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IFAC PapersOnLine 55-34 (2022) 31–36

Polytopic Robust Distributed Kalman Consensus Filter for Sensor Networks ⋆

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Abstract: Distributed estimation is a fundamental task performed by sensor networks, often achieved through the Kalman consensus filter. However, it requires exact models, which is seldom possible in practice since parametric uncertainties are usually unavoidable. We address in this paper the robust distributed filtering problem regarding linear discrete-time systems subject to polytopic uncertainties. We consider uncertainties in all parameter matrices of the target system and sensing models. First, we introduce a centralized filter, obtained as the solution to a min-max optimization problem whose cost function collectively weights all the underlying polytope vertices. Then, we derive a fully distributed version of this filter by applying the hybrid consensus on measurements and information approach. The estimators are recursive and do not rely on numerical solvers, appealing features for real-time applications. We validate our approach and assess its performance with a numerical example, comparing it with other robust distributed strategies.

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Keywords: Sensor networks, distributed estimation, consensus, robust estimation.

1. INTRODUCTION

The increasing complexity of modern engineering systems continuously motivates the emergence of distributed designs. They are common, for instance, in large industrial plants, smart grids, energy transmission systems, and vehicle platoons (Ding et al., 2014, 2019). Such systems greatly benefit from the flexibility and reliability of sensor networks, where each node relies on its local computing power to process measured data and transmit/receive information to/from neighboring sensors. These networks are the foundation of distributed filtering, in which the sensors observe a target system and collectively estimate its state (He et al., 2020; Modalavalasa et al., 2021).

One of the most successful paradigms in distributed filtering is the consensus protocol (Olfati-Saber et al., 2007), whereby, through averaging, the sensors reach an agreement on a certain quantity. The literature features a variety of consensus-based techniques. Some of the most common are the consensus on measurements (CM) (Olfati-Saber, 2007), consensus on information (CI) (Battistelli and Chisci, 2014; Kamal et al., 2013), and hybrid consensus on measurements and information (HCMCI) (Battistelli et al., 2015). The latter combines the complementary features of the CM and CI approaches, such that the sensors exchange their information and innovation pairs to improve the overall estimation quality.

Many distributed filtering strategies combine the Kalman filter with the consensus protocol. These solutions inherit the dependence of the Kalman filter on exact models of the target system and sensors. However, these models are often subject to parametric uncertainties, which can severely degrade estimation performance. The literature generally classifies the uncertainty models into two categories: stochastic and deterministic. Rastgar and Rahmani (2018, 2020) address the distributed estimation of systems with uncertainties of the first type. The second type is usually further divided into norm-bounded and polytopic uncertainties. The works by Duan et al. (2020) and Rocha and Terra (2020, 2022), for instance, deal with the norm-bounded case. The polytopic case, however, has not been explored as thoroughly as the former ones. In this model, the system parameters arbitrarily vary within a convex polyhedron centered at the nominal parameters. Shen et al. (2010) and de Souza et al. (2016) are among the few works that handle this kind of uncertainty in distributed filtering. Nevertheless, they are not fully distributed, since both require knowledge of the whole network to obtain the filter gains. Moreover, they rely on the solution of computationally expensive linear matrix inequalities (LMIs).

In this paper, we address the robust distributed filtering problem for sensor networks. We consider polytopic uncertainties in all parameter matrices of the discrete-time linear target system and sensor models. First, we propose a robust centralized Kalman filter, which results from the solution to a regularized least-squares estimation problem with polytopic uncertainties (Rocha et al., 2022). It consists of a min-max optimization subject to

⋆ This work was supported by the São Paulo Research Foundation (FAPESP) (grant no. 2021/08103-0, 2017/16346-4, 2014/50851-0) and by the National Council for Scientific and Technological Development (CNPq) (grant no. 465755/2014-3).

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Peer review under responsibility of International Federation of Automatic Control.
10.1016/j.ifacol.2022.11.304

linear equality constraints obtained from each vertex of the polytope. Through the penalty function method (Luenberger and Ye, 2008), we include the constraints into the cost function, such that the vertices are collectively weighted by a penalty parameter, also used to adjust the estimation performance. The analytical solution provides a recursive filter, presented as a correction-prediction algorithm. Then, we derive a fully distributed version of the centralized estimator by applying the HCMCI protocol. For a sufficiently large number of consensus iterations, the distributed filter approaches its centralized counterpart. The proposed recursive filters do not depend on numerical solvers and LMIs, which is unprecedented in the literature. This reduced computational burden is a desirable feature in online applications.

Notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space, and $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices. I_n is the $n \times n$ identity matrix. The superscript T denotes transposition. For a real symmetric matrix P, $P > 0$ ($P \geq 0$) means that P is positive (semi)definite, and $P^{1/2}$ is the square-root factor of P. The **col** operator stacks its operands in a block-column matrix, whereas **diag** yields a block-diagonal matrix. The weighted squared Euclidean norm of x is $\|x\|_P^2 = x^T P x$. Whenever convenient, we adopt the notation $X^T P(\bullet) = X^T P X$.

