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Equivalences in selective topological games on the class of dense subsets in the space of continuous functions $C_k(X)$

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ABSTRACT

We obtain an equivalence in variations of selective topological games for the case of the class of k-covers in regular spaces and the class of dense subsets of the space of continuous functions with the compact-open topology.

1. Introduction

In recent years, the selection principles and topological games have become very important in describing many topological properties. The formal definition of selection principles was born in [11], and is as follows:

1. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of sets. We say that $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ holds if, for any sequence $\langle A_n : n \in \omega\rangle$ of elements in $\mathcal{A}$, there is a sequence $\langle b_n : n \in \omega\rangle$ with, for all $n \in \omega$, $b_n \in A_n$ and such that $\{b_n : n \in \omega\} \in \mathcal{B}$.
2. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of sets. We say that $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ holds if, for any sequence $\langle A_n : n \in \omega\rangle$ of elements in $\mathcal{A}$, there is a sequence $\langle F_n : n \in \omega\rangle$ with, for all $n \in \omega$, $F_n \in [A_n]^{<\aleph_0}$ and such that $\bigcup_{n\in\omega} F_n \in \mathcal{B}$.

For several years, many different results have involved selection principles, with the most diversity of classes $\mathcal{A}$ and $\mathcal{B}$ appearing in the literature.

We must emphasize that in the notation, the sub-index 1 and fin indicate the number of elements selected from each element of the sequence (fin indicates that is selected finitely many elements). Then, naturally, we can define variations of these selection principles, such as $\mathsf{S}_2(\mathcal{A},\mathcal{B})$, $\mathsf{S}_3(\mathcal{A},\mathcal{B})$,...; and, more generally, $\mathsf{S}_f(\mathcal{A},\mathcal{B})$, with $f : \omega \to \omega\backslash\{0\}$ being an arbitrary function.

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On the other hand, the term "topological property defined for a game" was introduced for Telgársky in [18]. The formal definitions of selective topological games are as follows:

1. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of sets. The game of two players $\mathsf{G}_1(\mathcal{A},\mathcal{B})$ is played as: in each round $n \in \omega$, Player I chooses $A_n \in \mathcal{A}$. Player II responds $b_n \in A_n$. Player II wins if $\{b_n : n \in \omega\} \in \mathcal{B}$. Otherwise, Player I wins.
2. Let $\mathcal{A}$ and $\mathcal{B}$ be classes of sets. The game for two players $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B})$ is played as: in each round $n \in \omega$, player I chooses $A_n \in \mathcal{A}$. Player II responds $F_n \in [A_n]^{<\aleph_0}$. Player II wins if $\bigcup_{n\in\omega} F_n \in \mathcal{B}$. Otherwise, Player I wins.

Informally, a strategy in a game G is a form of how a determinate player decides his play in a specific inning. We use the notation $I \uparrow \mathsf{G}$ to state that there is a winning strategy (that is, a strategy that cannot be defeated for any strategy of his opponent) for Player I in the game G (and $I \not\uparrow \mathsf{G}$ if not). For more information about topological games, see [3].

Note that we can analogously define a game by changing the number of elements chosen by Player II and we can define $\mathsf{G}_2(\mathcal{A},\mathcal{B})$, $\mathsf{G}_3(\mathcal{A},\mathcal{B})$,...; and $\mathsf{G}_f(\mathcal{A},\mathcal{B})$, with $f : \omega \to \omega\backslash\{0\}$ an arbitrary function.

We can easily see that if $I \not\uparrow \mathsf{G}_1(\mathcal{A},\mathcal{B})$ then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ holds and the same result applies to $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$. An interesting question is whether one will know if the reciprocal result is valid, too. There are some particular cases where this is true, as well as cases where it is not valid.

The Hurewicz's and Pawlikowski's theorems (the original versions were born in [8] and [10], respectively) state that the reciprocal result is true, in the case where $\mathcal{A}$ and $\mathcal{B}$ are both families of open covers. Interesting and recent applications in topological games, particularly of the results just mentioned, is to establish results with respect to colorings of edge sets of complete graphs with vertices in infinite semigroups (also called Ramsey results). For a little more of information, see [16], [17].

By a topological space (X,τ), we say that a family $\mathcal{U}$ of subsets of X is an open cover if any element of $\mathcal{U}$ is open and $\bigcup\mathcal{U} = X$. We denote by $\mathcal{O}_X$ a family of all open covers in X.

We say that a open cover $\mathcal{U}$ of X is a *k-cover* if for any compact $K \subset X$ there is $U \in \mathcal{U}$ such that $K \subset U$. We denote by $\mathcal{K}_X$ a family of all *k-covers* in X.

If $\mathfrak{B}$ is a family of subsets in X, then a family $\mathcal{U}$, of open subsets of X, is a $\mathfrak{B}$*-cover* if $X \notin \mathcal{U}$ and for all $B \in \mathfrak{B}$, there is $U \in \mathcal{U}$ such that $B \subset U$. For $\mathfrak{B}$ fixed, we denote by $\mathcal{O}^X_{\mathfrak{B}}$ the family of all $\mathfrak{B}$*-covers*.

We say that $\mathfrak{B}$ is a bornology if it is ideal and covers the entire space X. We say that a subset $\mathfrak{B}'$ is a compact base of $\mathcal{B}$, if it is cofinal and every element is a compact set.

The i-weight of X is the smallest cardinality $\mathrm{w}(Y)$ (recall that $\mathrm{w}(Y)$ is the smallest cardinality of a basis of Y), where Y is a continuous one-to-one image of X. Denote the i-weight of X as $\mathrm{iw}(X)$.

The set of all dense subsets of a topological space X is denoted by $\mathcal{D}_X$.

In [2] the following problem was proposed:

Question 1.1. *What can be said about the relation between the various games* $\mathsf{G}_k(\mathcal{A},\mathcal{B})$, $\mathsf{G}_f(\mathcal{A},\mathcal{B})$ *and* $\mathsf{G}_{\text{fin}}(\mathcal{A},\mathcal{B})$*-and their associated selective properties- for other pairs* $(\mathcal{A},\mathcal{B})$*?*

In this paper we focus in the study of relations about games. In the case that $\mathcal{A} = \mathcal{B} = \mathcal{O}_X$ an equivalence of the games $\mathsf{G}_1(\mathcal{O}_X,\mathcal{O}_X)$ and $\mathsf{G}_f(\mathcal{O}_X,\mathcal{O}_X)$, with $f : \omega \to \omega\backslash\{0\}$ be an arbitrary function, is obtained when X is a Hausdorff space (the case when X is not Hausdorff is still open!).

In the same work [2] was proved that the games $\mathsf{G}_k(\Omega_x,\Omega_x)$, $\mathsf{G}_{k+1}(\Omega_x,\Omega_x)$, $\mathsf{G}_f(\Omega_x,\Omega_x)$, with $k \in \omega$ are all different (recall that $\Omega_x = \{A \subset X : x \in \overline{A}\}$, for $x \in X$). Here, we proved an equivalence of these games in the case of $\mathcal{A} = \mathcal{B} = \mathcal{K}_X$ in the case when X is a regular space (section 2), and an equivalence in the case of $\mathcal{A} = \mathcal{B} = \mathcal{D}_{C_k(X)}$, when X is a Tychonoff space and $\mathrm{iw}(X) = \aleph_0$ (section 3). Finally, in section 4 we make emphasis in open problems and additional commentaries about equivalences in topological games.

By definitions that not include here, we cite: about general topology [7] and about topological games [3].

2. Equivalence in variations of selective topological games that involves k-covers

We start by recalling the following result establishes an equivalence for games with respect to Player I in the case of $\mathcal{B}$-covers

Theorem 2.1. *([4]) Let (X, τ) be a topological space, $\mathfrak{B}$ be a family of subsets of X and $f : \omega \to \omega \backslash \{0\}$ be a function. The following assertions are equivalent:*

1. $I \uparrow \mathsf{G}_1(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$;
2. $I \uparrow \mathsf{G}_f(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$.

In particular, $I \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$ if, and only if, $I \uparrow \mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$.

Let $\mathfrak{B}$ be a bornology. By $B \in \mathfrak{B}$, we define $\tau_B = \{U \in \tau : B \subseteq U\}$ and $\mathbf{B}_{\mathfrak{B}} = \{\tau_B : B \in \mathfrak{B}\}$. With this notation, we can define the following

Definition 2.2. The game $\mathfrak{B}$-*open* is played as follows: in each inning $n \in \omega$, Player I chooses $B \in \mathfrak{B}$ and Player II responds with $U_n \in \tau_B$. Player I wins if $\{B_n : n \in \omega\} \in \mathcal{O}_{\mathfrak{B}}$. Otherwise, Player II is the winner.

The following definitions were introduced in [5]

Definition 2.3. Let X and Y be two sets. Then X is called coinitial in Y with respect to $\subseteq$, denote this by $X \preceq Y$, if $X \subseteq Y$ and for all $y \in Y$, there is an $x \in X$ such that $x \subseteq y$.

