Initial energy-momentum to final flow: A general framework for heavy-ion collisions

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The evolution of a relativistic heavy-ion collision is typically understood as a process that transmutes the initial geometry of the system into the final momentum distribution of observed hadrons, which can be described via a cumulant expansion of the initial distribution of energy density and is represented at leading order as the well-known eccentricity scaling of anisotropic flow. We extend this framework to include the contribution from initial momentum-space properties, as encoded in other components of the energy-momentum tensor. We confirm the validity of the framework in state-of-the-art hydrodynamic simulations of large and small systems. With this framework, it is possible to separate the effects of early time dynamics from those of final-state evolution, even in the case when the distribution of energy does not fully determine subsequent evolution, as for example, in small systems. Specifically, we answer the question of when and how azimuthal correlations from the initial state survive to the final state. In very small systems such as p-p, for example, initial momentum degrees of freedom dominate over energy. Thus, even if the system forms a quark-gluon plasma that is well described by hydrodynamics, the usual hydrodynamic picture of the transmutation of initial geometry to final momentum anisotropy is broken. Nevertheless, we show that the hydrodynamic response to the full energy-momentum tensor can be well understood in a similar manner as larger systems. Additionally, this framework elucidates the generic features of the system's evolution that are responsible for the impressive success of hydrodynamic simulations, but which may still hold even in cases where hydrodynamics is not applicable.

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I. INTRODUCTION

Relativistic heavy-ion collisions probe a diverse range of exotic physical phenomena. While nominally governed by the fundamental theories of the standard model of particle physics, and especially quantum chromodynamics (QCD), it is usually exceptionally difficult to make direct first-principles computations to describe these phenomena. Instead, experimental measurements can be used to inform our general understanding of the underlying physics. That is, theoretical descriptions of these complex dynamical processes inevitably involve models or effective theories, with uncertain aspects and parameters that are constrained by measured observables, thus giving valuable insight into underlying physics. Due to the complexity of the collision system, these observables may be simultaneously sensitive to a large number of these unknown parameters, and it can be difficult to isolate and constrain them individually. Because of this, it is of considerable utility to characterize the dependence of observed quantities on particular aspects of the underlying processes. In this way, such effects can be separated and studied.

A notable example is the framework that directly relates geometric properties of the collision system at early times to the anisotropic momentum distribution of particles at the end of the system evolution, expressed in relations such as the famous eccentricity scaling of elliptic flow,

$$V_2 = \kappa_2 \varepsilon_2. \tag{1}$$

Here V_2 is the elliptic flow—a particular (vector) measure of the anisotropy of the final distribution of particles in a collision event. To the extent that this approximate relation holds, it is proportional to the eccentricity—a particular (vector) measure of the initial spatial anisotropy ε_2, which contains all relevant information from the early time state of the system that determines the final elliptic flow. All relevant information about the subsequent evolution of the system is then encoded in the (scalar) response coefficient κ_2. Such a relation is now understood as the leading order in a systematically improvable expansion framework, with subleading contributions that can clearly be seen in simulations [1,2]. Once established, these types of relations can be very powerful. Their validity alone gives nontrivial information about the behavior of the system. Furthermore, this clean separation of effects makes it possible to extract targeted information from experimental data [3].

Until now, this framework only contains information to connect the initial geometry (via the initial distribution of energy in the system) to the final particle distribution. Here we propose to extend and generalize the framework to include the effects of other degrees of freedom in the early time system, which are also expected to contribute to final observables—namely other components of the energy-momentum tensor $T^{\mu\nu}$. While these additional components are typically believed to have subdominant contribution to measured observables in typical heavy-ion collisions, it is important to characterize their effects for several reasons:

(1) Since the initial stages of a collision are not well understood, it is not actually known for certain how large or important these contributions are. It thus becomes

interesting to quantify their effects, in order to put limits on their possible value.

(2) Even if the effects are typically small, they may have outsized effect in certain situations—e.g., specially chosen observables or observables in certain systems such as collisions between smaller ions.

(3) This may give additional insight into the framework itself and the minimal requirements for its validity. While it is inspired by (and usually tested with) hydrodynamic models, it may in fact be more general, similar to how hydrodynamics itself may be valid even far from equilibrium [4–6]. Thus, when we see signatures of geometric scaling—or its generalization here including momentum degrees of freedom—we can better interpret the implications.

In Sec. II, we review this framework in its current form and its derivation, being careful to specify the minimal assumptions required. In Sec. III, we present an ansatz for the inclusion of additional components of $T^{\mu\nu}$; namely, the momentum density and stress tensor. We provide some additional motivation for this ansatz in Sec. IV. In Sec. V we test this proposal using state-of-the-art hydrodynamic simulations. Finally, we summarize our conclusions and our outlook for the future.

Notation. . We use natural units where $\hbar = c = 1$.

II. REVIEW: CUMULANT EXPANSION OF INITIAL ENERGY DENSITY

The starting point is the assumption that knowledge of the energy-momentum tensor at some time τ_0 is sufficient to predict a particular observable of interest to some desired accuracy. Specifically, we posit that

(1) the final momentum-space distribution function f of particles in a collision event is a deterministic functional of the energy momentum tensor $T^{\mu\nu}$ and relevant conserved currents $\{j_i^\mu\}$ at some time τ_0,

$$f(p^\mu, \tau \to \infty) \equiv E\frac{dN}{d^3p} = \mathcal{F}\left[T^{\mu\nu}(\vec{x}, \tau_0), \left\{j_i^\mu(\vec{x}, \tau_0)\right\}\right]. \tag{2}$$

It is typical to use a quantity such as rapidity y or pseudorapidity η to characterize the momentum along the direction of the beam, while the transverse momentum can be characterized with a magnitude p_T and azimuthal angle ϕ_p. The dependence on angle can be nicely characterized by a Fourier series

$$E\frac{dN}{d^3p} = N(\eta, p_T)\sum_{n=-\infty}^{\infty} V_n(\eta, p_T)e^{-in\phi_p}, \tag{3}$$

where the anisotropic flow coefficients V_n are two-dimensional (2D) vectors, written here as a number in the complex plane. Each rotational mode then has its own relation

$$V_n = \mathcal{F}_n\left[T^{\mu\nu}(\vec{x}, \tau_0), \left\{j_i^\mu(\vec{x}, \tau_0)\right\}\right]. \tag{4}$$

More specifically, we assume that this is a good approximation for describing a particular measurement, which may be true even if the assumption does not hold more generally. We note here that this assumption is not general—the evolution of an arbitrary physical system can in principle depend on an infinite number of initial quantum correlations, classical functions, etc. The assumption that only the (expectation value of) the energy-momentum tensor and conserved currents are sufficient to determine the evolution of the system is highly restrictive. Nevertheless, while this assumption is motivated by the success of hydrodynamic simulations, the validity of the hydrodynamic equations may not strictly be required. Nor does the validity of hydrodynamics automatically mean that it is a good assumption. For example any hydrodynamic fluctuations that might be present are neglected. Likewise, it is useful only to the extent that knowledge of some single-body distribution function $f(p^\mu)$ is useful. Either multiparticle correlations can be neglected, or are specified with their own analogous relation.

