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Summary

This paper discusses calibration in functional regression models. Classical and inverse type estimators are considered. First order approximation to the bias and to the mean squared error (MSE) of the estimators are considered. Numerical comparisons seem to indicate that the classical estimator obtained via maximum likelihood estimation performs better than the other estimators considered.

1. Introduction

The simple regression model relating the dependent variable y and the independent covariate x in a sample of size n is given by

$$y_i = \alpha + \beta x_i + e_i, \tag{1.1}$$

$i = 1, \ldots, n$, where, typically, e_i are independent and normally distributed with zero mean and variance σ_e^2, which we denote by $N(0, \sigma_e^2)$. However, in some situations, the covariate x is observed with error and the observed value is

$$X_i = x_i + u_i, \tag{1.2}$$

where $u_i \overset{iid}{\sim} N(0, \sigma_u^2)$, $i = 1, \ldots, n$ (*iid*: independent and identically distributed). In (1.1) and (1.2), e_i and u_i are considered to be independent. When the covariate x is a fixed quantity, the functional model follows. If x_i is a random variable, the structural model obtains (Fuller, 1987). This paper is devoted to the functional model, with the ratio of variances $\lambda = \sigma_e^2/\sigma_u^2$ known.

The linear calibration problem emcompasses two steps. In the first step (calibration stage), the linear regression model parameters are estimated. In the second step, a random sample of size k of the dependent variable y corresponding to an unknown value of x, which we denote by x_o, is observed. For this step, we have

$$y_{0j} = \alpha + \beta x_0 + e_{0j}, \tag{1.3}$$

where, as in the first stage, $e_j \overset{iid}{\sim} N(0, \sigma_e^2)$, $j = 1, \ldots, k$. The calibration problem for the ordinary regression model has been extensively studied in the literature. Bayesian and classical estimation procedures abound in the literature. The main results are summarized in Brown (1993). Two main estimators are considered in the literature for x_0 in the ordinary regression model. The classical, or maximum likelihood estimator (MLE) and the inverse estimator, obtained by regressing x in y. Bayesian justification for the inverse estimator is given in Hoadley (1970). Shukla (1972) derives first order approximation to the bias and MSE of the two estimators, under the assumption that $|\beta| > 0$. Comparisons between the bias and MSE conducted by Shukla show that the inverse estimator is preferable when the observable mean $\bar{X} = \sum_{i=1}^{n} X_i/n$ is close to the unknown x_0. However, the overall better performance of the classical estimator is indicated by Shukla for its properties for large samples and also in the absense of any prior information about the unknown x_0. Fuller (1987) discusses properties of the classical estimator in the functional model. Some comparisons between the classical and the inverse estimator in the regression model with measurement errors specified by (1.1)-(1.3) with λ known and $k = 1$ is reported in Lee and Yum (1989). They derive approximate expression for the MSE of the classical and inverse estimators in the functional measurement error model (1.1)-(1.3) with several combinations of the parameter estimates. However, as shown in this paper, their MSE expressions do not envolve all the n^{-1} terms, as considered by Shukla in the ordinary regression model.

In this paper, n^{-1} approximations are obtained for the MSE of the classical and inverse estimators, with several combinations of the parameter estimates. As is well known, the least squares estimators of α and β are not consistent under the measurement error model given by (1.1)-(1.2). To make the estimation problem feasible, additional assumptions are required. This paper discusses the situation where the ratio of variances σ_e^2/σ_u^2 is considered as known. Section 2 discusses maximum likelihood estimators for the model specified by equations (1.1)-(1.3). First order approximation for the asymptotic covariance matrix of the maximum likelihood estimators is presented. It is also shown that the covariance matrix of the unknown parameters (including x_0) do not coincides with the inverse of the expected information matrix, a result first noted by Patefied (1977) in the estimation of α and β. In Section 3 alternative estimators are considered and first order approximations are obtained for their expected values and variances. Section 4 presents some analytical and numerical comparisons between the mean squared erros of the estimators considered. Tables are presented which can give indications of the estimator to use in some particular situations. Graphical comparisons of the estimators are also considered. Section 5 is dedicated to applications with real data sets. Finally, Section 6 discusses consequences in terms of MSE of considering an erroneous value for λ. The main conclusion when comparing the estimators considered is that the classical estimator computed with the maximum likelihood estimators of the parameters α and β behaves better, in terms of MSE than the other estimators, particularly if n and k are large. This conclusion seems to take effect specially when the average of the unobserved x_i (and consequently that of X_i) values is far from x_0. However, this better performance may be diminished if a wrong value is considered for λ.

2. Maximum Likelihood estimation and covariance matrices

In this section, we consider the case where $\sigma_e^2 = \lambda\theta$ and $\sigma_u^2 = \theta$, with λ known. Thus, it follows from (1.1)-(1.3) that

$$(2.1) \qquad \begin{pmatrix} X_i \\ y_i \end{pmatrix} \sim N(\begin{pmatrix} x_i \\ \alpha + \beta x_i \end{pmatrix}; \begin{pmatrix} \theta & 0 \\ 0 & \lambda\theta \end{pmatrix}),$$

$i = 1, \ldots, n$, and

$$y_{0j} \sim N(\alpha + \beta x_0, \lambda\theta), \tag{2.2}$$

$j = 1, \ldots, k$. Thus, the log-likelihood function corresponding to the observed data is proportional to

$$\frac{1}{\theta^{n+k/2}} e^{-\frac{1}{2\lambda\theta}[\lambda \sum_{i=1}^{n}(X_i - x_i)^2 + \sum_{i=1}^{n}(y_i - \alpha - \beta x_i)^2 + \sum_{j=1}^{k}(y_{0j} - \alpha - \beta x_o)^2]},$$

$i = 1, \ldots, n$. After differentiating the log-likelihood function and equating the derivatives to zero, we arrive at the following likelihood estimators:

$$\hat{\alpha} = \bar{y} - \hat{\beta}\bar{X}, \tag{2.3}$$

$$\hat{\beta} = \frac{S_{yy} - \lambda S_{XX} + \sqrt{(S_{yy} - \lambda S_{XX})^2 + 4\lambda S_{Xy}^2}}{2S_{Xy}}, \tag{2.4}$$

$$\hat{x}_i = \frac{\lambda x_i + \hat{\beta}(y_i - \hat{\alpha})}{\lambda + \hat{\beta}^2}, \tag{2.5}$$

$i = 1, \ldots, n$,

$$\hat{\theta} = \frac{n(S_{yy} + \hat{\beta}^2 S_{XX} - 2\hat{\beta}S_{Xy})}{(2n + k)(\lambda + \hat{\beta}^2)} + \frac{kS_{yy}^0}{\lambda(n + k)}, \tag{2.6}$$

$$\hat{x}_{0C} = \frac{\bar{y}_0 - \hat{\alpha}}{\hat{\beta}}, \tag{2.7}$$

where $\bar{y} = \sum_{i=1}^{n} y_i/n$, $\bar{X} = \sum_{i=1}^{n} X_i/n$, $\bar{y}_0 = \sum_{j=1}^{k} y_{0j}/k$, $S_{yy} = \sum_{i=1}^{n}(y_i - \bar{y})^2/n$, $S_{XX} = \sum_{i=1}^{n}(X_i - \bar{X})^2/n$, $S_{Xy} = \sum_{i=1}^{n}(X_i - \bar{X})(y_i - \bar{y})/n$ and $S_{yy}^0 = \sum_{j=1}^{k}(y_{0j} - \bar{y}_0)^2/k$.

Notice that the maximum likelihood estimator of x_0 given in (2.7) is the well known classical estimator, but computed with respect to the maximum likelihood estimators of α and β (given by (2.3) and (2.4)) under model (1.1)-(1.2), which depends on the data only through the calibration stage.

The covariance matrix, up to first order terms, of the maximum likelihood estimators given above, is obtained under the assumption that $k = an$, for $a > 0$, and

$$\lim_{n\to\infty} \bar{x} = \mu, \quad \text{and} \quad \lim_{n\to\infty} \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{n} = \nu^2, \tag{2.8}$$

are both finite. The strong consistency of the maximum likelihood estimators is also proved.

