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QUALITATIVE PROPERTIES FOR SOLUTIONS TO CONFORMALLY INVARIANT FOURTH ORDER CRITICAL SYSTEMS

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Abstract. We study qualitative properties for nonnegative solutions to a conformally invariant coupled system of fourth order equations involving critical exponents. For solutions defined in the punctured space, there exist essentially two cases to analyze. If the origin is a removable singularity, we use an integral moving spheres method to prove that non-singular solutions are rotationally invariant. More precisely, they are the product of a fourth order spherical solution by a unit vector with nonnegative coordinates. If the origin is a non-removable singularity, we show that the solutions are radially symmetric and strongly positive. Furthermore, using a Pohozaev-type invariant, we prove the non-existence of semi-singular solutions, *i.e.*, all components equally blow-up in the neighborhood of the origin. Namely, they are classified as multiples of the Emden–Fowler solution.

1. Introduction.

We study qualitative properties for nonnegative p-map solutions $\mathcal{U} = (u_1, \dots, u_p) : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^p$ to the following fourth order system in the punctured space,

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2} u_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}, \tag{$\mathcal{S}_p$}$$

where $n \geq 5$, Δ^2 is the bi-Laplacian and $|\mathcal{U}|$ is the Euclidean norm, that is, $|\mathcal{U}| = (\sum_{i=1}^p u_i^2)^{1/2}$. System ($\mathcal{S}_p$) is strongly coupled by the Gross–Pitaevskii nonlinearity $f_i(\mathcal{U}) = c(n)|\mathcal{U}|^{2^{**}-2} u_i$ with associated potential $F(\mathcal{U}) = (f_1(\mathcal{U}), \dots, f_p(\mathcal{U}))$, where $2^{**} = 2n/(n-4)$ is the critical Sobolev exponent, and $c(n) = [n(n-4)(n^2-4)]/16$ a normalizing constant. We will always keep these notation throughout the paper.

By a (classical) solution to System ($\mathcal{S}_p$), we mean a p-map $\mathcal{U}$ such that each component $u_i \in C^{4,\zeta}(\mathbb{R}^n \setminus \{0\})$, for some $\zeta \in (0,1)$, and it satisfies ($\mathcal{S}_p$) in the classical sense.

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A solution may develop an isolated singularity when $x=0$, that is, some components may blow-up at the origin. More accurately, a solution to ($\mathcal{S}_p$) is said to be singular, if there exists $i \in I := \{1,\dots,p\}$ such that the origin is a non-removable singularity for u_i. Otherwise, if the origin is a removable singularity for all components, this solution is called non-singular, and it can be extended continuously to the whole domain. We also say that a p-map solution $\mathcal{U}$ is nonnegative (strongly positive) when $u_i \geq 0$ ($u_i > 0$) and $\mathcal{U}$ is superharmonic in case $-\Delta u_i > 0$ for all $i \in I$. When either $u_i > 0$ or $u_i \equiv 0$ for any $i \in I$, a solution $\mathcal{U}$ is called weakly positive.

Let us notice that when $p=1$, ($\mathcal{S}_p$) becomes the following fourth order equation,

$$\Delta^2 u = c(n)u^{2^{**}-1} \quad \text{in} \quad \mathbb{R}^n \setminus \{0\}. \tag{$\mathcal{S}_1$}$$

In this sense, the Gross–Pitaevskii nonlinearity is the more natural coupling term such that ($\mathcal{S}_p$) generalizes ($\mathcal{S}_1$). Our objective is to present classification results for both non-singular and singular solutions to the conformally invariant system ($\mathcal{S}_p$).

Our first main result is motivated by the classification theorem below due to C. S. Lin [31, Theorem 1.3] (see also [39, Theorem 1.1] and [38, Theorem 1.3]) for positive solutions to ($\mathcal{S}_1$) with a removable singularity at the origin

Theorem A. *Let u be a nonnegative non-singular solution to ($\mathcal{S}_1$). Then, there exist $x_0 \in \mathbb{R}^n$ and $\mu > 0$ such that u is radially symmetric about x_0 and*

$$u \equiv u_{x_0,\mu},$$

where

$$u_{x_0,\mu}(x) = \left(\frac{2\mu}{1+\mu^2|x-x_0|^2}\right)^{\frac{n-4}{2}}. \tag{1}$$

Let us call $u_{x_0,\mu}$ a fourth order spherical solution.

This $(n+1)$-parameter family of solutions can also be regarded as maximizers for the Sobolev embedding theorem $\mathcal{D}^{2,2}(\mathbb{R}^n) \hookrightarrow L^{2^{**}}(\mathbb{R}^n)$, that is,

$$\|u_{x_0,\mu}\|_{L^{2^{**}}(\mathbb{R}^n)} = S(n)\|u_{x_0,\mu}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)} \quad \text{with} \quad S(n) = \left(c(n)\omega_n^{4/n}\right)^{-1/2},$$

where ω_n is the volume of the unit ball in the Euclidean space. The existence of extremal functions for the last identity was obtained by P.-L. Lions [32, Section V.3]. Besides, these optimizers were found in a more general setting by E. Lieb [30, Theorem 3.1] (See also D. E. Edmunds et al. [17, Theorem 2.1]).

Our second main result yields a classification theorem for nonnegative singular solutions to ($\mathcal{S}_p$). On this subject, we should mention that when the origin is a non-removable singularity, C. S. Lin [31, Theorem 1.4] obtained radial symmetry for solutions to ($\mathcal{S}_1$) using the asymptotic moving planes technique. Recently, Z. Guo, X. Huang, L. Wang and J. Wei. [20, Theorem 1.3] proved the existence of periodic solutions applying a mountain pass theorem and conjectured that all solutions are periodic. Later on, R. L. Frank and T. König [18, Theorem 2] answered this conjecture, obtaining more accurate results concerning the classification for global singular solutions to ($\mathcal{S}_1$). Namely, they used the Emden–Fowler change coordinates (see Section 4.3) to transform ($\mathcal{S}_1$) into the fourth order Cauchy problem,

$$\begin{cases} v^{(4)} - K_2 v^{(2)} + K_0 v = c(n)v^{2^{**}-1} \quad \text{in} \quad \mathbb{R}, \\ v(0)=a, \ v^{(1)}(0)=0, \ v^{(2)}(0)=b(a), \ v^{(3)}(0)=0, \end{cases} \tag{2}$$

where K_2, K_0 are constants depending on the dimension (see (47)). In this work, positive periodic solutions $v_{a,T}$ to (2) are proved to exist using a topological shooting method based on the parameter $b(a)$. One needs to be restricted to the situation $a \in (0, a_0]$, where $a_0 = [n(n-4)/(n^2-4)]^{(n-4)/8}$ and $T \in (0, T_a]$ is the period with $T_a \in \mathbb{R}$ is the fundamental period of v_a. Now let us state their results

Theorem B. *Let u be a nonnegative singular solution to ($\mathcal{S}_1$). Then, u is radially symmetric about the origin. Moreover, there exist $a \in (0, a_0]$ and $T \in (0, T_a]$ such that*

$$u \equiv u_{a,T},$$

where

$$u_{a,T}(x) = |x|^{\frac{4-n}{2}} v_a(-\ln|x| + T), \tag{3}$$

with v_a is the unique T-periodic bounded solution to (2) and $T_a \in \mathbb{R}$ its fundamental period. Let us call both $u_{a,T}$ and $v_{a,T}$ Emden–Fowler (or Delaunay-type) solutions.

In the light of Theorems A and B, we present our main results.

Theorem 1.1. *Let $\mathcal{U}$ be a nonnegative non-singular solution to ($\mathcal{S}_p$). There exists $\Lambda \in \mathbb{S}^{p-1}_+ = \{x \in \mathbb{S}^{p-1} : x_i \geq 0\}$ and a fourth order spherical solution given by (1) such that*

$$\mathcal{U} \equiv \Lambda u_{x_0,\mu}.$$

Furthermore, we show that the non-singular solutions classified above are the extremal maps for a higher order Sobolev-type inequality.

Theorem 1.2. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_p$). There exists $\Lambda^* \in \mathbb{S}^{p-1}_{+,*} = \{x \in \mathbb{S}^{p-1} : x_i > 0\}$ and an Emden–Fowler solution given by (3) such that*

$$\mathcal{U} \equiv \Lambda^* u_{a,T}.$$

Since singular solutions to the blow-up limit equation ($\mathcal{S}_p$) are the natural candidates for asymptotic models of the same system in the punctured ball, the last theorem is the first step in describing the local asymptotic behavior for positive singular solutions to

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2} u_i \quad \text{in} \quad B_1^n \setminus \{0\}.$$

This asymptotic analysis would be a version of the celebrated results due to L. A. Caffarelli, B. Gidas and J. Spruck [4] and N. Korevaar, R. Mazzeo, F. Pacard and R. Schoen [27].

Remark 1.3. The existence of non-singular (singular) solutions to ($\mathcal{S}_p$) follows directly from Theorem A (Theorem B). In fact, for any $\Lambda \in \mathbb{S}^{p-1}_+$ ($\Lambda^* \in \mathbb{S}^{p-1}_{+,*}$), we observe that $\mathcal{U} = \Lambda u_{x_0,\mu}$ ($\mathcal{U} = \Lambda^* u_{a,T}$) is a non-singular (singular) solution to ($\mathcal{S}_p$). Roughly speaking, our results classify these solutions as the only possible expressions for nontrivial solutions to ($\mathcal{S}_p$).

Our results are natural generalizations of the famous classification due to L. A. Caffarelli, B. Gidas and J. Spruck [4] on the classical singular Yamabe equation (see also [35, 2]). Moreover, O. Druet, E. Hebey and J. Vétois [15] proved the Liouville-type theorem for the associated strongly coupled vectorial equation. Recently, R. Caju, J. M do Ó and A. Santos [5] generalized [4] to this class of systems. For more related results, we refer the reader to [22, 38, 40].

Strongly coupled fourth order systems appear in several important branches of mathematical physic. For instance, in hydrodynamics, for modeling the behavior of deep-water and Rogue waves in the ocean [33]. Also, in the Hartree–Fock theory for Bose–Einstein double condensates [1]. Moreover, in conformal geometry, ($\mathcal{S}_1$) is the limit equation of the conformally constant Q-curvature problem. Hence, in the same way of the singular Yamabe problem, solutions to ($\mathcal{S}_1$) give rise to complete conformal metrics with a constant Q-curvature. For more details on this and some applications, see, for instance, [23]. Motivated by its applications in nonlinear analysis, minimal surface theory, and differential geometry, classification for singular solutions to PDEs has been a topic of intense study in recent years. There exists a vast literature for problems this arising in conformal geometry. For instance, in prescribing different curvature types, such as the higher order Q-curvature, the fractional curvature, and the σ_k-curvature.

The primary sources of difficulties in seeking qualitative properties for fourth order systems like ($\mathcal{S}_p$) are the lack of maximum principle and the failure of truncation methods provoked by the fourth order operator on the left-hand side of ($\mathcal{S}_p$), the coupled setting caused by the Gross–Pitaevskii nonlinearity in the right-hand side of ($\mathcal{S}_p$). In both theorems, we study the PDE or ODE satisfied by the quotient of any two strictly positive components. We compute the linear fourth order equation satisfied by this quotient. For which we prove a strong Liouville-type result.

The proof of Theorems 1.1 is based on the integral moving sphere technique for the norm of a p-map solution. This integral technique allows us to avoid the use of a classical form of a maximum principle, which is not available in our fourth order setting. Another difficulty is that due to the Gross–Pitaevskii nonlinearity, it may occur that the process does not hold for some components, which we prove that is not the case. Our strategy relies on recovering regularity and superharmonicity properties for each component solution, based on a comparison with the norm of the vectorial solution. We then prove that the classification holds for weak solutions and that classical solutions satisfy an estimate of the L^2-norm of its Laplacian, and so they are weak. Theorem 1.2 is proved using the moving planes technique, which shows that all components solutions are rotationally invariant and radially monotonically decreasing. Then, we analyse the Pohozaev invariant, which provides a removable singularity classification theorem. In this case, we can prove that all components blow-up at the origin with the same prescribed asymptotic rate.

Here is a our plan for this paper. In Section 2, we summarize some basic definitions that will be used in this work. In Section 3, we prove that solutions to ($\mathcal{S}_p$) are non-singular and weakly positive. Also, we show that Theorem 1.1 holds for weak solutions to ($\mathcal{S}_p$). Hence, we apply an integral moving spheres method to prove the classification for the norm of a vectorial solution. Using the classification for the norm of a vectorial solution, we show that classical solutions are weak solutions as well. Hence, a direct integral method is used to prove the classification in Theorem 1.1 for weak solutions. We also prove that solutions from Theorem 1.1 are extremal functions for a Sobolev embedding theorem. In Section 4, we obtain that singular solutions are as well classical, and we employ an asymptotic moving planes method to show they are rotationally invariant about the origin. Hence, on the singular case, Syst. ($\mathcal{S}_p$) is equivalent to a one-dimensional fourth order ODE system. We define its Hamiltonian energy and Pohozaev invariant, which we use to perform a delicate ODE analysis to prove a removable-singularity classification for solutions to ($\mathcal{S}_p$). This in turns implies the proof of Theorem 1.2 follows.

2. **Preliminaries.** We introduce some basic definitions used in the remaining part of the text. Here and subsequently, we always deal with non-trivial nonnegative solutions $\mathcal{U}$ of ($\mathcal{S}_p$), that is, $u_i \geq 0$ for all $i \in I$ and $|\mathcal{U}| \not\equiv 0$, where we recall the notation $I = \{1, \dots, p\}$. We split the index set I into two parts $I_0 = \{i \in I : u_i \equiv 0\}$ and $I_+ = \{i \in I : u_i > 0\}$. Then, following standard notation for elliptic systems from [24], we divide solutions to ($\mathcal{S}_p$) into two types.

Definition 2.1. Let $\mathcal{U}$ be a nonnegative solution to ($\mathcal{S}_p$). We call $\mathcal{U}$ strongly positive if $I_+ = I$. On the other hand, when $I_0 \neq \emptyset$, we say that $\mathcal{U}$ is weakly positive.

Remark 2.2. For the proof of Theorems 1.1, it is crucial to show that solutions to ($\mathcal{S}_p$) are weakly positive. We need to guarantee that nontrivial solutions to ($\mathcal{S}_p$) do not develop zeros in the domain. Namely, our strategy is to prove that the so-called quotient function $q_{ij} = u_i/u_j$ is constant for all $i, j \in I_+$. First, for the quotient to be well defined, the denominator must be strictly positive.

When $\liminf_{|x|\to 0} |\mathcal{U}(x)| = \infty$, we call $\mathcal{U}$ a singular solution. In this case, some components might develop a non-removable singularity at the origin. Following [12], we will divide singular solutions into two classes. Namely, a solution to ($\mathcal{S}_p$) is called fully-singular, if the origin is a non-removable singularity for all component solution u_i. Otherwise, we say that $\mathcal{U}$ is semi-singular. More precisely, we present the following definitions.

Definition 2.3. For $\mathcal{U}$ a nonnegative singular solution to ($\mathcal{S}_p$), let us define its blow-up set by $I_\infty = \{i \in I : \liminf_{|x|\to 0} u_i(|x|) = \infty\}$.

It is easy to observe that $\mathcal{U}$ being a singular solution to ($\mathcal{S}_p$) is equivalent to $I_\infty \neq \emptyset$. Hence, in terms of the blow-up set's cardinality, we divide singular solutions to ($\mathcal{S}_p$) as follows.

Definition 2.4. Let $\mathcal{U}$ be a nonnegative singular solution to ($\mathcal{S}_p$). We say that $\mathcal{U}$ is fully-singular if $I_\infty = I$. Otherwise, if $I_\infty \neq I$ we call $\mathcal{U}$ semi-singular.

Definition 2.5. Let $\Omega = \mathbb{R}^n$ ($\Omega = \mathbb{R}^n \setminus \{0\}$ be the punctured space) be the whole space, and $\mathcal{U}$ be a nonnegative non-singular (singular) solution to ($\mathcal{S}_p$).
(i)We say that $\mathcal{U} \in \mathcal{D}^{2,2}(\Omega, \mathbb{R}^p)$ is a weak solution to ($\mathcal{S}_p$), if for all nonnegative $\Phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^p)$, one has

$$\int_{\mathbb{R}^n} \langle \Delta \mathcal{U}, \Delta \Phi \rangle \, \mathrm{d}x = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}-2} \langle \mathcal{U}, \Phi \rangle \, \mathrm{d}x. \tag{4}$$

(ii)We say that $\mathcal{U} \in L^1_{\mathrm{loc}}(\Omega, \mathbb{R}^p)$ is a distributional solution to ($\mathcal{S}_p$), if for all nonnegative $\Phi \in C_c^\infty(\mathbb{R}^n, \mathbb{R}^p)$, one has

$$\int_{\mathbb{R}^n} \langle \mathcal{U}, \Delta^2 \Phi \rangle \, \mathrm{d}x = \int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}-2} \langle \mathcal{U}, \Phi \rangle \, \mathrm{d}x. \tag{5}$$

Here $\mathcal{D}^{2,2}(\Omega, \mathbb{R}^p)$ is the classical Beppo–Levi space, defined as the completion of the space of compactly supported smooth p-maps, denoted by $C_c^\infty(\Omega, \mathbb{R}^p)$, under the Dirichlet norm $\|\mathcal{U}\|^2_{\mathcal{D}^{2,2}(\Omega,\mathbb{R}^p)} = \sum_{i=1}^p \|\Delta u_i\|^2_{L^2(\Omega)}$.

