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Total Curvature and Topological Structure of Complete Minimal Surfaces

by

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§1. INTRODUCTION

This work is mainly inspired by the paper of Chern and Osserman [3] to investigate complete minimal surfaces in euclidean spaces. Chern and Osserman [3] showed that the total curvature of a complete minimal surface in $\mathbf{R}^n$ is either $-\infty$ or is of the form $-2\pi N$ for some non-negative integer N. Therefore it is quite natural to study complete minimal surfaces by their total curvature. We will give a detailed account of some simplest cases, including previous works of C. C. Chen [1, 2], Osserman [6], Hoffman and Osserman [5], and Gackstatter [4]

Furthermore, since Gauss-Bonnet, it has always been observed the relation between the topological property and geometrical property of surfaces. As a matter of fact, for complete minimal surfaces, Chern and Osserman [3] have obtained such a relation, expressed in the following inequality

(1.1) $$C(\boldsymbol{S}) \leq 2\pi(\chi - \boldsymbol{r}).$$

where $C(\boldsymbol{S})$ is the total curvature, χ the Euler characteristic and $\boldsymbol{r}$ the number of boundary components of the surface $\boldsymbol{S}$. For complete minimal surfaces with finite total curvature, this relation can be interpreted in a numerical manner:

$$-2\pi N \leq 2\pi(2 - 2\gamma - 2\boldsymbol{r})$$

or

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$$N \geq 2\gamma + 2r - 2, \tag{1.2}$$

where γ is the genus of S (in this case S is conformally equivalent to a compact Riemann surface deleted with a finite number of points). Therefore we can deduce that for each fixed N, a complete minimal surface in $\mathbf{R}^n$ with total curvature $-2\pi N$ can only have genus $\gamma \leq [N/2]$, since a minimal surface can never be compact (this implies $r > 0$). In this work, we also study the problem of whether this upper limit can be attained by the genus of complete minimal surfaces with given total curvature. In fact, we obtained

THEOREM A: *For $N > 0$, even, the genus of any complete minimal surface with total curvature $-2\pi N$ is always less than $N/2$.*

THEOREM B: a) *For $N = 1$, there do exist complete minimal surfaces with total curvature $-2\pi N$ and genus $[N/2]$. And any one for them is congruent to a complex curve in $\mathbf{C}^2 = \mathbf{R}^4$;*

b) *For $N = 3$, there does not exist complete minimal surface with total curvature $-2\pi N$ and genus $[N/2]$.*

c) *For $N > 3$, odd, there may exist complete minimal surfaces with total curvature $-2\pi N$ and genus $[N/2]$. But any one of them must lie in some 4-dimensional affine subspace, and, in no way, they can be represented as complex curves in $\mathbf{C}^2$.*

§2. MINIMAL SURFACES

In this chapter we will discuss some elementary properties of minimal surfaces in euclidean spaces.

Let M be an orientable 2-dimensional manifold without boundary. An immersion of M

$$\vec{x} : M \longrightarrow \mathbf{R}^n \tag{2.1}$$

is said to be minimal if the mean curvature vector field $\vec{H}$ vanishes. In terms of local parameters (u, v), this means, with respect to any normal vector $\vec{N}$, the first and second fundamental forms satisfy

$$Eg - 2Ff + Ge = 0 \tag{2.2}$$

where

$$E = \left\langle \frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial u} \right\rangle,\ F = \left\langle \frac{\partial \vec{x}}{\partial u}, \frac{\partial \vec{x}}{\partial v} \right\rangle,\ G = \left\langle \frac{\partial \vec{x}}{\partial v}, \frac{\partial \vec{x}}{\partial v} \right\rangle$$

$$e = \left\langle \frac{\partial^2 \vec{x}}{\partial u^2}, \vec{N} \right\rangle,\ f = \left\langle \frac{\partial^2 \vec{x}}{\partial u\, \partial v}, \vec{N} \right\rangle,\ g = \left\langle \frac{\partial^2 \vec{x}}{\partial v^2}, \vec{N} \right\rangle$$

For non-parametric surface, $(x, y, \vec{f}(x, y))$, (2.2) can be written as

$$\left(1 + \left|\frac{\partial \vec{f}}{\partial y}\right|^2\right)\frac{\partial^2 \vec{f}}{\partial x^2} - 2\left\langle \frac{\partial \vec{f}}{\partial x}, \frac{\partial \vec{f}}{\partial y} \right\rangle \frac{\partial^2 \vec{f}}{\partial x\, \partial y} + \left(1 + \left|\frac{\partial \vec{f}}{\partial x}\right|^2\right)\frac{\partial^2 \vec{f}}{\partial y^2} = 0. \tag{2.3}$$