2. PROBLEM FORMULATION

2.1 System Model

Consider a sensor network featuring S sensors. They are connected according to the undirected graph $\mathbb{G} = (\mathbb{S}, \mathbb{E})$, where $\mathbb{S} = \{1, 2, \ldots, S\}$ is the node set and $\mathbb{E} \subseteq \mathbb{S} \times \mathbb{S}$ is the edge set. We denote by $\mathcal{N}_i = \{j \in \mathbb{S} \,|\, (i,j) \in \mathbb{E}\}$ the neighborhood of a sensor i, with cardinality N_i. We further define the inclusive neighborhood set $\mathcal{N}_i^+ = \mathcal{N}_i \cup \{i\}$. We assume that $\mathbb{G}$ has a fixed topology and is connected, i.e., there is a path between every pair of nodes.

Consider the discrete-time state-space description of a linear target system:

$$x_{k+1} = (F_k + \delta F_k)x_k + (H_k + \delta H_k)w_k, \tag{1}$$

which is observed by the set of S sensors $\mathbb{S} = \{1, \ldots, S\}$, each described by the model

$$y_k^i = (C_k^i + \delta C_k^i)x_k + (D_k^i + \delta D_k^i)v_k^i, \tag{2}$$

for $k = 0, \ldots, N$, with state vector $x_k \in \mathbb{R}^n$ and process noise $w_k \in \mathbb{R}^p$. For each sensor $i \in \mathbb{S}$, $y_k^i \in \mathbb{R}^r$ is a measurement vector and $v_k^i \in \mathbb{R}^q$ is the measurement noise. $F_k \in \mathbb{R}^{n\times n}$, $H_k \in \mathbb{R}^{n\times p}$, $C_k^i \in \mathbb{R}^{r\times n}$ and $D_k^i \in \mathbb{R}^{r\times q}$ are nominal parameter matrices, whereas $\delta F_k \in \mathbb{R}^{n\times n}$, $\delta H_k \in \mathbb{R}^{n\times p}$, $\delta C_k^i \in \mathbb{R}^{r\times n}$ and $\delta D_k^i \in \mathbb{R}^{r\times q}$ are unknown parametric uncertainties bounded to a convex polyhedral domain described by V vertices,

$$\mathbb{V}_k := \left\{ \left(\delta F_k,\, \delta H_k,\, \delta C_k^i,\, \delta D_k^i\right) = \sum_{\nu=1}^{V} \alpha_{\nu,k}\left(F_{\nu,k},\, H_{\nu,k},\, C_{\nu,k}^i,\, D_{\nu,k}^i\right) \right\}, \tag{3}$$

where $F_{\nu,k} \in \mathbb{R}^{n\times n}$, $H_{\nu,k} \in \mathbb{R}^{n\times p}$, $C_{\nu,k}^i \in \mathbb{R}^{r\times n}$ and $D_{\nu,k}^i \in \mathbb{R}^{r\times q}$ are known matrices, for all $i \in \mathbb{S}$, and $\alpha_k := [\alpha_{1,k} \cdots \alpha_{V,k}]^T$ belongs to the unit simplex

$$\Lambda_V := \left\{ \alpha \in \mathbb{R}^V \,:\, \sum_{\nu=1}^{V} \alpha_\nu = 1,\, \alpha_\nu \geq 0 \right\}. \tag{4}$$

2.2 Polytopic Robust Centralized Estimation Problem

In the centralized problem, we consider a central estimator with access to measurements from all sensors. The goal is to leverage all the information from the network to obtain the best estimates $\hat{x}_{k|k}^*$ of x_k and $\hat{x}_{k+1|k}^*$ of x_{k+1}.

As Bryson and Ho (1975) point out, stochastic estimation problems can also be interpreted as deterministic least-squares problems. Therefore, the random variables w_k and $\{v_k^i\}_{i=1}^S$ are treated as model fitting errors to be minimized. At each step k, we assume the availability of a prior central state estimate $\hat{x}_{k|k-1}^*$ and a set of sensor measurements $\{y_k^i\}_{i=1}^S$. Thus, based on Sayed (2001), Ishihara et al. (2015), and Rocha and Terra (2020), we formulate a min-max optimization problem where we minimize a cost function under the maximum influence of the polytopic uncertainties $\delta_k := \left\{\delta F_k, \delta H_k, \{\delta C_k^i\}_{i=1}^S, \{\delta D_k^i\}_{i=1}^S\right\}$, i.e., for $k = 0, \ldots, N$,

$$\min_{w_k, \{v_k^i\}_{i=1}^S, x_k, x_{k+1}} \max_{\delta_k} \|x_k - \hat{x}_{k|k-1}^*\|^2_{(P_{k|k-1}^*)^{-1}} + \|w_k\|^2_{Q_k^{-1}} + \sum_{i=1}^{S} \|v_k^i\|^2_{(R_k^i)^{-1}} \tag{5}$$

subject to the set of constraints

$$\begin{bmatrix} I_n \\ \vdots \\ I_n \end{bmatrix} x_{k+1} = \begin{bmatrix} F_k + \delta F_{1,k} \\ \vdots \\ F_k + \delta F_{V,k} \end{bmatrix} x_k + \begin{bmatrix} H_k + \delta H_{1,k} \\ \vdots \\ H_k + \delta H_{V,k} \end{bmatrix} w_k, \tag{6a}$$