Definition 2.4. A set $\mathcal{R}$ is called a reflection of a family $\mathcal{A}$ if $\{\mathrm{range}(f) : f \in \mathcal{C}(\mathcal{R})\} \preceq \mathcal{A}$, where $\mathcal{C}(X) = \{f \in (\bigcup X)^X : x \in X \Rightarrow f(x) \in x\}$ is the collection of all choice functions on X.

Theorem 2.5. *([5]) Let $\mathcal{R}$ be a reflection of a family $\mathcal{A}$. Then $\mathsf{G}_1(\mathcal{A}, \mathcal{B})$ and $\mathsf{G}_1(\mathcal{R}, \neg\mathcal{B})$ are dual games, where $\neg\mathcal{B}$ denotes $\mathcal{P}(\bigcup \mathcal{A}) \backslash \mathcal{B}$.*

Additionally we can note the following

Proposition 2.6. *$\mathbf{B}_{\mathfrak{B}}$ is a reflection of $\mathcal{O}^X_{\mathfrak{B}}$.*

Proof. Let $\mathcal{U} \in \mathcal{O}^X_{\mathfrak{B}}$ and $\tau_B \in \mathbf{B}_{\mathfrak{B}}$, with $B \in \mathfrak{B}$. So, there is $U_B \in \mathcal{U}$ such that $B \subseteq U_B$. Define $f(\tau_B) = U_B \in \tau_B$. It is clear that $\mathrm{range}(f) \in \mathcal{O}^X_{\mathfrak{B}}$ and $\mathrm{range}(f) \subseteq \mathcal{U}$. □

With the last result, we can obtain the following

Theorem 2.7. *The games* $\mathfrak{B}$-open *and $\mathsf{G}_1(\mathcal{O}_{\mathfrak{B}}, \mathcal{O}_{\mathfrak{B}})$ are dual.*

Proof. Note that the game $\mathfrak{B}$-*open* is equivalent to $\mathsf{G}_1(\mathbf{B}_{\mathfrak{B}}, \neg\mathcal{O}^X_{\mathfrak{B}})$. The result follows from the previous proposition and Theorem 2.5. □

In particular, when $\mathfrak{B} = \mathbf{K} := \{A \subset X : \overline{A} \text{ is compact}\}$, we call the game $\mathfrak{B}$-*open* as $\mathcal{K}_X$-*open*.

Now, we can obtain

Lemma 2.8. *Let (X, τ) be a regular topological space. The following assertions are equivalent:*

1. *X is compact;*
2. *For all $\mathcal{U} \in \mathcal{K}_X$, there is $\mathcal{U}' \subset \mathcal{U}$ finite such that $X \subset \overline{\bigcup \mathcal{U}'}$*

Observation 2.9. *Note that the previously result is true if we change, in the statement (2), $\mathcal{K}_X$ by $\mathcal{O}^X_{\mathfrak{B}}$, where $\mathfrak{B}$ is a bornology with a compact base.*

With the above lemma we obtain the following

Lemma 2.10. *Let (X, τ) be a regular space. Let σ be a strategy for Player II in $\mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$. For all $s \in (\mathcal{K}_X)^{<\omega}$, define:*

$$C_s = \bigcap_{\mathcal{U} \in \mathcal{K}_X} \overline{\bigcup \sigma(s^\frown \langle \mathcal{U} \rangle)}$$

Then C_s is a compact subset of X.

Proof. Let $\mathcal{U} \in \mathcal{K}_{C_s}$. According to the Lemma 2.8, it suffices to prove that there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that $C_s \subset \overline{\bigcup \mathcal{U}'}$.

Let a compact $K \subset X$. By the regularity, we can obtain $\{A_x : x \in K \cap (X \backslash C_s)\}$ an open cover of $K \cap (X \backslash C_s)$, where the closure of A_x is disjoint of C_s. Then,

$$K = (K \cap C_s) \cup [K \cap (X \backslash C_s)] \subset U_K \cup \left(\bigcup_{x \in K \cap (X \backslash C_s)} A_x \right).$$

Due to the compactness of K, we have

$$\mathcal{V} = \left\{ U_K \cup \left(\bigcup_{i=1}^{r_K} A_{x_i} \right) : K \subset X \text{ compact}, \{x_i : 1 \leq i \leq r_K\} \in [K \cap (X \backslash C_s)]^{<\aleph_0}, r_K \in \omega \backslash \{0\} \right\} \in \mathcal{K}_X.$$

So, $C_s \subset \overline{\bigcup \sigma(s^\frown \mathcal{V})}$. As $\bigcup_{i=1}^{r_{K_1}} \overline{A_{x_i^1}}$ and $\bigcup_{i=1}^{r_{K_2}} \overline{A_{x_i^2}}$ are disjoint from C_s, these elements can be removed from the set $\overline{\bigcup \sigma(s^\frown \mathcal{V})}$. So, $C_s \subset \overline{U_{K_1} \cup U_{K_2}}$. Therefore, C_s is a compact subset of X. □

Lemma 2.11. *Suppose that a topological space (X, τ) satisfies the requirement that for all $\mathcal{U} \in \mathcal{K}_X$ there is a countable $\mathcal{U}' \in \mathcal{K}_X$ such that $\mathcal{U}' \subseteq \mathcal{U}$. If $A \subseteq X$ is closed, then A satisfies the requirement that for all $\mathcal{V} \in \mathcal{K}_A$ there is a countable $\mathcal{V}' \in \mathcal{K}_A$ such that $\mathcal{V}' \subseteq \mathcal{V}$.*

Observation 2.12. *The previous lemma is also valid in the following form: If, in a topological space (X, τ), for all $\mathcal{U} \in \mathcal{O}^X_{\mathfrak{B}}$ there is a countable $\mathcal{U}' \in \mathcal{O}^X_{\mathfrak{B}}$ (resp. $\mathcal{O}_X$) with $\mathcal{U}' \subseteq \mathcal{U}$, then any closed subset A of X has the following property: for any $\mathcal{V} \in \mathcal{O}^A_{\mathfrak{C}}$, there is a countable $\mathcal{V}' \in \mathcal{O}^A_{\mathfrak{C}}$ (resp. $\mathcal{O}_A$) with $\mathcal{V}' \subseteq \mathcal{V}$ (here $\mathfrak{B}$ is a bornology with compact base and $\mathfrak{C} = \{B \cap A : B \in \mathfrak{B}\}$).*

Based on the proof in [6], we can obtain the following result.

Theorem 2.13. *Let (X, τ) be a regular space. Then $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$ and $\mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$ are equivalent.*

Proof. It is sufficient to prove $II \uparrow \mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X) \Rightarrow II \uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

Let σ be a winning strategy for Player II in the game $\mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$ We define a winning strategy ρ for Player I in the game $\mathcal{K}_X$-*open* as follows. Consider $C_0 := C_\emptyset$, where $C_\emptyset$ is, as in Lemma 2.10, the first move of the strategy ρ in the game $\mathcal{K}_X$-*open*.

Suppose that Player II responds with $V_0 \in \tau$ such that $C_0 \subseteq V_0$. Then $X \backslash V_0 \subseteq X \backslash C_0$. Let $K \subset X \backslash V_0$ be a compact set. For all $x \in K$, by definition of $C_\emptyset$, there is $\mathcal{U}_x \in \mathcal{K}_X$ such that $x \in X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}_x \rangle)}$. So

$$K \subseteq \bigcup_{x \in K} X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}_x \rangle)}.$$

As $\{X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}_x \rangle)} : x \in K\}$ is an open cover of K, it follows from the compactness of K that there exist $n_K \in \mathbb{N}$ and a finite set $A_K = \{x_i^K : 1 \leq i \leq n_K\} \subset K$ such that

$$K \subseteq \bigcup_{i \in F_K} X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}^K_{(i)} \rangle)},$$

where $\mathcal{U}^K_{(i)} := \mathcal{U}_{x_i^K}$ and $F_K := \{1, 2, ..., n_K\}$.

Then,

$$\left\{ \bigcup_{i \in F_K} X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}^K_{(i)} \rangle)} : K \subseteq X \backslash V_0 \text{ compact} \right\} \in \mathcal{K}_{X \backslash V_0}.$$

By Lemma 2.11, we can fix a set

$$\left\{ \bigcup_{i \in F_{(m)}} X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}^{(m)}_{(i)} \rangle)} : m \in \omega \right\} \in \mathcal{K}_{X \backslash V_0},$$

where $\mathcal{U}^{(m)}_{(i)} := \mathcal{U}^{K_{(m)}}_{(i)}$ and $F_{(m)} := F_{K_{(m)}}$.