Remark 2.6. In what follows, we use classical regularity theory to prove that any weak non-singular (singular) solution to ($\mathcal{S}_p$) is also a classical non-singular (singular) solution. Since we are working on unbounded domains, it is not direct, though, to verify that classical solutions to ($\mathcal{S}_p$) are also weak. In general, it is true that, by the Green identity, classical solutions $\mathcal{U} \in C^{4,\zeta}(\Omega, \mathbb{R}^p)$ also satisfy (4). Nevertheless, to show that $\mathcal{U} \in \mathcal{D}^{2,2}(\Omega, \mathbb{R}^p)$ is an entire solution to ($\mathcal{S}_p$), one needs to prove some suitable decay at both the origin and infinity.

3. **Liouville-type theorem for non-singular solutions.**

This section is devoted to present the proof of Theorem 1.1. First, using the regularity lifting theorem, we aim to obtain regularity results for solutions to ($\mathcal{S}_p$) with a removable singularity at the origin. Second, employing an iteration argument, we show that non-singular solutions to ($\mathcal{S}_p$) are weakly positive. Then, we perform an integral moving spheres technique to obtain the classification for non-singular solutions to ($\mathcal{S}_p$). This argument provides as a by-product an estimate for the Sobolev norm of solutions to ($\mathcal{S}_p$), yielding that classical solutions to ($\mathcal{S}_p$) are also weak (see Remark 2.6). Finally, as an application of our main result, we show that non-singular solutions to ($\mathcal{S}_p$) are indeed extremal maps for a vectorial Sobolev embedding. Our inspiration are the results in [25, 8, 3, 28, 15, 29, 9].

3.1. **Regularity.**

We prove that weak solutions to ($\mathcal{S}_p$) are as well as classical solutions. We should mention that De Giorgi–Nash–Moser bootstrap techniques combined with the Brézis–Kato method are standard strategies to produce regularity results for second order elliptic PDEs involving critical growth. Unfortunately, this tool does not work in our critical fourth order setting. More precisely, the nonlinearity on the right-hand side of ($\mathcal{S}_p$) has critical growth, so $|\mathcal{U}|^{2^{**}-2}u_i \in L^{2n/(n+4)}(\mathbb{R}^n)$. Notice that we cannot conclude, using the Sobolev embedding theorem, that $|\mathcal{U}|^{2^{**}-2}u_i$ belongs to $L^q(\mathbb{R}^n)$ for some $q > 2n/(n+4)$ and any $i \in I$. We can overcome this lack of integrability by applying the lifting method in [10, Theorem 3.3.1].

Since the origin is a removable singularity, System ($\mathcal{S}_p$) can be modeled in the entire space, in the sense that solutions can be smoothly extended to be defined in $\mathbb{R}^n$. In this situation, ($\mathcal{S}_p$) is reduced to

$$\Delta^2 u_i = c(n)|\mathcal{U}|^{2^{**}-2}u_i \quad \text{in} \quad \mathbb{R}^n.$$

Subsequently, the idea is to provide some properties for solutions to ($\mathcal{S}_p$) by writing this system as a nonlinear fourth order Schrödinger equation with potential $V : \mathbb{R}^n \to \mathbb{R}$ defined by $V(x) = c(n)|\mathcal{U}(x)|^{2^{**}-2}$.

In the next step, we show that it is possible to improve the Lebesgue class in which solutions to ($\mathcal{S}_p$) lie. Here our strategy is to prove that they indeed belong to the Lebesgue space $L^s(\mathbb{R}^n, \mathbb{R}^p)$ for any $s > 2^{**}$.

Proposition 3.1. *Let $\mathcal{U} \in \mathcal{D}^{2,2}(\mathbb{R}^n, \mathbb{R}^p)$ be a nonnegative weak non-singular solution to ($\mathcal{S}_p$). Then, $\mathcal{U} \in L^s(\mathbb{R}^n, \mathbb{R}^p)$ for all $s > 2^{**}$.*

Proof. Let us consider the spaces $Z = C_c^\infty(\mathbb{R}^n)$, $X = L^{2n/(n-4)}(\mathbb{R}^n)$ and $Y = L^q(\mathbb{R}^n)$ for $q > 2n/(n-4)$. Let $\Gamma_2(x,y) = C(n)|x-y|^{4-n}$ be the fundamental solution to Δ^2 in $\mathbb{R}^n$, where $C(n) = [(n-4)(n-2)\omega_{n-1}]^{-1}$. Thus, it is well-defined the inverse operator

$$(Tu)(x) = \int_{\mathbb{R}^n} \Gamma_2(x,y)u(y)\,\mathrm{d}y.$$

Hence, using the Hardy–Littlewood–Sobolev inequality (see [30]), we get that for any $q \in (1, n/4)$, there exists $C > 0$ such that

$$\|Tu\|_{L^{\frac{nq}{n-4q}}(\mathbb{R}^n)} = \|\Gamma_2 * u\|_{L^{\frac{nq}{n-4q}}(\mathbb{R}^n)} \leq C\|u\|_{L^q(\mathbb{R}^n)}.$$

For $M > 0$, let us define $\widetilde{V}_M(x) = V(x) - V_M(x)$, where

$$V_M(x) = \begin{cases} V(x), & \text{if } |V(x)| \geq M, \\ 0, & \text{otherwise.} \end{cases}$$

Applying the integral operator $T_M u := \Gamma_2 * V_M u$ on ($\mathcal{S}_p$), we obtain that $u_i = T_M u_i + \widetilde{T}_M u_i$, where

$$(T_M u_i)(x) = \int_{\mathbb{R}^n} \Gamma_2(x,y) V_M(y) u_i(y)\,\mathrm{d}y \text{ and } \widetilde{T}_M u_i(x) = \int_{\mathbb{R}^n} \Gamma_2(x,y) \widetilde{V}_M(y) u_i(y)\,\mathrm{d}y.$$

Claim 1. *For $n/(n-4) < q < \infty$, there exists $M \gg 1$ large such that $T_M : L^q(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is a contraction.*

In fact, for any $q \in (n/(n-4), \infty)$, there exists $m \in (1, n/4)$ such that $q = nm/(n-4m)$. Then, by the Hölder inequality, for any $u \in L^q(\mathbb{R}^n)$, we get that there exists $C > 0$ satisfying

$$\|T_M u\|_{L^q(\mathbb{R}^n)} \leq \|\Gamma_2 * V_M u\|_{L^q(\mathbb{R}^n)} \leq C \|V_M\|_{L^{n/4}(\mathbb{R}^n)} \|u\|_{L^q(\mathbb{R}^n)}.$$

Since $V_M \in L^{n/4}(\mathbb{R}^n)$ it is possible to choose $M \gg 1$ such that $\|V_M\|_{L^{n/4}(\mathbb{R}^n)} < 1/2C$. Therefore, we arrive at $\|T_M u\|_{L^q(\mathbb{R}^n)} \leq 1/2 \|u\|_{L^q(\mathbb{R}^n)}$, which yields T_M is a contraction.

Claim 2. *For any $n/(n-4) < q < \infty$, it follows that $\widetilde{T}_M u_i \in L^q(\mathbb{R}^n)$.*

Indeed, for any $n/(n-4) < q < \infty$, choose $1 < m < n/4$, satisfying $q = nm/(n-4m)$. Since $\widetilde{V}_M$ is bounded, we obtain

$$\|\widetilde{T}_M u_i\|_{L^q(\mathbb{R}^n)} = \|\Gamma_2 * \widetilde{V}_M u_i\|_{L^q(\mathbb{R}^n)} \leq C_1 \|\widetilde{V}_M u_i\|_{L^m(\mathbb{R}^n)} \leq C_2 \|u_i\|_{L^m(\mathbb{R}^n)}.$$

However, using the Sobolev embedding theorem, we have that $u_i \in L^m(\mathbb{R}^n)$ when $m = 2n/(n-4)$, which implies $q = 2n/(n-8)$. Thus, we find that $u_i \in L^q(\mathbb{R}^n)$ when

$$\begin{cases} 1 < q < \infty, & \text{if } 5 \leq n \leq 8 \\ 1 < q \leq \frac{2n}{n-8}, & \text{if } n \geq 9. \end{cases}$$

Now we can repeat the argument for $m = 2n/(n-8)$ to obtain that $u_i \in L^q(\mathbb{R}^n)$ for

$$\begin{cases} 1 < q < \infty, & \text{if } 5 \leq n \leq 12 \\ 1 < q \leq \frac{2n}{n-12}, & \text{if } n \geq 13. \end{cases}$$

Therefore proceeding inductively as in the last argument, the proof of the claim follows.

Combining Claims 1 and 2, we can apply [10, Theorem 3.3.1] to show that $u_i \in L^q(\mathbb{R}^n)$ for all $q > 2^{**}$ and $i \in I$. In particular, the proof of the proposition is concluded. □

Corollary 3.2. *Let $\mathcal{U} \in \mathcal{D}^{2,2}(\mathbb{R}^n, \mathbb{R}^p)$ be a nonnegative weak non-singular solution to ($\mathcal{S}_p$). Then, $\mathcal{U} \in C^{4,\zeta}(\mathbb{R}^n, \mathbb{R}^p)$ is a classical non-singular solution to ($\mathcal{S}_p$).*

Proof. Using the Proposition 3.1, we can apply Morrey embedding theorem to get $u_i \in C^{0,\zeta}(\mathbb{R}^n)$ for some $\zeta \in (0,1)$. Finally using Schauder estimates, one concludes $u_i \in C^{4,\zeta}(\mathbb{R}^n)$, which provides that $\mathcal{U} \in C^{4,\zeta}(\mathbb{R}^n, \mathbb{R}^p)$. □

3.2. **Superharmonicity.** We obtain a strong maximum principle for nonnegative solutions to ($\mathcal{S}_p$). We prove that any component solution to ($\mathcal{S}_p$) is superharmonic. Compared to its scalar counterpart, the main difference in our approach is the appearance of the strong coupling term on the right-hand side of ($\mathcal{S}_p$). This coupled nonlinearity could imply the failure of the method for some components. However, we can overcome this issue thanks to an inequality involving the norm of the p-map solution.

Proposition 3.3. *Let $\mathcal{U}$ be a nonnegative solution to ($\mathcal{S}_p$). Then, $-\Delta u_i \geq 0$ in $\mathbb{R}^n$ for all $i \in I$.*

Proof. Supposing by contradiction that the proposition does not hold, there exists $i \in I$ and $x_0 \in \mathbb{R}^n$ satisfying $-\Delta u_i(x_0) < 0$. Since the Laplacian is invariant under translations, we may suppose without loss of generality that $x_0 = 0$. Let us reformulate ($\mathcal{S}_p$) as the following system in the whole space

$$\begin{cases} -\Delta u_i &= h_i \\ -\Delta h_i &= c(n)|\mathcal{U}|^{2^{**}-2}u_i. \end{cases} \tag{6}$$

Let $B_r \subseteq \mathbb{R}^n$ be the ball of radius $r > 0$, and ω_{n-1} be the $(n-1)$-dimensional surface measure of the unit sphere, we consider

$$\overline{u}_i = \frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} u_i \, \mathrm{d}\sigma_r \quad \text{and} \quad \overline{h}_i = \frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} h_i \, \mathrm{d}\sigma_r,$$

the spherical averages of u_i and h_i, respectively. Now taking the spherical average on the first line of (6), and using that $\overline{\Delta u_i} = \Delta \overline{u}_i$, implies

$$\Delta \overline{u}_i + \overline{h}_i = 0.$$

Furthermore, we rewrite the second equality of (6) to get $\Delta h_i + c(n)|\mathcal{U}|^{2^{**}-2}u_i = 0$, from which, by taking again the spherical average in both sides, we conclude

$$\begin{aligned} 0 &= \frac{1}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} \left(\Delta h_i + c(n)|\mathcal{U}|^{2^{**}-2}u_i\right) \mathrm{d}\sigma_r \\ &= \Delta \overline{h}_i + \frac{c(n)}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} |\mathcal{U}|^{2^{**}-2}u_i \, \mathrm{d}\sigma_r. \end{aligned}$$

Hence,

$$\Delta \overline{h}_i = -\frac{c(n)}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} |\mathcal{U}(x)|^{2^{**}-2}u_i(x) \, \mathrm{d}\sigma_r, \tag{7}$$

which, by using that $0 \leq u_i(x) \leq |\mathcal{U}(x)|$ for any $x \in \mathbb{R}^n$, implies

$$\begin{aligned} -\frac{c(n)}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} |\mathcal{U}(x)|^{2^{**}-2}u_i(x) \, \mathrm{d}\sigma_r &\leq -\frac{c(n)}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} |u_i(x)|^{2^{**}-1} \, \mathrm{d}\sigma_r \\ &\leq \left(\frac{-c(n)}{n\omega_{n-1}r^{n-1}} \int_{\partial B_r} |u_i(x)| \, \mathrm{d}\sigma_r\right)^{2^{**}-1} \\ &= -c(n)\overline{u}_i^{2^{**}-1}, \end{aligned} \tag{8}$$

where on the second inequality, we used the Jensen inequality for the convex function $t \mapsto t^{2^{**}-1}$. Finally, combining (7) and (8), we get

$$\Delta \overline{h}_i + c(n)\overline{u}_i^{2^{**}-1} \leq 0.$$

Therefore, we can conclude the proof following the same steps as in [39, Theorem 2.1]. □

As a consequence of the last result, we can prove that solutions to ($\mathcal{S}_p$) are weakly positive.

Corollary 3.4. *Let $\mathcal{U}$ be a nonnegative solution to ($\mathcal{S}_p$). Then, for any $i \in I$ we have that either $u_i \equiv 0$ or $u_i > 0$. In other terms, $I = I_0 \cup I_+$ is a disjoint union.*

3.3. **Kelvin transform.** We define some type of transform suitable to explore the symmetries of ($\mathcal{S}_p$), which is called the fourth order Kelvin transform of a p-map. This map is a key ingredient for developing a sliding method, namely the moving spheres or the moving planes techniques.

For $\Omega = \mathbb{R}^n$ or $\Omega = \mathbb{R}^n \setminus \{0\}$, we define the Kelvin transform. To this end, for given $x_0 \in \mathbb{R}^n$ and $\mu > 0$, we need to establish the concept of inversion about a sphere $\partial B_\mu(x_0)$, which is a map $\mathcal{I}_{x_0,\mu} : \Omega \to \Omega_{x_0,\mu}$ given by $\mathcal{I}_{x_0,\mu}(x) = x_0 + \mathcal{K}_{x_0,\mu}(x)^2(x - x_0)$, where $\mathcal{K}_{x_0,\mu}(x) = \mu/|x - x_0|$ and $\Omega_{x_0,\mu} := \mathcal{I}_{x_0,\mu}(\Omega)$ is the domain of the Kelvin transform. In particular, when $x_0 = 0$ and $\mu = 1$, we denote it simply by $\mathcal{I}_{0,1}(x) = x^*$ and $\mathcal{K}_{0,1}(x) = x|x|^{-2}$.

Definition 3.5. For any $\mathcal{U} : \Omega \to \mathbb{R}^p$, let us consider the fourth order Kelvin transform about the sphere with center at $x_0 \in \mathbb{R}^n$ and radius $\mu > 0$ defined on $\mathcal{U}_{x_0,\mu} : \Omega_{x_0,\mu} \to \mathbb{R}^p$ by

$$\mathcal{U}_{x_0,\mu}(x) = \mathcal{K}_{x_0,\mu}(x)^{n-4}\mathcal{U}\left(\mathcal{I}_{x_0,\mu}(x)\right).$$

In particular, when $p = 1$ we set the notation $u_{x_0,\mu}$.

Now we need to understand how ($\mathcal{S}_p$) behaves under the Kelvin transform's action.