The idea, then, is to characterize this "system response" $\mathcal{F}$ as precisely as possible. To do this, it is useful to first characterize the initial condition, in a way that important information can be separated from unimportant information.

The general principle that we use for this separation invokes the presence of a hierarchy of length scales. We posit that

(2) the structure of the initial conditions at small scales has less importance for the determination of final observables than structure at larger scales.

As with the first assumption, Eq. (2), this is motivated by hydrodynamics simulations—traditionally hydrodynamics is thought of as a description of long-wavelength modes in a system, such that short-wavelength modes are not relevant. Again, this does not necessarily mean that the validity of the framework is restricted to hydrodynamic systems, nor is its applicability guaranteed by the validity of hydrodynamics. Nevertheless, its success in describing simulations has already proven its utility and suggests that the assumptions are justified, at least in typical applications.

This separation into a hierarchy of scales is naturally achieved with a spatial Fourier transform of the initial-state fields. To make this explicit we first make various approximations, which can later be relaxed. To start, we assume that additional conserved currents (such as the baryon current) can be neglected. This is typically a good approximation at the highest collision energies, where chemical potentials are close to zero in a large space-time region of the collision system. We leave the study of conserved currents to future work.

Second, we neglect the longitudinal dependence of the initial state. For information on this approximation and how to relax it, see Refs. [7,8], whose methods can be similarly applied to the results of this work.

It is usually assumed that the most important component of the energy-momentum tensor is the energy density $T^{\tau\tau}$. The goal of this work is to relax this assumption, which will be done in the following section. For now, we finally have

$$V_n = \mathcal{F}[T^{\tau\tau}(\vec{x}_\perp, \tau_0)], \tag{5}$$

and we need to characterize only a single scalar field ρ, which is a function of two spatial dimensions:

$$\rho(\vec{x}_\perp) = T^{\tau\tau}(\vec{x}_\perp). \tag{6}$$

We take a two-dimensional Fourier transform to obtain a cumulant[1] generating function $W(\vec{k}_\perp)$

$$e^{W(\vec{k}_\perp)} \equiv \int d^2x_\perp\, \rho(\vec{x}_\perp)\, e^{-i\vec{x}_\perp\cdot\vec{k}_\perp}. \tag{7}$$

In this way, the behavior of the generating function at small $k = |\vec{k}_\perp|$ represents properties of the initial condition at large length scales, and vice versa. So we can naturally create a hierarchical set of quantities as coefficients of a Taylor series, expanded around $k = 0$

$$W(\vec{k}_\perp) = \sum_{n,m} \frac{1}{m!} W_{n,m} k^m e^{-in\phi_k}, \tag{8}$$

where we simultaneously decompose the coefficients into rotational modes via Fourier series in angle ϕ_k. (Recall that k is the magnitude of the Fourier variable $\vec{k}_\perp$, while ϕ_k is its azimuthal orientation.)

Therefore, the initial density is fully characterized by the discrete set of cumulants $W_{n,m}$, which are cleanly ordered in terms of the length scales they represent. In fact, cumulants with smaller m represents larger scales, and are more important for determining the final V_n according to ansatz 2.

In addition, they are separated into rotational modes labeled by n, which will aid in constructing estimators for V_n, which also have specific rotation properties. Specifically, if the system is rotated by some azimuthal angle, $\phi \to \phi + \delta$, then

$$\begin{aligned} V_n &\to V_n e^{in\delta}, \\ W_{n,m} &\to W_{n,m} e^{in\delta}. \end{aligned} \tag{9}$$

We note also that the cumulants constructed this way are translation invariant,[2] like the momentum-space observables V_n. These properties will make it simpler to construct estimators with the correct symmetries.

So finally we can formally write the system response as a function of cumulants (rather than a functional of the initial density), with the sensitivity of the response ordered by index m

$$E\frac{dN}{d^3p} = f(\{W_{n,m}\}), \tag{10}$$

$$\frac{\partial f}{\partial W_{n,m}} \gg \frac{\partial f}{\partial W_{n',m'}} \text{ for } m < m'. \tag{11}$$

Note that this is not the same as assuming that the initial density has the property $W_{n,m} \gg W_{n',m'}$. It is instead a statement about the *system response* to the initial conditions.

In general each cumulant is a dimensionful quantity, which must be compared with some scale in order to construct estimators for the (dimensionless) flow coefficients V_n. In principle a collision system can have a number of relevant scales. However, for a given collision system (and especially in a fixed centrality interval) many of the scales do not vary greatly. So in practice it suffices to make the simple and natural choice of the transverse size $\mathcal{R}$ of the system as defined by the lowest cumulant

$$\mathcal{R}^2 = W_{0,2} = \langle|\vec{x}_\perp - \langle\vec{x}_\perp\rangle_E|^2\rangle_E, \tag{12}$$

with the brackets representing an energy-weighted average

$$\langle\ldots\rangle_E \equiv \frac{\int d^2x_\perp \cdots T^{\tau\tau}(\vec{x}_\perp)}{\int d^2x_\perp T^{\tau\tau}(\vec{x}_\perp)}. \tag{13}$$

We can then define dimensionless quantities for the anisotropic ($n \neq 0$) cumulants

$$\varepsilon_{n,m} \equiv -\frac{W_{n,m}}{\mathcal{R}^m}. \tag{14}$$

Finally, we posit that

(3) the system response $f(\{\varepsilon_{n,m}\})$ can be expressed as a power series in the anisotropy coefficients $\varepsilon_{n,m}$.