And for isothermal parameters (ζ, η), i.e.

$$E = G \text{ and } F = 0, \tag{2.4}$$

(2.2) becomes

$$e + g = 0 \tag{2.5}$$

that is,

$$\left\langle \frac{\partial^2 \vec{x}}{\partial \zeta^2} + \frac{\partial^2 \vec{x}}{\partial \eta^2}, \vec{N} \right\rangle = 0, \qquad \forall \vec{N} \in TM^{\perp} \tag{2.6}$$

Since it can be shown easily that $\partial^2 \vec{x}/\partial \zeta^2 + \partial^2 \vec{x}/\partial \eta^2$ is always normal to the surface, thus we have

PROPOSITION 2.1: *A surface* $\mathbf{R}^n$ *is minimal if and only if the immersion is harmonic, with respect to the conformal structure induced by isothermal parameters.*

For isothermal parameters (ζ, η), consider the complex parameter $\zeta = \xi + i\eta$ and set

$$\phi_k(\zeta) = \frac{\partial x_k}{\partial \xi} - i\frac{\partial x_k}{\partial \eta}, \qquad k = 1, \cdots, n. \tag{2.7}$$

From the identities

$$\sum_{k=1}^{n} \phi_k^2(\zeta) = \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial \xi}\right)^2 - \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial \eta}\right)^2 - 2i \sum_{k=1}^{n} \frac{\partial x_k}{\partial \xi} \frac{\partial x_k}{\partial \eta} = E - G - 2iF \tag{2.8}$$

and

$$\sum_{k=1}^{n} |\phi_k^2(\zeta)|^2 = \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial \xi}\right)^2 + \sum_{k=1}^{n} \left(\frac{\partial x_k}{\partial \eta}\right)^2 = E + G \tag{2.9}$$

we can deduce easily

PROPOSITION 2.2: a) *$\phi_k(\zeta)$ is analytic in ζ if and only if x_k is harmonic in ξ, η.*

b) *ξ, η are isothermal parameters if and only if*

$$\sum_{k=1}^{n} \phi_k^2(\zeta) = 0 \tag{2.10}$$

c) *if ξ, η are isothermal parameters, then $\vec{x}$ is an immersion if and only if*

$$\sum_{k=1}^{n} |\phi_k(\zeta)|^2 \neq 0 \tag{2.11}$$

and the induced metric ds^2 is given by $ds^2 = \lambda^2 |d\zeta|^2$ where

$$\lambda^2 = \frac{1}{2} \sum_{k=1}^{n} |\phi_k(\zeta)|^2 \tag{2.12}$$

d) *the immersion can be recaptured by the identity*

$$x_k = \operatorname{Re} \int \phi_k(\zeta)\, d\zeta. \tag{2.13}$$

The Gaussian curvature K of the surface (2.1) is given, in terms of the local parameter ζ, by

$$K = -\frac{\Delta \log \lambda}{\lambda^2} \tag{2.14}$$

where $\Delta = \partial^2/\partial\xi^2 + \partial^2/\partial\eta^2$ is the Laplacian operator.

Using (2.12), we can find the expression

$$K = -\frac{4|\phi \wedge \phi'|^2}{|\phi|^6} \tag{2.15}$$

where

$$\phi = (\phi_1, \cdots, \phi_n), \qquad |\phi|^2 = \sum_{k=1}^{n} |\phi_k|^2, \tag{2.16}$$

and

$$|\phi \wedge \phi'|^2 = \sum_{1 \le j < k \le n} |\phi_j \phi_k' - \phi_j' \phi_k|^2. \tag{2.17}$$

Let D be a domain in the ζ-plane, and denote by S the corrsponding part of the surface (2.1). Then the area of S is

$$A(S) = \int_D\!\!\int \lambda^2\, d\xi\, d\eta = \frac{1}{2} \int_D\!\!\int |\phi|^2\, d\xi\, d\eta \tag{2.18}$$

and the total curvature of S is

$$(2.19)\quad C(S) = \int_D\int K dA = \int_D\int K\lambda^2\, d\xi\, d\eta = -2\int_D\int \frac{|\phi \wedge \phi'|^2}{|\phi|^4}\, d\xi\, d\eta.$$

The generalized Gauss map of the surface (2.1) is defined to be

$$(2.20)\qquad g : M \longrightarrow G_{n,2}$$

which assigns to each point of M the oriented tangent plane to the surface where $G_{n,2}$ is the Grassmannian of oriented planes in $\mathbf{R}^n$. The Grassmannian $G_{n,2}$ can be identfied with the hyperquadric Q_{n-2} in the complex projective space $P^{n-1}(\mathbf{C})$ in the following way.