Proposition 3.6. *System ($\mathcal{S}_p$) is conformally invariant, in the sense that it is invariant under the action of Kelvin transform, i.e., if $\mathcal{U}$ is a non-singular solution to ($\mathcal{S}_p$), then $\mathcal{U}_{x_0,\mu}$ is a solution to*

$$\Delta^2(u_i)_{x_0,\mu} = c(n)|\mathcal{U}_{x_0,\mu}|^{2^{**}-2}(u_i)_{x_0,\mu} \quad \text{in} \quad \mathbb{R}^n \setminus \{x_0\}, \tag{9}$$

where $\mathcal{U}_{x_0,\mu} = ((u_1)_{x_0,\mu}, \dots, (u_p)_{x_0,\mu})$.

Proof. For all $x \in \mathbb{R}^n \setminus \{x_0\}$, let us recall the formulas below

$$\Delta u_{x_0,\mu}(x) = \mathcal{K}_{x_0,\mu}(x)^{n+2}\Delta u\left(\mathcal{I}_{x_0,\mu}(x)\right) = \mathcal{K}_{x_0,\mu}(x)^4(\Delta u)_{x_0,\mu}(x)$$

and

$$\Delta^2 u_{x_0,\mu}(x) = \mathcal{K}_{x_0,\mu}(x)^{n+4}\Delta^2 u\left(\mathcal{I}_{x_0,\mu}(x)\right) = \mathcal{K}_{x_0,\mu}(x)^8(\Delta^2 u)_{x_0,\mu}(x). \tag{10}$$

Next, expanding the right-hand side of (9), we observe

$$|\mathcal{U}_{x_0,\mu}(x)|^{2^{**}-2}(u_i)_{x_0,\mu} = \mathcal{K}_{x_0,\mu}(x)^{n+4}|\mathcal{U}(x)|^{2^{**}-2}u_i(x). \tag{11}$$

Therefore, the proof of the proposition follows by a combination of (10) and (11). □

Remark 3.7. Proposition 3.6 is not a surprising conclusion since the Gross–Pitaevskii-type nonlinearity preserves the same conformal invariance enjoyed by the scalar case. Namely, ($\mathcal{S}_1$) is invariant under the conformal euclidean group's action.

3.4. **Integral moving spheres method.** We apply the moving sphere method to show that nonnegative solutions $\mathcal{U}$ to ($\mathcal{S}_p$) are radially symmetric, that is, u_i is radially symmetric for all $i \in I$. Furthermore, we provide the classification for each u_i, and, in particular, for the norm $|\mathcal{U}|$. The moving spheres method is an alternative variant of the moving planes method, which can also be used to obtain radial symmetry or more robust Liouville-type results for solutions to more general PDEs.

Initially, Notice that system ($\mathcal{S}_p$) is equivalent to the following vectorial integral equation

$$u_i(x) = \int_{\mathbb{R}^n} |x-y|^{4-n} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y \quad \text{in} \quad \mathbb{R}^n, \tag{$\mathcal{I}_p$}$$

where $\widehat{f}_i(\mathcal{U}) = [(n-2)(n-4)\omega_{n-1}]^{-1} c(n) |\mathcal{U}|^{2^{**}-2} u_i$. In the sense that every solution to ($\mathcal{S}_p$) is a solution ($\mathcal{I}_p$) plus a constant, and the reciprocal also holds.

The following lemma is the first step to apply the integral moving spheres method.

Lemma 3.8. *Let $\mathcal{U}$ be a nonnegative non-singular solution to ($\mathcal{I}_p$). For any $x \in \mathbb{R}^n$, $z \in \mathbb{R}^n \setminus (\{0\} \cup B_\mu(x))$, it holds that $u_i(z) - (u_i)_{x,\mu}(z) > 0$ for $i \in I$ and some $\mu \in (0,1)$.*

Proof. Let $\mathcal{U}$ be a nonnegative solution to ($\mathcal{I}_p$). Using the identities in [28, page 162], one has

$$\left(\frac{\mu}{|z-x|}\right)^{n-4} \int_{|y-x|\geq\mu} |\mathcal{I}_{x,\mu}(z) - y|^{n-4} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y = \int_{|y-x|\leq\mu} |z-y|^{n-4} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y$$

and

$$\left(\frac{\mu}{|z-x|}\right)^{n-4} \int_{|y-x|\leq\mu} |\mathcal{I}_{x,\mu}(z) - y|^{n-4} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y = \int_{|y-x|\geq\mu} |z-y|^{n-4} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y,$$

which yields

$$(u_i)_{x,\mu}(z) = \int_{\mathbb{R}^n} |z-y|^{n-4} \widehat{f}_i(\mathcal{U}(y)) \, \mathrm{d}y \quad \text{for} \quad z \in \mathcal{I}_{x,\mu}(\mathbb{R}^n), \tag{12}$$

Consequently, for any $x \in \mathbb{R}^n$ and $\mu < 1$, we have that for $z \in \mathbb{R}^n \setminus \{0\} \cup B_\mu(x)$,

$$u_i(z) - (u_i)_{x,\mu}(z) = \int_{|y-x|\geq\mu} E(x,y,\mu,z) \left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_{x,\mu}(y))\right] \, \mathrm{d}y,$$

where

$$E(x,y,z,\mu) := |z-y|^{4-n} - \left(\frac{|z-x|}{\mu}\right)^{4-n} |\mathcal{I}_{x,\mu}(z) - y|^{4-n} \tag{13}$$

is used to estimate the difference between a $\mathcal{U}$ and its Kelvin transform $\mathcal{U}_{x,\mu}$. Finally, using its decay properties, it is straightforward to check that $E(x,y,z,\mu) > 0$ for all $|z-x| > \mu > 0$, which is enough in the region on which the difference has the correct sign. Otherwise, one can apply the Hardy–Littlewood–Sobolev as in [11, Theorem 2.2] to conclude the proof. ∎

Next, let us introduce the critical sliding parameter as the supremum for which an inequality relating a component function and its Kelvin transform is satisfied.

Definition 3.9. Given $x \in \mathbb{R}^n$, for each $i \in I$, let us define

$$\mu_i^*(x) = \sup\left\{\mu > 0 : (u_i)_{r,x} \leq u_i \text{ in } \mathbb{R}^n \setminus B_r(x) \text{ for any } 0 < r < \mu\right\}. \tag{14}$$

Since each u_i is superharmonic for all $i \in I$, by using [29, Lemma 11.1], we get $\mu_i^*(x) > 0$ for $i \in I$. Thus, we can define

$$\mu^*(x) = \inf_{i \in I} \mu_i^*(x) > 0.$$

The next lemma is essentially the moving spheres technique in its integral form. This method provides the exact form for any blow-up limit solution to ($\mathcal{S}_p$), which depends on whether the critical sliding parameter $\mu^*(x)$ is finite or infinite. The main ingredient in the proof is the integral version of the moving spheres technique contained in [28, Lemma 3.2].

Lemma 3.10. *Let $\mathcal{U}$ be a nonnegative non-singular solution to ($\mathcal{S}_p$), $z \in \mathbb{R}^n$ and $\mu^*(z) > 0$ given by (14). Assume that the origin is a removable singularity. The following holds:*
(i) *if $\mu^*(x) < \infty$ is finite, then $\mathcal{U}_{x,\mu^*(x)} = \mathcal{U}$ in $\mathbb{R}^n \setminus \{x\}$.*
(ii) *if $\mu^*(x_0) = \infty$, for some $x_0 \in \mathbb{R}^n$, then $\mu^*(x) = \infty$ for all $x \in \mathbb{R}^n$.*

Proof. Without loss of generality, we may assume $x_0 = 0$. Let us fix $\mu^* = \mu^*(0)$ and $(u_i)_\mu = (u_i)_{0,\mu}$ for all $i \in I$. By the definition of $\mu^*(x)$, we have that $(u_i)_{\mu^*}(x) \le u_i(x)$, for all $|x| \ge \mu^*$. Thus, by (12), with $x = 0$ and $\mu = |x| \ge \mu^*$, and the positivity of the kernel $E(0, y, z, \mu)$ given by (13), either $u_{\mu^*}(y) = u(y)$ for all $|x| \ge \mu^*$ or $(u_i)_{\mu^*}(y) < (u_i)(y)$ for all $|x| > \mu^*$. In the former case, the conclusion easily follows. In the sequel, we assume that the last condition holds. Hence, using the integral representation in ($\mathcal{I}_p$) we have

$$\begin{aligned}
&\liminf_{|z|\to\infty} |z|^{n-4}\left[u_i(z) - (u_i)_{\mu^*}(z)\right] \\
&= \liminf_{|z|\to\infty} \int_{|y|\ge\mu^*} |z|^{n-4} E(0, y, z, \mu^*)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_{\mu^*}(y))\right] \mathrm{d}y \\
&\ge \int_{|y|\ge\mu^*} \left(1 - \left(\frac{\mu^*}{|y|}\right)^{n-4}\right)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_{\mu^*}(y))\right] \mathrm{d}y > 0,
\end{aligned}$$

which implies that there exists $\varepsilon_1 \in (0,1)$ satisfying $u_i(z) - (u_i)_{\mu^*}(z) \ge \varepsilon_1 |z|^{4-n}$ for all $|z| \ge \mu^* + 1$ and $i \in I$. Moreover, there exists $\varepsilon_2 \in (0, \varepsilon_1)$ such that, for $|z| \ge \mu^* + 1$ and $\mu^* \le \mu \le \mu^* + \varepsilon_2$, we find

$$\left(u_i - (u_i)_{\mu^*}\right)(z) \ge \varepsilon_1 |z|^{4-n} + \left((u_i)_{\mu^*} - (u_i)_\mu\right)(z) \ge \frac{\varepsilon_1}{2}|z|^{4-n}. \tag{15}$$

Whence, for any $\varepsilon \in (0, \varepsilon_2)$ (to be chosen later), $\mu^* \le \mu \le \mu^* + \varepsilon$, and $\mu \le |y| \le \mu^* + 1$, we have

$$\begin{aligned}
\left(u_i - (u_i)_{\mu^*}\right)(z) &= \int_{|y|\ge\mu} E(0, y, z, \mu)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y))\right] \mathrm{d}y \\
&\ge \int_{\mu\le|y|\le\mu+1} E(0, y, z, \mu)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y))\right] \mathrm{d}y \\
&+ \int_{\mu^*+2\le|z|\le\mu^*+3} E(0, y, z, \mu)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y))\right] \mathrm{d}y \\
&\ge \int_{\mu^*\le|y|\le\mu^*+1} E(0, y, z, \mu)\left[\widehat{f}_i(\mathcal{U}_{\mu^*}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y))\right] \mathrm{d}y \\
&+ \int_{\mu^*+2\le|y|\le\mu^*+3} E(0, y, z, \mu)\left[\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y))\right] \mathrm{d}y.
\end{aligned}$$

Now using (15), there exists $\delta_1 > 0$ such that $\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_\mu(y)) \geq \delta_1$ for $\mu^* + 2 \leq |y| \leq \mu^* + 3$. Since $E(0, y, z, \mu) = 0$ for all $|z| = \lambda$ and

$$\nabla_z E(0, y, z, \mu) \cdot z\big|_{|z|=\mu} = (n-4)|z-y|^{6-n}\left(|z|^2 - |y|^2\right) > 0 \text{ for all } \mu^* + 2 \leq |y| \leq \mu^* + 3,$$

where $\delta_2 > 0$ is a constant independent of ε. Then, there exists $C > 0$ such that, for $\mu^* \leq \mu \leq \mu^* + \varepsilon$, we get

$$\left|\widehat{f}_i(\mathcal{U}(y)) - \widehat{f}_i(\mathcal{U}_{\mu^*}(y))\right| \leq C(\mu - \mu^*) \leq C\varepsilon \quad \text{for all} \quad \mu^* \leq \mu \leq |y| \leq \mu^* + 1.$$

Furthermore, recalling that $\mu \leq |z| \leq \mu^* + 1$, we obtain

$$\begin{aligned}
&\int_{\mu \leq |y| \leq \mu^*} E(0, y, z, \mu)\, \mathrm{d}y \\
&\leq \left|\int_{\mu \leq |y| \leq \mu^*+1} \left[|y-z|^{4-n} - |\mathcal{I}_\mu(z) - y|^{4-n} + \left(\frac{\mu}{|z|} - 1\right)^{n-4} |\mathcal{I}_\mu(z) - y|^{n-4}\right] \mathrm{d}y\right| \\
&\leq C\, |\mathcal{I}_\mu(z) - z| + C(|z| - \mu) \leq C(|z| - \mu),
\end{aligned}$$

which, provides that for small $0 < \varepsilon \ll 1$, $\mu^* \leq \mu \leq \mu^* + \varepsilon$, and $\mu \leq |z| \leq \mu^* + 1$, it follows

$$\begin{aligned}
&\left(u_i - (u_i)_{\mu^*}\right)(z) \\
&\geq -C\varepsilon \int_{\mu \leq |y| \leq \mu^*+1} E(0, y, z, \mu)\, \mathrm{d}y + \delta_1\delta_2(|z| - \mu) \int_{\mu^*+2 \leq |z| \leq \mu^*+3} \mathrm{d}y \\
&\geq \left(\delta_1\delta_2 \int_{\mu^*+2 \leq |y| \leq \mu^*+3} \mathrm{d}z - C\varepsilon\right)(|z| - \mu) \geq 0.
\end{aligned}$$

This is a contradiction to the definition of $\mu^* > 0$. Therefore, the first part of the lemma is established.

Next, by the definition of $\mu^*(x)$, we know that $|\mathcal{U}_{x,\mu}(z)| \leq |\mathcal{U}(z)|$ for all $0 < \mu < \mu^*(x)$, $|z - x| \geq \mu$; thus, multiplying it by $|z|^{n-4}$, and taking the limit as $|z| \to \infty$, yields

$$l = \liminf_{|z| \to \infty} |z|^{n-4} |\mathcal{U}(z)| \geq \mu^{n-4} |\mathcal{U}(z)| \quad \text{for all} \quad 0 < \mu < \mu^*(x). \tag{16}$$

On the other hand, if $\mu^*(x_0) < \infty$, multiplying the identity obtained in (i) by $|z|^{n-4}$ and passing to the limit when $|z| \to \infty$, we obtain

$$l = \lim_{|z| \to \infty} |z|^{n-4} |\mathcal{U}(z)| = \mu^*(x_0)^{n-4} |\mathcal{U}(x_0)| < \infty. \tag{17}$$

Finally, by (16) and (17), if there exists $x_0 \in \mathbb{R}^n$ such that $\mu^*(x_0) < \infty$, then $\mu^*(x) < \infty$ for all $x \in \mathbb{R}^n$. □

3.5. **Classification for weak solutions.** We use the weak formulation of solution to ($\mathcal{S}_p$) to prove a version of Theorem 1.1 for weak solutions, which is based on the analysis of quotient functions $q_{ij} = u_i/u_j$. The main idea is use the integral representation for solutions to ($\mathcal{S}_p$), which yields a more quantitative estimate for the constants $\Lambda_{ij} = q_{ij}$ (see (18)). Before starting our method, we must be cautious that the quotient is well-defined since we may have solutions having zeros in the domain or even being identically null. By Proposition 3.4, we know that the latter situation does not occur. Moreover, we can avoid the former situation by assuming that component solutions u_i are strictly positive, that is, $i \in I_+$. Notice that Theorem 1.1 is now equivalent to proving that all quotient functions are identically constant, *i.e.*, component solutions are proportional to each other $u_j = \Lambda_{ij} u_j$.

Before, we need an auxiliary result, which is a variant of the classical strong Liouville-type result for biharmonic functions in [26, Theorem 1].

Lemma 3.11. *Let $q \in C^4(\mathbb{R}^n)$ be a nonnegative solution to the fourth linear problem*

$$\Delta^2 q + c_3(x)\nabla\Delta q + c_2(x)\Delta q + c_1(x)\nabla q = 0 \quad \text{in} \quad \mathbb{R}^n,$$

where $c_2 \in C^\infty(\mathbb{R}^n)$ is a nonpositive smooth function, and $c_1, c_3 \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ are smooth matrices. If q is bounded above and below, then q is constant.