Thus, in the end, we have a systematic double expansion, with terms ordered in importance according to the power series,[3] as well as the value of m in each factor $\varepsilon_{n,m}$. One cannot only determine a leading order estimator for each harmonic V_n, but also systematically improve it with higher-order corrections.

The most familiar estimator is for V_2. The lowest cumulant with $n \neq 0$ has $m = 2$,[4] and so the leading order estimator is a linear relation with a single power of the lowest cumulant with $n = 2$:

$$\mathcal{V}_2^{(\text{est})} = \kappa_{2,2}\,\varepsilon_{2,2} \tag{15}$$

$$= -\kappa_2 \frac{\langle(re^{i\phi} - \langle re^{i\phi}\rangle)^2\rangle_E}{\langle|re^{i\phi} - \langle re^{i\phi}\rangle|^2\rangle_E} \tag{16}$$

$$\xrightarrow{\text{centered}} -\kappa_2 \frac{\langle r^2 e^{2i\phi}\rangle_E}{\langle r^2\rangle_E}, \tag{17}$$

where the last line is written in a centered coordinate system with $\langle x\rangle_E = \langle y\rangle_E = 0$, and an unimportant numerical factor has been absorbed into the definition of κ_2. To leading order, all relevant information about the initial state is contained in $\varepsilon_{2,2}$, while all relevant information about the system response to this initial state is contained in the response coefficient κ.

The next correction is either the next-order linear term $\varepsilon_{2,4}$ or nonlinear terms with $m < 4$ such as $\varepsilon_{1,3}^2$ ($m = 3$) or even $\varepsilon_{2,2}|\varepsilon_{2,2}|^2$ (which involves only $m = 2$ cumulants but is order three in the power series). The general principles of the cumulant expansion do not dictate which type of correction is more important but must be verified for the system in question.

[1]The language of cumulants is borrowed from probability theory by analogy—if ρ were a probability density, then $W_{n,m}$ would be a cumulant in a more traditional sense.

[2]The exception is $W_{1,1} = \langle x + iy\rangle$ which represents the energy-weighted center of the system and contains all existing information about absolute position.

[3]The convergence properties of this series are not known. We assume that it is, at least, an asymptotic series.

[4]See the Appendix for more details about the cumulant expansion.

For V_3, the expansion is similar. There is no possible term involving only $m = 2$ cumulants (note that to have the correct rotation property, the sum of n values must add to 3, so we need at least one odd cumulant in each term[5]). So the leading estimator is again linear,

$$\mathcal{V}_3^{(\mathrm{est})} = \kappa_{3,3}\,\varepsilon_{3,3} \tag{18}$$

$$= -\kappa_3 \frac{\langle (re^{i\phi} - \langle re^{i\phi}\rangle)^3\rangle_E}{\langle |re^{i\phi} - \langle re^{i\phi}\rangle|^3\rangle_E} \tag{19}$$

$$\xrightarrow{\text{centered}} -\kappa_3 \frac{\langle r^3 e^{3i\phi}\rangle_E}{\langle r^2\rangle_E^{3/2}}, \tag{20}$$

where a centered coordinate system is again defined by $\langle r^{i\phi}\rangle_E = 0$. Corrections again include possible linear ($\varepsilon_{3,5}$) and nonlinear (e.g., $\varepsilon_{2,2}\varepsilon_{1,3}$) contributions.

Other predictors can be more complicated. For V_4, for example, one can have a linear estimator $\varepsilon_{4,4}$, which is order $m = 4$, or a quadratic estimator $\varepsilon_{2,2}^2$. Each possible term is lower in one part of the double expansion and higher in the other, and it is not obvious which is more important in a particular system. In general, both contributions can be important [1].

As a summary, by making three independent assertions (labeled 1–3 above), we were able to construct a systematic expansion for estimating the flow coefficients V_n, representing the rotational modes of the final particle spectrum. It is useful to note a few important properties of the resulting estimators:

(a) The symmetries are manifest:
  (i) *Translation invariance.* The flow coefficients V_n are translation invariant, as are the building block cumulants $W_{n,m}$,
  (ii) *Rotations.* Each cumulant, as well as each product of cumulants, have a well-defined rotation property [see Eq. (9)] that can be matched with the relevant flow harmonic.

(b) The terms are ordered in importance (according to hypotheses 2 and 3), so that the estimator can be systematically improved to arbitrary order.

III. ANSATZ FOR INCLUDING ADDITIONAL COMPONENTS OF $T^{\mu\nu}$

In addition to the initial geometric distribution of energy, the final-state momentum distribution of particles can also depend on momentum degrees of freedom in the initial state. Hydrodynamic evolution, for example, depends on the entire energy-momentum tensor as initial conditions for the equations of motion. We therefore would like to relax the assumption made in Eq. (5) of Sec. II, and include other components of the energy-momentum tensor $T^{\mu\nu}$.

Our proposal is to include the additional effects at the level of the scalar field ρ of Eq. (6). That is, we write

$$\rho(\vec{x}_\perp) = T^{\tau\tau}(\vec{x}_\perp) - \alpha\partial_i T^{\tau i}(\vec{x}_\perp) + \beta\partial_i\partial_j T^{ij}(\vec{x}_\perp) \tag{21}$$

and construct a cumulant expansion exactly as before, following Eqs. (7) through (14) without alteration and constructing estimators as an ordered power series in the generalized eccentricities $\varepsilon_{n,m}$

The new contributions each come with an associated (dimensionful) response coefficient that encodes information about the system response to these aspects of the initial conditions—α gauges the importance of momentum density relative to the energy density while β represents the relative importance of initial transverse stress. As with the coefficients κ that multiply each term in a given estimator, these new coefficients should depend only on the subsequent evolution of the system rather than any aspect of the initial state.