$$\begin{bmatrix} I_r \\ \vdots \\ I_r \end{bmatrix} y_k^i = \begin{bmatrix} C_k^i + \delta C_{1,k}^i \\ \vdots \\ C_k^i + \delta C_{V,k}^i \end{bmatrix} x_k + \begin{bmatrix} D_k^i + \delta D_{1,k}^i \\ \vdots \\ D_k^i + \delta D_{V,k}^i \end{bmatrix} v_k^i, \forall i \in \mathbb{S}, \tag{6b}$$

where, for each vertex $\nu = 1, \ldots, V$ and sensor $i \in \mathbb{S}$,

$$\left(\delta F_{\nu,k},\, \delta H_{\nu,k},\, \delta C_{\nu,k}^i,\, \delta D_{\nu,k}^i\right) = \alpha_{\nu,k} V\left(F_{\nu,k},\, H_{\nu,k},\, C_{\nu,k}^i,\, D_{\nu,k}^i\right). \tag{7}$$

Additionally, $P_{k|k-1}^* > 0$, $Q_k > 0$, and $R_k^i > 0$ are given weighting matrices respectively on the prior estimation error $x_k - \hat{x}_{k|k-1}^*$, w_k, and each v_k^i, $i \in \mathbb{S}$. The solution to problem (5)–(6) yields the posterior and predicted prior central state estimates, $\hat{x}_{k|k}^*$ and $\hat{x}_{k+1|k}^*$, which compose the Polytopic Robust Centralized Kalman Filter.

2.3 Polytopic Robust Distributed Estimation Problem

The goal of each sensor node $i \in \mathbb{S}$, at each step k, is to obtain the best estimates $\hat{x}_{k|k}^i$ of x_k and $\hat{x}_{k+1|k}^i$ of x_{k+1}, referred to as posterior and predicted prior state estimates, despite the model uncertainties. Moreover, these estimates should be computed in a distributed rather than centralized fashion. Thus, each sensor has access to its own data and to data provided by its neighbors. We achieve this objective by taking advantage of the distributed feature of the average consensus protocol (Olfati-Saber et al., 2007) to approximate the results of a robust centralized filter. Similar strategies are applied, e.g., in Kamal et al. (2013), Battistelli et al. (2015), and Rocha and Terra (2022).

3. POLYTOPIC ROBUST CENTRALIZED KALMAN FILTER

In order to solve the constrained min-max problem (5)–(6), we first transform it into an unconstrained problem. This is achieved via the penalty function method (Luenberger and Ye, 2008), such that the constraints are included in the cost function as quadratic terms multiplied by a penalty parameter $\mu > 0$. Hence, we rewrite problem (5)–(6) as

$$\min_{z_k} \max_{\delta A_k, \delta b_k} \|z_k\|^2_{\mathcal{Q}_k} + \|(A_k + \delta A_k)z_k - (b_k + \delta b_k)\|^2_{\mu I}, \quad (8)$$

for each $k = 0, 1, \ldots, N$, where

$$z_k := \begin{bmatrix} w_k \\ \boldsymbol{v}_k \\ x_k - \hat{x}^*_{k|k-1} \\ x_{k+1} \end{bmatrix}, \ \mathcal{Q}_k := \begin{bmatrix} Q_k^{-1} & 0 & 0 & 0 \\ 0 & \mathcal{R}_k^{-1} & 0 & 0 \\ 0 & 0 & (P^*_{k|k-1})^{-1} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

$$A_k := \begin{bmatrix} \boldsymbol{H}_k & 0 & \boldsymbol{F}_k & -\boldsymbol{I}_n \\ 0 & \mathscr{D}_k & \mathscr{C}_k & 0 \end{bmatrix}, \ b_k := \begin{bmatrix} -\boldsymbol{F}_k \hat{x}^*_{k|k-1} \\ \mathscr{Y}_k - \mathscr{C}_k \hat{x}^*_{k|k-1} \end{bmatrix},$$

in which we define the augmented vectors and matrices

$$\boldsymbol{v}_k := \mathbf{col}\left(v_k^1, \ldots, v_k^S\right), \ \mathcal{R}_k := \mathbf{diag}\left(R_k^1, \ldots, R_k^S\right),$$
$$\boldsymbol{F}_k := \mathbf{1}_V \otimes F_k, \quad \mathscr{Y}_k := \mathbf{col}\left(\mathbf{1}_V \otimes y_k^1, \ldots, \mathbf{1}_V \otimes y_k^S\right),$$
$$\boldsymbol{H}_k := \mathbf{1}_V \otimes H_k, \quad \mathscr{C}_k := \mathbf{col}\left(\mathbf{1}_V \otimes C_k^1, \ldots, \mathbf{1}_V \otimes C_k^S\right),$$
$$\boldsymbol{I}_n := \mathbf{1}_V \otimes I_n, \quad \mathscr{D}_k := \mathbf{diag}\left(\mathbf{1}_V \otimes D_k^1, \ldots, \mathbf{1}_V \otimes D_k^S\right),$$

where $\mathbf{1}_V := \left[1 \cdots 1\right]^T \in \mathbb{R}^V$ and $\otimes$ denotes the Kronecker product. In addition, the parametric uncertainties are