Theorem 2.1. *Under the regression model (1.1)-(1.3) and assumptions (2.8), with $k = an$ in (1.3), it follows that the maximum likelihood estimators $\hat{\alpha}, \hat{\beta}, \hat{x}_o$, and the estimator*

$$\tilde{\theta} = \left(\frac{2n+k}{n+k}\right)\hat{\theta},$$

are strongly consistent with covariance matrix given by

(2.9)

$$\theta(\lambda+\beta^2)\begin{pmatrix} \frac{1}{n} + \frac{\bar{x}^2(1+t)}{nS_{xx}} & -\frac{\bar{x}(1+t)}{nS_{xx}} & \frac{1}{\beta}\left[-\frac{1}{n} + \frac{(x_o-\bar{x})\bar{x}(1+t)}{nS_{xx}}\right] & 0 \\ & \frac{(1+t)}{nS_{xx}} & -\frac{(x_o-\bar{x})(1+t)}{n\beta S_{xx}} & 0 \\ & & \frac{1}{\beta^2}\left[\frac{1}{n} + \frac{\lambda}{k(\lambda+\beta^2)} + \frac{(x_0-\bar{x})^2(1+t)}{nS_{xx}}\right] & 0 \\ & & & \frac{2\theta}{(n+k)(\lambda+\beta^2)} \end{pmatrix} + O(n^{-2})$$

where $S_{xx} = \sum_{i=1}^n (x_i - \bar{x})^2/n$ and

(2.10)
$$t = \frac{\lambda\theta}{(\lambda+\beta^2)S_{xx}}.$$

Proof. First, note that $X_1, X_2, \ldots,$ are independent random variables with $E[X_i] = x_i$ and $Var[X_i] = \theta$, $i = 1, \ldots$. Thus, since

$$\sum_{n=1}^{\infty} \frac{Var[X_n]}{n^2} = \theta \sum_{n=1}^{\infty} \frac{1}{n^2} = \theta\frac{\pi^2}{6},$$

which is finite, it follows from Kolmogorov's strong law of the large numbers that $\bar{X} - \bar{x} \xrightarrow{a.s.} 0$, as $n \to \infty$. Thus, $\bar{X}$ is a consistent estimator of $\bar{x}$. From assumption (2.8) it then follows that

$$\bar{X} \xrightarrow{as} \mu,$$

as $n \to \infty$. Similarly, it can be shown that $\bar{y} \xrightarrow{a.s.} \alpha + \beta\mu$, $S_{XX} \xrightarrow{a.s.} \theta + \nu^2$, $S_{Xy} \xrightarrow{a.s.} \beta\nu^2$, $S_{yy} \xrightarrow{a.s.} \lambda\theta + \beta^2\nu^2$, $\bar{y}_0 \xrightarrow{a.s.} \alpha + \beta x_0$ and $S^0_{yy} \xrightarrow{a.s.} \lambda\theta$, as $n \to \infty$. Since the maximum likelihood estimators are continuous functions of the above sample moments, except when $S_{Xy} = 0$ and $\hat{\beta} = 0$, which has probability zero, it is easy to show that

$$\lim_{n\to\infty} (\hat{\alpha}, \hat{\beta}, \hat{x}_{0C}, \tilde{\theta}) = (\alpha, \beta, x_0, \theta),$$

with probability one, concluding the strong consistency of the maximum likelihood estimators. To obtain the first order approximation to the covariance matrix of the maximum likelihood estimators, we use the approach considered in Barnett (1969) (see also Patefield, 1977). Using standard expansions of the corresponding sample moments (see also Kendall and Stuart, 1952, 5th edition, pp 322), it can be shown that

$$Var[S_{XX}] = \frac{2\theta}{n}[\theta + 2S_{xx}] + O(n^{-2}),$$

$$Var[S_{Xy}] = \frac{\theta(\lambda + \beta^2)}{n}S_{xx} + \frac{\lambda\theta^2}{n} + O(n^{-2}),$$

$$Var[S_{yy}] = \frac{2\lambda\theta}{n}[\lambda\theta + 2\beta^2 S_{xx}] + O(n^{-2})$$

and

$$Var[S^0_{yy}] = \frac{2\lambda^2\theta^2}{k} + O(n^{-2}),$$

since, by assumption, $k = an$. Furthermore, simple algebraic manipulations yield $Cov[S_{XX}, S_{yy}$
0,

$$Cov[S_{XX}, S_{Xy}] = \frac{2\beta\theta}{n}S_{xx},$$

and

$$Cov[S_{yy}, S_{Xy}] = \frac{2\beta\lambda\theta}{n}S_{xx}.$$

In addition, standard properties of the normal distribution imply that the covariance between sample means and sample variances or covariances are all null, as for example, $Cov[\bar{y}, S_{yy}] = Cov[\bar{X}, S_{XX}] = 0$. Since the maximum likellihood estimators of α, β, x_0 and θ are continuos functions of the above sample moments, the result follows by using the approach considered in Barnett (1969). Being $\mathbf{m} = (m_1, \ldots, m_s)$ a vector of sample moments, it follows, up to first order terms, that

$$Cov[g(\mathbf{m}), h(\mathbf{m})] = \sum_{i,j;i\neq j}^{s} g^{(i)}h^{(j)}Cov[m_i, m_j] + \sum_{i=1}^{k} g^{(i)}h^{(j)}Var[m_i], \tag{2.11}$$

where $g^{(i)}$ and $h^{(j)}$ are partial derivatives of g and h, repectively, evaluated at $E[\mathbf{m}]$. The approach is illustrated by computing $Var[\hat{x}_{0C}]$. In this case, $\mathbf{m} = (\bar{X}, \bar{y}, \bar{y}_0, S_{XX}, S_{yy}, S_{Xy}, S^0_{yy})$. Thus, using (2.11), with $h = g$, it follows that

$$Var[\hat{x}_{0C}] = (g^{(1)})^2 Var[\bar{X}] + (g^{(2)})^2 Var[\bar{y}] + (g^{(3)})^2 Var[\bar{y}_0] + (g^{(4)})^2 Var[S_{XX}]$$
$$+(g^{(5)})^2 Var[S_{Xy}] + (g^{(6)})^2 Var[S_{yy}] + 2g^{(4)}g^{(5)}Cov[S_{XX}, S_{Xy}]$$
$$+2g^{(5)}g^{(6)}Cov[S_{yy}, S_{Xy}] + O(n^{-2}),$$

since all the other covariances are null, with

$$g^{(1)} = 1, \quad g^{(3)} = \frac{1}{\beta}, \quad g^{(5)} = -\frac{(x_0 - \bar{x})(\lambda - \beta^2)}{(\lambda + \beta^2)\beta S_{xx}},$$

$$g^{(2)} = -\frac{1}{\beta}, \quad g^{(4)} = \frac{(x_0 - \bar{x})\lambda}{(\lambda + \beta^2)S_{xx}}, \quad g^{(6)} = -\frac{(x_0 - \bar{x})}{(\lambda + \beta^2)S_{xx}}, \quad g^{(7)} = 0.$$

The result follows by replacing in the above expression, the corresponding first order approximations for the variances and covariances obtained above.

By computing the second derivative of the log-likelihood function with respect to the parameters $x_1, \ldots, x_n, \alpha, \beta, x_o, \theta$, and taking expectations, we can obtain the expected information matrix, which can be inverted by using standard properties of partitioned matrices. It can be shown that the part of the inverse of the expected information matrix corresponding to the parameters $\alpha, \beta, x_o, (2n+k)\theta/(n+k)$, is given by

$$(2.12) \qquad \theta(\lambda + \beta^2)\begin{pmatrix} \frac{1}{n} + \frac{\bar{x}^2}{nS_{xx}} & -\frac{\bar{x}}{nS_{xx}} & \frac{1}{\beta}[-\frac{1}{n} + \frac{\bar{x}(x_o - \bar{x})}{nS_{xx}}] & 0 \\ & \frac{1}{nS_{xx}} & -\frac{(x_0 - \bar{x})}{\beta n S_{xx}} & 0 \\ & & \frac{1}{\beta^2}[\frac{1}{n} + \frac{\lambda}{k(\lambda + \beta^2)} + \frac{(x_0 - \bar{x})^2}{nS_{xx}}] & 0 \\ & & & \frac{2\theta(2n+k)}{(n+k)^2(\lambda+\beta^2)} \end{pmatrix}.$$

Thus, by comparing the inverse of the information matrix given by (2.12) with the corresponding elements of the covariance matrix (2.9), it follows that the inverse of the expected information matrix is different from the asymptotic covariance matrix, a fact also noted by Patefield (1977) when estimating α and β under model (1.1)-(1.2).

The covariance matrix of the maximum likelihood estimators with k fixed is now discussed. A lemma is presented first, which is used to prove the main result.

Lemma 2.1. *Under model (1.1)-(1.2) it follows that*

$$(2.13) \qquad Cov[\hat{\beta}, \bar{X}] = Cov[\hat{\beta}, \bar{y}] = 0,$$

$$(2.14) \qquad Cov[\hat{\beta}, S_{XX}] = -\frac{2\theta\beta}{n}(1+t) + O(n^{-2}),$$

$$(2.15) \qquad Cov[\hat{\beta}, S_{Xy}] = \frac{\theta(\lambda - \beta^2)}{n}(1+t) + O(n^{-2}),$$

$$(2.16) \qquad Cov[\hat{\beta}, S_{yy}] = \frac{2\lambda\theta\beta}{n}(1+t) + O(n^{-2}).$$

Moreover,

$$(2.17) \qquad E[\hat{\beta}] = \beta + \frac{\beta\theta}{nS_{xx}}(1+t) + O(n^{-2}),$$

with t as given in (2.10).