To illustrate the results of this we list the lowest-order estimators for elliptic and triangular flow. We first define the notation

$$U \equiv T^{\tau x} + iT^{\tau y}, \tag{22}$$

$$C \equiv T^{xx} - T^{yy} + 2iT^{xy}, \tag{23}$$

$$\langle\cdots\rangle_U = \frac{\int d^2x_\perp \cdots U(\vec{x}_\perp)}{\int d^2x_\perp T^{\tau\tau}(\vec{x}_\perp)}, \tag{24}$$

$$\langle\cdots\rangle_C = \frac{\int d^2x_\perp \cdots C(\vec{x}_\perp)}{\int d^2x_\perp T^{\tau\tau}(\vec{x}_\perp)} \tag{25}$$

so that U is a complex representation of momentum density while C is a complex representation of (the two degrees of freedom of) the traceless part of the 2D stress tensor. Note that, unlike the brackets of Eq. (13), these subscripted brackets do not represent weighted averages but just convenient ratios that show up in the final cumulants.

The relevant cumulants are then

$$W_{2,2} \propto \langle r^2 e^{i2\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 2\beta\langle 1\rangle_C - (\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U)^2, \tag{26}$$

$$\begin{aligned} W_{3,3} \propto{}& \langle r^3 e^{i3\phi}\rangle_E - 3\alpha\langle r^2 e^{i2\phi}\rangle_U - 6\beta\langle re^{i\phi}\rangle_C - (\langle re^{i\phi}\rangle_E \\ &- \alpha\langle 1\rangle_U)(3[\langle r^2 e^{i2\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 2\beta\langle 1\rangle_C] \\ &- 2[\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U]), \end{aligned} \tag{27}$$

and the final estimators are constructed as in the previous section, from Eq. (13). A more detailed derivation of cumulants, including numerical factors is shown in the Appendix.

As before, we can always simplify these expressions with a judicious choice of coordinate center. For example, one can choose a coordinate system where $W_{1,1} = \langle re^{i\phi}\rangle - \alpha\langle 1\rangle_U = 0$. Note that if the net momentum is not zero, then $\langle 1\rangle_U \neq 0$, and this is not the same center of coordinates where $\langle re^{i\phi}\rangle = 0$ that simplifies the expression for the denominator $\mathcal{R}$. This coordinate center is then also dependent on response coefficient α. Typically, however, the net transverse momentum is negligible, so the two choices of coordinate center coincide. Nevertheless, the recentered estimators can then be written as

$$\mathcal{V}_2^{(est)} = -\kappa_2 \frac{\langle r^2 e^{i2\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 2\beta\langle 1\rangle_C}{\langle |re^{i\phi} - \langle re^{i\phi}\rangle_E|^2\rangle_E}, \tag{28}$$

$$\mathcal{V}_3^{(est)} = -\kappa_3 \frac{\langle r^3 e^{i3\phi}\rangle_E - 3\alpha\langle r^2 e^{i2\phi}\rangle_U - 6\beta\langle re^{i\phi}\rangle_C}{\langle |re^{i\phi} - \langle re^{i\phi}\rangle_E|^3\rangle_E}. \tag{29}$$

[5]Recall that $W_{1,1}$ represents the center of the system and is not an appropriate estimator for V_n. The lowest translation-invariant cumulant with $n = 1$ is $W_{1,3}$.

In this form one can more easily see the role of the response coefficients κ_n, α, β, and the factors that represent properties of the initial state. It is interesting to compare the contribution from initial transverse stress to elliptic flow, to the quantity known as "momentum eccentricity," which has been used in the past as a proxy for elliptic flow itself at the end of the system's evolution [9] and recently as an initial-state estimator for elliptic flow [10]:

$$\varepsilon_p = \frac{\int d^2x_\perp (T^{xx} - T^{yy})}{\int d^2x_\perp (T^{xx} + T^{yy})}. \tag{30}$$

One can see that it almost coincides with the term $\langle 1 \rangle_C$ in the estimator proposed here.

With this ansatz we have retained all the important properties of the estimators listed at the end of the previous section (manifest symmetries and systematic improvability), as well as one additional desired property. While the energy density is nonzero anywhere there is matter, this is not true for momentum and stress, which can be large in principle but can also be negligible. Therefore we expect that if we uniformly rescale momentum density or stress to zero, its contribution to the final anisotropic flow should vanish. This is in contrast with the case of energy density, where a uniform rescaling should not necessarily result in a vanishing anisotropic flow. The proposed framework has this property, and in addition is well behaved in the case where net momentum vanishes.

We finally note that this proposal, in order to have all the generic properties that one expects for a good predictor, is quite restrictive. Specifically, while each term in an estimator in the original framework comes with an independent unknown response coefficient, only two additional response coefficients have been added here, and a single value for each of the two coefficients should correctly describe all azimuthal harmonics, and any higher-order corrections. This is a very restrictive condition, which can be used to test the validity of the framework. We return to this in Sec. V.

IV. A THOUGHT EXPERIMENT

This ansatz has all the expected and desired properties for the construction of estimators, including systematic improvements, and we see that there is evidence from numerical simulations that the leading-order estimators predict well the results of hydrodynamic simulations. However, it may not be obvious where it comes from. In this section we illustrate why Eq. (21) represents a natural quantity to consider by way of the following example:

Consider the case where at some time τ_0 there are no off-diagonal elements of the energy-momentum tensor, so that the initial energy density alone is sufficient to predict the final particle distribution in a given event. As reviewed in Sec. II, we can make accurate predictions of the final flow coefficients by performing a cumulant decomposition of the initial energy density $\rho = T^{\tau\tau}(\tau_0)$, and making ratios to construct estimators.

Now imagine that we do not know exactly what is τ_0, and so we want to construct estimators from the state of the system at some time that might be slightly different, $\tau = \tau_0 + \delta\tau$. The final particle distribution is the same, and so the estimators constructed at time τ should be close to those that we know work at time τ_0.

That is, we want cumulants of the energy density at time τ_0, but written in terms of quantities at time τ. Using local conservation of energy we can write

$$\partial_\tau T^{\tau\tau}(\tau) = -\partial_i T^{\tau i}(\tau) - \partial_\eta T^{\tau\eta} - \tau T^{\eta\eta} - \frac{1}{\tau} T^{\tau\tau} \tag{31}$$

$$\simeq \frac{1}{\delta\tau}[T^{\tau\tau}(\tau) - T^{\tau\tau}(\tau_0)]. \tag{32}$$

So our generating function can be written as

$$\rho(\vec{x}_\perp) = T^{\tau\tau}(\tau_0) \tag{33}$$

$$\simeq \left(1 + \frac{\delta\tau}{\tau}\right) T^{\tau\tau}(\tau) + \delta\tau \partial_i T^{\tau i}(\tau) + \tau\delta\tau T^{\eta\eta}(\tau). \tag{34}$$

Here we assume an approximate boost invariance to neglect the derivative of the longitudinal momentum density $\partial_\eta T^{\tau\eta}$.

