{\rm l}) f(q) = f'(q) = f(g_{\rm l}^{-1} q),\ \ g_{\rm l} \in G, \tag{A.2}$$

which corresponds to a change of the srf; see (5), whereas the right regular representation $T_{\rm R}(g)$ acts in the same space as follows:

$$T_{\rm R}(g_{\rm r}) f(q) = f'(q) = f(q g_{\rm r}),\quad g_{\rm r} \in G, \tag{A.3}$$

which corresponds to a change of the brf; see (6). The decomposition of the left (and right) regular representation contains any irrep of the group.

Each set of the left and right transformations forms the group $SO(3)$. Since these two transformation sets commute with each other, we can consider them as the direct product $\Pi = SO(3)\times SO(3)$. The transformations from Π act in the space of functions depending on three parameters (on the rotator orientation) as follows:

$$T_\Pi(g_{\rm l}, g_{\rm r}) f(q) = f(g_{\rm l}^{-1} q g_{\rm r}) = f'(q). \tag{A.4}$$

For Hermitian generators that correspond to the one-parameter subgroup $\omega(t)$ in the left $T_{\rm L}(g)$ (A.2) and right $T_{\rm R}(g)$ (A.3) regular representations, we obtain, respectively:

$$\hat{J}_\omega f(q) = -\mathrm{i}\lim_{t\to 0} \frac{f(\omega^{-1}(t)q) - f(q)}{t},$$
$$\hat{J}_\omega^{\rm R} f(q) = -\mathrm{i}\lim_{t\to 0} \frac{f(q\omega(t)) - f(q)}{t}.$$

Let us choose one-parameter subgroups as follows:

$$\omega_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \end{pmatrix},\quad \omega_2 = \begin{pmatrix} \cos t & 0 & -\sin t \\ 0 & 1 & 0 \\ \sin t & 0 & \cos t \end{pmatrix},$$
$$\omega_3 = \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The transformations $\omega_k^{-1}(t)q$ correspond to rotations about the axes $\boldsymbol{e}_a$, whereas $q\omega_k(t)$ correspond to rotations about the axes $\boldsymbol{\xi}_a$. The straightforward calculations yield the following expressions for generators of the external transformations:

$$\hat{J}_1 = -\mathrm{i}\left(\frac{\cos\phi}{\sin\theta}\frac{\partial}{\partial\psi} - \sin\phi\frac{\partial}{\partial\theta} - \cos\phi\cot\theta\frac{\partial}{\partial\phi}\right),$$
$$\hat{J}_2 = -\mathrm{i}\left(\frac{\sin\phi}{\sin\theta}\frac{\partial}{\partial\psi} + \cos\phi\frac{\partial}{\partial\theta} - \sin\phi\cot\theta\frac{\partial}{\partial\phi}\right),$$
$$\hat{J}_3 = -\mathrm{i}\frac{\partial}{\partial\phi} \tag{A.5}$$

and

$$\hat{J}_1^{\rm R} = -\mathrm{i}\left(\frac{\cos\psi}{\sin\theta}\frac{\partial}{\partial\phi} - \sin\psi\frac{\partial}{\partial\theta} - \cos\psi\cot\theta\frac{\partial}{\partial\psi}\right),$$
$$\hat{J}_2^{\rm R} = \mathrm{i}\left(\frac{\sin\psi}{\sin\theta}\frac{\partial}{\partial\phi} + \cos\psi\frac{\partial}{\partial\theta} - \sin\psi\cot\theta\frac{\partial}{\partial\psi}\right),$$
$$\hat{J}_3^{\rm R} = \mathrm{i}\frac{\partial}{\partial\psi} \tag{A.6}$$

for the generators of the internal transformations.

It is easy to see that all the internal generators commute with all the external generators. This follows from the associativity of the group multiplication: in the product $g^{-1}qh$, the result does not depend on whether one multiplies q first from the right or from the left.

The following commutation relations hold:

$$[\hat{J}_i, v^j_{\ a}] = \mathrm{i}\varepsilon_{ijk} v^k_{\ a},\quad [\hat{J}_a^{\rm R}, v^i_{\ b}] = \mathrm{i}\varepsilon_{abc} v^i_{\ c}, \tag{A.7}$$

where v_a^j are elements of matrix V (A.1).

It should be noted that the quantities $\hat{J}_a^{\rm R}$ are invariant under the external transformations; however, they transform as vector components under the internal transformations. In turn, $\hat{J}_k$ are 'external' vectors and 'internal' invariants.