Proof. Equation (2.13) follows directly from the independence of the sample mean and the sample variance in normal samples. Equations (2.14) to (2.16) can be proved by using the same techniques as in Barnett (1969). To prove (2.17), notice first that $\hat{\beta}$ is a continuous function of (S_{XX}, S_{yy}, S_{Xy}). Taking Expectation of a Taylor expansion of $\hat{\beta}$ at $\mathbf{a} = (E[S_{XX}], E[S_{Xy}], E[S_{yy}])$, leads to

$$E[\hat{\beta}] = \beta + \frac{1}{2}\frac{\partial^2 \hat{\beta}}{\partial S_{XX}^2}|_{\mathbf{a}} Var[S_{XX}] + \frac{1}{2}\frac{\partial^2 \hat{\beta}}{\partial S_{Xy}^2}|_{\mathbf{a}} Var[S_{Xy}] + \frac{1}{2}\frac{\partial^2 \hat{\beta}}{\partial S_{yy}^2}|_{\mathbf{a}} Var[S_{yy}]$$

$$+ \frac{\partial^2 \hat{\beta}}{\partial S_{XX}\partial S_{yy}}|_{\mathbf{a}} Cov[S_{XX}, S_{Xy}] + \frac{\partial^2 \hat{\beta}}{\partial S_{Xy} S_{yy}}|_{\mathbf{a}} Cov[S_{Xy}, S_{yy}] + O(n^{-2}).$$

The result follows by replacing the expressions for the variances and covariances of the sample moments obtained in the proof of Theorem 2.1.

In the next theorem, the covariance matrix of the maximum likelihood estimators is discussed for the case of a large n and k fixed. The case $k = 1$ is investigated in Lee and Yum (1989), although the expressions they obtain for the mean squared error are not of order n^{-1}.

Theorem 2.2. *Under model (1.1)-(1.3), with λ known and k fixed, it follows that the covariance matrix of the estimators $\hat{\alpha}, \hat{\beta}, \hat{x}_{0C}, \tilde{\theta}$ is given by*

$$(2.18)\quad \theta(\lambda+\beta^2)\begin{pmatrix} \frac{1}{n} + \frac{\bar{x}^2(1+t)}{nS_{xx}} & -\frac{\bar{x}(1+t)}{nS_{xx}} & \frac{1}{\beta}[-\frac{1}{n} + \frac{(x_0-\bar{x})\bar{x}(1+t)}{nS_{xx}} & 0 \\ & \frac{(1+t)}{nS_{xx}} & -\frac{(x_0-\bar{x})(1+t)}{n\beta S_{xx}} & 0 \\ & & v_{xx} & 0 \\ & & & \frac{2\theta}{(n+k)(\lambda+\beta)} \end{pmatrix} + O(n^{-2}),$$

where

$$v_{xx} = \frac{1}{\beta^2}[\frac{1}{n} - \frac{\lambda}{k(\lambda+\beta^2)} + ((x_0 - \bar{x})^2 + \frac{3\lambda\theta}{k\beta^2})\frac{(1+t)}{nS_{xx}} - \frac{2\lambda\theta(1+t)}{nk(\lambda+\beta^2)S_{xx}})]$$

and t is as given in (2.10). Moreover,

$$(2.19)\qquad E[\hat{x}_{0C}] = x_0 + \frac{(x_0 - \bar{x})(1+t)\lambda\theta}{n\beta^2 S_{xx}} + O(n^{-2}).$$

Proof. The part of the covariance matrix corresponding to the maximum likelihood estimators $\hat{\alpha}$ and $\hat{\beta}$ is as given in Theorem 2.1. With respect to $\hat{x}_{0C}$ notice that we can write

$$(2.20)\qquad \hat{x}_{0C} = \bar{X} + \frac{\beta(x_{0C} - \bar{x}) + \bar{e}_0 - \bar{e}}{\hat{\beta}},$$

where $\bar{X} = \sum_{i=1}^{n} X_i/n$, $\bar{e} = \sum_{i=1}^{n} e_i/n$ and $\bar{e}_0 = \sum_{j=1}^{k} e_{0j}/k$, so that variances and covariances involving $\hat{x}_{0C}$ depend on $1/\hat{\beta}$, which can be written, after a Taylor series expansion, as

$$\frac{1}{\hat{\beta}} = \frac{1}{E[\hat{\beta}]} - \frac{(\hat{\beta} - E[\hat{\beta}])}{E^2[\hat{\beta}]} + \frac{(\hat{\beta} - E[\hat{\beta}])^2}{E^3[\hat{\beta}]} + O_p(n^{-2}). \tag{2.21}$$

The combination of (2.20), (2.21) and (2.17) yield result (2.19) after conveniently taking expectations. The remaining terms in the covariance matrix (2.18) can be obtained in a similar fashion.

Note that the expected value given in (2.19) hold also for the case of k large, since it is independent of k. Moreover, since the bias of $\hat{x}_{0C}$ is of order n^{-1}, the first order approximation to the mean squared error of this estimator coincides with its variance given in Theorem 2.2 for k fixed and in Theorem 2.1 for $k = an$. The next theorem presents a consistent estimator for the $MSE[\hat{x}_{0C}]$ for $k = an$.

Theorem 2.3. *Under the assumptions considered in Theorem 2.1, a consistent estimator for the $MSE[\hat{x}_{0C}]$ is given by*

$$\widehat{MSE[\hat{x}_{0C}]} = \frac{\tilde{\theta}(\lambda + \hat{\beta}^2)}{n\hat{\beta}^2} + \frac{\lambda\tilde{\theta}}{k\hat{\beta}^2} - \frac{2\lambda\tilde{\theta}}{nk\hat{\beta}^2\hat{t}^2(\lambda + \hat{\beta}^2)}(\lambda(\hat{l}+1) + \hat{\beta}^2\hat{l})$$

$$+((\hat{x}_0 - \bar{x})^2 + \frac{3\lambda\tilde{\theta}}{k\hat{\beta}^2})(\frac{\lambda(\hat{l}+1) + \hat{\beta}^2\hat{l}}{n\hat{\beta}^2\hat{t}^2}),$$

where

$$\hat{l} = \frac{S_{X_y}}{\hat{\beta}\tilde{\theta}} \quad \text{and} \quad \hat{t} = \frac{\lambda\hat{\beta}\tilde{\theta}}{(\lambda + \hat{\beta}^2)S_{X_y}}.$$

Proof. The result follows basically from Theorem 2.1, which imply that

$$\hat{l} \xrightarrow{as} \frac{\nu^2}{\theta} \quad \text{and} \quad \hat{t} \xrightarrow{as} \frac{\lambda\theta}{(\lambda + \beta^2)\nu^2},$$

as $n \to \infty$.

3. Some other estimators

Another type of estimator that we can consider is the inverse estimator, which can be obtained by regressing x_i on y_i, as justified, for example, in Krutchokoff (1969). As considered in the literature, the inverse estimator is given by

$$\hat{x}_{0I} = \hat{\alpha}_I + \hat{\beta}_I\bar{y}_0,$$

with $\hat{\alpha}_I = \bar{X} - \hat{\beta}_I \bar{y}$ and

$$\hat{\beta}_I = \frac{S_{XX} - \lambda^{-1} S_{yy} + \sqrt{(S_{yy} - \lambda^{-1} S_{XX})^2 + 4\lambda^{-1} S_{Xy}^2}}{2S_{Xy}},$$

which can obtained from $\hat{\beta}$ by interchanging the roles of X and y. It can be shown that $\hat{\beta}_I = 1/\hat{\beta}$, with $\hat{\beta}$ given in (2.4), which implies that

$$\hat{x}_{0I} = \hat{\alpha}_I + \frac{\bar{y}_0}{\hat{\beta}} = \bar{X} + \frac{\bar{y}_0 - \bar{y}}{\hat{\beta}} = \frac{\bar{y}_0 - \hat{\alpha}}{\hat{\beta}} = \hat{x}_{0C},$$

as also noted by Lee and Yum (1989). Thus, both estimators have similar biases, variances and mean squared errors.

By considering the classical and inverse estimators with respect to the ordinary least squares estimators, we can define two other estimators, which we denote by

$$\hat{x}_{0CL} = \frac{\bar{y}_0 - \hat{\alpha}_L}{\hat{\beta}_L},$$

and

$$\hat{x}_{0IL} = \hat{\alpha}_{IL} + \hat{\beta}_{IL}\bar{y},$$

where $\hat{\alpha}_L = \bar{y} - \bar{X}\hat{\beta}_L$, $\hat{\beta}_L = S_{Xy}/S_{XX}$, $\hat{\alpha}_{IL} = \bar{X} - \bar{y}\hat{\beta}_{IL}$ and $\hat{\beta}_{IL} = S_{Xy}/S_{yy}$. The inverse estimators is also considered in Fuller (1987). We present next first order expressions for the mean and variances related to the estimators presented above. The results in the two lemmas presented next follow by taking Taylor series expansions of the estimators $\hat{\beta}_L$ and $\hat{\beta}_{IL}$ as considered in (2.21) for $\hat{\beta}$.