Again, we know that a decomposition of this generating function gives a good estimation of the final flow. In particular, we note that the divergence of the transverse momentum density $\partial_i T^{\tau i}$ appears naturally as an additive contribution to the generating function.

Next, we can use the conservation of momentum to relate the momentum density to the stress tensor

$$\partial_\tau T^{\tau i}(\tau) = -\partial_j T^{ji}(\tau) - \partial_\eta T^{\eta i} - \frac{1}{\tau} T^{\tau i} \tag{35}$$

$$\simeq \frac{1}{\delta\tau}[T^{\tau i}(\tau) - T^{\tau i}(\tau_0)] \tag{36}$$

$$\simeq T^{\tau i}(\tau)/\delta\tau, \tag{37}$$

$$\Rightarrow \delta\tau \partial_i T^{\tau i}(\tau) \simeq -\partial_i \partial_j T^{ij} - \frac{1}{\tau} \partial_i T^{\tau i}. \tag{38}$$

So considering a higher order in δ_τ we can replace some or all of the divergence term with an expression involving the divergence of the transverse stress T^{ij}:

$$\rho(\vec{x}_\perp) \simeq \left(1 + \frac{\delta\tau}{\tau}\right) T^{\tau\tau}(\tau) + \tau\delta\tau T^{\eta\eta}(\tau) + \delta\tau\left(\gamma - \frac{\delta\tau}{\tau}\right) \partial_i T^{\tau i}(\tau) - (1-\gamma)\delta\tau^2 \partial_i \partial_j T^{ij}, \tag{39}$$

for some value of $0 \leqslant \gamma \leqslant 1$.

Except for the term involving longitudinal pressure $T^{\eta\eta}$, the generating function indeed takes the form of our ansatz (21).

From this exercise we can make a few observations. In general, it will not be the case that the system is dominated by energy density at a time infinitesimally close to the considered initial time (if ever). But, nevertheless, it serves as an illustration of why the three particular terms of our ansatz (21) can naturally appear in an additive combination—while the different components of $T^{\mu\nu}$ can rapidly vary individually, conservation of energy and momentum ensure that a sum of this form evolves more slowly and forms a more stable function from which to construct final-state estimators.

We also note that this argument suggests that incorporating the longitudinal pressure $T^{\eta\eta}$ could result in an even better estimator. However, we note that this quantity is nonzero even in previous studies that do not consider initial momentum degrees of freedom. Since the energy density alone predicts very accurately the final flow in this case, we can guess that the anisotropy of $T^{\eta\eta}$ is typically highly correlated with $T^{\tau\tau}$, and so including it may not significantly improve flow estimators. In any case, in this work we focus on including momentum degrees of freedom, and so we leave to future work the effect of initial longitudinal pressure.

V. NUMERICAL VALIDATION

To test the proposed flow estimators in various collision systems, we perform state-of-the-art hydrodynamic simulations of Pb-Pb, *p*-Pb, and *p*-*p* collisions, as performed at the CERN Large Hadron Collider (LHC). For initial conditions we use the IP-Glasma model [11,12], which provides a full (2D) energy-momentum tensor that can be used to initialize hydrodynamic evolution, simulated with MUSIC [13]. All hydrodynamic parameters were taken from a comprehensive Bayesian analysis [14]. The hydrodynamic stage is followed by the UrQMD hadronic afterburner [15,16], with oversampled events to accurately reconstruct the underlying particle distribution in every hydrodynamic event.

In each simulated event, one can compare the initial-state estimator $\mathcal{V}_n^{(est)}(\kappa_n, \alpha, \beta)$ proposed here with the actual flow harmonic calculated from produced hadrons V_n. A natural way to quantify the event-by-event success of the estimator is to compute the Pearson correlation coefficient between the two quantities [17]:

$$Q_n(\alpha, \beta) = \frac{\mathrm{Re}\langle V_n \mathcal{V}_n^{(\mathrm{est})*}\rangle}{\sqrt{\langle |V_n|^2\rangle \langle |\mathcal{V}_n^{(\mathrm{est})}|^2\rangle}}, \tag{40}$$

where in this section the angle brackets are defined as

$$\langle \cdots \rangle = \frac{1}{N_{\mathrm{events}}} \sum_{\mathrm{events}} \cdots . \tag{41}$$

A maximal value of $Q_n = 1$ implies a perfect estimator in every event $V_n \propto \mathcal{V}_n^{(est)}$, while a value of zero means they have no (linear) correlation, indicating a very poor estimator.

The estimators (28) depend on the usual response coefficients κ_n, associated with the response of each harmonic to the initial energy distribution, as well as two new response coefficients that represent the (relative) importance of initial momentum α and stress β to the hydrodynamic response. However, the coefficient κ_n is only an overall multiplicative factor, and so it does not affect the correlation coefficient [it cancels in the numerator and denominator of Eq. (40)].

To determine the best value of κ_n, we introduce the event-by-event error of the estimator of each harmonic

$$\xi_n(\kappa_n, \alpha, \beta) = V_n - \mathcal{V}_n^{(\mathrm{est})}(\kappa_n, \alpha, \beta). \tag{42}$$

We can choose κ_n to minimize the rms error of that respective harmonic over all the events in a centrality class. Using the notation

$$V_n^{(\mathrm{est})}(\kappa_n, \alpha, \beta) \equiv \kappa_n\, \varepsilon_n(\alpha, \beta), \tag{43}$$

the optimal value for a fixed α and β is then

$$\kappa_n(\alpha, \beta) = \frac{\mathrm{Re}\langle V_n \varepsilon_n\rangle}{\langle |\varepsilon_n|^2\rangle}. \tag{44}$$

The rms error in this case can be written as

$$\langle |\xi_n(\alpha, \beta)|^2\rangle = \langle |V_n|^2\rangle - \kappa_n^2(\alpha, \beta)\langle |\varepsilon_n(\alpha, \beta)|^2\rangle, \tag{45}$$

In Fig. 1, we show Q_2 and Q_3 in simulated Pb-Pb, *p*-Pb, and *p*-*p* systems for various choices of response coefficients α and β. The choice $\alpha = \beta = 0$ corresponds closely to the usual eccentricity, which neglects momentum degrees of freedom in the hydrodynamic initial condition. We can see that, in large systems, this eccentricity already gives an excellent estimator for the final flow coefficient, event-by-event, with $Q_n(0, 0) \simeq 1$ for most centralities. However, in smaller collision systems the usual eccentricity becomes less and less accurate as a predictor for final flow, such that $Q_n(0, 0)$ is generally below 0.4 in proton-proton collisions. This is consistent with the results of Ref. [10], which found that momentum degrees of freedom can become more important than spatial eccentricity in smaller systems.