The quadric is defined by

$$(2.21)\qquad Q_{n-2} = \left\{Z = [z_1,\cdots, z_n] \in P^{n-1}(\mathbf{C}) \;\middle|\; \sum_{k=1}^{n} z_k^2 = 0\right\}$$

For each oriented plane Σ in $G_{n,2}$, choose $\vec{v}$, $\vec{w}$ a positive base with $|\vec{v}| = |\vec{w}|$ and $\langle\vec{v}, \vec{w}\rangle = 0$, then $[\vec{v} + i\vec{w}]$ represents an elements in Q_{n-2}. This can be seen easily to be the desired identification. Under this identification the generalized Gauss map is given by

$$(2.22)\qquad g(\zeta) = [\bar{\phi}(\zeta)].$$

The relation between the Gauss map and the total curvature can be made explicit by introducing the canonical hermitian metric on $P^{n-1}(\mathbf{C})$

$$(2.23)\qquad d\hat{S}^2 = 2\frac{|Z \wedge dZ|^2}{|Z|^4}$$

so that, in the case $n = 3$, the induced metric on Q_1 becomes that of the unit sphere, and the generalized Gauss map reduces to the classical one.

If D is a domain in the ζ-plane, and $\hat{S}$ is the image surface under the Gauss map (2.22), then the metric (2.23) takes the form

$$(2.24)\qquad d\hat{S}^2 = \hat{\lambda}^2|d\zeta|^2, \qquad \hat{\lambda}^2 = 2\frac{|\phi \wedge \phi'|^2}{|\phi|^4}$$

and the area of S with respect to this metric is

$$(2.25)\qquad A(\hat{S}) = \int_D\int \hat{\lambda}^2\, d\xi\, d\eta = -C(S),$$

by (2.19). Thus we have

PROPOSITION 2.3: *The total curvature of any portion of a minimal surface* *equals the negative of the area of its image under the Gauss map.*

And the area of the image can be related to the "value distribution" of the analytic curve (2.22). Consider an arbitrary hyperplane

$$H : \sum_{k=1}^{n} a_k z_k = 0 \tag{2.26}$$

in $P^{n-1}(\mathbf{C})$. To this hyperplane corresponds the analytic differential

$$w = \sum_{k=1}^{n} \bar{a}_k \phi_k(\zeta)\, d\zeta \tag{2.27}$$

which is defined globally on M. There are two possibilities. Either $w \equiv 0$, in which case the image of M under the Gauss map lies entirely in the hyperplane H, or else the zeros of w are isolated and each of them has a given order. This order is called the order of intersection of $g(M)$ with H at the corresponding point. For any compact subdomain D of M, we may therefore define an integer $n(D, H)$, the total order of intersection of $g(M)$ with H at points of D, equal to the total order of zeros of w in D.

The space of all hyperplanes in $P^{n-1}(\mathbf{C})$ has a measure, to be denoted by da, which is invariant under the isometries of $P^{n-1}(\mathbf{C})$ with its standard hermitian metric. We normalize da so that the total measure

$$\int da = 1. \tag{2.28}$$

Then the area of the image $\hat{D} = g(D)$ can be expressed by the formula [7]

$$A(\hat{D}) = 2\pi \int n(D, a)\, da, \tag{2.29}$$

where the integration is extended over all the hyperplanes not containing $g(D)$.

Finally, we say that the minimal surface S given by (2.1) is degenerate if its image $g(M)$ under the Gauss map lies in a hyperplane of $P^{n-1}(\mathbf{C})$. And we say, further, that S is h-degenerate if h is the largest integer such that the image under the Gauss map, $g(M)$, lies in a projective subspace of condimension h.

§3. TOTAL CURVATURE AND THE CHERN-OSSERMAN INEQUALITY

In this chapter we recall some results of Chern-Osserman [3] on the total curvature and topological structure of complete minimal surfaces. For complete minimal surfaces with finite total curvature, Chern and Osserman [3] found some significant conformal properties.