Lemma 3.1. *Under model (1.1) and (1.2) it follows when λ is known that*

$$E[\hat{\beta}_L] = \frac{\beta l}{1+l}\left(1 + \frac{1+3l}{n(1+l)^2}\right) + O(n^{-2})$$

and

$$Var[\hat{\beta}_L] = \frac{\lambda}{n(1+l)} + \frac{\beta^2 l(1+l^2)}{n(1+l)^4} + O(n^{-2}),$$

where $l = S_{xx}/\theta$.

Davies and Hulton (1975) report also some studies on the asymptotic bias and MSE of the least squares estimator $\hat{\beta}_L$. However they emphasize only the leading terms of the approximations.

Lemma 3.2. *Under the model defined by (1.1) and (1.2), when λ is known, it follows that*

$$E[\hat{\beta}_{IL}] = \frac{Z}{\beta(1+Z)} + \frac{Z(1+3Z)}{n\beta(1+Z)^3} + O(n^{-2})$$

and

$$Var[\hat{\beta}_{IL}] = \frac{1}{n\lambda(1+Z)} + \frac{Z(1+Z^2)}{n\beta^2(1+Z)^4} + O(n^{-2}),$$

where

$$Z = \frac{\beta^2 S_{xx}}{\lambda\theta}.$$

The next result deals with estimator $\hat{x}_{0CL}$.

Theorem 3.1. *Under the assumptions in Lemma 3.1, it follows with k fixed that*

$$E[\hat{x}_{0CL}] = x_0 + \frac{(x_0-\bar{x})}{l} + \frac{(x_0-\bar{x})}{nl}\left[\frac{(1+l^2)}{l(1+l)} + \frac{\lambda(1+l)^2}{\beta^2 l^2} - \frac{(1+3l)}{1+l}\right] + O(n^{-2})$$

and

$$Var[\hat{x}_{0CL}] = \frac{\theta}{n}\left(1+\frac{\lambda(1+l)^2}{\beta^2 l^2}\right) + \frac{\lambda\theta}{k\beta^2 l^2}(1+l)^2 - \frac{2\lambda\theta}{nk\beta^2 l^2}(1+3l)$$

$$+\left((x_0-\bar{x})^2 + \frac{3\lambda\theta}{k\beta^2}\right)\left[\frac{\lambda(1+l)^3}{n\beta^2 l^4} + \frac{(1+l^2)}{nl^3}\right] + O(n^{-2}),$$

with l as given in Lemma 3.1.

Proof. The proof follows by considering the expressions

$$E[\hat{x}_{0CL}] = E[\bar{X}] + \beta(x_0-\bar{X})\left\{E\left[\frac{1}{\hat{\beta}_L}\right] + \frac{Var[\hat{\beta}_L]}{E^3[\hat{\beta}_L]}\right\} + O(n^{-2})$$

and

$$Var[\hat{x}_{0CL}] = Var[\bar{X}] + \beta^2(\bar{x}_0-\bar{x})^2 Var\left[\frac{1}{\hat{\beta}_L}\right] + Var\left[\frac{\bar{e}_0}{\hat{\beta}_L}\right] + Var\left[\frac{\bar{e}}{\hat{\beta}_L}\right],$$

after computing the required expectations and variances, as for example,

$$Var\left[\frac{1}{\hat{\beta}_L}\right] = \frac{(1+l)^4 Var[\hat{\beta}_L]}{\beta^4 l^4} + O(n^{-2}),$$

where $Var[\hat{\beta}_L]$ is given in Lemma 3.1.

The next theorem provides results related to the inverse estimator.

Theorem 3.2. *Under the assumptions considered in Lemma 3.2 with k fixed, it follows that*

$$E[\hat{x}_{0IL}] = x_0 - \frac{(x_0-\bar{x})}{(1+Z)} + \frac{(x_0-\bar{x})Z(1+3Z)}{n(1+Z)^3} + O(n^{-2})$$

and

$$Var[\hat{x}_{0IL}] = \frac{\theta}{n}\left(1+\frac{\lambda Z^2}{\beta^2(1+Z)^2}\right)+\frac{\lambda\theta Z^2}{k\beta^2(1+Z)^2}+\frac{(x_0-\bar{x})^2}{n(1+Z)}\left(\frac{\beta^2}{\lambda}+\frac{Z(1+Z^2)}{(1+Z)^3}\right)$$
$$+\frac{\theta}{nk(1+Z)}\left(1+\frac{\lambda Z(3Z^2-2Z+1)}{\beta^2(1+Z)^3}\right)+O(n^{-2}),$$

with Z as given in Lemma 3.2.

Proof. The proof follows by writting

$$\hat{x}_{0IL} = \bar{X} + \hat{\beta}_{IL}(\beta(\bar{x}_0 - \bar{x}) + \bar{e}_0 - \bar{e}),$$

and considering the required expansions, as considered in Lemma 3.2.

4. Comparisons between the estimators

The results is Sections 2 and 3 imply, up to order n^{-1}, that all three estimators are biased, since

(4.1) $$Bias(\hat{x}_{0C}) = Bias(\hat{x}_{0I}) = \frac{(x_0-\bar{x})\lambda}{n(\lambda+\beta^2)l}\left(1+\frac{\lambda(1+l)}{\beta^2 l}\right),$$

$$Bias(\hat{x}_{0CL}) = \frac{(x_0-\bar{x})}{l}+\frac{(x_0-\bar{x})}{nl}\left\{\frac{\lambda(1+l)^2}{\beta^2 l^2}+\frac{(1+l^2)}{l(1+l)}-\frac{(1+3l)}{1+l}\right\}$$

and

$$Bias(\hat{x}_{0IL}) = -\frac{(x_0-\bar{x})}{(1+Z)}+\frac{(x_0-\bar{x})Z(1+3Z)}{n(1+Z)^3}.$$

However, as $n \to \infty$ $(k \geq 1)$,

$$Bias(\hat{x}_{0C}) \to 0,$$
$$Bias(\hat{x}_{0CL}) \to \frac{x_0-\bar{x}}{l}$$
$$Bias(\hat{x}_{0IL}) \to \frac{x_0-\bar{x}}{1+Z}.$$

Thus, only $\hat{x}_{0C}$ is asymptotically unbiased. Note also that, the three estimators are unbiased when $\bar{x} = x_0$ and the bias may be small when $\bar{x}$ is close to x_0. Furthermore, as $n \to \infty$ and $k \to \infty$ it follows that

$$MSE[\hat{x}_{0C}] \longrightarrow 0,$$

$$MSE[\hat{x}_{0CL}] \longrightarrow \frac{(x_0-\bar{x})^2}{l^2} = \frac{(x_0-\bar{x})^2\beta^4}{\lambda^2 Z^2}$$

and

$$MSE[\hat{x}_{0IL}] \longrightarrow \frac{(x_0-\bar{x})^2}{(1+Z)^2} = \frac{(x_0-\bar{x})^2\lambda^2}{(\lambda+\beta^2 l)^2}.$$

Thus, the above results imply that $\hat{x}_{0C}$ is consistent (see also Theorem 2.1), while the others are not. When $x_0 = \bar{x}$, the three estimators are consistent. Morever, as $n \to \infty$ and $k \to \infty$, with respect to the mean squared error, we can write

$$MSE[\hat{x}_{0CL}] - MSE[\hat{x}_{0IL}] \longrightarrow \frac{(x_0 - \bar{x})^2}{\lambda^2 Z^2 (1+Z)^2}((\beta^4 - \lambda^2)Z^2 + 2\beta^4 Z + \beta^4).$$

Thus, since $Z > 0$, if $\beta^2 > \lambda$ then $MSE[\hat{x}_{0CL}] > MSE[\hat{x}_{0IL}]$, for large n and k. Otherwise, $MSE[\hat{x}_{0CL}] < MSE[\hat{x}_{0IL}]$.