We also show in Fig. 1 the Pearson correlation coefficients for the full estimators when the response coefficients are tuned to maximize each harmonic individually, which we notate as $Q_2(\alpha_2, \beta_2)$ and $Q_3(\alpha_3, \beta_3)$. We can see that unlike the usual eccentricities the proposed estimators are excellent in all cases, achieving $Q_2 > 0.9$ and $Q_3 > 0.7$ even in low-multiplicity *p*-*p* collisions.

We note, however, that our framework does not have independent response coefficients for each harmonic but instead shares a single set of new coefficients α, β. Therefore, an important nontrivial test of the proposed estimator is whether all harmonics are compatible with the same value of α and β. One way to test this is to compute Q_n with the optimal values of α and β for a different harmonic—i.e., $Q_2(\alpha_3, \beta_3)$ and $Q_3(\alpha_2, \beta_2)$. These are also shown in Fig. 1. Remarkably, the estimators still give an excellent description of the simulation results, giving very strong evidence of the validity of the proposed framework.

Finally, we define a harmonic-summed rms error

$$\langle |\xi(\alpha, \beta)|^2\rangle \equiv \langle |\xi_2(\alpha, \beta)|^2\rangle + \langle |\xi_3(\alpha, \beta)|^2\rangle, \tag{46}$$

which we can minimize by choosing optimal values $\bar{\alpha}$ and $\bar{\beta}$. We show $Q_2(\bar{\alpha}, \bar{\beta})$ and $Q_3(\bar{\alpha}, \bar{\beta})$ in Fig. 1. One can see that we still have an excellent estimator in all cases.

For further illustration we define a sort of likelihood function, which quantifies how good the estimator is for various values of α and β, using the energy-only estimator as a reference ($\alpha = \beta = 0$),

$$\mathcal{L}(\alpha, \beta) = \exp\left(-\frac{\langle |\xi(\alpha, \beta)|^2\rangle}{\langle |\xi(0, 0)|^2\rangle}\right). \tag{47}$$

The likelihood is maximized when the error is minimized, with an exponential decrease when the error becomes large.

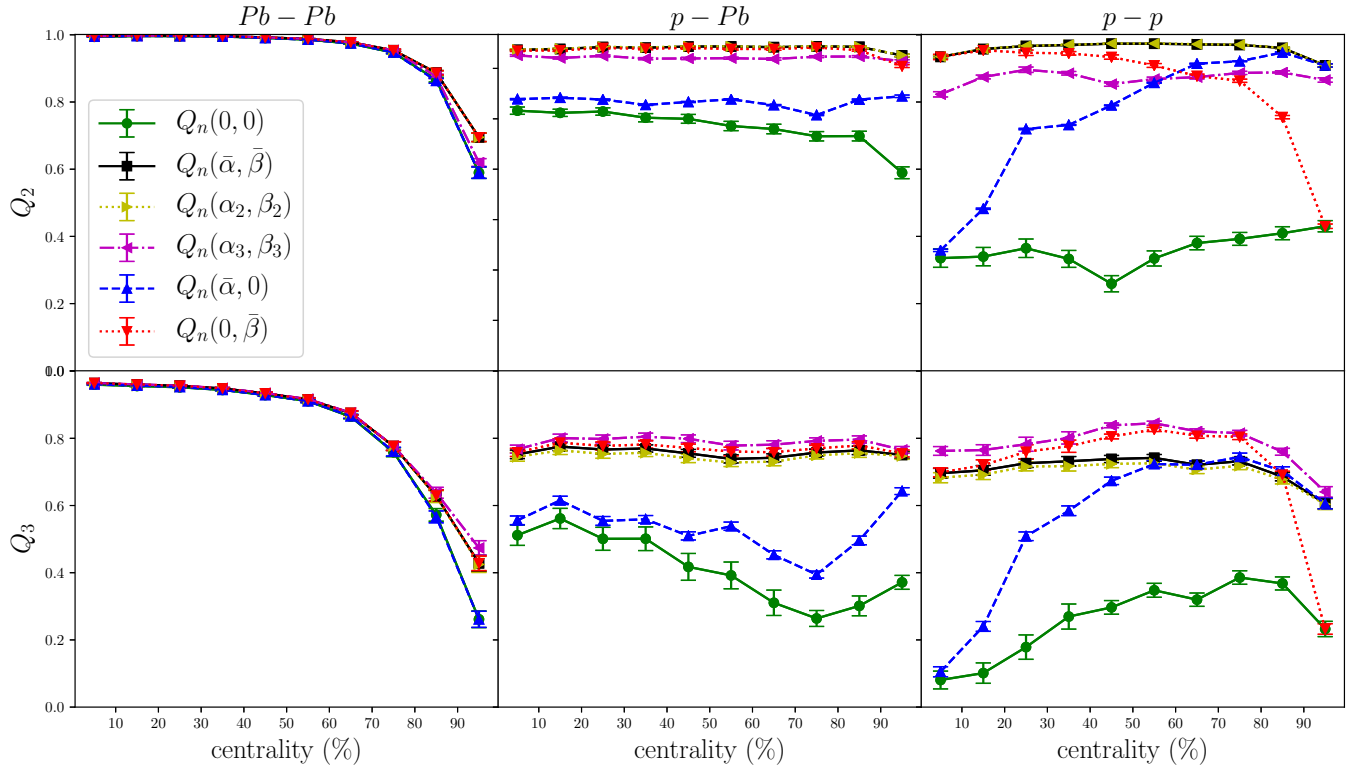

FIG. 1. Pearson correlation coefficient $Q_n(\alpha,\beta)$ (40) between final flow V_n and flow estimators $\mathcal{V}_n^{(est)}$ constructed from initial energy-momentum of hybrid hydrodynamic simulations with IP-Glasma initial conditions. Response coefficients (α_2,β_2) and (α_3,β_3) are the values that minimize the error (45) (and maximize Q_n) for each harmonic, respectively, while values $(\bar{\alpha},\bar{\beta})$ maximize the harmonic-combined error (43). $Q_n(0,0)$ represents the usual energy-only estimator, for reference, while $Q_n(\alpha,0)$ and $Q_n(0,\beta)$ neglect only the initial stress or momentum density, respectively, to quantify the importance of each individual contribution. Error bars are statistical, estimated from a jackknife resampling.