PROPOSITION 3.1: *Let S be a complete minimal surface in* $\mathbf{R}^n$, *defined by a map* (2.13) *on a Riemann surface M. The following conditions are equivalent*:

a) *S has finite total curvature.*

b) *there exists an integer N such that the image of M under the Gauss map intersects at most N times with all hyperplanes which do not contain it.*

c) *the surface M is conformally equivalent to a compact surface W punctured at a finite number of points* $p_1, \cdots, p_r$ *and the differentials* $\alpha_k = \phi_k(\zeta)\, d\zeta$, $k = 1, \cdots, n$, *are either regular or have a pole at each* p_j, $j = 1, \cdots, r$; *therefore the Gauss map extends analytically to W.*

For the proof, see [3].

From this proposition, the following result can be easily deduced.

PROPOSITION 3.2: *The total curvature of a complete minimal surface in* $\mathbf{R}^n$ *is either* $-\infty$ *or* $-2\pi N$ *for some integer* $N \geq 0$.

Proof: Since $K \leq 0$, the total curvature is either $-\infty$ or else finite. In the latter case, by combining Proposition 3.1 and formula (2.29), one sees that the area of the image of the compact Riemann surface W under the extendend Gauss map is equal to $2\pi N$. Since the removal of a finite number of points does not change the area, the result follows immediately from (2.25). Q. E. D.

Remark: Therefore it's quite natural to study complete minimal surfaces by their total curvature.

Furthermore, Chern and Osserman [3] observed that there is a significant relation between the total curvature and the topological structure of a complete minimal surface. In fact, they proved:

PROPOSITION 3.3: *Let S be a complete minimal surface with Euler characteristic* χ *and r boundary components. Then the total curvature satisfies*

$$C(S) \leq 2\pi(\chi - r). \tag{3.1}$$

Proof: If $C(S) = -\infty$, there is nothing to prove. Otherwise we apply Proposition 3.1 and condition (c) holds. If W is a compact surface of genus γ, then $\chi = 2 - 2\gamma - r$. For each $j = 1, \cdots, r$, let m_j be the maximum order of poles of α_k at p_j. It is easy to see that one can choose $a_1, \cdots, a_n \in \mathbf{C}$ so that

$$w = \sum_{k=1}^{n} a_k \alpha_k \tag{3.2}$$

has a pole of order m_j at p_j. By the Riemann relation, if w has N zeros in W, then

$$N - \sum_{j=1}^{r} m_j = 2\gamma - 2. \tag{3.3}$$

But, it's known [3] that $m_j \geq 2$ for $j = 1, \cdots, r$. Hence $N \geq 2r + 2\gamma - 2 = r - \chi$ and $C(S) = -2\pi N \leq 2\pi(\chi - r)$. Q.E.D.

Remark: From (3.1), we see that, for a complete minimal surface, if the topological structure is complicated then the total curvature is small, or, for a given total curvature, the topological structure is limited. In fact, we have the inequality

$$N \geq -2 + 2\gamma + 2r \tag{3.4}$$

and, in particular, with r always positive, we have

$$N \geq 2\gamma \quad \text{or} \quad [N/2] \geq \gamma. \tag{3.5}$$

In the next section, we will discuss the general problem of interest to characterize complete minimal surfaces by their total curvature, and, the problem whether the genus can achieve the upper limit $[N/2]$.

§4. Complete Minimal Surfaces with Finite Total Curvature

We first discuss some simplest examples of complete minimal surfaces by their total curvatue. From Proposition 3.2, we know that the total curvature of complete minimal surfaces are of the form $0, -2\pi, -4\pi, -6\pi, -8\pi, \cdots$. And from Proposition 3.3, we see that, topologically, the first examples are the simplest. Now we give a detailed account of these examples.

I. $C(S)=0$

In this case, $K=0$ and it's well-known that the only examples are planes.

II. $C(S)=-2\pi$

From (3.4), we know that

$$\gamma=0 \quad \text{and} \quad r=1, \tag{4.1}$$

that is, S is conformally equivalent to the complex plane. And, in fact, any such surface is congruent to the graph of a function of the form

$$w=cz^2 \tag{4.2}$$

in $\mathbf{C}^2=\mathbf{R}^4$ with $c>0$. For details see C. C. Chen [1], or Hoffman and Osserman [5].