4.1. Numerical evaluations of the MSE

As seen from the expressions of their MSE for small n, it is not simple to compare analitically the three estimators. To obtain numerical comparisons between the estimators, we write the three first order approximations to their mean squared errors as quadratic forms as considered in Lwin and Maritz (1982). After some algebraic manipulations. we write

(4.1)
$$MSE[\hat{x}_{0CL}] = A_1 w^2 + C_1,$$

where $w = (x_0 - \bar{x})$,

$$A_1 = \frac{1}{l^2} + \frac{(1+l)^2(l+3)}{nl^4} + \frac{(l+3)(1+l^2) - 2l(1+3l)}{nl^3(1+l)}$$

and

$$C_1 = \frac{\theta}{n}(1 + \frac{T(1+l)^2}{l^2}) + \frac{T\theta(1+l)^2}{kl^2} - \frac{2T\theta(1+3l)}{nkl^2} + \frac{3T\theta}{nkl^3}((1+l^2) + \frac{T(1+l)^3}{l}),$$

where $T = \lambda/\beta^2$;

(4.2)
$$MSE[\hat{x}_{0C}] = A_2 w^2 + C_2,$$

where

$$A_2 = \frac{1}{nl} + \frac{T(1+l)}{nl^2},$$

and

$$C_2 = \frac{\theta}{n}(1+T) + \frac{T\theta}{k} - \frac{2T\theta}{nkl^2}(l + \frac{T}{T+1}) + \frac{3T\theta}{nkl}(1 + \frac{T(1+l)}{l})$$

and

(4.3)
$$MSE[\hat{x}_{0IL}] = A_3 w^2 + C_3,$$

where

$$A_3 = \frac{1}{(1+Z)^2} + \frac{1}{nT(1+Z)}$$

and

$$C_3 = \frac{\theta}{n}(1 + \frac{TZ^2}{(1+Z)^2}) + \frac{T\theta}{k(1+Z)^2} + \frac{\theta}{nk(1+Z)}(1 + \frac{TZ(3Z^2 - 2Z + 1)}{(1+Z)^3}.$$

The approach used to compare the three estimators is illustrated with estimators $\hat{x}_{0CL}$ and $\hat{x}_{0IL}$. Similar comparisons hold with respect to the other estimators. After some algebraic manipulations, it can be shown that $A_1 > 0$ if $n \geq 3$, $C_1 > 0$, if $n + k \geq 3$, $A_2 > 0$, $C_2 > 0$, $A_3 > 0$ if $n \geq 6$ and $C_3 > 0$ so that the following situations can happen:

I. $C_1 > C_3$ and $A_1 > A_3$

II. $C_1 < C_3$ and $A_1 < A_3$

III. $C_1 > C_3$ and $A_1 < A_3$

IV. $C_1 < C_3$ and $A_1 > A_3$

In situation I., $MSE[\hat{x}_{0IL}] < MSE[\hat{x}_{0CL}]$, while in II., $MSE[\hat{x}_{0CL}] < MSE[\hat{x}_{0IL}]$. Further, in III., $MSE[\hat{x}_{0IL}] < MSE[\hat{x}_{0CL}]$ if

$$x_0 \in (\bar{x} - (\frac{C_1 - C_3}{A_3 - A_1})^{1/2}; \bar{x} + (\frac{C_1 - C_3}{A_3 - A_1})^{1/2});$$

while, in situation IV., $MSE[\hat{x}_{0CL}] < MSE[\hat{x}_{0IL}]$, if

$$x_0 \in (\bar{x} - (\frac{C_3 - C_1}{A_1 - A_3})^{1/2}; \bar{x} + (\frac{C_3 - C_1}{A_1 - A_3})^{1/2}).$$

It can be shown that $A_1 > A_2$ if $n \geq 4$ and $C_1 > C_2$ if $n + k \geq 2$. This is the situation expressed by situation I. above. Thus, if $n \geq 4$ it follows that $MSE[\hat{x}_{0C}] < MSE[\hat{x}_{0CL}]$.

Direct analytical comparisons between estimators $\hat{x}_{0C}$ and $\hat{x}_{0IL}$ and this with $\hat{x}_{0CL}$ are complicated. Thus, numerical evaluations of the MSE of the estimators are performed for the parameter values given in Table 4.1, with the restriction that $n > k$.

Table 4.1. Parameter values used in numerical evaluations

θ	λ	l	β	n	k
0.01	0.1	1.0	0.5	10	1
0.1	1.0	10.0	1.0	15	2
1.0	10.0	100.0	5.0	30	5
	20.0	1000.0		100	10
					15

The numerical results for comparing estimators $\hat{x}_{0IL}$ and $\hat{x}_{0CL}$ are reported in Table 4.2. The numerical results for comparing estimators $\hat{x}_{0IL}$ and $\hat{x}_{0C}$ are reported in Table 4.3 below. In all numerical evaluations it was noted that $C_1 < C_3$.

Table 4.2. Numerical evaluations of A_1 and A_3

l		1			10				100				1000			
β	n \ λ	0.1	1	10	0.1	1	10	20	0.1	1	10	20	0.1	1	10	20
	10										x	x			x	x
0.5	15					x	x			x	x	x		x	x	x
	30					x	x	x		x	x	x		x	x	x
	100					x	x	x		x	x	x		x	x	x
	10						x	x			x	x				
1.0	15						x	x			x	x			x	x
	30						x	x			x	x			x	x
	100						x	x			x	x			x	x
	10															
5.0	15															
	30															
	100															

(x:situations where A3 > A1)

Note that in situations where $A_1 > A_3$, situation IV. follows, that is, $MSE[\hat{x}_{0IL}] < MSE[\hat{x}_{0CL}]$. Otherwise, there is an interval $(x_{0I}; x_{0S})$, with a better performance of predictor $\hat{x}_{0IL}$ in this region.

The numerical evaluations also show that $C_2 > C_3$ in all evaluations. The results of the numerical evaluations corresponding to A_2 and A_3 are presented in Table 4.3.

Table 4.3. Numerical evaluations of A_2 and A_3

l		1			10				100				1000			
β	n \ λ	0.1	1	10	0.1	1	10	20	0.1	1	10	20	0.1	1	10	20
	10					x	x				x	x			x	x
0.5	15		x		x	x	x	x	x	x	x	x	x	x	x	x
	30	x	x		x	x	x	x	x	x	x	x	x	x	x	x
	100	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	10						x	x			x	x				x
1.0	15		x			x	x	x		x	x	x		x	x	x
	30		x			x	x	x		x	x	x		x	x	x
	100	x	x	x	x	x	x	x	x	x	x	x	x	x	x	x
	10															
5.0	15						x	x			x	x			x	x
	30			x			x	x			x	x			x	x
	100		x	x		x	x	x		x	x	x		x	x	x

(x: situations where A3 > A2)

Note that in situations where $A_2 > A_3$, $MSE[\hat{x}_{0IL}] < MSE[\hat{x}_{0C}]$. Otherwise, if $A_2 < A_3$, which happens in most of the entries of the table, specially when l and λ are large, then $MSE[\hat{x}_{0IL}] < MSE[\hat{x}_{0C}]$ in an interval $(x_{0I}; x_{0S})$, with reversing inequality outside the interval. It is worth remarking that the numerical evaluations reported above are independent of the values of θ and k.

4.2. Graphical representations

Some graphical comparisons are reported next. In Figures 4.1.a, 4.1.b and 4.1.c, we compare the asymptotic MSE of estimators $\hat{x}_{0CL}$, given in expression (4.1) with the asymptotic MSE of the estimator $\hat{x}_{0IL}$, given by (4.3). In all evaluations we took $\theta = 0.01$, $l = 10.0$, $\beta = 0.5$ and $k = 2$. Figure 4.1.a shows the behavior of the two MSEs for $w = x_0 - \bar{x} = 0.5$, being this value increased in Figures 4.1.b ($w = 3.0$) and 4.1.c ($w = 10.0$).

Figures 4.1.a, 4.1.b and 4.1.c show that the distance w has fundamental influence on the behavior of the estimators. When $w = 0.5$, Figure 4.1.a seems to indicate that the inverse estimator is better for small sample sizes with greater improvement as λ increases. For larger sample sizes, and λ in a narrow strip, the behavior of the two estimator is similar. However, as w increases, $\hat{x}_{0CI}$ looses its supremacy over the classical estimator, unless λ is small. This corresponds to the situation where the response variable y is measured with small error.

Figures 4.2.a, 4.2.b and 4.2.c report the asymptotic MSE of estimators $\hat{x}_{0C}$ (given in (4.2)) and the estimator $\hat{x}_{0IL}$ (given in (4.3)) for the three different values of w and the same parameter values considered in Figures 4.1.a, 4.1.b and 4.1.c. The main conclusion is somewhat similar to the case reported in Figure 4.1, that is, for small values of w the inverse estimator behaves better and as w increases, the classical estimator presents better behavior. This is in accordance with the numerical illustrations reported in Tables 4.2 and 4.3.

5. Applications

In this section we consider applications of the results obtained so far to two real data sets presented in the literature.

Application 5.1. The first data set considered appears in Miller (1980), where simultaneous pairs of measurements of serum kanamycin level in blood samples drawn from 20 babies are presented. One of the measurements was obtained by a heelstick method (x) and the other by using an umbilical cateter (y).