Proof. Since at critical point $x_0 \in \mathbb{R}^n$ of q, one has that $\nabla q(x_0) = 0$. The proof follows by noticing that since ∇q is a solution a second order uniformly elliptic operator, a uniqueness result follows by the weak maximum principle from [16, Theorem 1] and the Harnack inequality from [6, Theorem 3.6]. □

We state the main result of this part, which is a fourth order version of [14, Proposition 3.1]

Theorem 1' *Let $\mathcal{U}$ be a weak nonnegative non-singular solution to $(\mathcal{S}_p)$. Then, there exist $x_0 \in \mathbb{R}^n$, $\mu > 0$, and $\Lambda \in \mathbb{S}^{p-1}_+$ such that $\mathcal{U} = \Lambda u_{x_0,\mu}$, where $u_{x_0,\mu}$ is a fourth order spherical solution given by (1).*

Proof. For a weak nonnegative non-singular solution $\mathcal{U}$ to $(\mathcal{S}_p)$ and $i, j \in I_+$, let us consider the quotient function $q_{ij} : \mathbb{R}^n \to (0, \infty)$ given by $q_{ij} := u_i/u_j$. Besides, by the smoothness result in Corollary 3.2, we get $q_{ij} \in C^\infty(\mathbb{R}^n)$ for all $i, j \in I_+$. Moreover, there exists $C > 0$ satisfying $0 \leq q_{ij}(x) \leq C$ for all $x \in \mathbb{R}^n$.

In what follows, we divide the argument into two claims. The first one provides a strong classification for any quotient function.

Claim 1. *For all $i, j \in I_+$, there exists a constant $\Lambda_{ij} > 0$ such that $q_{ij} \equiv \Lambda_{ij}$.*

As a matter of fact, a straightforward computation yields

$$\Delta^2 q_{ij} = \frac{u_j\Delta^2 u_i - u_i\Delta^2 u_j}{u_j^2} - \frac{4}{u_j}\nabla\Delta q_{ij}\nabla u_j - \frac{6}{u_j}\Delta q_{ij}\Delta u_j - \frac{4}{u_j}\nabla q_{ij}\nabla\Delta u_j.$$

Notice that since $\mathcal{U}$ solves $(\mathcal{S}_p)$, the first term on the right-hand side of the last equation is zero. Thus, we are left with

$$\Delta^2 q_{ij} + \frac{4}{u_j}\nabla u_j\nabla\Delta q_{ij} + \frac{6}{u_j}\Delta u_j\Delta q_{ij} + \frac{4}{u_j}\nabla\Delta u_j\nabla q_{ij} = 0.$$

The conclusion is a consequence of Lemma 3.11.

Claim 2.*For all $i, j \in I_+$, it follows*

$$\Lambda_{ij} = \frac{\displaystyle\int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}-2} u_i \, \mathrm{d}x}{\displaystyle\int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}-2} u_j \, \mathrm{d}x}. \tag{18}$$

In fact, this equivalent to prove that $\min_{\partial B_R(0)} q_{ij} \to \Lambda_{ij}$ and $\max_{\partial B_R(0)} q_{ij} \to \Lambda_{ij}$ as $R \to \infty$. To this end, we divide the proof into three steps. The first one concerns the behavior at infinity of component solutions to $(\mathcal{S}_p)$.

Step 1: $|x|^{(n-4)/2} u_i(x) = o_R(1)$ as $R \to \infty$.

For $R > 0$, let us consider the rescaling of $\mathcal{U}$ given by $\mathcal{W}_R(x) = R^{(n-4)/2}\mathcal{U}(Rx)$, which in terms of component takes the form $(w_R)_i = R^{(n-4)/2} u_i(Rx)$. Since $u_i \in$

$L^{2^{**}}(\mathbb{R}^n)$, we get

$$\Delta^2(w_R)_i = c(n)|\mathcal{W}|^{2^{**}-2}(w_R)_i \quad \text{as} \quad R \to \infty.$$

and

$$\int_{B_2(0)\setminus B_{1/2}(0)} |\mathcal{W}_R|^{2^{**}} \, \mathrm{d}x = o_R(1) \quad \text{as} \quad R \to \infty.$$

Thus, $(w_R)_i \to 0$ in $C^\infty_{\text{loc}}(B_{3/2}(0) \setminus B_{3/4}(0))$ as $R \to \infty$.

In the next step, we obtain an upper bound for component solutions to ($\mathcal{S}_p$), which provides an interpolation estimate, showing that $u_i \in L^p(\mathbb{R}^n)$ for $2 < p < 2^{**}$.

Step 2: For any $0 < \varepsilon < 1/2$, there exists $C_\varepsilon > 0$ such that $u_i(x) \leq C_\varepsilon |x|^{(4-n)(1-\varepsilon)}$ for all $x \in \mathbb{R}^n$.

First, by Step 1 for a given $0 < \varepsilon < 1/2$, there exists $R_\varepsilon \gg 1$ sufficiently large satisfying

$$\sup_{\mathbb{R}^n \setminus B_{R_\varepsilon}(0)} |x|^2 |\mathcal{U}(x)|^{2^{**}-2} < \frac{(n-4)^2}{2}\varepsilon(1-\varepsilon). \tag{19}$$

For $R \geq R_\varepsilon$, let us consider $\sigma(R) = \max_{i\in I_+} \max_{\partial B_R(x_0)} u_i$ and the auxiliary function

$$G_\varepsilon(x) = \sigma(R_\varepsilon)\left(\frac{|x|}{R_\varepsilon}\right)^{(4-n)(1-\varepsilon)} + \sigma(R)\left(\frac{|x|}{R}\right)^{(4-n)\varepsilon}.$$

Notice that, by construction, we clearly have that $u_i \leq G_\varepsilon$ on $\partial B_R(0) \cup \partial B_{R_\varepsilon}(0)$. Let us suppose that there exists $x_0 \in B_R(0) \setminus \bar{B}_{R_\varepsilon}(0)$, a maximum point of u_i/G_ε, which would imply that $\Delta(u_i G_\varepsilon^{-1}(x_0)) \leq 0$, and then

$$\frac{\Delta u_i(x)}{u_i(x)} \geq \frac{\Delta G_\varepsilon^{-1}(x)}{G_\varepsilon^{-1}(x)}. \tag{20}$$

Furthermore, a direct computation implies

$$\Delta G_\varepsilon^{-1}(x) = G_\varepsilon^{-1}(x)\frac{(n-4)^2}{2}\varepsilon(1-\varepsilon)|x|^{-2}. \tag{21}$$

Therefore, by Proposition 3.3 we obtain that $\Delta^2 u_i(x) - \Delta u_i(x) \geq 0$, which combined with (20) and (21) yields

$$|x|^2|\mathcal{U}(x)|^{2^{**}} = \frac{\Delta^2 u_i(x)}{u_i(x)} \geq \frac{\Delta u_i(x)}{u_i(x)} \geq \frac{\Delta G_\varepsilon^{-1}(x)}{G_\varepsilon^{-1}(x)} = \frac{(n-4)^2}{2}\varepsilon(1-\varepsilon).$$

This is a contradiction with (19) since our choice of $R_\varepsilon > 0$. Then, applying the strong maximum principle, we have

$$u_i(x) \leq \sigma(R_\varepsilon)\left(\frac{|x|}{R_\varepsilon}\right)^{(4-n)(1-\varepsilon)} + \sigma(R)\left(\frac{|x|}{R}\right)^{(4-n)\varepsilon} \quad \text{in} \quad B_R(0) \setminus \bar{B}_{R_\varepsilon}(0), \tag{22}$$

for all $R > R_\varepsilon$. Thus, using (22) combined with Step 1, and taking the limit as $R \to \infty$, we get

$$u_i(x) \leq \sigma(R_\varepsilon)\left(\frac{|x|}{R_\varepsilon}\right)^{(4-n)(1-\varepsilon)} \quad \text{in} \quad \mathbb{R}^n.$$

Step 3: $|x|^{4-n}u_i(x) = \int_{\mathbb{R}^n} \widehat{f}_i(\mathcal{U}(x)) \, \mathrm{d}x + o_R(1)$ as $R \to \infty$.

First, since $u_i \in L^{2^{**}}(\mathbb{R}^n)$, we have $|\mathcal{U}|^{2^{**}-2}u_i \in L^{2n/(n+4)}(\mathbb{R}^n)$ for all $i \in I$, which implies $|\mathcal{U}|^{2^{**}-2}u_i \in W^{-2,2}(\mathbb{R}^n)$. Recall that ($\mathcal{S}_p$) can be reduced to the following integral system ($\mathcal{I}_p$), from which follows

$$|x|^{n-4}u_i(x) = \int_{\mathbb{R}^n} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y = C_n(I_1 + I_2),$$

where

$$I_1 = \int_{B_R(0)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y.$$

and

$$I_2 = \int_{\mathbb{R}^n\setminus B_R(0)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y.$$

To control I_1, we observe that since

$$\int_{B_R(0)} \left[\left(\frac{|x|}{|x-y|}\right)^{n-4} - 1\right] \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y = o_R(1), \tag{23}$$

the following asymptotic identity holds

$$I_1 = \int_{B_R(0)} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y + o_R(1) \quad \text{as} \quad R \to \infty, \tag{24}$$

where the identity (23) holds because the integrand is bounded.

Now it remains to estimate I_2. Accordingly, using Step 2, we can write

$$\begin{aligned} I_2 &= \int_{\mathbb{R}^n\setminus B_R(0)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y \qquad (25)\\ &\leq \int_{B_{|x|/2}(x)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y + \int_{\mathbb{R}^n\setminus B_{|x|/2}(x)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y \\ &\leq C_\varepsilon^{2^{**}-1} \int_{B_{|x|/2}(x)} \left(\frac{|x|}{|x-y|}\right)^{n-4} \left(\frac{|x|}{2}\right)^{-(n+4)(1-\varepsilon)} \mathrm{d}y + 2^{n-4} \int_{\mathbb{R}^n\setminus B_R(0)} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y \\ &\leq C_\varepsilon^{2^{**}-1} 2^{(n+4)(1-\varepsilon)-2}\omega_{n-1}|x|^{n-(n+4)(1-\varepsilon)} + 2^{n-4}\int_{\mathbb{R}^n\setminus B_R(x)} \widehat{f}_i(\mathcal{U}(y))\,\mathrm{d}y, \end{aligned}$$

Choosing $\varepsilon = 4/(n+4)$ in (25), we obtain that $n-(n+4)(1-\varepsilon) \leq 0$, and so

$$I_2 = o_R(1) \quad \text{as} \quad R \to \infty,$$

which combined with (25) and (24), concludes the proof of Step 3.

Now using Step 3, we obtain that for all $i,j \in I_+$, it holds

$$q_{ij}(x) = \frac{u_i(x)}{u_j(x)} = \frac{|x|^{n-4}u_i(x)}{|x|^{n-4}u_j(x)} = \frac{\int_{\mathbb{R}^n} \widehat{f}_i(\mathcal{U}(x))\,\mathrm{d}x + o_R(1)}{\int_{\mathbb{R}^n} \widehat{f}_j(\mathcal{U}(x))\,\mathrm{d}x + o_R(1)},$$

which by taking the limit as $R \to \infty$ yields (18).

Finally, combining Claims 1 and 2, we find that $u_i = \Lambda_{ij}u_j$, which concludes the proof using the same argument as in the other proof in the last section. □

3.6. **Proof of Theorem 1.1.** Using Lemma 3.10 (i) and the classification in [29, Lemma 11.1], we can compute the $\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p)$-norm of any classical solution to ($\mathcal{S}_p$), which enables us to conclude that classical solutions are weak solutions, then Theorem 1' can be applied to give the proof for Theorem 1.1 (See Remark 2.6).

Proof. By Lemma 3.10 (ii), we may assume $\mu^*(y)<\infty$ for any $y\in\mathbb{R}^n$. Moreover, using [29, Lemma 11.1], there exist $x_0\in\mathbb{R}^n$ and $\mu'>0$ and $\mu''\geq 0$ such that

$$|\mathcal{U}(x)|=\left(\frac{\mu'}{\mu''+|x-x_0|^2}\right)^{\frac{n-4}{2}} \quad \text{for all} \quad x\in\mathbb{R}^n. \tag{26}$$

Let us consider a smooth cut-off function satisfying $\eta\equiv 1$ in $[0,1]$, $0\leq\eta\leq 1$ in $[1,2)$ and $\eta\equiv 0$ in $[2,\infty)$. For $R>0$, setting $\eta_R(x)=\eta(R^{-1}x)$, and multiplying the equation ($\mathcal{S}_p$) by $\eta_R u_i$, we obtain $\Delta^2 u_i\eta_R u_i=|\mathcal{U}|^{2^{**}-2}\eta_R u_i^2$, which gives us

$$\sum_{i=1}^p \Delta^2 u_i\eta_R u_i=c(n)\sum_{i=1}^p|\mathcal{U}|^{2^{**}-2}\eta_R u_i^2=c(n)|\mathcal{U}|^{2^{**}}\eta_R.$$

Thus,

$$\int_{\mathbb{R}^n}\sum_{i=1}^p\Delta^2 u_i\eta_R u_i\ \mathrm{d}x=c(n)\int_{\mathbb{R}^n}|\mathcal{U}|^{2^{**}}\eta_R\ \mathrm{d}x. \tag{27}$$

Using integration by parts on the left-hand side,

$$\int_{\mathbb{R}^n}\sum_{i=1}^p\Delta^2 u_i\eta_R u_i\ \mathrm{d}x=\sum_{i=1}^p\int_{\mathbb{R}^n}u_i\Delta^2(\eta_R u_i)\ \mathrm{d}x. \tag{28}$$

Applying the formula for the bi-Laplacian of the product on the right-hand side of (28),

$$\begin{aligned}\sum_{i=1}^p\int_{\mathbb{R}^n}u_i\Delta^2(\eta_R u_i)\ \mathrm{d}x=&\sum_{i=1}^p\int_{\mathbb{R}^n}\left[u_i\Delta^2(\eta_R)u_i+4u_i\nabla\Delta\eta_R\nabla u_i\right]\ \mathrm{d}x\\ &+\sum_{i=1}^p\int_{\mathbb{R}^n}\left[6u_i\Delta\eta_R\Delta u_i+4u_i\nabla\eta_R\nabla\Delta u_i+u_i\eta_R\Delta^2 u_i\right]\ \mathrm{d}x,\end{aligned}$$

which combined with (28) provides

$$\sum_{i=1}^p\int_{\mathbb{R}^n}\left[u_i\Delta^2(\eta_R)u_i+4u_i\nabla\Delta\eta_R\nabla u_i+6u_i\Delta\eta_R\Delta u_i+4u_i\nabla\eta_R\nabla\Delta u_i\right]\ \mathrm{d}x=0. \tag{29}$$

Again, we use integration by parts in (29) to find

$$\begin{aligned}&\sum_{i=1}^p\Bigg[\int_{\mathbb{R}^n}u_i^2\Delta\eta_R\ \mathrm{d}x-4\left(\int_{\mathbb{R}^n}\Delta\eta_R|\nabla u_i|^2\ \mathrm{d}x+\int_{\mathbb{R}^n}u_i\Delta\eta_R\Delta u_i\ \mathrm{d}x\right)\\ &+6\int_{\mathbb{R}^n}u_i\Delta\eta_R\Delta u_i\ \mathrm{d}x-4\left(\int_{\mathbb{R}^n}u_i\eta_R\Delta^2 u_i\ \mathrm{d}x+\int_{\mathbb{R}^n}\eta_R\nabla u_i\nabla\Delta u_i\ \mathrm{d}x\right)\Bigg]=0,\end{aligned}$$

which yields

$$4\sum_{i=1}^p\int_{\mathbb{R}^n}\Delta^2 u_i\eta_R u_i\ \mathrm{d}x$$

$$= \sum_{i=1}^{p} \left(\int_{\mathbb{R}^n} (u_i)^2 \Delta^2 \eta_R \, \mathrm{d}x - 4 \int_{\mathbb{R}^n} \Delta \eta_R |\nabla u_i|^2 \, \mathrm{d}x \right. \tag{30}$$
$$\left. +2 \int_{\mathbb{R}^n} u_i \Delta \eta_R \Delta u_i \, \mathrm{d}x + 4 \int_{\mathbb{R}^n} \Delta u_i \nabla u_i \nabla \eta_R \, \mathrm{d}x + 4 \int_{\mathbb{R}^n} \eta_R |\Delta u_i|^2 \, \mathrm{d}x \right).$$

As a result of (27) and (30), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}} \eta_R \, \mathrm{d}x \\ &= \frac{1}{4} \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R \, \mathrm{d}x - \int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 \Delta \eta_R \, \mathrm{d}x \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^n} \langle \mathcal{U}, \Delta \mathcal{U} \rangle \Delta \eta_R \, \mathrm{d}x + \int_{\mathbb{R}^n} \langle \Delta \mathcal{U}, \nabla \mathcal{U} \rangle \nabla \eta_R \, \mathrm{d}x + \int_{\mathbb{R}^n} |\Delta \mathcal{U}|^2 \eta_R \, \mathrm{d}x. \end{aligned} \tag{31}$$

Moreover, we have

$$\int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R \, \mathrm{d}x = \mathcal{O}(R^{4-n}) \quad \text{as} \quad R \to \infty.$$