III. $C(S)=-4\pi$

From the Chern-Osserman inequality (2.4), we have that

$$\gamma=0, \quad \text{and} \quad r=1 \text{ or } 2 \tag{4.3}$$

or

$$\gamma=1, \quad \text{and,} \quad r=1 \tag{4.4}$$

We [2] have used the elliptic function theory to show that the second case cannot happen. This technique will be shown in the following case. And this fact can also be deduced from the work of Gackstatter [4]. Therefore the only possible complete minimal surfaces with total curvature -4π are simply-connected or doubly connected. Recently, Hoffman and Osserman [5] gave a complete description of these surfaces. (it's well known [6] that such surfaces in $\mathbf{R}^3$ are the Enneper's surface and the Catenoid). In fact, they found

PROPOSITION 4.1: *Let S be a complete simply connected minimal surface in $\mathbf{R}^n$ with total curvature -4π. Then S lies in some affine* 6*-dimensional subspace, and, in terms of suitable coordinates, may be rerpesented by $\chi=\operatorname{Re}\int\phi(\zeta)\,d\zeta$, where $\phi(\zeta)$ up to a constant factor, has one of the following two forms*:

a)

$$\phi = (\phi_1,\ i\phi_1,\ \phi_3,\ i\phi_3,\ \phi_5,\ i\phi_5) \tag{4.5}$$

with

$$\phi_1 = \zeta^2 + c_1,\ \phi_3 = b_3\zeta + c_3,\ \phi_5 = c_5 \tag{4.6}$$

where c_1 ,c_3, c_5, b_3 *are arbitrary complex constants subjects only to the contraints that if* $c_5 = 0$, *then* $c_3^2 + c_1 b_3^2 \neq 0$. *In this case, S is a complex analytic curve in* $\mathbf{C}^3$.

b)

$$\phi = (\zeta^2 + c_1,\ i\zeta^2 + c_2,\ b_3\zeta - \mu b_4,\ b_4\zeta + \mu b_3,\ c_5,\ c_6) \tag{4.7}$$

where

$$c_1 = A - B, \qquad c_2 = i(A + B) \tag{4.8}$$

with

$$A = \mu^2 + \frac{c_5^2 + c_6^2}{b_3^2 + b_4^2}, \qquad b = \frac{1}{4}(b_3^2 + b_4^2) \tag{4.9}$$

and μ, b_3, b_4, c_5, c_6 *are arbitrary complex constants satisfying only* $b_3^2 + b_4^2 \neq 0$.

PROPOSITION 4.2: *Let S be a complete doubly-connected minimal surface in* $\mathbf{R}^n$ *with total curvature* -4π. *Then S lies in some affine* 5-*dimensional subspace and is one of the two following forms*

a)

$$x = \mathrm{Re}\left\{\left(d_1\zeta - \frac{c}{\zeta},\ d_2\zeta - \frac{c}{\zeta},\ \alpha\log\zeta,\ d_4\ \zeta,\ d_5\zeta\right)\right\} \tag{4.10}$$

in $0 < |\zeta| < \infty$, *where* c, d_4, d_5 *are arbitrary complex constants,* α *is an arbitrary positive real constant, and* d_1,d_2 *are defined by*

$$\begin{aligned} d_1 &= \frac{c}{\alpha^2}(d_4^2 + d_5^2) - \frac{\alpha^2}{4c} \\ d_2 &= i\left[\frac{c}{\alpha^2}(d_4^2 + d_5^2) + \frac{\alpha^2}{4c}\right]. \end{aligned} \tag{4.11}$$

b) *S lies in a* 4-*dimensional subspace and in terms of suitable coordinates, S is the graph in* $\mathbf{C}^2$ *of the function*

$$w = az + \frac{b}{z},\ b \neq 0,\ 0 < |z| < \infty. \tag{4.12}$$

IV. $C(S) = -6\pi$

In this case, the topoplogical options are the same as (4.3) and (4.4) we will show that the second case cannot happen either. This fact cannot be deduced from the work of Gackstatter [4]. We will give 2 proofs. In the the first one we extend our technique in [1] by using the elliptic function theory. And in the second proof we adapt recent results of Hoffman and Osserman [5] so that the general case will be treated later. Before giving the proofs, we need establish two lemmas:

LEMMA 4.3: *Let* $\vec{A}^\nu = \vec{B}^\nu + i\vec{C}^\nu \in \mathbf{C}^n$, $\nu = 1, \cdots, m$ *be vectors linearly independent over* $\mathbf{C}$. *If*

$$\sum_{k=1}^{n} A_k^\nu A_k^\mu = 0, \qquad \forall \nu,\ \mu = 1, \cdots, m \tag{4.13}$$

or equivalently.

$$\begin{aligned} &\langle \vec{B}^\nu, \vec{B}^\mu \rangle = \langle \vec{C}^\nu, \vec{C}^\mu \rangle \text{ and} \\ &\langle \vec{B}^\nu, \vec{C}^\mu \rangle = -\langle \vec{B}^\mu, \vec{C}^\nu \rangle, \ \forall \nu,\ \mu = 1, \cdots, m, \end{aligned} \tag{4.14}$$

then $\vec{B}^\nu$, $\vec{C}^\nu$, $\nu = 1, \cdots, m$, *are linearly independent over* $\mathbf{R}$.