Fix any bijection $\varphi : \omega^{<\omega} \longrightarrow \omega$ such that if $s \subset t$ then $\varphi(s) \leq \varphi(t)$ and we define $(F_{t \upharpoonright k})^{dom(t)} := F_{t \upharpoonright 1} \times F_{t \upharpoonright 2} \times ... \times F_{t \upharpoonright dom(t)}$. Suppose that up to the inning $n \in \omega$ in the game $\mathcal{K}_X$-*open*, the sequence C_0, V_0,...,C_{n-1}, V_{n-1} has been played, where V_j is an open set that contains C_j, for all $0 \leq j \leq n-1$, and $\mathcal{U}_r^{\varphi^{-1}(j)^\frown m}$, with $r \in (F_{t \upharpoonright k})^{dom(\varphi^{-1}(j)^\frown m)}$, were also defined and satisfies the following, for all $m \in \omega$. If $s = \varphi^{-1}(j)$ and $r \in (F_{t \upharpoonright k})^{dom(s)}$, then:

1.

$$C_j^r = \bigcap_{\mathcal{U} \in \mathcal{K}_X} \overline{\bigcup \sigma(\langle \mathcal{U}^{s \upharpoonright 1}_{r \upharpoonright 1}, \mathcal{U}^{s \upharpoonright 2}_{s \upharpoonright 2}, ..., \mathcal{U}^{s \upharpoonright \mathrm{dom}(s)}_{r \upharpoonright dom(r)}, \mathcal{U} \rangle)},$$

for $0 \leq j \leq n-1$. Note that this set is a compact subset of X by Lemma 2.10. So,

$$C_j = \bigcup_{r \in (F_{t \upharpoonright k})^{dom(s)}} C_j^r$$

is a compact subset of X.

2. By Lemma 2.11, there is

$$\left\{ \bigcup_{i \in F_{s^\frown m}} \bigcap_{r \in (F_{t \upharpoonright k})^{dom(s)}} X \backslash \overline{\bigcup \sigma(\langle \mathcal{U}^{s \upharpoonright 1}_{r \upharpoonright 1}, \mathcal{U}^{s \upharpoonright 2}_{r \upharpoonright 2}, ..., \mathcal{U}^{s \upharpoonright \mathrm{dom}(s)}_{r \upharpoonright \mathrm{dom}(r)}, \mathcal{U}^{s^\frown m}_{r^\frown i} \rangle)} \right\}_{m \in \omega} \in \mathcal{K}_{X \backslash V_j}.$$

Now, we define the choice of Player I using ρ in this inning. Let $t = \varphi^{-1}(n)$. For all $r \in (F_{t\restriction k})^{dom(t)}$, we define:

$$C_n^r = \bigcap_{\mathcal{U}\in\mathcal{K}_X} \overline{\bigcup \sigma(\langle \mathcal{U}_{r\restriction 1}^{t\restriction 1}, \mathcal{U}_{r\restriction 2}^{t\restriction 2}, ..., \mathcal{U}_{r\restriction dom(r)}^{t\restriction \mathrm{dom}(t)}, \mathcal{U}\rangle)}.$$

Note that this set is compact by Lemma 2.10. So, if we define

$$C_n = \bigcup_{r\in(F_{t\restriction k})^{dom(t)}} C_n^r,$$

is a compact subset.

If V_n is a choice of Player II, by Lemma 2.11, it follows that there is:

$$\left\{ \bigcup_{i\in F_{t^\frown m}} \bigcap_{r\in(F_{t\restriction k})^{dom(t)}} X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_{r\restriction 1}^{t\restriction 1}, \mathcal{U}_{r\restriction 2}^{t\restriction 2}, ..., \mathcal{U}_{r\restriction \mathrm{dom}(r)}^{t\restriction \mathrm{dom}(t)}, \mathcal{U}_{r^\frown(i)}^{t^\frown m}\rangle)} \right\}_{m\in\omega} \in \mathcal{K}_{X\setminus V_n}.$$

This completes the definition of the strategy $\rho : {}^{<\omega}(\mathbf{B_K}) \to \mathbf{K}$ for Player I in the game $\mathcal{K}_X$*-open*. We now prove that ρ is a winning strategy. In fact, suppose that C_0, V_0, C_1, V_1,... is a play in the $\mathcal{K}_X$*-open* game, where Player I uses strategy ρ.

Suppose that $\{V_n : n \in \omega\} \notin \mathcal{K}_X$. Then, there is a compact $K \subset X$ such that $K \not\subset V_n$, for all $n \in \omega$. In particular, there is $x_0 \in K$ and $x_0 \notin V_0$. So, there is $m_0 \in \omega$ such that

$$x_0 \in \bigcup_{i\in F_{(m_0)}} X\setminus \overline{\bigcup \sigma(\langle \mathcal{U}_{(i)}^{(m_0)}\rangle)}.$$

Then, there is $i_0 \in F_{(m_0)}$ such that

$$x_0 \in X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_{(i_0)}^{(m_0)}\rangle)}.$$

Let $n_1 = \varphi((m_0))$. There is $x_1 \in K$ such that $x_1 \notin V_{n_1}$. So, there is $m_1 \in \omega$ such that

$$x_1 \in \bigcup_{i\in F_{(m_0,m_1)}} \bigcap_{r\in(F_{(m_0)})^1} X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_r^{(m_0)}, \mathcal{U}_{r^\frown(i)}^{(m_0,m_1)}\rangle)}.$$

So, there is $i_1 \in F_{(m_0,m_1)}$ such that

$$x_1 \in \bigcap_{r\in(F_{(m_0)})^1} X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_r^{(m_0)}, \mathcal{U}_{r^\frown(i_1)}^{(m_0,m_1)}\rangle)}.$$

In particular:

$$x_1 \in X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_{(i_0)}^{(m_0)}, \mathcal{U}_{(i_0,i_1)}^{(m_0,m_1)}\rangle)}.$$

In general, suppose that we have defined $m_0, m_1, ..., m_{l-1} \in \omega$ and $i_0, i_1, ..., i_{l-1}$, with $i_k \in F_{(m_0,m_1,...,m_k)}$, $0 \le k \le l-1$, such that

$$x_k \in X\setminus\overline{\bigcup \sigma(\langle \mathcal{U}_{(i_0)}^{(m_0)}, \mathcal{U}_{(i_0,i_1)}^{(m_0,m_1)}, ..., \mathcal{U}_{(i_0,i_1,...,i_k)}^{(m_0,m_1..,m_k)}\rangle)},$$

for all $0 \leq k \leq l-1$. Let $n_l = \varphi((m_0, m_1, ...m_{l-1})) = \varphi(t)$. As there is $x_l \in K$ such that $x_l \notin V_{n_l}$, it follows that there is $m_l \in \omega$ such that

$$x_l \in \bigcup_{i \in F_{(m_0,m_1,...,m_l)}} \bigcap_{r \in (F_{t\upharpoonright k})^{dom(t)}} X \setminus \overline{\bigcup \sigma(\langle \mathcal{U}^{(m_0)}_{r\upharpoonright 1}, ..., \mathcal{U}^{(m_0,m_1,...,m_{l-1})}_{r\upharpoonright l}, \mathcal{U}^{(m_0,m_1,...,m_l)}_{r^\frown i} \rangle)}.$$

Then there is $i_l \in F_{(m_0,...,m_{l-1},m_l)}$ such that

$$x_l \in \bigcap_{r \in (F_{t\upharpoonright k})^{dom(t)}} X \setminus \overline{\bigcup \sigma(\langle \mathcal{U}^{(m_0)}_{r\upharpoonright 1}, ..., \mathcal{U}^{(m_0,m_1,...,m_{l-1})}_{r\upharpoonright l}, \mathcal{U}^{(m_0,m_1,...,m_l)}_{r^\frown i_l} \rangle)}.$$

In particular:

$$x_l \in X \setminus \overline{\bigcup \sigma(\langle \mathcal{U}^{(m_0)}_{(i_0)}, ..., \mathcal{U}^{(m_0,m_1,...,m_{l-1})}_{(i_0,i_1...,i_{l-1})}, \mathcal{U}^{(m_0,m_1,...,m_l)}_{(i_0,i_1,...,i_{l-1},i_l))} \rangle)}.$$

So, we obtain $\{\mathcal{U}^{(m_0,m_1...,m_l)}_{(i_0,i_1,...,i_l}\}_{l\in\omega}$, a sequence of $\mathcal{K}_X$-*covers* such that there is a $K \subset X$ compact, with the property that

$$K \not\subset \bigcup \sigma(\langle \mathcal{U}^{(m_0)}_{(i_0)}, ..., \mathcal{U}^{(m_0,...,m_l)}_{(i_0,...,i_l)} \rangle),$$

for all $l \in \omega$. That is, this sequence defines a strategy of Player I to defeat σ in the game $\mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$. But this contradicts the fact that σ is a winning strategy for Player II in the game $\mathsf{G}_2(\mathcal{K}_X, \mathcal{K}_X)$.