Indeed, we observe

$$\begin{aligned} \left| \int_{\mathbb{R}^n} |\mathcal{U}|^2 \Delta^2 \eta_R \, \mathrm{d}x \right| &\leq \int_{\mathbb{R}^n} |\mathcal{U}|^2 |\Delta^2 \eta_R| \, \mathrm{d}x \\ &\leq \|\Delta^2 \eta_R\|_{C^0(\mathbb{R}^n)} \int_{B_{2R}(0) \setminus B_R(0)} |\mathcal{U}|^2 \, \mathrm{d}x \\ &\leq \frac{\|\Delta^2 \eta\|_{C^0(\mathbb{R}^n)}}{R^4} \int_R^{2R} |\mathcal{U}(r)|^2 r^{n-1} \, \mathrm{d}r \\ &\leq \frac{\|\Delta^2 \eta\|_{C^0(\mathbb{R}^n)} \|\mathcal{U}\|^2_{L^\infty(\mathbb{R}^n)}}{R^4} \int_R^{2R} r^{n-1} \, \mathrm{d}r \\ &= C(n) R^{n-4}. \end{aligned}$$

Analogously to the others terms, we get the following estimates

$$\int_{\mathbb{R}^n} |\nabla \mathcal{U}|^2 \Delta \eta_R \, \mathrm{d}x = \mathcal{O}(R^{2-n}) \quad \text{as} \quad R \to \infty$$

and

$$\int_{\mathbb{R}^n} \langle \mathcal{U}, \Delta \mathcal{U} \rangle \Delta \eta_R \, \mathrm{d}x = \int_{\mathbb{R}^n} \langle \Delta \mathcal{U}, \nabla \mathcal{U} \rangle \nabla \eta_R \, \mathrm{d}x = \mathcal{O}(R^{1-n}) \quad \text{as} \quad R \to \infty,$$

which, by taking $R \to \infty$ in (31), we find that $\eta_R \to 1$ in the $C^0(\mathbb{R}^n)$-topology, and

$$\int_{\mathbb{R}^n} |\Delta \mathcal{U}|^2 \, \mathrm{d}x = c(n) \int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}} \, \mathrm{d}x < \infty.$$

Since $|\mathcal{U}|$ has the classification (26), a direct computation yields

$$\int_{\mathbb{R}^n} |\mathcal{U}|^{2^{**}} \, \mathrm{d}x = S(n)^{-n},$$

where $S(n) = \sqrt{c(n) \omega_n^{4/n}}$. Hence, $\mathcal{U} \in \mathcal{D}^{2,2}(\mathbb{R}^n, \mathbb{R}^p)$ is a weak solution to ($\mathcal{S}_p$), and the proof follows as a direct application of Theorem 1'. □

3.7. **Maximizers for a vectorial Sobolev inequality.** We show that solutions obtained in Theorem 1.1 are the extremal p-maps for a type of vectorial higher order Sobolev embedding. As usual, let us denote by $\mathcal{D}^{k,q}(\mathbb{R}^n,\mathbb{R}^p)$ the Beppo–Levi space defined as the completion of $C_c^\infty(\mathbb{R}^n,\mathbb{R}^p)$ with respect to the norm provided by the highest derivative term. Notice that if $q=2$, then $\mathcal{D}^{k,2}(\mathbb{R}^n,\mathbb{R}^p)$ is a Hilbert space as the usual scalar product given by $\langle\mathcal{U},\mathcal{V}\rangle=\sum_{i=1}^p\langle u_i,v_i\rangle_{\mathcal{D}^{k,2}(\mathbb{R}^n)}$. Moreover, for the fourth order critical Sobolev exponent $2^{**}=2n/(n-4)$, we have the continuous embedding, $\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p)\hookrightarrow L^{2^{**}}(\mathbb{R}^n,\mathbb{R}^p)$ with

$$\|\mathcal{U}\|_{L^{2^{**}}(\mathbb{R}^n,\mathbb{R}^p)}\leq S(n,p)\|\mathcal{U}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p)}. \tag{32}$$

Our result states that the solutions to ($\mathcal{S}_p$) are the extremal functions for (32). Remarkably, the best constant in (32) coincides with the one when $p=1$, that is, $S(n,1)=S(n,p)$ for all $p>1$.

Proposition 3.12. *Let $\mathcal{U}_{x_0,\mu}$ be a spherical solution to ($\mathcal{S}_p$). Then, up to constant, $\mathcal{U}_{x_0,\mu}$ is the unique extremal family of extremal p-maps for the Sobolev inequality (32), that is,*

$$\|\mathcal{U}_{x_0,\mu}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p)}=S(n,p)\|\mathcal{U}_{x_0,\mu}\|_{L^{2^{**}}(\mathbb{R}^n,\mathbb{R}^p)}. \tag{33}$$

Moreover, $S(n,p)=S(n)$ for all $p>1$.

Proof. Initially, we observe

$$S(n,p)^{-2}=\inf_{\mathcal{H}^p(\mathbb{R}^n)}\sum_{i=1}^p\int_{\mathbb{R}^n}|\Delta u_i|^2\,\mathrm{d}x, \tag{34}$$

where $\mathcal{H}^p(\mathbb{R}^n)=\{\mathcal{U}\in\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p):\|\mathcal{U}\|_{L^{2^{**}}(\mathbb{R}^n,\mathbb{R}^p)}=1\}$. When $p=1$, this result is a consequence of Theorem A.

Claim 1. *$S(n,p)=S(n)$ for all $p>1$.*

In fact, by taking $u\in\mathcal{D}^{2,2}(\mathbb{R}^n)$ satisfying $\|u\|_{L^{2^{**}}(\mathbb{R}^n)}=1$, we have that $\mathcal{U}=u\mathbf{e_1}$ belongs to $\mathcal{H}^p(\mathbb{R}^n)$, where $\mathbf{e_1}=(1,0,\dots,0)$. Substituting $\mathcal{U}$ in (34), we get that $S(n,p)\leq S(n)$.

Conversely, we have

$$\left(\sum_{i=1}^p\int_{\mathbb{R}^n}|u_i|^{2^{**}}\,\mathrm{d}x\right)^{2/2^{**}}\leq S(n)^{-1}\sum_{i=1}^p\int_{\mathbb{R}^n}|\Delta u_i|^2\,\mathrm{d}x. \tag{35}$$

Therefore, by (35) we find that $S(n,p)^{-1}\leq S(n)^{-1}$, which gives us the proof of the claim.

Also, using the following computation

$$\frac{\|\mathcal{U}_{x_0,\mu}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n,\mathbb{R}^p)}}{\|\mathcal{U}_{x_0,\mu}\|_{L^{2^{**}}(\mathbb{R}^n,\mathbb{R}^p)}}=\frac{\|u_{x_0,\mu}\|_{\mathcal{D}^{2,2}(\mathbb{R}^n)}}{\|u_{x_0,\mu}\|_{L^{2^{**}}(\mathbb{R}^n)}}=S(n).$$

To prove the uniqueness, observe that if $\mathcal{U}$ satisfies that (33), then, up to constant, $\mathcal{U}$ satisfy $\Delta^2\mathcal{U}=c(n)|\mathcal{U}|^{2^{**}-2}\mathcal{U}$ in $\mathbb{R}^n$, which, by Theorem 1.1 concludes the proof of the proposition. □

4. **Classification result for singular solutions.** The objective of this section is to present the proof of Theorem 1.2. We prove that each component of a singular solutions to ($\mathcal{S}_p$) are superharmonic. Then, we show that singular solutions to ($\mathcal{S}_p$) are radially symmetry about the origin. We obtain radial symmetry via an asymptotic moving planes technique; this property turns ($\mathcal{S}_p$) into a fourth order ODE system. Eventually, we define a Pohozaev-type invariant by integrating the

Hamiltonian energy of the associated Emden–Fowler system. Moreover, we prove that the Pohozaev invariant sign provides a removable singularity classification for strongly positive solutions to ($\mathcal{S}_p$), which combined with a delicate ODE analysis completes our argument. Here, as in Proposition 3.2, we emphasize that weak singular solutions to ($\mathcal{S}_p$) are classical solutions as well (*c.f.* [36, Proposition 3.1]). Here our approach follows the ones in [39, 38, 31, 29, 28, 22, 21, 20, 19, 18, 12].

4.1. **Superharmonicity.** We prove some superharmonicity result for solutions to ($\mathcal{S}_p$), which will be use in the moving planes technique. For this, we need to establish some preliminary results concerning the integrability of solutions to ($\mathcal{S}_p$). Namely, we show that any nonnegative singular solution to ($\mathcal{S}_p$) is distributional in the sense of (5).

Lemma 4.1. *Let $\mathcal{U}$ be a nonnegative singular solution to ($\mathcal{S}_p$). Then, it holds $\mathcal{U} \in L^{2^{**}-1}(\mathbb{R}^n, \mathbb{R}^p)$. In particular, $\mathcal{U}$ is a distribution solution to ($\mathcal{S}_p$).*

Proof. For any $0 < \varepsilon \ll 1$, let us consider $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ with $0 \leq \eta_\varepsilon \leq 1$ satisfying

$$\eta_\varepsilon(x) = \begin{cases} 0, & \text{if } |x| \leq \varepsilon \\ 1, & \text{if } |x| \geq 2\varepsilon, \end{cases} \tag{36}$$

and $|D^{(j)}\eta_\varepsilon(x)| \leq C\varepsilon^{-j}$ for $j \geq 1$. Define $\xi_\varepsilon = (\eta_\varepsilon)^{\frac{n+4}{2}}$. Multiplying ($\mathcal{S}_p$) by ξ_ε, and integrating by parts in B_r with $r \in (1/2, 1)$, we obtain

$$\int_{B_r} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x = \int_{\partial B_r} \partial_\nu \Delta u_i \, \mathrm{d}\sigma_r + \int_{B_r} u_i \Delta^2 \xi_\varepsilon \, \mathrm{d}x \quad \text{for all} \quad i \in I.$$

On the other hand, there exists $C > 0$ such that

$$\left|\Delta^2 \xi_\varepsilon\right| \leq C\varepsilon^{-4} (\eta_\varepsilon)^{\frac{n-4}{2}} \chi_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} = C\varepsilon^{-4} (\xi_\varepsilon)^{\frac{n-4}{n+4}} \chi_{\{\varepsilon \leq |x| \leq 2\varepsilon\}},$$

which, by Hölder's inequality, gives us

$$\begin{aligned} \left| \int_{B_r} u_i \Delta^2 \xi_\varepsilon \, \mathrm{d}x \right| &\leq C\varepsilon^{-4} \int_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} u_i \xi_\varepsilon^{\frac{n-4}{n+4}} \, \mathrm{d}x \\ &\leq C\varepsilon^{-4} \varepsilon^{\frac{8n}{n+4}} \left(\int_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x \right)^{\frac{n-4}{n+4}} \\ &\leq C \left(\int_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x \right)^{\frac{n-4}{n+4}}. \end{aligned}$$

Thus, it follows

$$\int_{B_r} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x \leq \int_{\partial B_r} \partial_\nu \Delta u_i \, \mathrm{d}\sigma_r + C \left(\int_{\{\varepsilon \leq |x| \leq 2\varepsilon\}} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x \right)^{\frac{n-4}{n+4}},$$

which provides a constant $C > 0$ (independent of ε) such that

$$\int_{B_r} |\mathcal{U}|^{\frac{8}{n-4}} u_i \xi_\varepsilon \, \mathrm{d}x \leq C.$$

Now letting $\varepsilon \to 0$, since $u_i \leq |\mathcal{U}|$, we conclude that $u_i \in L^{\frac{n+4}{n-4}}(B_r)$ for all $i \in I$ and the integrability follows.

For any nonnegative $\Phi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^p)$, we multiply ($\mathcal{S}_p$) by $\widetilde{\Phi} = \eta_\varepsilon\Phi$, where η_ε is given by (36). Then, using that $|\mathcal{U}| \in L^{\frac{n+4}{n-4}}(B_r)$ and integrating by parts twice, we get

$$\int_{\mathbb{R}^n} \langle \mathcal{U}, \Delta^2(\eta_\varepsilon\Phi)\rangle \, \mathrm{d}x = \int_{\mathbb{R}^n} \langle |\mathcal{U}|^{\frac{8}{n-4}}\mathcal{U}, \eta_\varepsilon\Phi\rangle \, \mathrm{d}x. \tag{37}$$

By a direct computation, we find that $\Delta^2(\eta_\varepsilon\phi_i) = \eta_\varepsilon\Delta^2\phi_i + \varsigma_i^\varepsilon$, where

$$\varsigma_i^\varepsilon = 4\langle\nabla\eta_\varepsilon, \nabla\Delta\phi_i\rangle + 2\Delta\eta_\varepsilon\Delta\phi_i + 4\Delta\eta_\varepsilon\Delta\phi_i + 4\langle\nabla\Delta\eta_\varepsilon, \nabla\phi_i\rangle + \phi_i\Delta^2\eta_\varepsilon.$$

Furthermore, using Hölder's inequality again, we find

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \langle \mathcal{U}, \Psi_\varepsilon\rangle \, \mathrm{d}x\right| &\leq C\left(\int_{\{\varepsilon\leq|x|\leq2\varepsilon\}} |\mathcal{U}|^{\frac{8}{n-4}}u_i \, \mathrm{d}x\right)^{\frac{n-4}{n+4}} \\ &\leq C\left(\int_{\{\varepsilon\leq|x|\leq2\varepsilon\}} |\mathcal{U}|^{\frac{n+4}{n-4}} \, \mathrm{d}x\right)^{\frac{n-4}{n+4}} \to 0 \quad \text{as} \quad \varepsilon\to 0,\end{aligned}$$

where $\Psi_\varepsilon = (\varsigma_1^\varepsilon, \dots, \varsigma_p^\varepsilon) \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^p)$. Finally, letting $\varepsilon \to 0$ in (37), and applying the dominated convergence theorem the proof follows. □

Finally, we present the most important result of this subsection

Proposition 4.2. *Let $\mathcal{U}$ be a nonnegative singular solution to ($\mathcal{S}_p$). Then, $\mathcal{U}$ is a superharmonic p-map in the distributional sense, that is, for all nonnegative $\Phi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^p)$, one has $\int_{\mathbb{R}^n}\langle\Delta\mathcal{U},\Delta\Phi\rangle \, \mathrm{d}x \geq 0$. Moreover, $\mathcal{U}$ is superharmonic in $\mathbb{R}^n\setminus\{0\}$.*

Proof. Proceeding similarly to Lemma 4.1, one can prove that $|\mathcal{U}| \in L^{\frac{n+4}{n-4}}_{\mathrm{loc}}(\mathbb{R}^n)$. Let $\eta_\varepsilon \in C^\infty(\mathbb{R}^n)$ be the cut-off function given by (36) and $\Phi \in C_c^\infty(\mathbb{R}^n,\mathbb{R}^p)$ be a nonnegative test p-map. Then, multiplying ($\mathcal{S}_p$) by $\eta_\varepsilon\phi_i$ for each $i \in I$, and integrating by parts twice, we get

$$\begin{aligned}0 &\leq \int_{\mathbb{R}^n} \eta_\varepsilon\phi_i|\mathcal{U}|^{\frac{8}{n-4}}u_i \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \Delta(\eta_\varepsilon\phi_i)\Delta u_i \, \mathrm{d}x \\ &= \int_{\mathbb{R}^n} \Delta u_i\left(\Delta\phi_i\eta_\varepsilon + \varsigma_i^\varepsilon\right) \, \mathrm{d}x,\end{aligned}$$

where $\varsigma_i^\varepsilon := 2\langle\nabla\phi_i,\nabla\eta_\varepsilon\rangle + \phi_i\Delta\eta_\varepsilon$. Notice that $\varsigma_i^\varepsilon(x) \equiv 0$ when $|x| \leq \varepsilon$ or $|x| \geq 2\varepsilon$, and $|\Delta\varsigma_i^\varepsilon(x)| \leq C\varepsilon^{-4}$, for some $C > 0$.