$$[\delta A_k \ \delta b_k] := \Gamma_k \left[\tilde{A}_k \ \tilde{b}_k\right], \quad (9)$$

$$\Gamma_k := \begin{bmatrix} \tilde{\boldsymbol{\alpha}}_k \otimes I_n & 0 \\ 0 & I_S \otimes (\tilde{\boldsymbol{\alpha}}_k \otimes I_r) \end{bmatrix}, \ \tilde{\boldsymbol{\alpha}}_k := \mathbf{diag}\left(\alpha_k\right),$$

$$\tilde{A}_k := V \begin{bmatrix} \tilde{\boldsymbol{H}}_k & 0 & \tilde{\boldsymbol{F}}_k & 0 \\ 0 & \widetilde{\mathscr{D}}_k & \widetilde{\mathscr{C}}_k & 0 \end{bmatrix}, \ \tilde{b}_k := -V \begin{bmatrix} \tilde{\boldsymbol{F}}_k \\ \widetilde{\mathscr{C}}_k \end{bmatrix} \hat{x}^*_{k|k-1},$$

for $\alpha_k \in \Lambda_V$, as in (4), such that $\|\Gamma_k\| \leq 1$. In addition, we define the following augmented matrices, which gather the corresponding polytope vertices:

$$\tilde{\boldsymbol{F}}_k := \mathbf{col}\left(F_{1,k}, \ldots, F_{V,k}\right), \ \tilde{\boldsymbol{H}}_k := \mathbf{col}\left(H_{1,k}, \ldots, H_{V,k}\right),$$
$$\tilde{\boldsymbol{C}}_k^i := \mathbf{col}\left(C_{1,k}^i, \ldots, C_{V,k}^i\right), \ \tilde{\boldsymbol{D}}_k^i := \mathbf{col}\left(D_{1,k}^i, \ldots, D_{V,k}^i\right),$$
$$\widetilde{\mathscr{C}}_k := \mathbf{col}\left(\tilde{\boldsymbol{C}}_k^1, \ldots, \tilde{\boldsymbol{C}}_k^S\right), \quad \widetilde{\mathscr{D}}_k := \mathbf{diag}\left(\tilde{\boldsymbol{D}}_k^1, \ldots, \tilde{\boldsymbol{D}}_k^S\right).$$

Problem (8)–(9) is a regularized least-squares problem with polytopic uncertainties (Rocha et al., 2022). The next lemma presents the solution to this class of problems.

Lemma 1. (Rocha et al., 2022) Consider the following optimization problem with uncertain polytopic data

$$\min_{z} \max_{\delta A, \delta b} \|z\|^2_{\mathcal{Q}} + \|(A + \delta A)z - (b + \delta b)\|^2_{\mu I}, \quad (10)$$

where $\mathcal{Q} = \begin{bmatrix} Q & 0 \\ 0 & 0 \end{bmatrix} \geq 0$, $Q > 0$, $\mu > 0$ is a fixed penalty parameter, and

$$[\delta A \ \delta b] = \Gamma \left[\tilde{A} \ \tilde{b}\right], \ \|\Gamma\| \leq 1. \quad (11)$$

The solution $\hat{z}$ to problem (10)–(11) is then given by

$$\hat{z} = \left(\mathcal{Q} + \hat{\omega} A^T A + \hat{\lambda} \tilde{A}^T \tilde{A}\right)^{-1} \left(\hat{\omega} A^T b + \hat{\lambda} \tilde{A}^T \tilde{b}\right), \quad (12)$$

where $\hat{\omega} = \dfrac{\hat{\lambda}\mu}{\hat{\lambda} - \mu}$, and $\hat{\lambda}$ is a scalar parameter such that

$$\hat{\lambda} := \arg\min_{\lambda > \mu} \ \Upsilon(\lambda), \quad (13)$$

with auxiliary objective function

$$\Upsilon(\lambda) = \|z(\lambda)\|^2_{\mathcal{Q}} + \|Az(\lambda) - b\|^2_{\omega(\lambda) I} + \lambda \|\tilde{A} z(\lambda) - \tilde{b}\|^2, \quad (14)$$

in which $\omega(\lambda) := \dfrac{\lambda\mu}{\lambda - \mu}$, and

$$z(\lambda) := \left(\mathcal{Q} + \omega(\lambda) A^T A + \lambda \tilde{A}^T \tilde{A}\right)^{-1} \left(\omega(\lambda) A^T b + \lambda \tilde{A}^T \tilde{b}\right).$$

If $\left[\bar{I}^T \ A^T \ \tilde{A}^T\right]^T$ has full column rank, with $\bar{I} = [I \ 0]$, for I with the same dimensions as Q, then $\hat{z}$ in (12) is unique.

Remark 1. The solution in Lemma 1 depends on the optimal $\hat{\lambda}$ parameter, which is the result of problem (13)–(14). Nevertheless, based on Sayed (2001), we can adopt the practical approximation $\hat{\lambda} = (1 + \xi)\,\mu$, for some $\xi > 0$.

Remark 2. Rocha and Terra (2021) point out that in a robust estimation context, the penalty parameter μ works as a robustness measure of the estimator. When the models are subject to major uncertainties, small values of μ increase the robustness. In contrast, larger values of μ can be used for mild uncertainties.