Proof: Let

$$c_1 \vec{B}^1 + \cdots + c_m \vec{B}^m + d_1 \vec{C}^1 + \cdots + d_m \vec{C}^m = 0, \qquad c_j,\ d_j \in \mathbf{R}. \tag{4.15}$$

Take the inner product of (4.15) with $\vec{B}^j$ and $\vec{C}^j$, respectively, we get

$$c_1\langle \vec{B}^1, \vec{B}^j \rangle + \cdots + c_m\langle \vec{B}^m, \vec{B}^j \rangle + d_1\langle \vec{C}^1, \vec{B}^j \rangle + \cdots + d_m\langle \vec{C}^m, \vec{B}^j \rangle = 0 \tag{4.16}$$

and

$$c_1\langle \vec{B}^1, \vec{C}^j \rangle + \cdots + c_m\langle \vec{B}^m, \vec{C}^j \rangle + d_1\langle \vec{C}^1, \vec{C}^j \rangle + \cdots + d_m\langle \vec{C}^m, \vec{C}^j \rangle = 0 \tag{4.17}$$

(4.14) and (4.16) imply that

$$-d_1 \vec{B}^1 - \cdots - d_m \vec{B}^m + c_1 \vec{C}^1 + \cdots + c_m \vec{C}^m \perp \vec{C}^j,\ j = 1, \cdots, m. \tag{4.18}$$

(4.14) and (4.17) imply that

$$-d_1 \vec{B}^1 - \cdots - d_m \vec{B}^m + c_1 \vec{C}^1 + \cdots + c_m \vec{C}^m \perp \vec{B}^j,\ j = 1, \cdots, m. \tag{4.19}$$

From (4.18) and (4.19), we get

$$-d_1 \vec{B}^1 - \cdots - d_m \vec{B}^m + c_1 \vec{C}^1 + \cdots + c_m \vec{C}^m = 0 \tag{4.20}$$

From (4.15) and (4.20). we have

$$(c_1 - id_1) \vec{A}^1 + \cdots + (c_m - id_m) \vec{A}^m = 0 \tag{4.21}$$

By hypothese, we get $c_1 - id_1 = \cdots = c_m - id_m = 0$, or $c_1 = \cdots = d_1 = \cdots = d_m = 0$. Q. E. D.

LEMMA 4.4: *Let $x: M \to \mathbf{R}^n$ be a non-flat minimal surface S with $M = W - \{p\}$, where W is a compact Riemann surface. Suppose the generalized Gauss map ϕ is regular or has a pole at p. Let ν be the order of pole of ϕ at p, i.e., the maximum order of pole of $\phi_k(\zeta)\, d\zeta$, $k = 1, \cdots, n$, at p. Then*

a) $\nu > 2$ *and*

b) *when $\nu = 3$, S lies in a 4-dimensional affine surface.*

Proof: a) if $\phi_k(\zeta)\, d\zeta$ is regular at p, then the harmonic function

$$x_k = \mathrm{Re} \int \phi_k(\zeta)\, d\zeta \tag{4.22}$$

extends to the compact surface W. Hence x_k is constant. Therefore ϕ can't be regular and since the residue of each $\phi_k(\zeta)\, d\zeta$ at p is zero, the order of pole of ϕ at p has to be at least 2. And if the order of ϕ is 2, we may write

$$\phi_k(\zeta) = \frac{a_k}{\zeta^2} + b_k + \cdots, \qquad k = 1, \cdots, n \tag{4.23}$$

around p. Since $\phi(\zeta)$ represents the oriented tangent plane to S at $x(\zeta)$, we may assume the limiting tangent plane at p is the regular $x_1 x_2$-plane, that means, $[a_1, a_2, \cdots, a_n] = [1, i, 0, \cdots, 0]$ which implies $\phi_3(\zeta)\, d\zeta, \cdots, \phi_n(\zeta)\, d\zeta$ are regular at p. As we have just observed, this would force $x_3, \cdots, x_n$ to be constant, contradicting the hypothese that S is non-flat.

b) if $\nu = 3$, we may write

$$\phi_k(\zeta) = \frac{a_k}{\zeta^3} + \frac{b_k}{\zeta^2} + c_k + \cdots, \qquad k = 1, \cdots, n \tag{4.24}$$

around p. Thus we have

$$x_k = \mathrm{Re} \int \phi_k(\zeta)\, d\zeta = -\,\mathrm{Re}\left\{\frac{-a_k}{2\zeta^2} + \frac{-b_k}{\zeta} + c_k\, \zeta + \cdots\right\}. \tag{4.25}$$

And we can see that the singular part of $x = (x_1, \cdots, x_n)$ is generated by the real and the imaginary parts of the vectors $A = (a_1, \cdots, a_n)$ and

$B=(b_1,\cdots,b_n)$. Thus we may assume that, after a rigid transformations in $\mathbf{R}^n$, $x_5, x_6, \cdots, x_n$ are regular at p, hence are constants. Q. E. D.