In addition, since $n - 4 - \frac{n(n-4)}{n+4} > 0$, the following estimate holds,

$$\begin{aligned}\left|\int_{\mathbb{R}^n} \Delta u_i\varsigma_i^\varepsilon \, \mathrm{d}x\right| &\leq \int_{\mathbb{R}^n} u_i|\Delta\varsigma_i^\varepsilon| \, \mathrm{d}x \\ &\leq C\varepsilon^{-4}\left(\int_{\{\varepsilon\leq|x|\leq2\varepsilon\}} |\mathcal{U}|^{\frac{8}{n-4}}u_i \, \mathrm{d}x\right)^{\frac{n-4}{n+4}} \varepsilon^{n\left(1-\frac{n-4}{n+4}\right)} \\ &\leq C\varepsilon^{-4}\left(\int_{\{\varepsilon\leq|x|\leq2\varepsilon\}} |\mathcal{U}|^{s} \, \mathrm{d}x\right)^{\frac{n-4}{n+4}} \varepsilon^{n\left(1-\frac{n-4}{n+4}\right)} \\ &\leq C\varepsilon^{n-4-\frac{n(n-4)}{n+4}} \to 0 \quad \text{as} \quad \varepsilon\to 0,\end{aligned}$$

which implies

$$\int_{\mathbb{R}^n} \Delta u_i \Delta \phi_i \ \mathrm{d}x = \lim_{\varepsilon \to 0} \int_{\mathbb{R}^n} \left(\Delta u_i \Delta \varsigma_i^{\varepsilon} \ \mathrm{d}x + \eta_\varepsilon \Delta \phi_i \Delta u_i \right) \ \mathrm{d}x = \int_{\mathbb{R}^n} \phi_i |\mathcal{U}|^{\frac{8}{n-4}} u_i \ \mathrm{d}x \geq 0.$$

Thus, $-\Delta u_i$ is superharmonic in the whole space $\mathbb{R}^n$ in the distributional sense for all $i \in I$, which gives the first part of the proof.

To prove the second statement, given $0 < \varepsilon \ll 1$, let us consider $\widetilde{u}_i^{\varepsilon} := -\Delta u_i + \varepsilon$. Using Lemma 4.1, there exists a constant $C > 0$, depending only on n and s, such that for all $|x| \geq 4$, it holds

$$\sum_{j=0}^{3} |x|^{\gamma(s)+j} \left| D^{(j)} u_i(x) \right| \leq C,$$

which yields that $\lim_{|x|\to\infty} |\Delta u_i(x)| = 0$ for all $i \in I$. Whence, for any $0 < \varepsilon \ll 1$, there exists $R_\varepsilon \gg 1$ such that $\widetilde{u}_i^{\varepsilon} > \varepsilon/2$ for $|x| \geq R_\varepsilon$. Finally, using that $\widetilde{u}_i^{\varepsilon}$ is superharmonic in $\mathbb{R}^n$ in the distributional sense, we have that $\widetilde{u}_i^{\varepsilon} \geq 0$ in $\mathbb{R}^n \setminus \{0\}$, which, by passing to the limit as $\varepsilon \to 0$, provides $-\Delta u_i \geq 0$ in $\mathbb{R}^n \setminus \{0\}$. The last inequality concludes the proof of the lemma. ☐

4.2. **Asymptotic moving planes technique.** In this subsection, using a variant of the moving planes technique, we prove that singular solutions to ($\mathcal{S}_p$) are radially symmetric about the origin. In our case, solutions are singular at the origin, thus, to show that they are rotationally invariant, we need to perform an adaptation of Aleksandrov's method.

Proposition 4.3. *Let $\mathcal{U}$ be a strongly positive singular solution to equation ($\mathcal{S}_p$). Then, $|\mathcal{U}|$ is radially symmetric about the origin and monotonically decreasing.*

Proof. Since $\mathcal{U}$ is a singular solution, we may suppose without loss of generality that the origin is a non-removable singularity of u_1. Fixing $z \neq 0$ a non-singular point of $\mathcal{U}$, that is, $\lim_{|x|\to z} |\mathcal{U}(x)| < \infty$, we perform the fourth order Kelvin transform with center at the z and unitary radius,

$$(u_i)_{z,1}(x) = |x|^{4-n} u_i \left(z + \frac{x}{|x|^2} \right) \quad \text{for} \quad i \in I.$$

Denoting $\widetilde{u}_i = (u_i)_{z,1}$, we observe that $\widetilde{u}_1$ is singular at zero and $z_0 = -z/|z|^2$, whereas the others components are singular only at zero. Furthermore, using the conformal invariance of ($\mathcal{S}_p$), we get

$$\Delta^2 \widetilde{u}_i = c(n) |\widetilde{\mathcal{U}}|^{2^{**}-2} \widetilde{u}_i \quad \text{in} \quad \mathbb{R}^n \setminus \{0, z_0\}.$$

Let us set $\vartheta_i(x) = -\Delta \widetilde{u}_i(x)$, thus $\vartheta_i(x) = \mathcal{O}(|x|^{2-n})$ as $|x| \to \infty$. Using Proposition 4.2, we have that $\vartheta_i > 0$ in $\mathbb{R}^n \setminus \{0\}$ has the harmonic asymptotic expansion as $|x| \to 0$,

$$\begin{cases} \vartheta_i(x) = a_{i0}|x|^{2-n} + a_{ij} x_j |x|^{-n} + \mathcal{O}(|x|^{-n}) \\ \partial_{x_j} \vartheta_i(x) = (2-n) a_{i0} x_j |x|^{-n} + \mathcal{O}(|x|^{-n}) \\ \partial_{x_k x_j} \vartheta_i(x) = \mathcal{O}(|x|^{-n}), \end{cases}$$

where $a_{i0} = -\Delta \vartheta_i(z)$ and $a_{ij} = \partial_{y_j} - \Delta \vartheta_i(z)$.

Considering the axis defined by 0 and z as the reflection direction, we can suppose that this axis is orthogonal to the positive x_n direction, that is, given the unit vector $\mathbf{e}_n = (0, 0, \ldots, 1)$. For $\lambda > 0$, we consider the sets

$$\Sigma_\lambda := \{x \in \mathbb{R}^n : x_n > \lambda\} \quad \text{and} \quad T_\lambda := \partial \Sigma_\lambda,$$

and we define the reflection about the plane T_λ by

$$x = (x_1, \dots, x_{n-1}, x_n) \mapsto x_\lambda = (x_1, \dots, x_{n-1}, 2\lambda - x_n).$$

Let us also introduce the notation $(w_i)_\lambda(x) = \widetilde{u}_i(x) - (\widetilde{u}_i)_\lambda(x)$, where $(\widetilde{u}_i)_\lambda(x) = \widetilde{u}_i(x_\lambda)$. Then, showing radial symmetry about the origin for singular solutions to ($\mathcal{S}_p$) is equivalent to proving the following

$$(w_i)_\lambda \equiv 0 \quad \text{for} \quad \lambda = 0. \tag{38}$$

Subsequently, we divide the proof of (38) into three claims.

Claim 1. *There exists $\bar{\lambda}_1 > 0$ such that $(w_i)_\lambda < 0$ in Σ_λ for all $\lambda < \bar{\lambda}_1$ and $i \in I$.*

In fact, notice that $(w_i)_\lambda$ satisfies the following Navier problem

$$\begin{cases} \Delta^2 (w_i)_\lambda = (b_i)_\lambda (w_i)_\lambda & \text{in} \quad \Sigma_\lambda \\ \Delta (w_i)_\lambda = (w_i)_\lambda = 0 & \text{on} \quad T_\lambda, \end{cases} \tag{39}$$

where

$$(b_i)_\lambda = \frac{c(n)|\widetilde{\mathcal{U}}_\lambda|^{2^{**}-2}(\widetilde{u}_i)_\lambda - c(n)|\widetilde{\mathcal{U}}|^{2^{**}-2}\widetilde{u}_i}{\widetilde{u}_i - (\widetilde{u}_i)_\lambda} > 0 \quad \text{in} \quad \bar{\Sigma}_\lambda.$$

Then, as a consequence of [31, Lemma 3.1], there exist $\bar{\lambda}_0 < 0$ and $R > |z_0| + 10^{10} \gg 1$ such that

$$\Delta (w_i)_\lambda(x) = (\vartheta_i)_\lambda(x) - \vartheta_i(x) < 0 \quad \text{for} \quad x \in \Sigma_\lambda, \quad \lambda \le \bar{\lambda}_0 \quad \text{and} \quad |x| > R. \tag{40}$$

In addition, by [7, Lemma 2.1] we can find $C > 0$ satisfying

$$\vartheta_i(x) \ge C \quad \text{for} \quad x \in \bar{B}_R \setminus \{0, z_0\}. \tag{41}$$

Since $\vartheta_i(x) \to 0$ as $|x| \to \infty$, combining (40) and (41), there exists $\bar{\lambda}_1 < \bar{\lambda}_0$ such that

$$\Delta (w_i)_\lambda(x) = (\vartheta_i)_\lambda(x) - \vartheta_i(x) < 0 \quad \text{for} \quad x \in \Sigma_\lambda \quad \text{and} \quad \lambda \le \bar{\lambda}_1 \quad \text{for all} \quad i \in I. \tag{42}$$

Using that $\lim_{|x|\to\infty} (w_i)_\lambda(x) = 0$, we can apply the strong maximum principle to conclude that $(w_i)_\lambda(x) > 0$ for all $\lambda \le \bar{\lambda}_1$ and $i \in I$, which implies the proof of the claim.

Now thanks to Claim 1, we can define the critical sliding parameter given by

$$\lambda^* = \sup\{\lambda < 0 : \Delta(w_i)_{\bar{\lambda}} < 0 \text{ on } \Sigma_{\bar{\lambda}} \text{ for each } \bar{\lambda} \le \lambda \text{ and } i \in I\}.$$

Claim 2. *$(w_i)_{\lambda^*} \equiv 0$ for all $i \in I$.*

Fix $i \in I$ and suppose by contradiction that $(w_i)_{\lambda^*}(x_0) \neq 0$ for some $x_0 \in \Sigma_{\lambda^*}$. By continuity, we have that $\Delta(w_i)_{\lambda^*} \le 0$ in Σ_{λ^*}. Since $\lim_{|x|\to\infty} (w_i)_{\lambda^*}(x) = 0$, a strong maximum principles yields that $(w_i)_{\lambda^*} > 0$ in Σ_{λ^*}. Also, by ($\mathcal{S}_p$), we get $\Delta^2 (w_i)_{\lambda^*} = |\widetilde{\mathcal{U}}|^{2^{**}-2}\widetilde{u}_i - |\mathcal{U}_{\lambda^*}|^{2^{**}-2}(\widetilde{u}_i)_{\lambda^*}(x) > 0$. Hence, $\Delta(w_i)_{\lambda^*}$ is subharmonic. By employing the strong maximum principle again, we obtain that $\Delta(w_i)_{\lambda^*} < 0$. In addition, by the definition of λ^*, there exists a sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ such that, $\lambda_k \nearrow \lambda^*$ and $\sup_{\Sigma_{\lambda_k}} \Delta(w_i)_{\lambda_k}(x) > 0$. Observing that $\lim_{|x|\to\infty} \Delta(w_i)_{\lambda_k}(x) = 0$, we can find $x_k \in \Sigma_{\lambda_k}$ satisfying

$$\Delta(w_i)_{\lambda_k}(x_k) = \sup_{\Sigma_{\lambda_k}} \Delta(w_i)_{\lambda_k}(x). \tag{43}$$

By [31, Lemma 3.2], we observe that $\{x_k\}_{k\in\mathbb{N}}$ is bounded. Thus, up to a subsequence, we may assume that $x_k \to x_0$. If $x_0 \in \Sigma_{\lambda^*}$, passing to the limit in (43), we obtain $\Delta(w_i)_{\lambda^*}(x_0) = 0$, which is a contradiction with $\Delta(w_i)_{\lambda^*}(x_0) \le 0$. If

$x_0 \in T_{\lambda^*}$ we have that $\nabla(\Delta(w_i)_{\lambda^*}(x_0)) = 0$. This contradicts the Hopf boundary Lemma, because $\Delta(w_i)_{\lambda^*}$ is negative and subharmonic in Σ_{λ^*}.

Claim 3. $\lambda^* = 0$.

Let us assume that the claim is not valid, that is, $\lambda^* < 0$. Then, for $\lambda = \lambda^*$, it holds $(w_i)_{\lambda^*} < 0$ for all $i \in I$. In particular, it follows that $(w_1)_{\lambda^*} < 0$. Since $\lim_{|x|\to z_0} u_1(x) = \infty$, we observe that $\widetilde{u}_1$ cannot be invariant under the reflection x_{λ^*}. Thus, using a strong maximum principle for (39), we conclude

$$\widetilde{u}_1(x) < u_1(x_\lambda) \quad \text{for} \quad x \in \Sigma_{\lambda^*} \quad \text{and} \quad x_{\lambda^*} \notin \{0, z_0\}. \tag{44}$$

Notice that as a consequence of $\lambda^* < 0$, we have that $\{0, z_0\} \notin T_{\lambda^*}$. Whence, applying the Hopf boundary Lemma, we get

$$\partial_{x_k}(\widetilde{u}_1(x_{\lambda^*}) - \widetilde{u}_1(x)) = -2\partial_{x_k}\widetilde{u}_1(x) > 0. \tag{45}$$

Now choose $\{\lambda_k\}_{k\in\mathbb{N}}$ such that $\lambda_k \nearrow \lambda^*$ as $k \to \infty$ and $x_k \in \Sigma_{\lambda_k}$ such that $\widetilde{u}_1(x_{k\lambda_k}) < \widetilde{u}_1(x_k)$. Then, by [31, Lemma 3.2], we obtain that $\{x_k\}_{j\in\mathbb{N}}$ is bounded. Whence, $x_k \to \bar{x} \in \bar{\Sigma}_{\lambda^*}$ with $\widetilde{u}_1(\bar{x}_{\lambda^*}) \leq \widetilde{u}_1(\bar{x})$. By (44) we know that $\bar{x} \in \partial\Sigma_{\lambda^*}$ and then $\partial_{x_k}\widetilde{u}_1(\bar{x}) \geq 0$, a contradiction with (45), which proves (38). □

4.3. **Cylindrical transformation.** Let us introduce the so-called cylindrical transformation. Using this device, we convert singular solutions to ($\mathcal{S}_p$) in the punctured space into non-singular solutions in a cylinder.

Considering the vectorial Emden–Fowler change of variables (or logarithm coordinates) given by $\mathcal{V}(t,\theta) = r^\gamma \mathcal{U}(r,\sigma)$, where $r = |x|$, $t = -\ln r$, $\sigma = \theta = x/|x|$, and $\gamma = (n-4)/2$ is the Fowler rescaling exponent, sends the problem to the entire cylinder $\mathcal{C}_\infty = \mathbb{R} \times \mathbb{S}^{n-1}$. In the geometric setting, this change of variables corresponds to the conformal diffeomorphism between the cylinder $\mathcal{C}_\infty$ and the punctured space $\varphi : (\mathcal{C}_\infty, g_{\rm cyl}) \to (\mathbb{R}^n \setminus \{0\}, \delta_0)$ defined by $\varphi(t,\sigma) = e^{-t}\sigma$. Here $g_{\rm cyl} = \mathrm{d}t^2 + \mathrm{d}\sigma^2$ stands for the cylindrical metric with $\mathrm{d}\theta = e^{-2t}(\mathrm{d}t^2 + \mathrm{d}\sigma^2)$ its volume element obtained via the pullback $\varphi^*\delta_0$, where δ_0 is the standard flat metric.

Using this coordinate system, and performing a lengthy computation, we arrive at the following fourth order nonlinear PDE on the cylinder,

$$\Delta^2_{\rm cyl} v_i = c(n)|\mathcal{V}|^{2^{**}-2} v_i \quad \text{on} \quad \mathcal{C}_\infty. \tag{46}$$

Here $\mathcal{V} = (v_1, \ldots, v_p)$ and $\Delta^2_{\rm cyl}$ is the bi-Laplacian in cylindrical coordinates given by

$$\Delta^2_{\rm cyl} = \partial_t^{(4)} - K_2\partial_t^{(2)} + K_0 + \Delta^2_\theta + 2\partial_t^{(2)}\Delta_\theta - J_0\Delta_\theta,$$

where K_0, K_2, J_0 are constants depending only in the dimension defined by

$$K_0 = \frac{n^2(n-4)^2}{16}, \quad K_2 = \frac{n^2-4n+8}{2} \quad \text{and} \quad J_0 = \frac{n(n-4)}{4}. \tag{47}$$

Along this lines let us consider the cylindrical transformation of a p-map as follows

$$\mathfrak{F} : C_c^\infty(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to C_c^\infty(\mathcal{C}_\infty, \mathbb{R}^p) \quad \text{given by} \quad \mathfrak{F}(\mathcal{U}) = r^\gamma \mathcal{U}(r,\sigma).$$

Remark 4.4. The transformation $\mathfrak{F}$ is a continuous bijection with respect to the Sobolev norms $\|\cdot\|_{\mathcal{D}^{2,2}(\mathbb{R}^n\setminus\{0\},\mathbb{R}^p)}$ and $\|\cdot\|_{H^2(\mathcal{C}_\infty,\mathbb{R}^p)}$, respectively. Furthermore, this transformation sends singular solutions to ($\mathcal{S}_p$) into solutions to (46), and, by density, $\mathfrak{F} : \mathcal{D}^{2,2}(\mathbb{R}^n \setminus \{0\}, \mathbb{R}^p) \to H^2(\mathcal{C}_\infty, \mathbb{R}^p)$.