In Fig. 2, we show a contour plot of $\mathcal{L}(\alpha,\beta)$ in central collisions along with the maximal point $(\bar{\alpha},\bar{\beta})$ and the points that optimize each individual harmonic, (α_2,β_2), (α_3,β_3).

It is interesting to know which contribution is more important—the anisotropy due to initial stress, or that of the spatial distribution of the initial momentum density (at least in this particular model of IP-Glasma + hydro). We test this in Fig. 1 by plotting $Q_n(0,\bar{\beta})$ and $Q_n(\bar{\alpha},0)$. We see that neglecting the momentum density (setting $\alpha=0$) has a much smaller effect than the initial stress, which seems to be the dominant momentum-space hydrodynamic response except for low-multiplicity proton-proton collisions, where momentum density becomes dominant.

VI. CONCLUSIONS

We proposed a systematically improvable framework for estimating final flow coefficients V_n in relativistic heavy-ion collisions from the initial energy-momentum tensor. This extends previous work which only considered the initial energy (or entropy) density, and instead estimators are derived that consider the system response to these additional aspects of the early time state of the system. Similarly to how the system response to the initial distribution of energy is contained in coefficients for each harmonic κ_n, the additional information about the system response to these other aspects of the initial condition is contained in only two additional response coefficients, α and β.

Using state-of-the-art hybrid hydrodynamic simulations with IP-Glasma initial conditions we found that, while the usual eccentricities become poor estimators in smaller collision systems, our estimators remain excellent, even at lowest order in the smallest proton-proton collision systems. Remarkably, estimators for different harmonics defined simultaneously with the same values of response coefficients α and β still provide an excellent description of simulation results, providing powerful evidence of not only the utility of the estimators, but also of the correctness of the framework as a whole.

These results have several important implications. First is the practical matter that full simulations are computationally costly. Knowing that one is able to accurately predict final-state observables by calculating only the early time condition of the system can potentially save a significant amount of computing time. Beyond this, it provides insight into how different aspects of the initial condition manifest in final observables, perhaps allowing for clean separation of initial-state properties, as done with the usual eccentricity estimators in large collision systems.

Finally, the success of this framework provides some insight into the meaning of the success of hydrodynamic models in heavy-ion collisions. While inspired by hydrodynamics, the

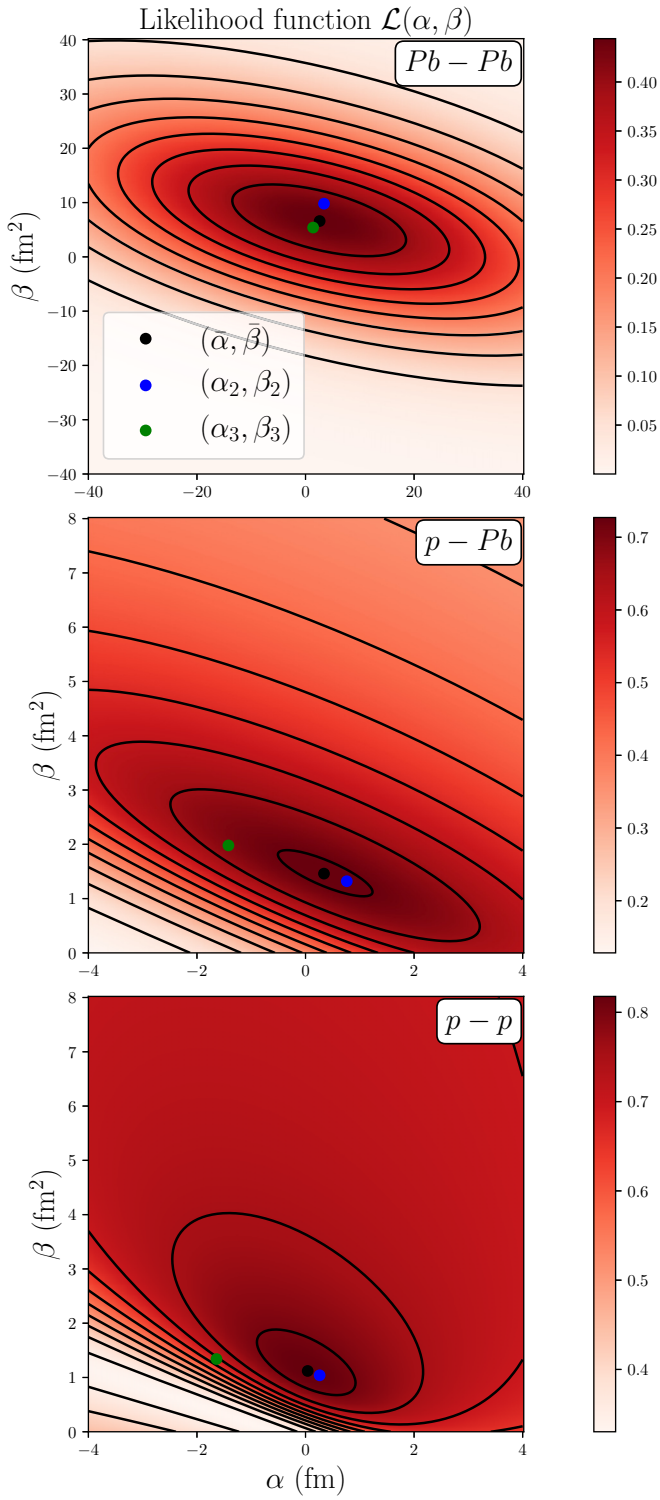

FIG. 2. The "likelihood" function, Eq. (47), as a function of response coefficients α and β for the 5%-highest multiplicity events in three collision systems—p-p, p-Pb, and Pb-Pb. The optimal single-harmonic values (α_2, β_2), (α_3, β_3) have similar likelihood to the harmonic-combined optimal value $(\bar{\alpha}, \bar{\beta})$, giving powerful evidence for the proposed framework.

clearly stated postulates underlying the framework may well be more general, and it will be interesting to see whether other physical models also satisfy the postulates, and whether the various aspects of the "collective" paradigm of high-energy nuclear collisions are more general than hydrodynamics itself.