Therefore, ρ is a winning strategy for Player I in the $\mathcal{K}_X$-*open* game. By duality, there is a winning strategy for Player II in $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$. This concludes the proof. □

By Observation 2.9, we can obtain the following results

Lemma 2.14. *Let (X, τ) be a regular space and $\mathfrak{B}$ be a bornology with compact base. Let σ be a strategy for Player II in $\mathsf{G}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. For $s \in^\omega \mathcal{O}^X_{\mathfrak{B}}$, define:*

$$C_s = \bigcap_{\mathcal{U} \in \mathcal{O}^X_{\mathfrak{B}}} \overline{\bigcup \sigma(s^\frown \langle \mathcal{U} \rangle)}.$$

Then C_s is a compact subset of X.

Lemma 2.15. *Let (X, τ) be a regular space, $\mathfrak{B}$ be a bornology with a compact base, and $f : \omega \to \omega \setminus \{0\}$ be a function. Let σ be the strategy of Player II in $\mathsf{G}_f(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. For all $s \in^\omega \mathcal{O}^X_{\mathfrak{B}}$, define:*

$$C_s = \bigcap_{\mathcal{U} \in \mathcal{O}^X_{\mathfrak{B}}} \overline{\bigcup \sigma(s^\frown \langle \mathcal{U} \rangle)}.$$

Then C_s is a compact subset of X.

From these results and with a few modifications to Theorem 2.13, we can obtain the following results.

Corollary 2.16. *Let (X, τ) be a regular space. Then the games $\mathsf{G}_{fin}(\mathcal{K}_X, \mathcal{K}_X)$ and $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$ are equivalent for to Player II.*

Remark 2.17. If (X, τ) is not a regular space the corollary above is false.

Indeed, let $\tau_{\mathbb{R}}$ be the usual topology in $\mathbb{R}$ and consider X as the set of real numbers equipped with the topology generated by the following basis:

$$\mathfrak{B} = \{U \backslash C : U \in \tau_{\mathbb{R}} \text{ and } C \text{ is countable}\}$$

Note that X is a Hausdorff space, but it is not a regular space (because $X \backslash \mathbb{Q}$ is open and the closure of any open set in X must contain an element in $\mathbb{Q}$).

We have that every compact subset K in X is finite, because otherwise we can obtain the open cover $\{K \backslash \{x_j : j \geq i\} : i \in \omega\}$ of K, where $\{x_i : i \in \omega\}$ is a countable subset of K, which does not contain a finite subcover. So, $\Omega_X = \mathcal{K}_X$. By Theorem 17 in [13] we have that $\mathsf{S}_1(\mathcal{K}_X, \mathcal{O}_X)$ does not hold ($\mathbb{R}$ does not hold $\mathsf{S}_1(\mathcal{O}_{\mathbb{R}}, \mathcal{O}_{\mathbb{R}})$). Then $\mathsf{S}_1(\mathcal{K}_X, \mathcal{K}_X)$ does not hold, and so $II \not\uparrow \mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$.

We claim the following statement: let (X, τ) be a topological space and $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}_{X^n}, \mathcal{O}_{X^n})$ for all $n \in \omega \backslash \{0\}$, then $II \uparrow \mathsf{G}_{fin}(\Omega_X, \Omega_X)$. The technique used in the demonstration of this fact is the same as that used in the analogous result for selective principles, we will perform the proof here for the interested reader. Indeed, consider σ_n a winning strategy for Player II in the game $\mathsf{G}_{\text{fin}}(\mathcal{O}_{X^n}, \mathcal{O}_{X^n})$, with $n \in \omega \backslash \{0\}$.

Let $t = \langle \mathcal{U}_0, ..., \mathcal{U}_k \rangle \in {}^{<\omega}\Omega_X$, with $k \in \omega$ and $\{A_n : n \in \omega \backslash \{0\}\}$ be a partition of ω into infinite sets.

We claim that if $\mathcal{U}_k \in \Omega$ and $n \in \omega \backslash \{0\}$ then $\mathcal{V}_k = \{U^n : U \in \mathcal{U}_k\} \in \mathcal{O}_{X^n}$. Indeed, let $(x_1, .., x_n) \in X^n$. As $F = \{x_1, ..., x_n\} \in [X]^{<\aleph_0}$, it follows that there is $U \in \mathcal{U}_k$ such that $F \subset U$. So, $(x_1, ..., x_n) \in U^n$. Then $\mathcal{V}_k \in \mathcal{O}_{X^n}$.

Now, suppose that $k \in A_{n_k}$, with $n_k \in \omega \backslash \{0\}$. Consider

$$t' = \langle \mathcal{V}_j : j \in A_{n_k}, j \leq k \rangle \in {}^{<\omega}\mathcal{O}_{X^{n_k}}$$

Define

$$\sigma(t) = \{U : U^{n_k} \in \sigma_{n_k}(t')\}.$$

As $\sigma_{n_k}(t')$ is finite then $\sigma(t)$ is finite. So σ defines a strategy by Player II in the game $\mathsf{G}_{\text{fin}}(\Omega_X, \Omega_X)$.

We have that σ is a winning strategy. Indeed, consider the following complete play in $\mathsf{G}_{\text{fin}}(\Omega_X, \Omega_X)$:

$$\langle \mathcal{U}_0, \sigma(\langle \mathcal{U}_0 \rangle), \mathcal{U}_1, \sigma(\langle \mathcal{U}_0, \mathcal{U}_1 \rangle), ..., \mathcal{U}_k, \sigma(\langle \mathcal{U}_0, ..., \mathcal{U}_k \rangle), ... \rangle.$$

So,

$$\langle \mathcal{V}_{a_1^n}, \sigma_n(\langle \mathcal{V}_{a_0^n} \rangle), \mathcal{V}_{a_1^n}, \sigma_n(\langle \mathcal{V}_{a_1^n}, \mathcal{V}_{a_2^n} \rangle), ... \mathcal{V}_{a_j^n}, \sigma_n(\langle \mathcal{V}_{a_1^n}, ..., \mathcal{V}_{a_j^n} \rangle), ... \rangle,$$

is a complete play in $\mathsf{G}_{\text{fin}}(\mathcal{O}_{X^n}, \mathcal{O}_{X^n})$. for all $n \in \omega \backslash \{0\}$, where $A_n = \{a_j^n : j \in \omega\}$. As σ_n is a winning strategy then

$$\bigcup_{j \in \omega} \sigma_n(\langle \mathcal{V}_{a_1^n}, ..., \mathcal{V}_{a_j^n} \rangle) \in \mathcal{O}_{X^n}.$$

So

$$\bigcup_{k \in \omega} \sigma(\langle \mathcal{U}_0, ..., \mathcal{U}_k \rangle) = \bigcup_{n \in \omega \backslash \{0\}} \bigcup_{j \in \omega} \left\{ U : U^n \in \sigma_n(\langle \mathcal{V}_{a_1^n}, ..., \mathcal{V}_{a_j^n} \right\}) \in \Omega_X$$

Thus, σ is a winning strategy by Player II in $\mathsf{G}_{\text{fin}}(\Omega_X, \Omega_X)$. Therefore, $II \uparrow \mathsf{G}_{\text{fin}}(\Omega_X, \Omega_X)$. This concludes our claim.

We know that $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}_X, \mathcal{O}_X)$ (see Example 3.5 in [1]). As $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}_{\mathbb{R}^n}, \mathcal{O}_{\mathbb{R}^n})$ (because $\mathcal{R}^n$ is σ-compact), with a few modifications in the argument in Example 3.5 in [1] we can obtain that $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}_{X^n}, \mathcal{O}_{X^n})$, for all $n \in \omega\backslash\{0\}$. By the last claim proved, we have $II \uparrow \mathsf{G}_{\text{fin}}(\Omega_X, \Omega_X)$. Therefore, $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{K}_X, \mathcal{K}_X)$.

Corollary 2.18. *Let (X, τ) be a regular space and $f : \omega \to \omega\backslash\{0\}$ be a function. Then, the games $\mathsf{G}_f(\mathcal{K}_X, \mathcal{K}_X)$ and $\mathsf{G}_1(\mathcal{K}_X, \mathcal{K}_X)$ are equivalent.*

Additionally, the following result is a slight modification of the proof performed by Scheepers in [12]. We will leave the proof here for the purposes of the interested reader.

Theorem 2.19. *Let (X, τ) be a separable metrizable space, and let $\mathfrak{B}$ be a bornology with a compact base. If $II \uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$ then X is σ-*compact*.*

Proof. Let $\mathcal{C}$ be a countable basis of X and σ be a winning strategy of Player II in $\mathsf{G}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$. We denote by $O_{\mathcal{C}}$ the family of all families in $\mathcal{O}^X_{\mathfrak{B}}$ whose elements belong to $\mathcal{C}$. Note that $\{\sigma(\langle \mathcal{U} \rangle) : \mathcal{U} \in O_{\mathcal{C}}\}$ is countable. In the same way as in the proof of Lemma 2.10, we can prove that

$$C_\emptyset = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle n \rangle} \rangle)}$$

is a compact subset of X.