4.4. **Pohozaev invariant.** The Pohozaev invariant is a homological constant related to the existence and classification of solutions to a large class of PDEs. Let us also emphasize that the existence of the Pohozaev-type invariant is closely related to conservation laws for the Hamiltonian energy of the ODE system (48).

Since we already know that solutions are rotationally invariant, the cylindrical transformation converts ($\mathcal{S}_p$) into a fourth order ODE system with constant coefficients. More specifically, using Proposition 4.3, we arrive at

$$\begin{cases} v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|\mathcal{V}|^{2^{**}-2} v_i \quad \text{in} \quad \mathbb{R} \quad \text{for} \quad i \in I, \\ v_i(0) = a_i, \quad v_i^{(1)}(0) = 0, \quad v_i^{(2)}(0) = b_i, \quad v_i^{(3)}(0) = 0, \end{cases} \tag{48}$$

where $\mathcal{V} = (v_1, \dots, v_p)$ and $a_i, b_i \in \mathbb{R}$ for all $i \in I$.

Remark 4.5. Notice that the vanishing of the first and third derivatives of the component solutions to (48) at origin is a consequence of the fact $v_i : \mathcal{C}_\infty \to \mathbb{R}$ for any $i \in I$ is an even function and invariant by translations. This happens for two reasons. First, there are only even-order derivatives on the right-hand side of (48). Second, the strongly coupling nonlinearity in the left-hand side of is even in the sense that $f_i(-\mathcal{V}) = f_i(\mathcal{V})$ for all p-map $\mathcal{V} : \mathcal{C}_\infty \to \mathbb{R}^p$ and $i \in I$, where we recall $f_i(\mathcal{V}) = c(n)|\mathcal{V}|^{2^{**}-2} v_i$. A formal proof involves a comparison principles which is based on a double application of the strong maximum principle. The interested reader may consult [18, 37] for more details.

Let us define an energy conserved in time for all p-map solutions $\mathcal{V}$ to (48).

Definition 4.6. For any $\mathcal{V}$ strongly positive solution to (48), let us consider its Hamiltonian Energy given by

$$\mathcal{H}(t, \mathcal{V}) = -\langle \mathcal{V}^{(3)}(t), \mathcal{V}^{(1)}(t) \rangle + \frac{1}{2}|\mathcal{V}^{(2)}(t)|^2 + \frac{K_2}{2}|\mathcal{V}^{(1)}(t)|^2 - \frac{K_0}{2}|\mathcal{V}(t)|^2 + \hat{c}(n)|\mathcal{V}(t)|^{2^{**}}, \tag{49}$$

where $\hat{c}(n) = (2^{**})^{-1} c(n)$.

Let us remark that this quantity satisfies

$$\partial_t \mathcal{H}(t, \mathcal{V}) = 0. \tag{50}$$

In other words, the Hamiltonian energy is invariant on the variable t. In addition, we can integrate (49) over $\mathbb{S}_t^{n-1}$ to define another conserved quantity.

Definition 4.7. For any $\mathcal{V}$ strongly positive solution to (48), let us define its cylindrical Pohozaev functional by

$$\mathcal{P}_{\text{cyl}}(t, \mathcal{V}) = \int_{\mathbb{S}_t^{n-1}} \mathcal{H}(t, \mathcal{V}) \, \mathrm{d}\theta.$$

Here $\mathbb{S}_t^{n-1} = \{t\} \times \mathbb{S}^{n-1}$ is the cylindrical ball with volume element given by $\mathrm{d}\theta = e^{-2t} \, \mathrm{d}\sigma$, where $\mathrm{d}\sigma_r$ is the volume element of the euclidean ball of radius $r > 0$.

By definition, $\mathcal{P}_{\text{cyl}}$ also does not depend on $t \in \mathbb{R}$. Then, let us consider the cylindrical Pohozaev invariant $\mathcal{P}_{\text{cyl}}(\mathcal{V}) := \mathcal{P}_{\text{cyl}}(t, \mathcal{V})$. Thus, by applying the inverse of cylindrical transformation, we can recover the classical spherical Pohozaev functional defined by $\mathcal{P}_{\text{sph}}(r, \mathcal{U}) := (\mathcal{P}_{\text{cyl}} \circ \mathfrak{F}^{-1})(\mathcal{V})$.

Remark 4.8. We do not provide the formula explicitly for the spherical Pohozaev, because it is too lengthy and is not required in the rest of this manuscript (for an expression in the scalar case, see [13, Proposition 4.1]). The cylindrical Pohozaev-invariant is enough to perform our methods. Indeed, fixing $\mathcal{H}(t,\mathcal{V}) \equiv H$ and $\mathcal{P}_{\rm sph}(r,\mathcal{U}) = P$, we have that $\omega_{n-1}H = P$. In other words, the Hamiltonian energy H and spherical Pohozaev invariant P have the same sign.

Remark 4.9. There exists a natural relation between the derivatives of $\mathcal{P}_{\rm sph}$ and $\mathcal{H}$ respectively,

$$\partial_r \mathcal{P}_{\rm sph}(r,\mathcal{U}) = r\partial_t \mathcal{H}(t,\mathcal{V}).$$

Thus, for any solution $\mathcal{U}$, the value $\mathcal{P}_{\rm sph}(r,\mathcal{U})$ is also radially invariant.

Now it is convenient to introduce an essential ingredient of our next results.

Definition 4.10. For any $\mathcal{U}$ strongly positive solution to ($\mathcal{S}_p$), let us define its spherical Pohozaev invariant given by $\mathcal{P}_{\rm sph}(r,\mathcal{U}) := \mathcal{P}_{\rm sph}(\mathcal{U})$.

Remark 4.11. For easy reference, let us summarize the following facts:
(i) There exists a type of equivalence between the cylindrical and spherical Pohozaev invariants, $\mathcal{P}_{\rm sph}(\mathcal{U}) = \omega_{n-1}\mathcal{P}_{\rm cyl}(\mathcal{V})$, where ω_{n-1} is the Lebesgue measure of the unit sphere in $\mathbb{R}^{n-1}$.
(ii) The Pohozaev invariant of the vectorial solutions are equal to the Pohozaev invariant in the scalar case, which can be defined in a similar way using the Hamiltonian energy associated to ($\mathcal{S}_1$). More precisely, we define $\mathcal{P}_{\rm sph}(u) = \mathcal{P}_{\rm cyl}(r^\gamma u)$, where

$$\mathcal{P}_{\rm cyl}(v) = \int_{\mathbb{S}_t^{n-1}} \left[-v^{(3)}v^{(1)} + \frac{1}{2}|v^{(2)}|^2 + \frac{K_2}{2}|v^{(1)}|^2 - \frac{K_0}{2}|v|^2 + \hat{c}(n)|v|^{2^{**}} \right] \mathrm{d}\theta.$$

Hence, if the non-singular solution is $\mathcal{U}_{x_0,\mu} = \Lambda u_{x_0,\mu}$ for some $\Lambda \in \mathbb{S}^{p-1}_{+}$ and $u_{x_0,\mu}$ a spherical solution from Theorem A, we obtain that $\mathcal{P}_{\rm sph}(\mathcal{U}_{x_0,\mu}) = \mathcal{P}_{\rm sph}(u_{x_0,\mu}) = 0$. Analogously, if the singular solution has the form $\mathcal{U}_{a,T} = \Lambda u_{a,T}$ for some $\Lambda \in \mathbb{S}^{p-1}_{+,*}$ and $u_{a,T}$ a Emden–Fowler solution from Theorem B, we get that $\mathcal{P}_{\rm sph}(\mathcal{U}_{a,T}) = \mathcal{P}_{\rm sph}(u_{a,T}) < 0$.

4.5. ODE system analysis.

In this subsection, we perform an asymptotic analysis program due to Z. Chen and C. S. Lin [18, Section 3]. This analysis is based on the Pohozaev invariant sign, which combined with some results from [20, 12] determines whether a solution to ($\mathcal{S}_p$) has a removable or a non-removable singularity at the origin.

Before studying how this invariant classifies solutions to ($\mathcal{S}_p$), we need to set some background results concerning the asymptotic behavior for solutions to (48) and their derivatives.

Definition 4.12. For any $\mathcal{V}$ solution to (48), let us define its asymptotic set given by

$$\mathcal{A}(\mathcal{V}) := \bigcup_{i=1}^{p} \mathcal{A}(v_i) \subset [0,\infty], \quad \text{where} \quad \mathcal{A}(v_i) := \left\{ l \in [0,\infty] : \lim_{t\to\pm\infty} v_i(t) = l \right\}.$$

In other words, $\mathcal{A}(\mathcal{V})$ is the set of all possible limits at infinity of the component solutions v_i.

The first of our lemmas states that the asymptotic set of $\mathcal{V}$ is quite simple, in the sense that it does not depend on $i \in I$, and coincides with the one in the scalar case.

Lemma 4.13. *Let $\mathcal{V}$ be a strongly positive solution to (48). Suppose that for all $i \in I$ there exists $l_i \in [0,\infty]$ such that $\lim_{t\to\pm\infty} v_i(t) = l_i$. Thus, $l_i \in \{0, l^*\}$, where $l^* = p^{-1}K_0^{\frac{n-4}{8}}$; in other terms, $\mathcal{A}(\mathcal{V}) = \{0, l^*\}$. Moreover, if $\mathcal{P}_{\rm cyl}(\mathcal{V}) \geq 0$, then $l^* = 0$.*

Proof. Here it is only necessary to consider the case $t \to \infty$ since when $t \to -\infty$, taking $\tau = -t$, and observing that $\widetilde{\mathcal{V}}(\tau) := \mathcal{V}(t)$ also satisfies (48), the result follows equally.

Suppose by contradiction that the lemma does not hold. Thus, for some fixed $i \in I$, one of the following two possibilities shall happen: either the asymptotic limit of v_i is a finite constant $l_i > 0$, which does not belong to the asymptotic set $\mathcal{A}(\mathcal{V})$, or the limit blows-up, that is, $l_i = +\infty$.

Subsequently, we consider these two cases separately:

Case 1: $l_i \in [0,\infty) \setminus \{0, l^*\}$.

By assumption, we have

$$\lim_{t\to\infty}\left(c(n)|\mathcal{V}|^{\frac{8}{n-4}}v_i(t) - K_0v_i(t)\right) = \kappa, \quad \text{where} \quad \kappa := c(n)pl_i^{\frac{n+4}{n-4}} - K_0l_i \neq 0, \tag{51}$$

which implies

$$c(n)|\mathcal{V}|^{\frac{8}{n-4}}v_i(t) - K_0v_i(t) = v_i^{(4)}(t) - K_2v_i^{(2)}(t). \tag{52}$$

A combination of (51) and (52) implies that for any $\varepsilon > 0$ there exists $T_i \gg 1$ sufficiently large satisfying

$$\kappa - \varepsilon < v_i^{(4)}(t) - K_2v_i^{(2)}(t) < \kappa + \varepsilon \quad \text{for} \quad t > T_i. \tag{53}$$

Now, integrating (53), we obtain

$$\int_{T_i}^t (\kappa-\varepsilon)\,\mathrm{d}\tau < \int_{T_i}^t \left[v_i^{(4)}(\tau) - K_2v_i^{(2)}(\tau)\right]\,\mathrm{d}\tau < \int_{T_i}^t (\kappa+\varepsilon)\,\mathrm{d}\tau,$$

which provides

$$(\kappa-\varepsilon)(t-T_i) + C_1(T_i) < v_i^{(3)}(t) - K_2v_i^{(1)}(t) < (\kappa+\varepsilon)(t-T_i) + C_1(T_i), \tag{54}$$

where $C_1(T_i) > 0$ is a constant. Defining $\delta := \sup_{t\geq T_i}|v_i(t) - v_i(T_i)| < \infty$, we obtain

$$\left|\int_{T_i}^t K_2v_i^{(1)}(\tau)\,\mathrm{d}\tau\right| \leq |K_2|\delta.$$

Hence, integrating (54) provides

$$\frac{(\kappa-\varepsilon)}{2}(t-T_i)^2 + L(t) < v_i^{(2)}(t) < \frac{(\kappa+\varepsilon)}{2}(t-T_i)^2 + R(t), \tag{55}$$

where $L(t), R(t) \in \mathcal{O}(t^2)$, namely

$$L(t) = C_1(T_i)(T_i-t) - |K_2|\delta + C_2(T_i) \quad \text{and} \quad R(t) = C_1(T_i)(T_i-t) + |K_2|\delta + C_2(T_i).$$

Then, repeating the same integration procedure in (55), we find

$$\frac{(\kappa-\varepsilon)}{24}(t-T_i)^4 + \mathcal{O}(t^4) < v_i(t) < \frac{(\kappa+\varepsilon)}{2}(t-T_i)^4 + \mathcal{O}(t^4) \quad \text{as} \quad t\to\infty. \tag{56}$$

Therefore, since $\kappa \neq 0$ we can choose $0 < \varepsilon \ll 1$ sufficiently small such that $\kappa - \varepsilon$ and $\kappa+\varepsilon$ have the same sign. Finally, by passing to the limit as $t \to \infty$ on inequality (56), we obtain that v_i blows-up and $l_i = \infty$, which is contradiction. This concludes the proof of the claim.

Case 2: $l_i = \infty$.

This case is more delicate, and it requires a suitable choice of test functions from [34]. More precisely, let $\phi_0 \in C^\infty([0,\infty])$ be a nonnegative function satisfying $\phi_0 > 0$ in $[0,2)$,

$$\phi_0(z) = \begin{cases} 1, & \text{for } 0 \leq z \leq 1, \\ 0, & \text{for } z \geq 2, \end{cases}$$

and for $j \in \{1,2,3,4\}$, let us fix the positive constants

$$M_j := \int_0^2 \frac{|\phi_0^{(j)}(z)|}{|\phi_0(z)|} \, \mathrm{d}z. \tag{57}$$

Using the contradiction assumption, we may assume that there exists $T_i > 0$ such that for $t > T_i$, it follows

$$\begin{aligned} v_i^{(4)}(t) - K_2 v_i^{(2)}(t) &= \hat{c}(n)|\mathcal{V}(t)|^{\frac{8}{n-4}} v_i(t) - K_0 v_i(t) \\ &\geq v_i(t)^{\frac{n+4}{n-4}} - K_0 v_i(t) \geq \frac{c(n)}{2} v_i(t)^{\frac{n+4}{n-4}} \end{aligned} \tag{58}$$

and

$$v_i^{(3)}(t) - K_2 v^{(1)}(t) = \frac{1}{2} \int_{T_i}^t v_i(\tau)^{\frac{n+4}{n-4}} \, \mathrm{d}\tau + C_1(T_i). \tag{59}$$

Besides, as a consequence of (59), we can find $T_i^* > T_i$ satisfying $v_i^{(3)}(T_i^*) - K_2 v^{(1)}(T_i^*) := \upsilon > 0$. Furthermore, since (48) is autonomous, we may suppose without loss of generality that $T_i^* = 0$. Then, multiplying inequality (58) by $\phi(t) = \phi_0(\tau/t)$, and by integrating, we find

$$\int_0^{T'} v_i^{(4)}(\tau)\phi(\tau) \, \mathrm{d}\tau - K_2 \int_0^{T'} v_i^{(2)}(\tau)\phi(\tau) \, \mathrm{d}\tau \geq \frac{1}{2} \int_0^{T'} v_i(\tau)^{\frac{n+4}{n-4}} \, \mathrm{d}\tau,$$

where $T' = 2T$. Moreover, integration by parts combined with $\phi^{(j)}(T') = 0$ for $j = 0,1,2,3$ implies

$$\int_0^{T'} v_i(\tau)\phi^{(4)}(\tau) v_i(\tau) \, \mathrm{d}\tau - K_2 \int_0^{T'} v_i(\tau)\phi^{(2)}(\tau) \, \mathrm{d}\tau \geq \frac{c(n)}{2} \int_0^{T'} v_i(\tau)^{\frac{n+4}{n-4}} \, \mathrm{d}\tau + \upsilon. \tag{60}$$

On the other hand, applying the Young inequality on the right-hand side of (60), it follows

$$v_i(\tau)|\phi^{(j)}(\tau)| = \varepsilon v_i^{\frac{n+4}{n-4}}(\tau)\phi(\tau) + C_\varepsilon \frac{|\phi^{(j)}(\tau)|^{\frac{n+4}{8}}}{\phi(\tau)^{\frac{n-4}{8}}}. \tag{61}$$

Hence, combining (61) and (60), we have that for $0 < \varepsilon \ll 1$ sufficiently small, it follows that there exists $\widetilde{C}_1 > 0$ satisfying

$$\widetilde{C}_1 \int_0^{T'} \left[\frac{|\phi^{(4)}(\tau)|^{\frac{n+4}{8}}}{\phi(\tau)^{\frac{n-4}{8}}} + \frac{|\phi^{(2)}(\tau)|^{\frac{n+4}{8}}}{\phi(\tau)^{\frac{n-4}{8}}} \right] \mathrm{d}\tau \geq \frac{c(n)}{4} \int_0^{T'} v_i(\tau)^{\frac{n+4}{n-4}} \, \mathrm{d}\tau + \upsilon.$$

Now by (57), one can find $\widetilde{C}_2 > 0$ such that

$$\widetilde{C}_2 \left(M_4 T^{-\frac{n+2}{2}} - M_2 T^{-\frac{n}{4}} \right) \geq \frac{c(n)}{4} \int_0^T v_i(\tau)^{\frac{n+4}{n-4}} \, \mathrm{d}\tau. \tag{62}$$

Therefore, passing to the limit in (62) the left-hand side converges, whereas the right-hand side blows-up; this is a contradiction.