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APPENDIX: LIST OF CUMULANTS

We take the 2D Fourier transform of the generating function Eq. (21),

$$\rho(\vec{k}_\perp) = \int d^2x_\perp [T^{\tau\tau} - i\alpha k_i T^{\tau i} - \beta k_i k_j T^{ij}] e^{-i\vec{k}_\perp \cdot \vec{x}_\perp}. \quad \text{(A1)}$$

and expand in a Maclaurin series around $|\vec{k}_\perp| = \vec{0}$. Assuming a hierarchy of importance of length scales, we truncate at some maximum m_{max}

$$\rho(\vec{k}_\perp) = \frac{1}{m!} \sum_{m=0}^{m_{max}} \rho_m(\phi_k) k^m. \quad \text{(A2)}$$

To separate rotation modes, we decompose its Maclaurin coefficients in a Fourier series with respect to azimuthal angle (ϕ_k):

$$\rho(\vec{k}_\perp) = \frac{1}{m!} \sum_{m=0}^{m_{max}} \sum_{n=-m}^{m} \rho_{n,m} k^m e^{-in\phi_k}. \quad \text{(A3)}$$

The general moment can thus be expressed as

$$\begin{aligned}\rho_{n,m} = \frac{2\pi(-i)^m m!}{2^m \left(\frac{m+n}{2}\right)! \left(\frac{m-n}{2}\right)!} \int d^2x_\perp \Big[& r^m e^{in\phi} T^{\tau\tau} \\ & - \alpha \left(\tfrac{m+n}{2}\right) r^{(m-1)} e^{i(n-1)\phi} U - \alpha \left(\tfrac{m-n}{2}\right) r^{(m-1)} e^{i(n+1)\phi} U^* \\ & - \beta \left(\tfrac{m+n}{2}\right)\left(\tfrac{m+n}{2}-1\right) r^{(m-2)} e^{i(n-2)\phi} C \\ & - \beta \left(\tfrac{m+n}{2}\right)\left(\tfrac{m-n}{2}-1\right) r^{(m-2)} e^{i(n+2)\phi} C^* \\ & - \beta \left(\tfrac{m+n}{2}\right)(m-n) r^{(m-2)} e^{in\phi} T \Big], \end{aligned} \quad \text{(A4)}$$

where T is the following trace written as

$$T = T^{xx} + T^{yy}. \quad \text{(A5)}$$

Note, however, that the trace does not contribute to the leading estimators, where $n = m$.

To obtain translation-invariant cumulants, we define a logarithmic function of $\rho(\vec{x}_\perp)$:

$$W(\vec{k}_\perp) \equiv \ln[\rho(\vec{k}_\perp)], \quad \text{(A6)}$$

and, similarly, we expand it in Maclaurin and Fourier series

$$W(\vec{k}_\perp) = \sum_{m=0}^{m_{max}} \sum_{n=-m}^{m} W_{n,m} k^m e^{-in\phi_k}. \quad \text{(A7)}$$

Then, some of the lowest cumulants can be written as

$$W_{1,1} = \frac{(-i)}{2}[\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U], \tag{A8}$$

$$W_{2,2} = \frac{(-i)^2}{4}[\langle r^2 e^{i2\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 2\beta\langle 1\rangle_C - (\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U)^2], \tag{A9}$$

$$W_{3,3} = \frac{(-i)^3}{8}[\langle r^3 e^{i3\phi}\rangle_E - 3\alpha\langle r^2 e^{i2\phi}\rangle_U - 6\beta\langle re^{i\phi}\rangle_C - (\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U) \times (3(\langle r^2 e^{i2\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 2\beta\langle 1\rangle_C) - 2(\langle re^{i\phi}\rangle_E - \alpha\langle 1\rangle_U)^2)]. \tag{A10}$$

Cumulants of higher order have a rapidly increasing number of terms. We include a few here, simplified by writing in terms of moments $\rho_{n,m}$:

$$W_{1,3} = \frac{3(-i)^3}{8}\big[\langle r^3 e^{i\phi}\rangle_E - 2\alpha\langle re^{i\phi}\rangle_U - 4\beta\langle re^{-i\phi}\rangle_C - 4\beta\langle re^{i\phi}\rangle_T - \rho_{2,2}\rho_{-1,1} - 2\rho_{0,2}\rho_{1,1} + 2\rho_{1,1}^2\rho_{-1,1}\big], \tag{A11}$$

$$W_{4,4} = \frac{(-i)^4}{16}\big[\langle r^4 e^{i4\phi}\rangle_E - 4\alpha\langle r^3 e^{i3\phi}\rangle_U - 12\beta\langle r^2 e^{-i2\phi}\rangle_C - 4\rho_{3,3}\rho_{1,1} - 3\rho_{2,2}^2 + 12\rho_{2,2}\rho_{1,1}^2 - 6\rho_{1,1}^4,\big], \tag{A12}$$

$$W_{5,5} = \frac{(-i)^5}{32}\big[\langle r^5 e^{i5\phi}\rangle_E - 5\alpha\langle r^4 e^{i4\phi}\rangle_U - 20\beta\langle r^3 e^{-i3\phi}\rangle_C + 20\rho_{3,3}\rho_{2,2}^2 - 60\rho_{2,2}\rho_{1,1}^3 + 5\big(6\rho_{2,2}^2 - \rho_{4,4}\big)\rho_{1,1} - 10\rho_{2,2}\rho_{3,3} + 24\rho_{1,1}^5\big]. \tag{A13}$$

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