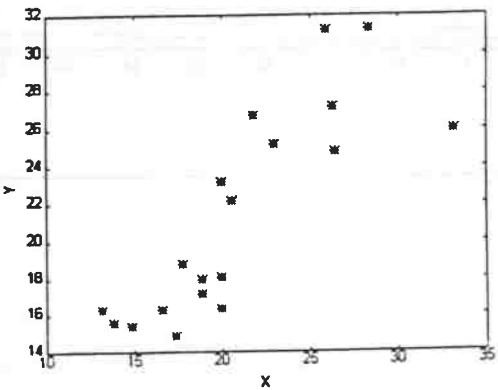


Figure 5.1. Serum kanamycin levels of 20 babies

Figure 5.1 presents the dispersion diagram of the data set. Suppose that we want to estimate $x_0 = x_2$, so that $k = 1$ and $n = 19$. In this case, $X_0 = 33.20$ and $\bar{X} = 20.20$. Some numerical evaluations show the following estimates: $\hat{x}_{0C} = 29.17$, $\hat{x}_{0CL} = 31.24$ and

Figure 4.1. $MSE[\hat{x}_{0CL}] : MSE[\hat{x}_{0IL}]$ **where (a)** $w = 0.5$; **(b)** $w = 3.0$ **and (c)** $w = 10.0$

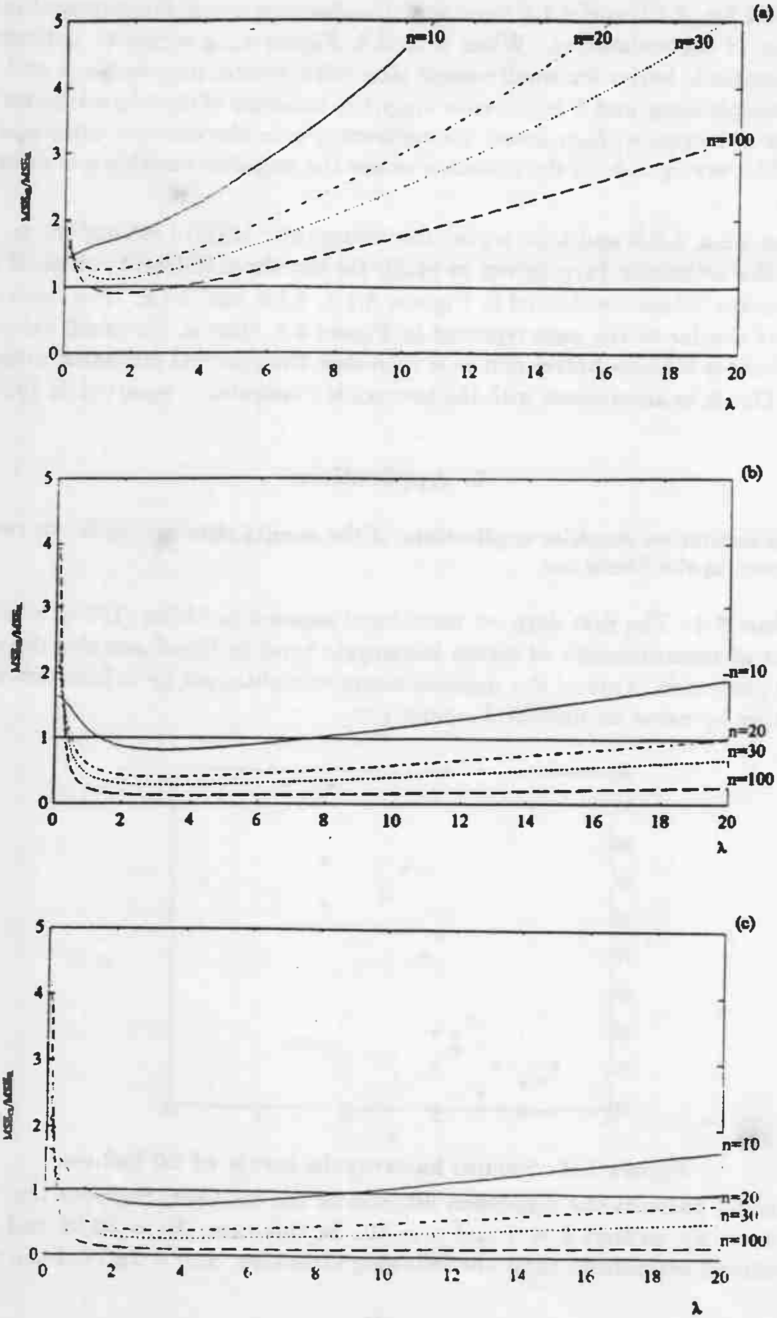
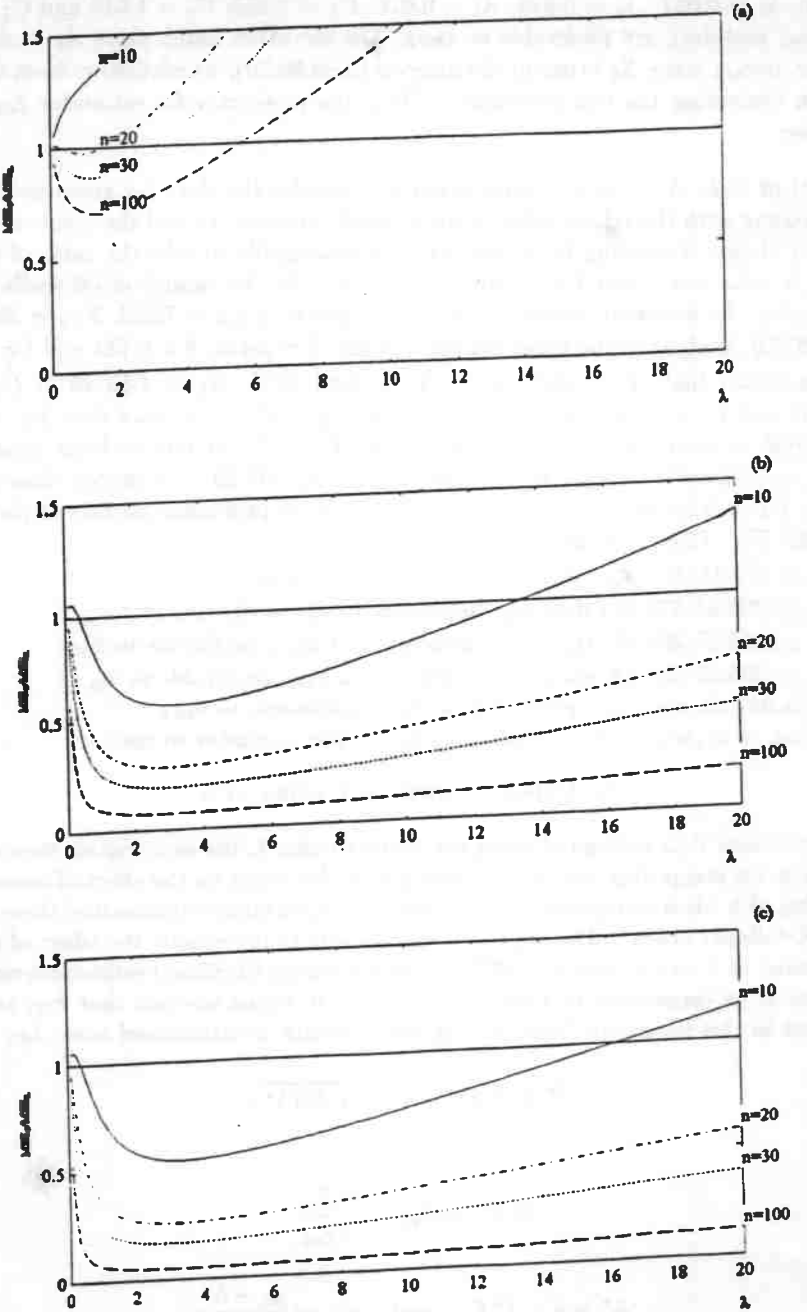


Figure 4.2. $MSE[\hat{x}_{0C}] : MSE[\hat{x}_{0IL}]$**, where (a)** $w = 0.5$**; (b)** $w = 3.0$ **and (c)** $w = 10.0$



$\hat{x}_{0IL} = 28.87$, $\hat{\beta} = 1.29$, $\hat{\beta}_L = 1.11$, $\hat{\beta}_{IL} = 0.70$, $\tilde{\theta} = 2.32$, $\hat{l} = 6.52$, $\hat{w} = \hat{x}_{0C} - \bar{X} = 8.97$. Moreover, $\hat{A}_1 = 0.037$, $\hat{A}_2 = 0.014$, $\hat{A}_3 = 0.016$, $\hat{C}_1 = 2.055$, $\hat{C}_2 = 1.612$ and $\hat{C}_3 = 1.369$, so that $\hat{x}_{0C}$ and $\hat{x}_{0IL}$ are preferable to $\hat{x}_{0CL}$. On the other hand, since $A_2 < A_3$, $\hat{x}_{0C}$ is preferable to $\hat{x}_{0IL}$ since X_0 is not in the interval (9.90;30.50), which follows from situtation III., when comparing the two estimators. Thus, the preference for estimator $\hat{x}_{0C}$ is clear in this case.