Remark: Gackstatter [4] has proved a more general formula relating the principal part of ϕ and the least dimension of the affine subspaces S can lie in. But, in our case, we need only a simpler treatment.

Now we proceed to prove

THEOREM 4.5: *Let $x: M^2 \to \mathbf{R}^n$ be a complete minimal surface with total curvature $C(S) = -6\pi$. Then M is simply connected or doubly connected.*

Proof 1: As we have discussed, we only need to show that the case $\gamma=1$ and $r=1$ cannot happen. Assume that it happened and let

$$M = \mathbf{C}/L - \{0\} \tag{4.26}$$

where

$$L = \{mw_1 + nw_2 \mid w_1, w_2\ R\text{-linearly independent in } \mathbf{C}\text{-plane, and, } m, n \in \mathbf{Z}\} \tag{4.27}$$

is a lattice in the complex plane.

With $C(S) = -6\pi$, we know that the extended generalized Gauss map

$$\tilde{\phi} : \mathbf{C} \longrightarrow P^{n-1}(\mathbf{C}) \tag{4.28}$$

should be of the form

$$\tilde{\phi}(\zeta) = [\vec{a} + p(\zeta)\vec{b} + p'(\zeta)\vec{c}] \tag{4.29}$$

where

$$p(\zeta) = \frac{1}{\zeta^2} + \sum_{w \in L-\{0\}} \left[\frac{1}{(\zeta - w)^2} - \frac{1}{w^2}\right] \tag{4.30}$$

is the Weierstrass P function and $\vec{a}, \vec{b}, \vec{c}$ are vectors in $\mathbf{C}^n$. The detailed argument of this part can be found in [2].

Set $\vec{a} = (a_1, \cdots, a_n)$, $\vec{b} = (b_1, \cdots, b_n)$, $\vec{c} = (c_1, \cdots, c_n)$. From

$$\tilde{\phi}^2 = \sum_{k=1}^n \tilde{\phi}_k^2 = 0$$

and the fundamental properties of the Weierstrass P function, we have

$$\sum_{k=1}^n a_k^2 = \sum_{k=1}^n b_k^2 = \sum_{k=1}^n c_k^2 = \sum_{k=1}^n a_k b_k = \sum_{k=1}^n b_k c_k = \sum_{k=1}^n a_k c_k = 0 \tag{4.31}$$

If the rank $(\vec{a}, b, \vec{c}) = 1$, then $\tilde{\phi}$ is constant and S would be flat. And if the rank ≥ 2, then from the fact $\tilde{x} = \mathrm{Re}\int \tilde{\phi}(\zeta)\, d\zeta$ is doubly periodic and Lemma 4.3, we will have that either

$$\int 1 d\zeta \quad \text{or} \quad \int p(\zeta)\, d\zeta \tag{4.32}$$

is an elliptic function, which is impossible. Q. E. D.

Proof 2: Here we avoid the elliptic function theory to show Thm. 4.5, so that general cases can be treated later. Still, let's assume $\gamma = 1$ and $r = 1$, i. e., $M = W - \{p\}$.

From the proof of Proposition 3.3, we see that the differential $\phi(\zeta)d\zeta$ is a pole of order 3 at p. Therefore, from Lemma 4.4, we know that S must lie in a 4-dimensional affine subspace. Thus, we may assume that $n = 4$. Using the results in [5] and the hypothese $\gamma = 1$ and $C(S) = -6\pi$, we can see that S is 2-degenerate or equivalently, S is a complex curve in $\mathbf{C}^2$ (this fact has been observed by Hoffman & Osserman). Now write $x(\zeta) = (\psi(\zeta), \mu(\zeta)) \in \mathbf{C}^2$ with ψ, μ holomorphic in M. Then $\phi = [\psi', i\psi', \mu', i\mu']$ with order of pole at $p = 3$. Say, $\psi'(\zeta)\, d\zeta$ has order 3 at p, hence $\psi(\zeta)$ has degree 2. So does $\mu(\zeta)$, because the genus of M is 1.