For all $m \in \omega$ fixed, we see that $\{\sigma(\langle \mathcal{U}_{\langle m \rangle}, \mathcal{U} \rangle) : \mathcal{U} \in O_{\mathcal{C}}\}$ is countable. Then

$$C_{\langle m \rangle} = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle m \rangle}, \mathcal{U}_{\langle m,n \rangle} \rangle)}$$

is a compact subset of X.

In general, given $s = \langle s_0, ..., s_k \rangle \in^{<\omega} \omega$, with $k \in \omega\backslash\{0\}$, we have that the following set

$$\{\sigma(\langle \mathcal{U}_{\langle s_0 \rangle}, \mathcal{U}_{\langle s_0, s_1 \rangle}, ..., \mathcal{U}_s, \mathcal{U} \rangle) : \mathcal{U} \in O_{\mathcal{C}}\}$$

is countable. Then

$$C_s = \bigcap_{n\in\omega} \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle s_0 \rangle}, \mathcal{U}_{\langle s_0, s_1 \rangle}, ..., \mathcal{U}_s, \mathcal{U}_{s^\frown n} \rangle)}$$

is a compact subset of X.

We claim that $X = \bigcup_{s\in^{<\omega}\omega} C_s$. In fact, suppose that there is $x \in X\backslash(\bigcup_{s\in^{<\omega}\omega} C_s)$. In particular, $x \notin C_\emptyset$. So, there is $n_0 \in \omega$ such that $x \notin \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle n_0 \rangle} \rangle)}$. Also, $x \notin C_{\langle n_0 \rangle}$. Then, there is $n_1 \in \omega$ such that $x \notin \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle n_0 \rangle}, \mathcal{U}_{\langle n_0, n_1 \rangle} \rangle)}$. Suppose that for all $k \in \omega\backslash\{0\}$, we have defined $n_0, ..., n_k \in \omega$. As $x \notin C_{\langle n_0,...,n_k \rangle}$, it follows that there is $n_{k+1} \in \omega$ such that

$$x \notin \overline{\bigcup \sigma(\langle \mathcal{U}_{\langle n_0 \rangle}, ..., \mathcal{U}_{\langle n_0,...,n_k \rangle}, \mathcal{U}_{\langle n_0,...,n_k,n_{k+1} \rangle} \rangle)}.$$

Then

$$\mathcal{U}_{\langle n_0 \rangle}, \mathcal{U}_{\langle n_1 \rangle}, ..., \mathcal{U}_{\langle n_0,...,n_k \rangle}, ...$$

is a play for Player I in $\mathsf{G}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$ that defeats σ, a contradiction. Therefore, X is σ-compact. □

3. Equivalent games in $C_k(X)$

The following result shows some translations of a topological space (X, τ) into the space of continuous functions $C_{\mathfrak{B}}(X)$

Theorem 3.1. *Let (X, τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with a compact base. Let $f : \omega \to \omega\backslash\{0\}$ be a function. The following assertions are equivalent*

1. $\mathsf{S}_f(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$ *holds;*
2. $\mathsf{S}_f(\Omega_g, \Omega_g)$ *holds, for all $g \in C_{\mathfrak{B}}(X)$;*
3. $\mathsf{S}_f(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \Omega_g)$ *holds, for all $g \in C_{\mathfrak{B}}(X)$.*

The game version of this result is given in the following result.

Theorem 3.2. *Let (X, τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with a compact base, and $f : \omega \to \omega\backslash\{0\}$ be a function. Then, the game $\mathsf{G}_f(\mathcal{O}_{\mathfrak{B}}, \mathcal{O}_{\mathfrak{B}})$, and the games $\mathsf{G}_f(\Omega_g, \Omega_g)$ and $\mathsf{G}_f(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \Omega_g)$ in $C_{\mathfrak{B}}(X)$ are equivalent for all $g \in C_{\mathfrak{B}}(X)$.*

In particular, it follows that the game $\mathsf{G}_f(\mathcal{K}_X, \mathcal{K}_X)$ is equivalent to $\mathsf{G}_f(\Omega_o, \Omega_o)$ in $C_k(X)$, for all function $f : \omega \to \omega\backslash\{0\}$.

Based in the proof of [15], we can obtain the following

Theorem 3.3. *Let (X, τ) be a topological space and $\mathfrak{B}$ be a family of subsets of X. The following assertions are equivalent:*

1. $\mathsf{S}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$ *holds;*
2. $I \not\uparrow \mathsf{G}_{\text{fin}}(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$.

So, we can obtain the following result.

Theorem 3.4. *Let (X, τ) be a Tychonoff space, $\mathfrak{B}$ be a bornology with compact base, $f : \omega \to \omega\backslash\{0\}$ be a function and $g \in C_{\mathfrak{B}}(X)$. The following assertions are equivalent:*

1. $\mathsf{S}_f(\Omega_g, \Omega_g)$ *holds in $C_{\mathfrak{B}}(X)$;*
2. $I \not\uparrow \mathsf{G}_f(\Omega_g, \Omega_g)$ *in $C_{\mathfrak{B}}(X)$;*

Proof. The result follows from Theorems 3.1, 3.2, and 3.3. □

In addition, from Corollary 2.18 and Theorem 3.2, the following result follows.

Corollary 3.5. *Let (X, τ) be a Tychonoff space and $f : \omega \to \omega\backslash\{0\}$ be a function. Then, the games $\mathsf{G}_1(\Omega_g, \Omega_g)$ and $\mathsf{G}_f(\Omega_g, \Omega_g)$ are equivalent in $C_k(X)$, for all $g \in C_k(X)$.*

From this last result also follows:

Corollary 3.6. *Let (X, τ) be a Tychonoff space and $f : \omega \to \omega\backslash\{0\}$ be a function. Then, the games $\mathsf{G}_1(\mathcal{D}_{C_k(X)}, \Omega_g)$ and $\mathsf{G}_f(\mathcal{D}_{C_k(X)}, \Omega_g)$ are equivalent in $C_k(X)$, for all $g \in C_k(X)$.*

The following results were obtained in [14]

Theorem 3.7. *Let (X, τ) be a separable metrizable space and $g \in C_p(X)$. The following assertions are equivalent*

1. $\mathsf{S}_{\text{fin}}(\Omega_g, \Omega_g)$;
2. $I \not\uparrow \mathsf{G}_{\text{fin}}(\Omega_g, \Omega_g)$;
3. $I \not\uparrow \mathsf{G}_{\text{fin}}(\mathcal{D}_{C_p(X)}, \mathcal{D}_{C_p(X)})$;
4. $\mathsf{S}_{\text{fin}}(\mathcal{D}_{C_p(X)}, \mathcal{D}_{C_p(X)})$.

Theorem 3.8. *Let (X, τ) be a separable metrizable space and $g \in C_p(X)$. The following assertions are equivalent*

1. $\mathsf{S}_1(\Omega_g, \Omega_g)$ *holds;*
2. $I \not\uparrow \mathsf{G}_1(\Omega_g, \Omega_g)$;
3. $I \not\uparrow \mathsf{G}_1(\mathcal{D}_{C_p(X)}, \mathcal{D}_{C_p(X)})$;
4. $\mathsf{S}_1(\mathcal{D}_{C_p(X)}, \mathcal{D}_{C_p(X)})$ *holds.*

We can obtain the versions of Theorems 3.7 and 3.8 in $C_{\mathfrak{B}}(X)$, with $\mathfrak{B}$ a bornology with a compact base. We have the following

Theorem 3.9. *[9] Let (X, τ) be a Tychonoff space. Then*

$$d(C_p(X)) = d(C_k(X)) = \text{iw}(X).$$

That result can be generalized to the following

Theorem 3.10. *Let (X, τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with a compact base. Then $d(C_{\mathfrak{B}}(X)) = \text{iw}(X)$. In particular if $\text{iw}(X) = \aleph_0$ then $C_{\mathfrak{B}}(X)$ is separable.*

The proof is practically the same given in [9], we only need the following

Theorem 3.11. *Let (X, τ) be a Tychonoff space and $\mathfrak{B}$ be a bornology with a compact base. Let $D \subseteq C_{\mathfrak{B}}(X)$ be a family that separates points and contains the constant function 1. Therefore, the subalgebra generated by D is dense in $C_{\mathfrak{B}}(X)$.*

This result can be obtained from

Theorem 3.12. *(Stone-Weierstrass) Let (X, τ) be a Hausdorff and compact topological space. If $D \subset C(X)$ separates points and contains a constant function 1, then the algebra generated by D is dense in $C(X)$ ($C(X)$ with the uniform topology).*

The following lemma is a particular property obtained on selection principle S_1 in the case of $\mathcal{A} = \mathcal{B} = \Omega_p$.