Now, we use the result in Lemma 1 to find a solution to the proposed robust centralized estimation problem (8)–(9). Ultimately, we obtain a set of correction-prediction algebraic expressions that compose the Polytopic Robust Centralized Kalman Filter, as Theorem 1 states.

Theorem 1. Consider problem (8)–(9), with given initial conditions $\hat{x}^*_{0|-1}$, $P^*_{0|-1} > 0$, $Q_k > 0$, $R_k^i > 0$, $\forall i \in \mathbb{S}$, and fixed parameters $\mu > 0$ and $\xi > 0$. For each $k = 0, \ldots, N$, its solution recursively yields the posterior and predicted prior central state estimates of system (1), $\hat{x}^*_{k|k}$ and $\hat{x}^*_{k+1|k}$, along with their corresponding error weighting matrices, $P^*_{k|k}$ and $P^*_{k+1|k}$, according to Algorithm 1.

Proof. Problem (8)–(9) is a regularized least-squares problem with polytopic uncertainties, as in (10)–(11). Therefore, we use Lemma 1 to obtain its solution. We also use the approximation for $\hat{\lambda}$ mentioned in Remark 1. We omit the details that lead to Algorithm 1 for brevity.

4. POLYTOPIC ROBUST DISTRIBUTED KALMAN FILTER

As mentioned previously, the main idea is to use the PRCKF (Algorithm 1) as a benchmark for a distributed formulation. This section shows how we use the average consensus protocol to derive a distributed approximation of the PRCKF.

4.1 Average Consensus Protocol

The average consensus protocol is an algorithm used to compute the arithmetic mean of a set of numerical entities (scalars, vectors or matrices) in a distributed manner (Olfati-Saber et al., 2007). Consider a connected undirected network of S nodes. Each node i has a state a_i and exchanges data with a set $\mathcal{N}_i$ of neighbors. The goal is to compute the average value of these states in a distributed fashion, which is achieved through Algorithm 2.

Suppose we choose the consensus weights $\pi_{i,j}$ such that $\pi_{i,j} \geq 0$ and $\sum_{j \in \mathcal{N}_i^+} \pi_{i,j} = 1$, for every node i. Thus,

Algorithm 1 Polytopic Robust Centralized Kalman Filter (PRCKF)

Initialization: Set $\hat{x}^*_{0|-1}$, $P^*_{0|-1} > 0$, $Q_k > 0$, $R^i_k > 0$, $\mu > 0$, and $\xi > 0$.
for $k = 0, 1, \ldots, N$ **do**:

1: Obtain measurements y^i_k from all sensors $i \in \mathbb{S}$.

2: Compute φ using the approximation for $\hat{\lambda}$:

$$\varphi = \hat{\lambda} V^2 = (1+\xi)\,\mu V^2$$

3: Compute the modified sensor model matrices, $\forall i \in \mathbb{S}$:

$$\Phi^i_2 = \xi V \varphi^{-1} I_r$$
$$\bar{R}^i_k = \varphi^{-1} I_{rV} + \tilde{\boldsymbol{D}}^i_k R^i_k (\tilde{\boldsymbol{D}}^i_k)^T$$
$$\widehat{R}^i_k = \Phi^i_2 + D^i_k \big[(R^i_k)^{-1} + \varphi (\tilde{\boldsymbol{D}}^i_k)^T \tilde{\boldsymbol{D}}^i_k\big]^{-1} (D^i_k)^T$$
$$\widehat{C}^i_k = C^i_k - D^i_k R^i_k (\tilde{\boldsymbol{D}}^i_k)^T (\bar{R}^i_k)^{-1} \tilde{\boldsymbol{C}}^i_k$$

4: Compute the modified system model matrices:

$$\Phi_1 = \xi V \varphi^{-1} I_n$$
$$\bar{Q}_k = \varphi^{-1} I_{nV} + \tilde{\boldsymbol{H}}_k Q_k \tilde{\boldsymbol{H}}^T_k$$
$$\widehat{Q}_k = \Phi_1 + H_k (Q^{-1}_k + \varphi \tilde{\boldsymbol{H}}^T_k \tilde{\boldsymbol{H}}_k)^{-1} H^T_k$$
$$\widehat{F}_k = F_k - H_k Q_k \tilde{\boldsymbol{H}}^T_k \bar{Q}^{-1}_k \tilde{\boldsymbol{F}}_k$$

5: Correction step:

5.1: Compute the posterior error weighting matrix:

$$P^*_{k|k} = \Bigg[(P^*_{k|k-1})^{-1} + \sum_{i=1}^{S} (\widehat{C}^i_k)^T (\widehat{R}^i_k)^{-1} \widehat{C}^i_k + \sum_{i=1}^{S} (\tilde{\boldsymbol{C}}^i_k)^T (\bar{R}^i_k)^{-1} \tilde{\boldsymbol{C}}^i_k + \tilde{\boldsymbol{F}}^T_k \bar{Q}^{-1}_k \tilde{\boldsymbol{F}}_k\Bigg]^{-1}$$

5.2: Compute the posterior central state estimate:

$$\hat{x}^*_{k|k} = P^*_{k|k} \left[(P^*_{k|k-1})^{-1} \hat{x}^*_{k|k-1} + \sum_{i=1}^{S} (\widehat{C}^i_k)^T (\widehat{R}^i_k)^{-1} y^i_k\right]$$