Write

$$\psi(\zeta) = \frac{a}{\zeta^2} + \cdots, \quad \mu(\zeta) = \frac{b}{\zeta^2} + \cdots, \qquad a, b \in \mathbf{C} - \{0\} \tag{4.33}$$

around p. Then $a\mu - b\psi$ is meromorphic at W with pole at p of order ≤ 1 and holomorphic elsewhere. Since $\gamma = 1$ (> 0), therefore $a\mu - b\psi$ $=$ constant, say, c, which implies

$$x(\zeta) = (\psi, \mu) = \mu\left(\frac{a}{b}, 1\right) + \left(\frac{-c}{b}, 0\right) \tag{4.34}$$

lies in an affine complex subspace of dimension 1, hence, S is flat. This is impossible. Q. E. D.

Now it remains to characterize the complete minimal surfaces with total curvature -6π. We believe that the same technique used by Hoffman and Osserman in Propositions, 4.1, 4.2 can be easily employed to solve this problem.

V. $C(S) = -8\pi$

In this case, from the Chern-Osserman inequality (3.5), we see that the genus satisfies

$$\gamma \leq 2. \tag{4.35}$$

But, for the case $\gamma = 2$, we have $r = 1$ and the order of pole of the Gauss map at the boundary, from the proof of Proposition 3.3, must be 2. From Lemma 4.4, we know that this is impossible. Therefore we have

THEOREM 4.6. *Let S be a complete minimal surface in $\mathbf{R}^n$ with total curvature* -8π. *Then the genus of S is* ≤ 1.

In fact, there do exist such surfaces with genus 1. For example

$$x(\zeta) = p'(\zeta)\,\vec{a} + \vec{p}(\zeta)\,b, \qquad \zeta \in \mathbf{C} - L \tag{4.36}$$

with suitably chosen $\vec{a}, \vec{b} \in \mathbf{C}^2$, will induce such an example with $\gamma = 1$. It seems to us that it will involve much more difficult technique to charcterize complete minimal surfaces with total curvature -8π.

From these simple cases we can see that the Chern-Osserman inequality, at least in special cases, is not sharp in terms of the genus of complete minimal surfaces. Now we like to investigate some general cases. From (3.5). we only have $N \geq 2\gamma$. The sharper results we found are:

THEOREM 4.7: *Let S be a non-flat complete minimal surface in $\mathbf{R}^n$ with total curvature* $-2\pi N$ *and genus* γ. *Then*

a) $N > 2\gamma$, *and*

b) *if N is odd and* $\gamma = (N-1)/2$, *then S lies in a* 4-*dimensional affine surface.*

Proof: a) If N is odd, then, from (3.5), obviously $N > 2\gamma$. If N is even and $N = 2\gamma$, then $r = 1$ and from the proof of Proposition 3.3, the order of pole of the Gauss map at the boundary is 2, which, from Lemma 4.4, is impossible.

b) If N is odd $\gamma = (N-1)/2$, then $r = 1$ and the order of pole of the Gauss map at the boundary is 3. Then, from Lemma 4.4, we know that S lies in a 4-dimensional affine subspace. Q. E. D.

Therefore it would be of interest to understand the case (b) in Theorem 4.7. What we found is

THEOREM 4.8: *Let S be a complete minimal surface in* $\mathbf{R}^4$ *with total curvature* $-2\pi N$, $N > 3$, *odd, and* $\gamma = (N-1)/2$. *Then S is nondegenerate.*

Proof: If S were degenerate, then S is either

a) 1-degenerate or b) 2-degenerate or c) 3-degenerate. From [5], we know that case a) implies that the total curvature $C(S)$ should be a multiple of 4π; case b) implies that S is a complex curve in $\mathbf{C}^2$. This fact, together with the fact the order of pole of the generalized Gauss map is 3 and the proof 2 of Theorem 4.5, will imply S to be flat. And the case c) is equivalent to S being flat. All cases are, therefore, impossible. Q.E.D

Remark: We have seen, so far, that when $N = 1$, there does exist complete minimal surface S, with total curvature $-2\pi N$ and genus $\gamma = (N-1)/2$, which lies in $\mathbf{R}^4$ as a complex curve. For $N > 1$, odd, $\gamma = (N-1)/2$, this case can't happen. In fact, for $N = 3$, as stated in Thm. 4.5, there does not exist complete minimal surface with total curvature $-2\pi N$ and $\gamma = (N-1)/2$. This leads us to wonder whether there do really exist such examples in general.

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