Application 5.2. As a second application we consider the data set given in Madansky (1959), dealing with the relationship of the Brinell hardness (y) and the yield strength (x) of artillery shells. According to Madansky, it is reasonable to take the ratio of variances $\lambda = 150$, a value not covered by Tables 4.2 and 4.3. In the sample of 50 shells reported by Madansky, the following statistics can be computed: $S_{XX} = 712.9$, $S_{Xy} = 2344.0$ and $S_{yy} = 13917.9$, leadind to the following estimators: $\hat{\beta} = 3.475$, $\tilde{\theta} = 0.754$ and $\hat{l} = 893.981$. It can be shown that $\hat{A}_1 = 3.02\ 10^{-4}$, $\hat{A}_2 = 3.00\ 10^{-4}$, $\hat{A}_3 = 4.50\ 10^{-4}$, $\hat{C}_1 = 9.59$ $\hat{C}_2 = 9.57$ and $\hat{C}_3 = 9.32$. Since $\hat{A}_1 > \hat{A}_2$ and $\hat{C}_1 > \hat{C}_2$, it follows that $\hat{x}_{0C}$ seems to be preferable to $\hat{x}_{0CL}$. Also, since $\hat{A}_1 < \hat{A}_3$ and $\hat{C}_1 > \hat{C}_3$, it follows from situation III. that $\hat{x}_{0IL}$ is preferable to $\hat{x}_{0CL}$ in the interval $(220.31, 307.53)$. Moreover, since $\hat{C}_2 > \hat{C}_3$ and $\hat{A}_2 < \hat{C}_3$, it follows from situation III that $\hat{x}_{0IL}$ is preferable to $\hat{x}_{0C}$ in the interval $(222.27, 305.57)$. Thus, it follows that:

If $x_0 < 220.31$ then $\hat{x}_{0C}$ preferable to $\hat{x}_{0CL}$ preferable to $\hat{x}_{0I}$;
If $x_0 \in (220.31, 222.27)$ then $\hat{x}_{0C}$ preferable to $\hat{x}_{0I}$ preferable to $\hat{x}_{0CL}$;
If $x_0 \in (222.27, 305.57)$ then $\hat{x}_{0IL}$ preferable to $\hat{x}_{0CL}$ preferable to $\hat{x}_{0C}$;
If $x_0 \in (305.57; 307.53)$ then $\hat{x}_{0C}$ preferable to $\hat{x}_{0IL}$ preferable to $\hat{x}_{0CL}$;
If $x_0 > 307.53$, then $\hat{x}_{0C}$ preferable to $\hat{x}_{0CL}$ preferable to $\hat{x}_{0IL}$.

Thus, unless x_0 is close to $\bar{x}$, it seems that $\hat{x}_{0C}$ is the estimator to use.

6. Using an incorrect value of λ

Suppose now that instead of using the correct value λ, the investigator uses an incorrect λ^* value for computing the classical estimator. An study on the effect of considering a wrong value of λ when estimating β is considered in Lakshminarayanan and Gunst (1984). See also Ketellaper (1984). The approach we follow is to investigate the effect of adopting a wrong value of λ on the bias and MSE of the maximum likelihood estimators considered in Sections 2, as considered in Ketellaper (1984). It seems obvious that $\hat{x}_{0C}$ should be inconsistent in this situation. This, among other results, is established next. Let

$$\hat{\beta}^* = S(\lambda^*) + \sqrt{\lambda^* + S^2(\lambda^*)},$$

where

$$S(\lambda^*) = S_{yy} - \frac{\lambda^* S_{xx}}{2S_{xy}},$$

$$\hat{\alpha}^* = \bar{y} - \hat{\beta}^* \bar{X}, \quad \text{and} \quad \hat{x}^*_{0C} = \frac{\bar{y}_0 - \hat{\alpha}^*}{\hat{\beta}^*}.$$

Lemma 6.1. *Under the model defined by (1.1)-(1.3), with λ (correct) replaced by λ^* (wrong), it follows that*

$$E[\hat{\beta}^*] = \frac{F}{F-1}g_\lambda(\lambda^*) + \frac{1}{nl\beta^2}\{\frac{\lambda^*}{4}(\frac{F-1}{g_\lambda(\lambda^*)})^3 J + F[g_\lambda(\lambda^*)(\lambda+\beta^2+\frac{\lambda}{l}) - (\lambda - \lambda^*\beta)]\} + O(n^{-2})$$

and

$$Var[\hat{\beta}^*] = \frac{F^2}{2nl\beta^2}J + O(n^{-2}),$$

where

$$g_\lambda(\lambda^*) = \frac{(\beta^2-\lambda^*)}{2\beta} + \frac{(\lambda-\lambda^*)}{2\beta l} + O(n^{-1}), \quad F = 1 + \frac{g_\lambda(\lambda^*)}{\sqrt{\lambda^* + g_\lambda^2(\lambda^*)}},$$

$$J = 2(\lambda^*)^2 + 2\lambda\beta^2 + 2\frac{\lambda\lambda^*}{l} - (\lambda-\lambda^*)(4\beta g_\lambda(\lambda^*) - \frac{(\lambda-\lambda^*)}{l}) + 2g_\lambda^2(\lambda^*)(\lambda+\beta^2+\frac{\lambda}{l}),$$

with l defined in Lemma 3.1.

The proof of the above lemma follows by taking Taylor series expansions of $\hat{\beta}^*$ at $(E[S_{XX}], E[S_{Xy}], E[S_{yy}])$ and then substituting for the required expectations, variances and covariances, which are given in Sections 2. The above lemma allows us to state the main result of the section.

Theorem 6.1. *Under the assumptions of Lemma 6.1 and assuming that k is a fixed integer, it follows that*

$$E[\hat{x}_{0C}^*] = x_0 + \frac{(x_0-\bar{x})}{Fg_\lambda(\lambda^*)}[F(\beta - g_\lambda(\lambda^*)) - \beta] - \frac{(x_0-\bar{x})(F-1)^2}{n\beta lF^2g_\lambda^2(\lambda^*)}I + \frac{(x_0-\bar{x})(F-1)^3}{2n\beta lFg_\lambda^3(\lambda^*)}J + O(n^{-2}) \tag{6.1}$$

and

$$Var[\hat{x}_{0C}^*] = \frac{\theta(F^2g_\lambda^2(\lambda^*) + \lambda(F-1)^2)}{nF^2g_\lambda^2(\lambda^*)} + \frac{\lambda\theta(F-1)^2}{kF^2g_\lambda^2(\lambda^*)} - \frac{2\lambda\theta(F-1)^3}{nk\beta^2lF^3g_\lambda^2(\lambda^*)}I + ((x_0-\bar{x})^2 + \frac{3\lambda\theta}{k\beta^2})\frac{(F-1)^4}{2nlF^2g_\lambda^4(\lambda^*)}J + O(n^{-2}), \tag{6.2}$$

where l is as given in Lemma 3.1, $g_\lambda(\lambda^*)$, F and J are as defined in Lemma 6.1 and

$$I = \frac{\lambda^*(F-1)^3}{4g_\lambda^3(\lambda^*)}J + F\{g_\lambda(\lambda^*)(\lambda+\beta^2+\frac{\lambda}{l}) - (\lambda-\lambda^*)\beta\}.$$

The mean squared error of estimator $\hat{x}_{0C}^*$ follows easily from (6.1) and (6.2). Figure 6.1 shows the performance of estimator $\hat{x}_{0C}^*$ computed with respect to the to λ^* with respect to the estimator $\hat{x}_{0C}$, computed with respect to the correct value of λ. As seen, the MSE of estimator $\hat{x}_{0C}$ can be severely affected by selecting a wrong value λ^*, specially if this value is somewhat far form the correct one. Moreover, the performance of $\hat{x}^*_{0C}$ becomes worse as w increases. Figure 6.2 compares the performance of $\hat{x}^*_{0C}$ with respect to the estimator $\hat{x}_{0IL}$. As seen from the figure, $\hat{x}_{0C}$ presents a better performance only close to the correct value of λ. Figure 6.3 compares the performance of $\hat{x}_{0CL}$ with respect to $\hat{x}_{0C}$ and $\hat{x}^*_{0C}$. As seen, estimator $\hat{x}^*_{0C}$ can be severely affected by large values of λ^*. We recall that it has been shown analytically in Section 4 that $MSE[\hat{x}_{0C}] < MSE[\hat{x}_{0CL}]$.

References

Barnett, V.D. (1969). Simultaneous pairwise linear structural relationship. *Biometrics*, 25, 129-142.

Brown, P. J. (1993). *Measurement, regression and calibration*. Oxford University Press. Oxford.

Davies, R.B. and Hulton, B. (1975). The effect of errors in the independent variable. *Biometrika*, 62, 383-391.

Fuller, W.A. (1987). *Measurement error models*. Wiley. New York.

Hoadley, B. (1970). A Bayesian look at inverse linear regression. *Journal of the American Statistical Association*, 65, 356-369.

Kendall, M.G. and Stuart, A. (1952). *The advanced theory of satatistics*. Vol. 1, 5th Edition. Griffin. London.

Ketellaper, R.H. (1983). On estimating parameters in a simple linear error in variables model. *Technometrics*, 25, 43-47.