Lemma 3.13. *Let (X, τ) be a Tychonoff space such that $\mathsf{S}_1(\Omega_o, \Omega_o)$ holds in $C_{\mathfrak{B}}(X)$. Then, for all sequences $\langle A_n : n \in \omega \rangle$ of elements in Ω_o, there is a pairwise disjoint sequence $\langle B_n : n \in \omega \rangle$ of elements in Ω_o and such that $B_n \subseteq A_n$.*

Proof. Let $\langle A_n : n \in \omega \rangle$ be a sequence of elements in Ω_o. By hypotheses, we can assume that each A_n is countable. Note that $f \in \Omega_o$ if, and only if, $|f| \in \Omega_o$. Then we can also assume that the elements of A_n are positives. Suppose that, for all $n \in \omega$, $A_n = \{f_m^n : m \in \omega\}$.

We define a strategy σ for I in $\mathsf{G}_1(\Omega_o, \Omega_o)$. In the first inning, we define $\sigma(\emptyset) = A_0 \in \Omega_0$. Suppose that Player II chooses the element $f^0_{m^0_0}$.

Define $\sigma(\langle f^0_{m^0_0}\rangle) = \{f^0_{k_0} + f^1_{k_1} : k_o, k_1 \in \omega, |\{f^0_{m^0_0}, f^0_{k_0}, f^1_{k_1}\}| = 3\}$. To see that it belongs to Ω_o, let $[o, B, \epsilon]$ be a basic neighborhood, with $B \in \mathfrak{B}$ and $\epsilon > 0$. So, there are $f^0_{k_0} \in (A_0 \backslash \{f^0_{m_0}\}) \cap [o, B, \frac{\epsilon}{2}]$ and $f^1_{k_1} \in (A_1 \backslash \{f^0_{m_0}, f^0_{k_0}\}) \cap [o, B, \frac{\epsilon}{2}]$. Then $f^0_{k_0} + f^1_{k_1} \in [o, B, \epsilon]$. Suppose that Player II responds with the element $f^0_{m^1_0} + f^1_{m^1_1}$.

Now, define

$$\sigma(\langle f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}\rangle) = \{f^0_{k_0} + f^1_{k_1} + f^2_{k_2} : k_o, k_1, k_2 \in \omega, |\{f^0_{m^0_0}, f^0_{m^1_0}, f^1_{m^1_1}, f^0_{k_0}, f^1_{k_1}, f^2_{k_2}\}| = 6\}.$$

Let $[o, B, \epsilon]$ be a basic neighborhood, with $B \in \mathfrak{B}$ and $\epsilon > 0$. Then, there are

$$f^0_{k_0} \in (A_0 \backslash \{f^0_{m_0} . f^0_{m^1_0}, f^1_{m^1_1}\}) \cap [o, B, \frac{\epsilon}{3}],$$
$$f^1_{k_1} \in (A_1 \backslash \{f^0_{m_0}, f^0_{m^1_0}, f^0_{k_0}, f^1_{m^1_1}\}) \cap [o, B, \frac{\epsilon}{3}] \text{ and}$$
$$f^2_{k_2} \in (A_2 \backslash \{f^0_{m_0}, f^0_{m^1_0}, f^0_{k_0}, f^1_{m^1_1}, f^1_{k_1}\}) \cap [o, B, \frac{\epsilon}{3}].$$

So, $f^0_{k_0} + f^1_{k_1} + f^2_{k_2} \in [o, B, \epsilon]$. Then $\sigma(\langle f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}\rangle) \in \Omega_o$. This way, we define for all inning $n \in \omega$.

By Theorem 3.4, we see that σ is not a winning strategy. So, there is a set C in Ω_o, with elements of the form

$$f^0_{m^0_0}, f^0_{m^1_0} + f^1_{m^1_1}, f^0_{m^2_0} + f^1_{m^2_1} + f^2_{m^2_2}, \ldots$$

Then we can consider, for all $n \in \omega$, the sets $B_n = \{f^n_{m^i_n} : i \geq n\}$. As $C \in \Omega_o$, it follows that, for all $n \in \omega$, $B_n \in \Omega_o$, and by the construction performed, all sets B_n are pairwise disjoint. □

With the lemma above we can obtain the following

Theorem 3.14. *Let (X, τ) be a Tychonoff space and with* $\mathrm{iw}(X) = \aleph_0$. *Let $g \in C_{\mathfrak{B}}(X)$, the following assertions are equivalent:*

1. *$\mathsf{S}_1(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;*
2. *$I \not\uparrow \mathsf{G}_1(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;*
3. *$I \not\uparrow \mathsf{G}_1(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$;*
4. *$\mathsf{S}_1(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ holds.*

Proof. (1) ⇔ (2). It follows from Theorem 3.4. As $C_{\mathfrak{B}}(X)$ is homogeneous, it follows that it is sufficient to prove (2) ⇒ (3) and (4) ⇒ (1), for the case $g = o$.

(2) ⇒ (3). Let σ be a strategy for Player I in game $\mathsf{G}_1(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$. As $C_{\mathfrak{B}}(X)$ is separable we can assume that σ chooses countable subsets, and we fix $\{g_n : n \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)}$. Let us define a strategy ρ for Player I in the game $\mathsf{G}_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$.

Suppose that $\sigma(\emptyset) = \{f_n : n \in \omega\} \in \mathcal{D}_{C_{\mathcal{B}}(X)}$. Defining $\rho(\emptyset) = \{|f_n - g_0| : n \in \omega\}$. We claim that $\rho(\emptyset) \in \Omega_o$. In fact, let $[o, B, \epsilon]$ be a basic neighborhood, with $B \in \mathfrak{B}$ and $\epsilon > 0$. As $[g_0, B, \epsilon]$ is an open subset of $C_{\mathfrak{B}}(X)$ and $\sigma(\emptyset) \in \mathcal{D}_{C_{\mathfrak{B}}(X)}$, it follows that there is $k \in \omega$ such that $f_k \in [g_0, B, \epsilon]$. So, $|f_k - g_0| \in [o, B, \epsilon]$.

Suppose that Player II chooses, in the game $\mathsf{G}_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$, the element $|f_{n^0_0} - g_0|$, and that $\sigma(\langle f_{n^0_0}\rangle) = \{f_{n^0_0,n} : n \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)}$. We define $\rho(\langle |f_{n^0_0} - g_0|\rangle) = \{|f_{n^0_0,i} - g_0| + |f_{n^0_0,j} - g_1| : i, j \in \omega\}$. Similarly to the previous case (in this case, consider the open set $[g_i, B, \frac{\epsilon}{2}]$, with $B \in \mathfrak{B}$ e $i = 0, 1$), it follows that $\rho(\langle |f_{n^0_0} - g_0|\rangle) \in \Omega_o$.

Suppose that Player II chose, in the game $\mathsf{G}_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$, the element $|f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|$. Also, suppose that

$$\sigma(\langle f_{n_0^0}, f_{n_0^0,n_0^1}\rangle) = \{f_{n_0^0,n_0^1,n} : n \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)} \text{ and}$$
$$\sigma(\langle f_{n_0^0}, f_{n_0^0,n_1^1}\rangle) = \{f_{n_0^0,n_1^1,n} : n \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)}.$$

So, we can define $\rho(\langle |f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|\rangle) = \{|f_{n_0^0,n_0^1,i_1} - g_0| + |f_{n_0^0,n_0^1,i_2} - g_1| + |f_{n_0^0,n_0^1,i_3} - g_2| + |f_{n_0^0,n_1^1,j_1} - g_0| + |f_{n_0^0,n_1^1,j_2} - g_1| + |f_{n_0^0,n_1^1,j_3} - g_2| : i_1, i_2, i_3, j_1, j_2, j_3 \in \omega\}$. Similarly to the previous case (in this case, consider the open set $[g_i, B, \frac{\epsilon}{6}]$, with $B \in \mathfrak{B}$ and $i = 0, 1, 2$), it follows that $\rho(\langle |f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|\rangle) \in \Omega_o$.

Following the construction above in the entire game $n \in \omega$, it follows that $\rho : {}^{<\omega}(\bigcup \Omega_o) \to \Omega_o$ is a strategy for Player I in the game $\mathsf{G}_1(\Omega_o, \Omega_o)$ in $C_{\mathfrak{B}}(X)$. By (2), we can choose a sequence of Player II choices, which form a set $C \in \Omega_o$, with elements of the form:

$$|f_{n_0^0} - g_0|, |f_{n_0^0,n_0^1} - g_0| + |f_{n_0^0,n_1^1} - g_1|, |f_{n_0^0,n_0^1,n_0^2} - g_0| + |f_{n_0^0,n_0^1,n_1^2} - g_0| +$$
$$|f_{n_0^0,n_0^1,n_2^2} - g_2| + |f_{n_0^0,n_1^1,n_3^2} - g_0| + |f_{n_0^0,n_1^1,n_4^2} - g_1| + |f_{n_0^0,n_1^1,n_5^2} - g_2|, \ldots$$

Then, by Lemma 3.13, we can obtain a partition of C in countable many pairwise disjoint $B_n \in \Omega_o$. For all $n \in \omega$, we define J_n as the set of all $m \in \omega$ such that Player II has chosen an element of B_n in the inning m. Note that these sets are pairwise disjoint and we can assume, for all $n \in \omega$, that $\min(J_n) \geq n$.