In terms of the Euler angles, expressions for generators (A.5) and (A.6) as well as the composition law look quite complicated. It is more simple to use the Cayley–Klein parameters (4) that are transformed under the spinor representation of the group $SU(2) \sim SO(3)$. By introducing 2×2 matrices, $E = \sigma^i \boldsymbol{e}_i$ and $\Xi = \sigma^a \boldsymbol{\xi}_a$, equation (2), which gives the relation between srf and brf, can be represented in the following form:

$$\Xi = Z^\dagger E Z, \quad Z^\dagger = Z^{-1},$$
$$Z = \begin{pmatrix} z^1{}_1 & z^1{}_2 \\ z^2{}_1 & z^2{}_2 \end{pmatrix} = \begin{pmatrix} z_1 & z_2 \\ -\bar{z}_2 & \bar{z}_1 \end{pmatrix} \in SU(2). \quad \text{(A.8)}$$

Rotations of srf (5) and brf (6) correspond to transformations in terms of unitary matrices U and $\underline{U}$

$$Z' = U^\dagger Z \underline{U}, \quad U,\ \underline{U} \in SU(2)$$

and therefore, according to (A.8), elements of the matrix Z have two kinds of spinor indices: the left one (external) and right one (internal).

The coordinates x^i of the vector $\boldsymbol{x} = x^i \boldsymbol{e}_i$ change under the external transformations according to the relation

$$X' = U^\dagger X U, \quad X = \sigma_i x^i, \quad \text{(A.9)}$$

where the 2×2 matrices U and $-U$ correspond to the same transformation.

Using equation (A.9) and the relation $\overline{U} = \sigma_2 U \sigma_2$, it is easy to see that $\sigma^\alpha_{k\,\beta} = (\sigma_k)^\alpha_\beta$ is an invariant tensor under $SU(2)$. A consequence of the unimodularity of the matrix U is the existence of the invariant antisymmetric tensor $\varepsilon^{\alpha\beta} = -\varepsilon^{\beta\alpha}$, $\varepsilon^{12} = \varepsilon_{21} = 1$. This fact allows one to lower and rise the spinor indices, $z_\alpha = \varepsilon_{\alpha\beta} z^\beta$, $z^\alpha = \varepsilon^{\alpha\beta} z_\beta$.

In terms of the variables z^α_a and derivatives $\partial^a_\alpha = \partial/\partial z^\alpha_a$, the external and internal generators take the form

$$\hat{J}_k = \frac{1}{2}(\sigma_k)^a{}_b z^\beta{}_a \partial_\beta{}^a, \quad \hat{I}_k = -\hat{J}^{\mathrm{R}}_k = \frac{1}{2}(\sigma_k)^a{}_b z_\alpha{}^b \partial^\alpha{}_a\,.$$

An explicit form of the states $|j\,m\,k\rangle$ is given by polynomials of $2J$th degree placed in the following tables:

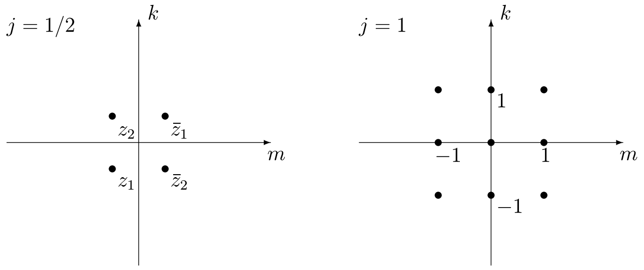
$j = 1/2$:

m k	$-1/2$	$1/2$
$-1/2$	z_1	z_2
$1/2$	$\bar{z}_2$	$\bar{z}_1$

$j = 1$: (A.10)

m k	-1	0	1
-1	$(z_1)^2$	$z_1 z_2$	$(z_2)^2$
0	$z_1 \bar{z}_2$	$z_1\bar{z}_1 - z_2\bar{z}_2$	$\bar{z}_1 z_2$
1	$(\bar{z}_2)^2$	$\bar{z}_1 \bar{z}_2$	$(\bar{z}_1)^2$

The polynomial of second degree $(1/2) z^\beta{}_a z_\beta{}^a = z_1\bar{z}_1 + z_2\bar{z}_2 = 1$, being absent from (A.10), is a group invariant. A scalar product, defined by integration with the invariant measure $\mathrm{d}\mu(z)$ on the group $SU(2)$ (22), allows one to verify the orthogonality of the states (A.10) and obtain the normalization coefficients. The functions (A.10) are Wigner functions (19) for $j = 1/2$ and 1.



For the sake of clarity, the figure shows the weight diagrams of representations with $j = 1/2$ and 1. For the left transformations, one mixes the states horizontally, and for the right transformations, vertically. In particular, at $j = 1$, considering only the left or only the right transformations (respectively, at fixed eigenvalues $\hat{I}_3$ and $\hat{J}_3$), we obtain two different sets of three equivalent irreps (in the general case, the number of equivalent irreps in the expansion will be obviously equal to the dimension of this irrep). However, if one examines both kinds of transformations at the same time, then all the nine states with m, $k = -1$, 0, 1 turn out to be related by the rising and lowering operators $\hat{J}_\pm$, $\hat{I}_\pm$. That is, the diagram of states of a rotator with a fixed total momentum j coincides with the weight diagram of the representation $T_{j,j}$ of the direct product $SU(2) \times SU(2)$.

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