6: Prediction step:

6.1: Obtain the prior error weighting matrix:

$$P^*_{k+1|k} = \widehat{F}_k P^*_{k|k} \widehat{F}^T_k + \widehat{Q}_k$$

6.2: Obtain the predicted prior central state estimate:

$$\hat{x}^*_{k+1|k} = \widehat{F}_k \hat{x}^*_{k|k}$$

end for

Algorithm 2 Average consensus (each node i) (Olfati-Saber et al., 2007)

Initialization: Set initial consensus state $a_i(0) = a_i$.
for $\ell = 0, 1, \ldots, L-1$ **do**:

1: Send current state $a_i(\ell)$ to neighbors $j \in \mathcal{N}_i$.

2: Receive current state $a_j(\ell)$ from neighbors $j \in \mathcal{N}_i$.

3: Update consensus state:

$$a_i(\ell+1) = \sum_{j \in \mathcal{N}^+_i} \pi_{i,j} a_j(\ell)$$

end for

we obtain a convex combination of the node states. Olfati-Saber et al. (2007) show that for a sufficiently large number of consensus iterations L, after applying Algorithm 2, the state of all nodes converge to the average of their initial states, i.e., $a_i(L) = 1/S \sum_{j=1}^{S} a_j(0)$. One usual choice of weights satisfying these conditions are the Metropolis weights (Xiao et al., 2005).

4.2 Distributed Filter

At this point, we are ready to employ the average consensus protocol to derive a polytopic robust distributed filter that approximates the PRCKF (Algorithm 1). To achieve this approximation, we specifically adopt the hybrid consensus on measurements and information (HCMCI) strategy developed in Battistelli et al. (2015).

We initialize each sensor $i \in \mathbb{S}$ with the same prior state estimate $\hat{x}^i_{0|-1}$ and prior error weighting matrix $P^i_{0|-1} > 0$. By communicating with its neighbors $j \in \mathcal{N}_i$, each sensor can receive their specific data.

The proposed Polytopic Robust Distributed Kalman Consensus Filter (PRDKCF) is presented in Algorithm 3. Based on the HCMCI approach, in steps 4 and 5 we simultaneously apply the average consensus protocol to the so-called prior information $(\bar{\Omega}^i_k, \bar{\omega}^i_k)$ and innovation $(\delta\bar{\Omega}^i_k, \delta\bar{\omega}^i_k)$ pairs of each sensor $i \in \mathbb{S}$. This way, by performing enough consensus iterations L, at convergence, we expect that, for all sensors $i \in \mathbb{S}$,

$$\bar{\Omega}^i_k(L) \to (P^*_{k|k-1})^{-1}, \quad \bar{\omega}^i_k(L) \to (P^*_{k|k-1})^{-1} \hat{x}^*_{k|k-1},$$
$$\delta\bar{\Omega}^i_k(L) \to \frac{1}{S} \sum_{i=1}^{S} \Big[(\widehat{C}^i_k)^T (\widehat{R}^i_k)^{-1} \widehat{C}^i_k + (\tilde{\boldsymbol{C}}^i_k)^T (\bar{R}^i_k)^{-1} \tilde{\boldsymbol{C}}^i_k\Big],$$
$$\delta\bar{\omega}^i_k(L) \to \frac{1}{S} \sum_{i=1}^{S} (\widehat{C}^i_k)^T (\widehat{R}^i_k)^{-1} y^i_k. \tag{15}$$

Then, in step 7 of Algorithm 3 we have the correction stage, which approximates step 5 of Algorithm 1. Note that we include a corrective scalar weight ρ^i_k to compensate for the $1/S$ factor in the outcome of $(\delta\bar{\Omega}^i_k(L), \delta\bar{\omega}^i_k(L))$, shown in (15). Finally, the prediction stage in step 8 of Algorithm 3 is based on step 6 of Algorithm 1. Assuming that $(P^i_{k|k}, \hat{x}^i_{k|k}) \to (P^*_{k|k}, \hat{x}^*_{k|k})$, we expect that $(P^i_{k+1|k}, \hat{x}^i_{k+1|k}) \to (P^*_{k+1|k}, \hat{x}^*_{k+1|k})$.

Remark 3. If $\rho^i_k = S$ in step 7 of Algorithm 3, we correctly approximate the centralized entities. However, S is usually not available to each sensor. Nonetheless, Garin and Schenato (2010) show that one can use average consensus to compute S in a distributed fashion. Initialize the consensus state of all sensors as $a_1(0) = 1$ and $a_i(0) = 0$, $i = 2, \ldots, S$. Each sensor then performs Algorithm 2, such that $a_i(L) \to 1/S$. Thus, we can choose $\rho^i_k = 1/a_i(L)$, if $a_i(L) > 0$, or $\rho^i_k = 1$, otherwise. This procedure can be performed along with steps 4 and 5.

5. NUMERICAL EXAMPLE

In this section, we assess the performance of the proposed PRCKF and PRDKCF with a numerical example adapted from Rocha et al. (2022) and Rocha and Terra (2022). We also compare our results with two other robust distributed estimation strategies from the literature.