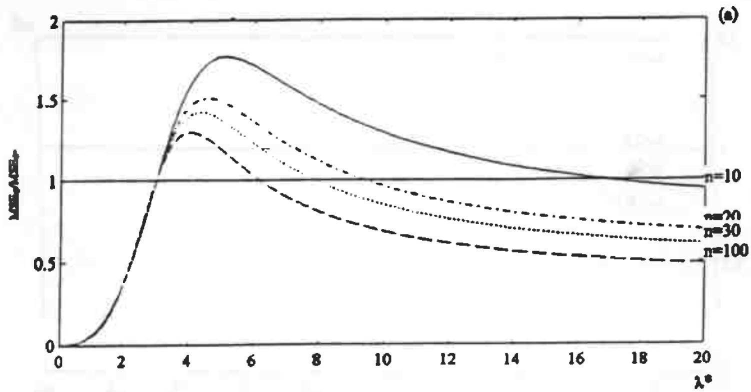
Krutchokoff, R.G. (1969). Classical and inverse regression methods of calibration in extrapolation. *Technometrics*, 11, 605-609.

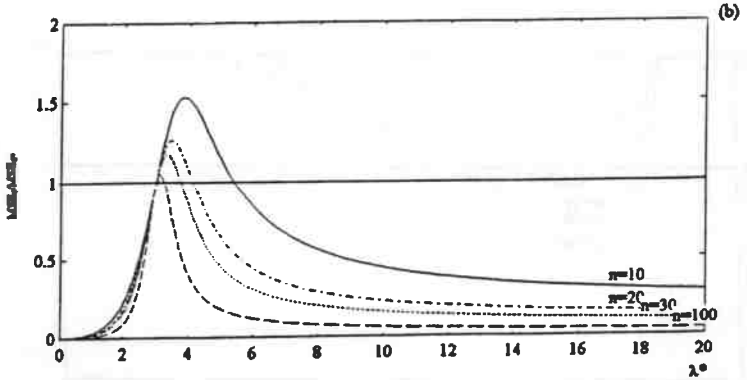
Lakshminarayanan, M.Y. and Gunst, R.F. (1984). Estimation of parameters in linear structural relationships: sensitivity to the choice of the ratio of error variances. *Biometrika*, 71, 569-573.

Lee, S.H. and Yum, B.J. (1989). Large sample comparisons of calibration procedures when both measurements are subject to error: The unreplicated case. *Communication in Statitics, Theory and Methods*, 18, 3821-3840.

Lwin, T. and Maritz, J.S. (1982). An analysis of the linear calibration controversy from the perspective of compound estimation. *Technometrics*, 24, 235-242.

Figure 6.1. $MSE[\hat{x}_{0C}] : MSE[\hat{x}^*_{0C}]$, **where (a)** $w = 0.5$; **(b)** $w = 3.0$ **and (c)** $w = 10.0$. **True** $\lambda = 3.0$





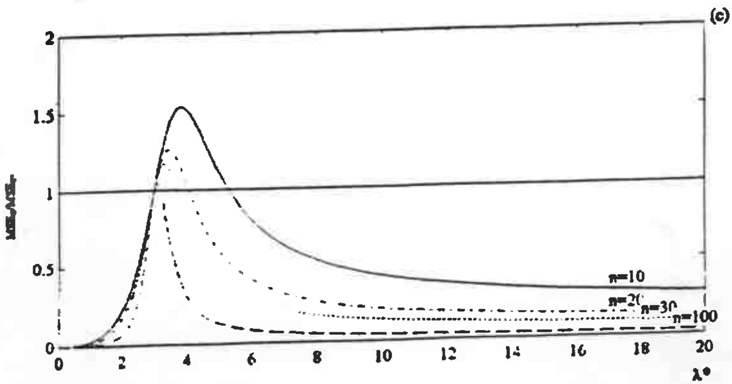
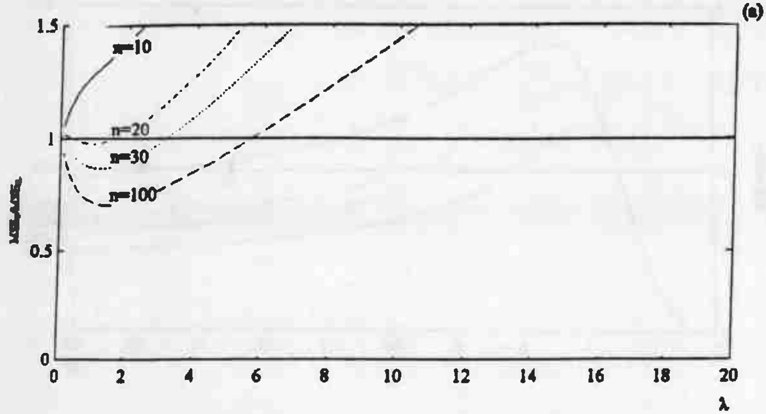


Figure 6.2. (a) $MSE[\hat{x}_{0C}] : MSE[\hat{x}_{0IL}]$ and (b) $MSE[\hat{x}^*_{0C}] : MSE[\hat{x}_{0IL}]$, where $w = 0.5$. True $\lambda = 3.0$



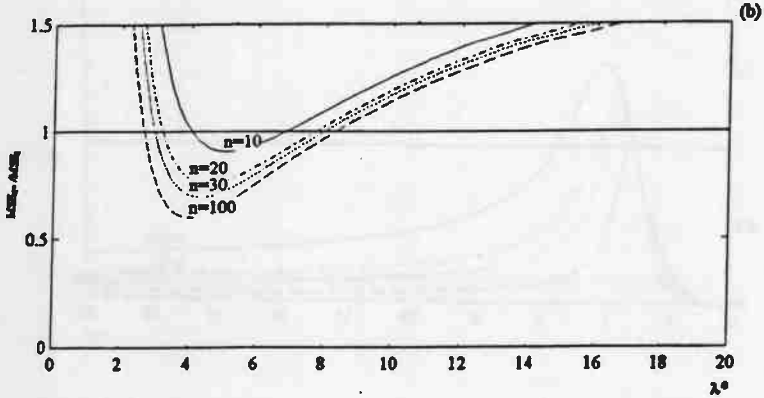
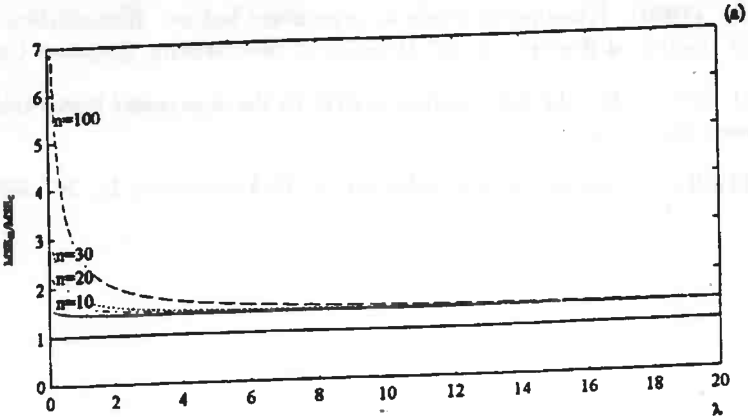
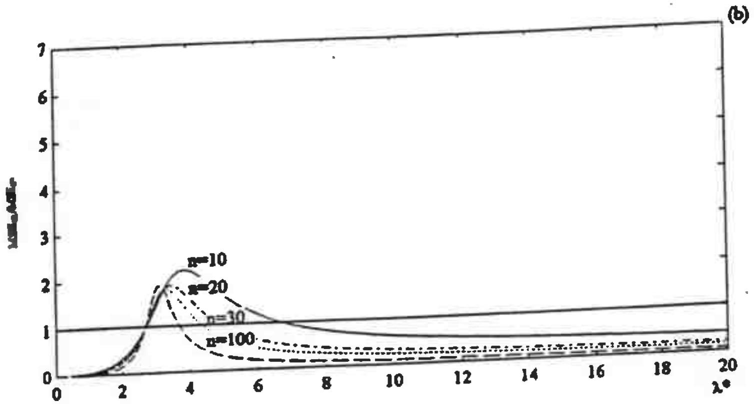


Figure 6.3. (a) $MSE[\hat{x}_{0CL}] : MSE[\hat{x}_{0C}]$, (b) $MSE[\hat{x}_{0CL}] : MSE[\hat{x}^*_{0C}]$, where $w = 3.0$. True $\lambda = 3.0$





Madansky, A. (1959). The fitting of straight lines when both variables are subject to error. *Journal of the American Statistical Association*, 54, 173-205.

Miller, R.G., Jr. (1980). Kanamycin levels in premature babies. *Biostatistics Casebook*, Vol. 3, 127-142. Technical Report No. 57, Division of Biostatistics, Stanford University.

Patefield, W.M. (1977). On the information matrix in the functional linear relationship. *Applied Statistics*, 26, 69-70.

Shukla, G.K. (1972). On the problem of calibration. *Technometrics*, 14, 547-553.

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