Thus, we define $m_0 = n_0^0$. Now, since the only possibilities are $1 \in I_0$ or $1 \in J_1$, define $m_1 = n_j^1$, where $j \in \{0, 1\}$ is the term $|f_{m_0,n_j^1} - g_j|$ of the choice of Player II in the inning 1. In general, for all $k \geq 2$, since the unique possibilities are $k \in J_i$, com $i \leq k$, we define $m_k = n_j^k$, where $j \leq k$ is the term $|f_{m_0,\ldots,m_{k-1},n_j^k} - g_j|$ of the choice of Player II in the inning k. So, for all $k \in \omega$, $\{|f_{m_0,\ldots,m_j} - g_k| : j \in I_k\} \in \Omega_0$. In fact, let $[o, B, \epsilon]$ be a basic neighborhood, with $B \in \mathfrak{B}$ and $\epsilon > 0$. As $B_k \in \Omega_o$, it follows that there is $r \in \omega$ such that $|f_{m_1,\ldots,m_r} - g_k| \in [o, B, \epsilon]$.

Finally, we claim that $\{f_{m_0,\ldots,m_j} : j \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)}$. In fact, let $[h, B, \epsilon]$ be a basic neighborhood, with $h \in C_{\mathfrak{B}}(X)$, $B \in \mathfrak{B}$ e $\epsilon > 0$. As $\{g_k : k \in \omega\} \in \mathcal{D}_{C_{\mathfrak{B}}(X)}$, it follows that there is $l \in \omega$ such that $g_l \in [h, B, \frac{\epsilon}{2}]$. So, there is $r \in \omega$, such that $|f_{m_0,\ldots,m_r} - g_l| \in [o, B, \frac{\epsilon}{2}]$. Therefore, $f_{m_0,\ldots,m_r} \in [h, B, \epsilon]$. So, we obtain a sequence of choices of Player II in the game $\mathsf{G}_1(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$ that defeats the strategy σ.

(4) $\Rightarrow$ (1). It follows from the implication (3) $\Rightarrow$ (1) in Theorem 3.1, changing Ω_o by $\mathcal{D}_{C_{\mathfrak{B}}}(X)$. □

With a few modifications to the previous theorem, we can obtain the following results.

Theorem 3.15. *Let (X, τ) be a Tychonoff space such that $\mathrm{iw}(X) = \aleph_0$. Let $f : \omega \to \omega \backslash \{0\}$ be a function and $g \in C_{\mathfrak{B}}(X)$, the following assertions are equivalent:*

1. *$\mathsf{S}_f(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;*
2. *$I \not\uparrow \mathsf{G}_f(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;*
3. *$I \not\uparrow \mathsf{G}_f(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$;*
4. *$\mathsf{S}_f(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ holds.*

Theorem 3.16. *Let (X, τ) be a Tychonoff space such that $\mathrm{iw}(X) = \aleph_0$. Let $g \in C_{\mathfrak{B}}(X)$, the following assertions are equivalent:*

1. *$\mathsf{S}_{\mathrm{fin}}(\Omega_g, \Omega_g)$ holds in $C_{\mathfrak{B}}(X)$;*
2. *$I \not\uparrow \mathsf{G}_{\mathrm{fin}}(\Omega_g, \Omega_g)$ in $C_{\mathfrak{B}}(X)$;*
3. *$I \not\uparrow \mathsf{G}_{\mathrm{fin}}(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ in $C_{\mathfrak{B}}(X)$;*

4. $\mathsf{S}_{\text{fin}}(\mathcal{D}_{C_{\mathfrak{B}}(X)}, \mathcal{D}_{C_{\mathfrak{B}}(X)})$ *holds in* $C_{\mathfrak{B}}(X)$*.*

With the following result we can obtain a equivalence of games G_f for the Player II

Theorem 3.17. *Let* (X, τ) *be a separable space and let* $f : \omega \to \omega\backslash\{0\}$ *be an increasing function. The following assertions are equivalent:*

1. $II \uparrow \mathsf{G}_f(\mathcal{D}_X, \mathcal{D}_X)$*;*
2. $II \uparrow \mathsf{G}_f(\mathcal{D}_X, \Omega_x)$*, for all* $x \in X$*.*

Finally, by Corollary 3.5, Corollary 3.6, Theorem 3.17, and Theorem 3.15, the following result follows:

Corollary 3.18. *Let* (X, τ) *be a Tychonoff space such that* $\mathrm{iw}(X) = \aleph_0$*. Let* $f : \omega \to \omega\backslash\{0\}$ *be an increasing function. Then the games* $\mathsf{G}_f(\mathcal{D}_{C_k(X)}, \mathcal{D}_{C_k(X)})$ *and* $\mathsf{G}_1(\mathcal{D}_{C_k(X)}, \mathcal{D}_{C_k(X)})$ *are equivalent.*

4. Additional commentaries and induced questions

We see that, in general, selection games in the class of dense subsets in topological spaces are different. In fact, consider the space $X = {}^{<\omega}\omega$ with the topology generated by the basis $\mathfrak{B} = \{X\backslash \bigcup_{f\in F}\{f \restriction n : n \in \omega\} : F \subset {}^{\omega}\omega \text{ is finite }\}$.

Firstly, let $k_0, k_1, ..., k_{m-1} \in \omega$, with $m \in \omega$ (here $k_{-1} = \emptyset$), we have that the set $D = \{(k_0, k_1..., k_{m-1}, k) : k \in \omega\}$ is dense in X, because for any $F \subset {}^{\omega}\omega$ finite the set $\{f \restriction m : f \in F\}$ is finite, and then there is $(k_0, .., k_{m-1}, k_m) \in D$ such that $(k_1, .., k_{n-1}, k_m) \neq f \restriction m$, for all $f \in F$. So $D \cup \Big(X\backslash \bigcup_{f\in F}\{f \restriction n : n \in \omega\}\Big) \neq \emptyset$.

Now, in the game $\mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_X)$, in the inning 0, Player I chooses $D_0 = \{(k) : k \in \omega\}$. If Player II chooses $x_0 = (k_0) \in D_0$, then Player I chooses $D_1 = \{(k_0, k) : k \in \omega\}$. If Player II chooses $x_1 = (k_0, k_1) \in D_1$, then Player I chooses $D_2 = \{(k_0, k_1, k) : k \in \omega\}$, and so on. Taking $f = (k_0, ..., k_n, ...)$ we have $\{x_n : n \in \omega\} \cap X\backslash\{f \restriction n : n \in \omega\} = \emptyset$, that is, $\{x_n : n \in \omega\} \notin \mathcal{D}_X$. So, $I \uparrow G_1(\mathcal{D}_X, \mathcal{D}_X)$, and then $II \not\uparrow G_1(\mathcal{D}_X, \mathcal{D}_X)$.

On the other hand, suppose that Player I chooses $A_0 \in \mathcal{D}_X$ in the inning 0 in the game $\mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X)$. As A_0 is dense, we can choose $a^0_1, a^0_2 \in A_1$ such that they do not belong to a same branch (branch is a set of the form $\{f \restriction n : n \in \omega\}$, with $f \in {}^{\omega}\omega$). Then, Player II chooses $\{a^0_1, a^0_2\}$. It is clear that $\{a^0_1\}$ or $\{a^0_2\}$ is a set such that no branch contains two elements of it. Namely $\{t_0\}$ as the set $\{a^0_1\}$.

In the next inning, suppose that Player I chooses $A_1 \in \mathcal{D}_X$. If there is an element a^1_1 in A_1 such that it is not in any branch that does not intersect $\{t_0\}$, then Player II chooses $\{a^1_1, a^1_2\}$, with a^1_2 an arbitrary element in A_1. Note that the set $\{t_0, a^1_1\}$ is a set such that no branch contains two elements of it. If all the elements in A_1 are in a branch that intersects $\{t_0\}$, since A_1 is dense, we can choose a^1_1 and a^1_2 incompatible elements (that is, $a^1_1 \not\subset a^1_2$ and $a^1_2 \not\subset a^1_1$) in A_2 such that $t_0 \subset a^1_1$ and $t_0 \subset a^1_2$. So, Player II chooses $\{a^1_1, a^1_2\}$. Note that the set is such that no branch contains two elements of it. So, in any of the cases, we have a set with 2 elements, namely $\{t_0, t_1\}$, such that no branch contains two elements of it.