Consider a discrete-time linear system as described in (1)–(3) with constant nominal parameter matrices and uncertainties bounded by a 2-vertex polytope given by

$$F_k = \begin{bmatrix} 0 & -0.5 \\ 1 & 1 \end{bmatrix}, \; H_k = \begin{bmatrix} -6 \\ 1 \end{bmatrix}, \; F_{1,k} = \begin{bmatrix} 0 & 0 \\ 0.1 & 0.3 \end{bmatrix}, \; H_{1,k} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix},$$
$$\big(F_{2,k}, H_{2,k}\big) = -\big(F_{1,k}, H_{1,k}\big).$$

Algorithm 3 Polytopic Robust Distributed Kalman Consensus Filter (PRDKCF) (each sensor i)

Initialization: Set $\hat{x}^i_{0|-1}$, $P^i_{0|-1} > 0$, $Q_k > 0$, $R^i_k > 0$, $\mu > 0$, $\xi > 0$, and $L \geq 1$.

for $k = 0, 1, \ldots, N$ **do**:

1: Obtain a measurement y^i_k.

2: Compute φ as in step 2 of Algorithm 1.

3: Compute the modified sensing model matrices as in step 3 of Algorithm 1.

4: Initialize consensus states:

$$\bar{\omega}^i_k(0) = (P^i_{k|k-1})^{-1}\hat{x}^i_{k|k-1}, \quad \bar{\Omega}^i_k(0) = (P^i_{k|k-1})^{-1}$$

$$\delta\bar{\omega}^i_k(0) = (\widehat{C}^i_k)^T(\widehat{R}^i_k)^{-1}y^i_k$$

$$\delta\bar{\Omega}^i_k(0) = (\widehat{C}^i_k)^T(\widehat{R}^i_k)^{-1}\widehat{C}^i_k + (\tilde{\boldsymbol{C}}^i_k)^T(\bar{R}^i_k)^{-1}\tilde{\boldsymbol{C}}^i_k$$

5: Simultaneously perform average consensus on states $\{\bar{\omega}^i_k(0),\ \bar{\Omega}^i_k(0),\ \delta\bar{\omega}^i_k(0),\ \delta\bar{\Omega}^i_k(0)\}$ using Algorithm 2 with L consensus iterations.

6: Compute the modified target system model matrices as in step 4 of Algorithm 1.

7: Correction step:

7.1: Compute the posterior error weighting matrix:

$$P^i_{k|k} = \left[\bar{\Omega}^i_k(L) + \rho^i_k\delta\bar{\Omega}^i_k(L) + \tilde{\boldsymbol{F}}^T_k\bar{Q}^{-1}_k\tilde{\boldsymbol{F}}_k\right]^{-1}$$

7.2: Compute the posterior state estimate:

$$\hat{x}^i_{k|k} = P^i_{k|k}\left[\bar{\omega}^i_k(L) + \rho^i_k\delta\bar{\omega}^i_k(L)\right]$$

8: Prediction step:

8.1: Obtain the prior error weighting matrix:

$$P^i_{k+1|k} = \widehat{F}_k P^i_{k|k}\widehat{F}^T_k + \widehat{Q}_k$$

8.2: Compute the predicted prior state estimate:

$$\hat{x}^i_{k+1|k} = \widehat{F}_k\hat{x}^i_{k|k}$$

end for

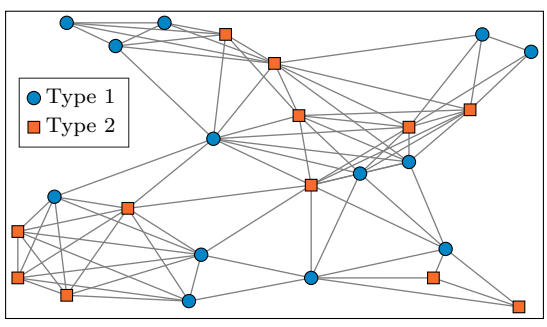

Fig. 1. Sensor network with 25 nodes and 81 edges.

Assume that w_k is a zero-mean white Gaussian noise signal with variance $Q_k = 1$. The initial state is $x_0 = [1\ 0]^T$.

A set of $S = 25$ sensors arranged in a random geometric undirected network, shown in Fig. 1, observe the target system. The sensing model is described as in (2)–(3), with v^i_k as zero-mean white Gaussian noise signals with variances R^i_k. Two distinct types of sensors are considered. Sensors of type 1 have $i = 1, 3, \ldots, 25$ and matrices

$$C^i_k = [-100\ \ 10],\ D^i_k = 1,\ C^i_{1,k} = [0.1\ \ 0.3],\ D^i_{1,k} = 0.1,$$
$$\left(C^i_{2,k}, D^i_{2,k}\right) = -\left(C^i_{1,k}, D^i_{1,k}\right),\ R^i_k = 1.$$

Sensors of type 2 have $i = 2, 4, \ldots, 24$ and matrices

$$C^i_k = [-95\ \ 16],\ D^i_k = 1,\ C^i_{1,k} = [0.3\ \ 0.5],\ D^i_{1,k} = 0.2,$$
$$\left(C^i_{2,k}, D^i_{2,k}\right) = -\left(C^i_{1,k}, D^i_{1,k}\right),\ R^i_k = 1.$$