In the next inning, suppose that Player I chooses $A_2 \in \mathcal{D}_X$. If there is an element a^2_1 in A_2 such that it is not in any branch that does not intersect $\{t_0, t_1\}$, then Player II chooses $\{a^2_1, a^2_2\}$, with a^2_2 an arbitrary element in A_1. Note that the set $\{t_0, t_1, a^2_1\}$ is a set such that no branch contains two elements of it. If all the elements in A_1 are in a branch that intersects $\{t_0, t_1\}$, since A_1 is dense, there is t_i such that we can choose a^2_1 and a^2_2 incompatible elements in A_2 with $t_i \subset a^2_1$ and $t_i \subset a^2_2$. So, Player II chooses $\{a^2_1, a^2_2\}$. Note that for $j \neq i$, the set $\{t_j, a^2_1, a^2_2\}$ is such that no branch contains two elements of it. So, in any of the cases, we have a set with 3 elements, namely $\{t_0, t_1, t_2\}$, such that no branch contains two elements of it.

In general, in the inning $n \geq 1$, suppose that Player I chooses $A_n \in \mathcal{D}_X$. If there is an element a^n_1 in A_n such that it is not in any branch that does not intersect $\{t_0, t_1, .., t_{n-1}\}$, then Player II chooses $\{a^n_1, a^n_2\}$, with a_2^n an arbitrary element in A_n. Note that the set $\{t_0, t_1, ..., t_{n-1}, a_1^n\}$ is a set such that no branch contains two elements of it. If all the elements in A_n are in a branch that intersects $\{t_0, t_1, ..., t_{n-1}\}$, since A_n is dense, there is t_i such that we can choose a_1^n and a_2^n incompatible elements in A_n with $t_i \subset a_1^n$ and $t_i \subset a_2^n$. So, Player II chooses $\{a_1^n, a_2^n\}$. Note that the set $\{t_j : j \neq i\} \cup \{a_1^n, a_2^n\}$ is such that no branch contains two elements of it. So, in any of the cases, we have a set with $n+1$ elements, namely $\{t_0, t_1, ..., t_n\}$, such that no branch contains two elements of it.

In summary, we obtain a strategy σ for Player II such that, for each $n \in \omega$, the set of answers played includes a set $\{t_0, ..., t_n\}$ with the property that no branch contains two elements of it.

We have that σ is a winning strategy. In fact, let D be the set of all the answers of Player II in a play using σ. If $D \notin \mathcal{D}_X$, then there is a basic open $U = X \setminus \bigcup_{F \in {}^\omega\omega} \{f \restriction n : n \in \omega\}$, with F finite, such that $D \cap U = \emptyset$. So, $D \subset \bigcup_{F \in {}^\omega\omega} \{f \restriction n : n \in \omega\}$. Suppose that $|F| = m$. Then, since the set $\{t_0, ..., t_m\} \subset D$ is such that no branch contains two elements of it, there is a t_i such that $t_i \notin \bigcup_{F \in {}^\omega\omega} \{f \restriction n : n \in \omega\}$. So, $D \not\subset \bigcup_{F \in {}^\omega\omega} \{f \restriction n : n \in \omega\}$, a contradiction. Then $D \in \mathcal{D}_X$. Therefore, $II \uparrow \mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X)$.

We can see that X is a T_1 space that is not a Hausdorff space. The following question is still open:

Question 4.1. *Restricted to Hausdorff spaces, the selection games $\mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_X)$ and $\mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X)$ are equivalent?*

If X is a P-*space* and I-*countable* Problem 4.1 has a positive answer. In fact, if X is a Hausdorff space, then X is a discrete space and therefore all games $\mathsf{G}_k(\mathcal{D}_X, \mathcal{D}_X)$, $\mathsf{G}_f(\mathcal{D}_X, \mathcal{D}_X)$ and $\mathsf{G}_{\text{fin}}(\mathcal{D}_X, \mathcal{D}_X)$ are equivalent, where $k \in \omega$ and $f : \omega \to \omega \setminus \{0\}$ is a function.

On the other hand, if X is not a Hausdorff space, then we have $II \uparrow \mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X) \Rightarrow II \uparrow \mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_X)$. In fact, let σ be a winning strategy for Player II in $\mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X)$. Suppose that Player I chooses $D_0 \in \mathcal{D}_X$ in the first inning of the game $\mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_X)$. Suppose that $\sigma(\langle D_0 \rangle) = \{x_0, y_0\}$. So, define $\varphi(\langle D_0 \rangle) = x_0$. Next, Player I chooses $D_1 \in \mathcal{D}_X$.

Recall that

$$II \uparrow \mathsf{G}_2(\mathcal{D}_X, \mathcal{D}_X) \Rightarrow \mathsf{S}_2(\mathcal{D}_X, \mathcal{D}_X) \Rightarrow \mathsf{S}_1(\mathcal{D}_X, \mathcal{D}_X) \Rightarrow \mathsf{S}_1(\mathcal{D}_X, \Omega_x),$$

for all $x \in X$.

Then, using that $\mathsf{S}_1(\mathcal{D}_X, \Omega_{y_0})$ holds, there is $\{z_n^1 : n \in \omega\} \subset D_1$ such that $\{z_n^1 : n \in \omega\} \in \Omega_{y_0}$. Let $\{U_n^0 : n \in \omega\}$ be a local base for y_0. Then $\bigcap_{n \in \omega} U_n^0$ is an open (because X is P-space) and contains y_0. Therefore, there is m_1 such that $z_{m_1}^1 \in \bigcap_{n \in \omega} U_n^0$. Define $\varphi(\langle D_0, D_1 \rangle) = z_{m_1}^1$. Thus, in each inning $2n$, $n \in \omega$, we define $\varphi(\langle D_0, D_1, ..., D_{2n} \rangle) = x_n$, where $\sigma(\langle D_0, D_2, ..., D_{2n} \rangle) = \{x_n, y_n\}$. On the other hand, in each inning $2n+1$, with $n \in \omega$, we define $\varphi(\langle D_0, D_1, ..., D_{2n+1} \rangle) = z_{m_{2n+1}}^{2n+1}$, where

$$z_{m_{2n+1}}^{2n+1} \in \bigcap_{m \in \omega} U_m^n$$

and $\{U_m^n : m \in \omega\}$ is a local base of y_n (here we use $\mathsf{S}_1(\mathcal{D}_X, \Omega_{y_n})$).

Then, we claim $\{x_n : n \in \omega\} \cup \{z_{m_{2n+1}}^{2n+1} : n \in \omega\} \in \mathcal{D}_X$. In fact, let $U \in \tau$. As σ is a winning strategy, we see that there is $k \in \omega$ such that $x_k \in U$ or $y_k \in U$. If the first case is true, we end. Suppose that $y_k \in U$. Then, there is an $l \in \omega$ such that $y_k \in U_l^k \in U$. So $z_{m_{2k+1}}^{2k+1} \in \bigcap_{m \in \omega} U_m^k \subset U$. Therefore, φ is a winning strategy for Player II in the game $\mathsf{G}_1(\mathcal{D}_X, \mathcal{D}_X)$.

As we have seen before, the selection games in the class of dense subsets in $C_k(X)$ (with X Tychonoff space) are equivalent. However, the following problems are still open:

Question 4.2. *If (X, τ) is a regular space and $f : \omega \to \omega \setminus \{0\}$ is a function. The games $\mathsf{G}_f(\Omega_X, \Omega_X)$ and $\mathsf{G}_1(\Omega_X, \Omega_X)$ are equivalent?*

Question 4.3. *Let (X, τ) be a regular space. If $s \in {}^{<\omega}\Omega_X$, then*

$$C_s = \bigcap_{\mathcal{U} \in \Omega_X} \overline{\bigcup \sigma(s^\frown \mathcal{U})}$$

is finite?

If the statements in either of the two problems are true, we can obtain a version of Corollary 3.18 for the function space $C_p(X)$.

In particular, when X is a P-*space*, Problems 4.2 and 4.3 have positive answers and therefore we have an equivalence of the topological games $\mathsf{G}_1(D_X, D_X)$ and $\mathsf{G}_f(D_X, D_X)$, with $f : \omega \to \omega \setminus \{0\}$ an increasing function.

More generally, we can formulate the following question:

Question 4.4. *Let (X, τ) be a regular or Tychonoff space and $\mathcal{B}$ a bornology with a compact basis. If $s \in {}^{<\omega}\mathcal{O}^X_{\mathfrak{B}}$ and σ is a strategy in $\mathsf{G}_f(\mathcal{O}^X_{\mathfrak{B}}, \mathcal{O}^X_{\mathfrak{B}})$, then*

$$C_s = \bigcap_{\mathcal{U} \in \mathcal{O}^X_{\mathfrak{B}}} \overline{\bigcup \sigma(s^\frown \mathcal{U})}$$

is an element of $\mathcal{B}$?

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