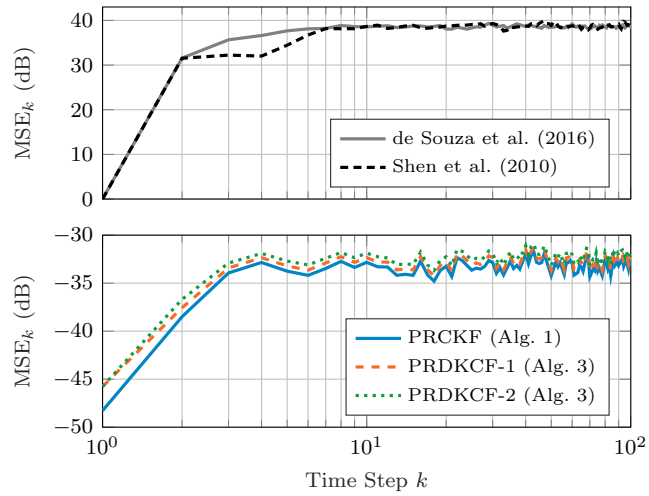

Fig. 2. Error variance curves for the simulated filters.

We apply Algorithm 3 with the following initialization data for all sensors $i \in \mathbb{S}$:

$$P^i_{0|-1} = I_2,\ \hat{x}^i_{0|-1} = [0\ 0]^T,\ \mu = 0.01,\ \xi = 0.01,\ L = 10.$$

For the consensus iterations, we adopt the Metropolis weights (Xiao et al., 2005). In addition, two scenarios are evaluated. In the first, named PRDKCF-1, the total number of sensors is available and $\rho^i_k = S$. In the second, named PRDKCF-2, we use the strategy in Remark 3 to approximate S. At each time step, the polytope coefficients $\alpha_k \in \Lambda_2$ (see (4)) are randomly selected.

To better evaluate the proposed PRDKCF, we compare its performance with that of the PRCKF (Algorithm 1), as well as with the H_∞ distributed consensus filter of Shen et al. (2010) and the robust mean square state estimator of de Souza et al. (2016). The simulation consists of performing $M = 1000$ Monte Carlo experiments, each with time horizon $N = 100$. At each time step k, we compute the mean squared estimation error (MSE) averaged along all experiments and sensors in the network, calculated as

$$\text{MSE}_k = \frac{1}{SM}\sum_{i=1}^{S}\sum_{m=1}^{M}\|x_k - \hat{x}^i_{k|k,m}\|^2.$$

This ensemble average is a reasonable approximation of the estimation error variance (Sayed, 2001).

The results are shown in Fig. 2. The PRCKF naturally exhibits the best performance, as it has access to all sensors at once, being considered a benchmark. The distributed counterparts closely follow the centralized filter. As expected, the PRDKCF-1 (S is known) presents a slightly smaller error compared to the PRDKCF-2 (S is estimated). The filters of Shen et al. (2010) and de Souza et al. (2016) exhibit similar results, yielding a much larger error compared to our proposed filters. The latter also assumes uncertainties in all parameter matrices, while the former considers uncertainties only in matrices F_k and C^i_k. Moreover, both depend on the solution of LMIs and are not fully distributed, as their gains are collectively computed, which may not scale well as the number of sensors increases.

Finally, we assess the effect of the number of consensus iterations L on the PRDKCF performance. Fig. 3 compiles a series of simulations with different values of L, considering

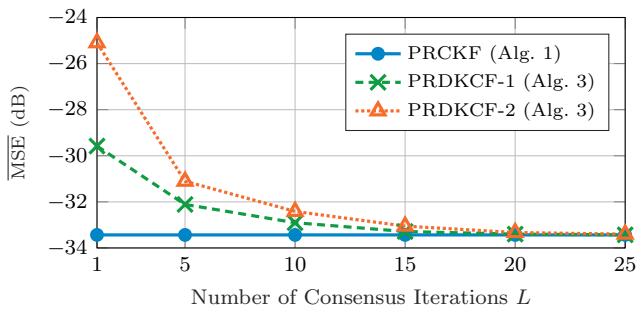

Fig. 3. Effect of the number of consensus iterations on the PRDKCF.

both scenarios of the PRDKCF. For each L, we compute the mean estimation error $\overline{\text{MSE}}$ over the full time horizon. We also show the PRCKF for comparison purposes. As L increases, the distributed filters approach the centralized filter. However, there is a performance trade-off, since more computation time is required.

6. CONCLUSION

In this paper, we proposed a novel robust consensus-based fully distributed Kalman filter for linear discrete-time systems. We assume that polytopic uncertainties are present in all parameter matrices of both the target system and sensing models. We first proposed the PRCKF, which gathers all sensors at once. Then, by applying the HCMCI protocol, we obtained a fully distributed filter that approaches the centralized behavior when a sufficiently large number of consensus iterations is considered. A numerical example showed that the proposed PRDKCF can indeed approximate the PRCKF. Moreover, it outperformed the other robust distributed filtering solutions in terms of MSE without needing to solve LMI-based optimization problems. This is a valuable computational advantage in real-time applications. In future works, we intend to study the stability properties of the proposed filters, as well as to investigate the case where the communication topology of the sensor network is described by a directed graph.

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