For proving the second part, let us notice that

$$\lim_{t\to\infty} \mathcal{P}_{\rm cyl}(t,\mathcal{V}) = \omega_{n-1}\left(\frac{K_0}{2}|l^*|^2 - \hat{c}(n)|l^*|^{\frac{2n}{n-4}}\right) \geq 0,$$

which implies $l^*=0$ and $\mathcal{P}_{\rm cyl}(\mathcal{V})=0$. ☐

The next lemma shows that if a component solution to ($\mathcal{S}_p$) blows-up, then it shall be in finite time. In this fashion, we provide an accurate higher order asymptotic behavior for singular solutions to (48), namely, $\bigcup_{j=1}^{\infty}\mathcal{A}\left(\mathcal{V}^{(j)}\right)=\{0\}$.

Lemma 4.14. *Let $\mathcal{V}$ be a strongly positive solution to (48) such that $\lim_{t\to\pm\infty} v_i(t)\in\mathcal{A}(\mathcal{V})$ for all $i\in I$. Then, for any $j\geq 1$, we have that $\lim\limits_{t\to\pm\infty} v_i^{(j)}(t)=0$.*

Proof. As before, we only consider the case $t\to\infty$. Since $\mathcal{A}(\mathcal{V})=\{0,l^*\}$ we must divide our approach into two cases:

Case 1: $\lim_{t\to\pm\infty} v_i(t)=0$.

For each ordinary derivative case $j=1,2,3,4$, we construct one step. When $j\geq 5$, the proof follows directly from lower order derivative cases, and it is omitted. We start with $j=2$:

Step 1: $\mathcal{A}(v_i^{(2)})=0$.

By assumption $v_i(t)<l^*$ for $t\gg 1$ large, one has

$$v_i^{(4)} - K_2 v_i^{(2)} = \left(c(n)|\mathcal{V}|^{\frac{8}{n-4}}v_i - K_0 v_i\right) < 0.$$

Defining $B_i(t)=v_i^{(2)}(t)+K_0v_i(t)$, it holds that $B_i^{(2)}(t)<0$ for all $t\in\mathbb{R}$, and thus, B_i is concave near infinity, which implies $\mathcal{A}(B_i)\neq\emptyset$. Hence, there exists $b_0^*\in[0,\infty]$ such that $b_0^*:=\lim_{t\to\infty}B_i(t)$ and $b_1^*:=\lim_{t\to\infty}v_i^{(2)}(t)$. Supposing that $b_1^*\neq 0$, there exist three possibilities: First, if we assume $b_1^*=\infty$, then we have that $\lim_{t\to\infty}v_i^{(1)}(t)=\infty$, which is contradiction with $\lim_{t\to\infty}v_i(t)=0$. Second, assuming $0<b_1^*<\infty$, it follows that $v_i^{(2)}(t)>b_1^*t/2$ for $t\gg 1$ sufficiently large; thus $v_i^{(1)}(t)>b_1^*t/4$, which is also a contradiction with the hypothesis. Third, $b^*<0$, then using the same argument as before, we obtain that $v_i^{(1)}(t)\leq b_1^*t/4$, leading to the same contradiction. Therefore $b_1^*=0$, which concludes the proof.

Step 2: $\mathcal{A}(v_i^{(1)})=0$.

Indeed, for $t\gg 1$ large, there exists $\tau\in[t,t+1]$ satisfying $v_i(t+1)-v_i(t)=v_i^{(1)}(t)+\frac{1}{2}v_i^{(2)}(\tau)$, which, by taking the limit, and since $\tau\to\infty$ if $t\to\infty$, one gets that $v_i(t+1)\to 0$ and $v_i(t)\to 0$, which provides $\lim_{\tau\to\infty}v_i^{(2)}(\tau)\to 0$. Consequently, one has that $v_i^{(1)}(t)\to 0$.

Step 3: $\mathcal{A}(v_i^{(3)})=0$.

Since H_i is concave for large $t\gg 1$ and $B_i(t)\to\infty$ as $t\to\infty$, we find $\lim_{t\to\infty}B_i^{(1)}(t)=0$. Consequently, $v_i^{(3)}(t)\to\infty$ as $t\to\infty$.

Step 4: $\mathcal{A}(v_i^{(4)})=0$.

By equation (48) and by Step 1, we observe that $v_i^{(4)}(t)\to\infty$ as $t\to\infty$.

As a combination of Step 1–4, we finish the proof of Case 1.

The second case has an additional difficulty. Precisely, since $v_i(t)\to l^*$ as $t\to\infty$ for sufficiently large $t\gg 1$, there exist two possibilities: either v_i is eventually decreasing or v_i is eventually increasing. In both situations, the proofs are similar; thus, we only present the first one.

Case 2: $\lim_{t\to\infty} v_i(t) = l^*$.

Here we proceed as before.

Step 1: $\mathcal{A}(v_i^{(2)}) = 0$.

Since we are considering v_i is eventually decreasing, there exists a large $T_i \gg 1$ such that $v_i(t) > l^*$ for $t > T_i$ and we get that $v_i^{(4)} - K_2 v_i^{(2)} = \left(c(n)|\mathcal{V}|^{\frac{8}{n-4}} v_i - K_0 v_i\right) \geq 0$. In this case, B_i is convex for sufficiently large $t \gg 1$. Hence, $\mathcal{A}(B_i) \neq \emptyset$ and there exists $b_0^* = \lim_{t\to\infty} B_i(t)$. Since $v_i(t) \to l^*$ as $t \to \infty$, we get that $\lim_{t\to\infty} v_i^{(2)}(t) = b_1^*$, where $b_1^* = b_0^* - K_2 l^*$. Now repeating the same procedure as before, we obtain that $b_1^* = 0$ and thus $\lim_{t\to\infty} B_i(t) = K_2 l^*$, which yields $\mathcal{A}(v_i^{(2)}) = 0$.

The remaining steps of the proof follow similarly to Claim 1, and so the proof of the lemma is finished. □

Before we continue our analysis, it is essential to show that any solution to (48) is bounded.

Lemma 4.15. *Let $\mathcal{V}$ be a strongly positive solution to (48). Then, $v_i(t) < l^*$ for all $i \in I$. In particular, $|\mathcal{V}|$ is bounded.*

Proof. For $i \in I$, let us define the set $Z_i = \left\{t \geq 0 : v_i^{(1)}(t) = 0\right\}$. We divide the proof of the lemma into two cases:

Case 1: Z_i is bounded.

In this case, we have that v_i is monotone for large $t \gg 1$ and $\mathcal{A}(v_i) \neq \emptyset$. Therefore, using Lemma 4.13 we obtain that v_i bounded by l^* for $t \gg 1$ sufficiently large.

Case 2: Z_i is unbounded.

Fixing $H > 0$, we define $F(\tau) = \hat{c}(n)|\tau|^{2^{**}} - \frac{1}{2}|\tau|^2$, which satisfies $\lim_{\tau\to\infty} F(\tau) = \infty$. Therefore, there exists $R_i > |v_i(0)|$ such that $F(\tau) > H$ for $\tau > R_i$.

Claim 1. $|v_i| < R_i$ *on* $[0, \infty)$.

Supposing by contradiction that $M_{R_i} = \{t \geq 0 : |v_i(t)| \geq R_i\}$ is non-empty, we can define $t_i^* = \inf_{M_{R_i}} v_i$, which is strictly positive by the choice of R_i. Thus, we obtain that $v_i(t_i^*) = R_i$ and also $v_i^{(1)}(t_i^*) \geq 0$. In addition, since Z_i is unbounded, we have that $Z_i \cap [t_i^*, \infty) \neq \emptyset$. Therefore, considering $T_i^* = \inf_{Z_i \cap [t_i^*, \infty)} v_i$. Hence, since solutions are classical and $v^{(1)}(T_i^*) = 0$ implies that $v_i^{(1)}(t) \geq 0$ for all $t \in [t_i^*, T_i^*]$. Eventually, we conclude that $v_i(T^*) > R_i$ and $\mathcal{H}(T_i^*, \mathcal{V}) = \frac{1}{2}|\mathcal{V}^{(2)}(T_i^*)|^2 + F(|\mathcal{V}(T_i^*)|) > H$, which is a contradiction with (50). To complete the proof lemma, one can check that $R_i = l^*$ for all $i \in I$. □

Lemma 4.16. *Let $\mathcal{V}$ be a strongly positive solution to (48). Then, it follows that $v_i^{(1)}(t) < \gamma v_i(t)$ for all $i \in I$ and $t \in \mathbb{R}$, where we recall that $\gamma = \frac{n-4}{2}$ is the Fowler rescaling exponent.*

Proof. Let us define

$$\widetilde{\gamma} = \sqrt{\frac{K_2}{2} - \sqrt{\frac{K_2^2}{4} - K_0}}.$$

Then, by a direct computation, we get that $\widetilde{\gamma}=\gamma$. Setting

$$\lambda_1=\frac{K_2}{2}-\sqrt{\frac{K_2^2}{4}-K_0} \quad \text{and} \quad \lambda_2=\frac{K_2}{2}+\sqrt{\frac{K_2^2}{4}-K_0},$$

we have that $\lambda_1+\lambda_2=K_2$ and $\lambda_1\lambda_2=K_0$. Defining the auxiliary function $\phi_i(t)=v_i^{(2)}-\lambda_2 v_i(t)$, we observe that $\phi_i^{(2)}-\lambda_2\phi_i=|\mathcal{V}|^{\frac{8}{n-4}}v_i$ and $-\phi_i^{(2)}+\lambda_2\phi_i\leq 0$. Hence, since $\mathcal{V}$ is a strongly positive solution to (2) by the strong maximum principle, we get that $\phi_i<0$, which implies that $w_i=v_i^{(1)}/v_i$ satisfies

$$w_i^{(1)}=-w_i^2+\lambda_1+\frac{\phi_i}{v_i} \quad \text{and} \quad \frac{v_i^{(2)}}{v_i}=\lambda_1+\frac{\phi_i}{v_i}. \tag{63}$$

Moreover, by Lemma 4.13, there exists $t_0\in\mathbb{R}$ such that $v_i^{(1)}(t_0)=0$, which provides $w_i(t_0)=0$. Setting $M:=\left\{t>t_0: w_i(t)\geq\sqrt{\lambda_1}\right\}$, the proof of the lemma is reduced to the next claim.

Claim 1. $M=\emptyset$.

Indeed, supposing the claim is not true, we set $t_1=\inf M$. Notice that $t_1>t_0$, $w_i^{(1)}(t_1)\geq 0$ and $w_i(t_1)=\sqrt{\lambda_1}$. On the other hand, by (63), we obtain that $w_i^{(1)}(t_1)=\frac{\phi_i(t_1)}{v_i(t_1)}<0$, which is a contradiction with the fact that v_i is positive since $\phi_i<0$. This finishes the proof of the claim. □

As an application of Lemma 4.16, we complete the proof of Proposition 4.3, which states that any component of $\mathcal{U}$ is radially monotonically decreasing.

Corollary 4.17. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_p$). Then, $\partial_r u_i(r)<0$ for all $r>0$ and $i\in I_+$.*

Proof. By a direct computation, we have that $\partial_r u_i(r)=-r^{\gamma-1}\left[v_i^{(1)}(t)-\gamma v_i(t)\right]$. Then, the proof of the corollary is a consequence of Lemma 4.16. □

4.6. **Removable singularity classification.** After establishing the previous lemmas concerning the asymptotic behavior of global solutions to the ODE system (48), we can prove the main results of the section, namely, the removable-singularity classification and the non-existence of semi-singular solutions to ($\mathcal{S}_p$). These results will be employed in the proof of Theorem 1.2. More precisely, we show that the Pohozaev invariant of any solution is always nonpositive, and it is zero, if, and only if, the origin is a non-removable, otherwise, for singular solutions to ($\mathcal{S}_p$) this invariant is always strictly negative.

To prove our removable singularity theorem, we need to define some auxiliary functions. For $i\in I$, let us set $\varphi_i:\mathbb{R}\to\mathbb{R}$ given by

$$\varphi_i(t)=v_i^{(3)}(t)v_i^{(1)}(t)-\frac{1}{2}|v_i^{(2)}(t)|^2-\frac{K_2}{2}|v_i^{(1)}(t)|^2+\frac{K_0}{2}|v_i(t)|^2-\hat{c}(n)|v_i(t)|^{2^{**}}.$$

Remark 4.18. By Lemma 4.14, we observe that

$$\varphi_i^{(1)}(t)=c(n)\left(|\mathcal{V}(t)|^{2^{**}-2}-|v_i(t)|^{2^{**}-2}\right)v_i(t)v_i^{(1)}(t).$$

Since $|\mathcal{V}|\geq|v_i|$, we have that $\operatorname{sign}(\varphi_i^{(1)})=\operatorname{sign}(v_i^{(1)})$. In other terms, the monotonicity of φ_i is the same as component function v_i. Moreover, it holds that $\sum_{i=1}^p\varphi_i(t)=-H$.

Proposition 4.19. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_p$). Then, $\mathcal{P}_{\rm sph}(\mathcal{U})\leq 0$ and $\mathcal{P}_{\rm sph}(\mathcal{U})=0$, if, and only if, $\mathcal{U}\in C^{4,\zeta}(\mathbb{R}^n,\mathbb{R}^p)$, for some $\zeta\in(0,1)$.*

Proof. Let us divide the proof into two claims as follows. The first one is concerned with the sign of the Pohozaev invariant. Namely, we show it is always nonpositive.

Claim 1. *If $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) \geq 0$, then $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) = 0$.*

Indeed, let us define the sum function $v_\Sigma : \mathbb{R} \to \mathbb{R}$ given by $v_\Sigma(t) = \sum_{i=1}^p v_i(t)$. Hence, by Lemma 4.13, for any v_i there exists a sufficient large $\hat{t}_i \gg 1$ such that $v_i^{(1)}(\hat{t}_i) = 0$. Furthermore, by Lemma 4.14 for any $i \in I$, we can find a sufficiently large $t_i \geq \hat{t}_i \gg 1$ such that $v_i^{(1)}(t) < 0$ for all $t > t_i$. Then, choosing $t_* > \max_{i\in I}\{t_i\}$, we have that $v_\Sigma^{(1)}(t) < 0$ for $t > t_*$, which implies $\lim_{t\to\infty} v_i(t) = 0$. Consequently, by Lemma 4.16, we conclude that $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) = 0$.

In the next claim, we use some arguments from [20, Lemma 2.4] to show that solutions with zero Pohozaev invariant have a removable singularity at the origin.

Claim 2.*If $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) = 0$, then $\mathcal{U} \in C^{4,\zeta}(\mathbb{R}^n, \mathbb{R}^p)$, for some $\zeta \in (0,1)$.*

In fact, note that v_Σ satisfies

$$v_\Sigma^{(4)} - K_2 v_\Sigma^{(2)} + K_0 v_\Sigma = c(n)|\mathcal{V}|^{2^{**}-2} v_\Sigma. \tag{64}$$

Setting $\widetilde{f}(\mathcal{V}) = c(n)|\mathcal{V}|^{2^{**}-2}v_\Sigma$, since $v_i(t) \to 0$ as $t \to \pm\infty$, it follows that $\lim_{t\to\infty} \widetilde{f}(\mathcal{V}(t)) = 0$. Then, we define $\tau = -t$ and $\widetilde{v}_\Sigma(\tau) = v_\Sigma(t)$, which implies that $\widetilde{v}_\Sigma$ also satisfies (64). Moreover, $\lim_{t\to-\infty} v_\Sigma(t) = \lim_{\tau\to\infty} \widetilde{v}_\Sigma(\tau) = 0$ and also

$$\lim_{\tau\to\infty} \widetilde{f}(\widetilde{\mathcal{V}}(\tau)) = 0. \tag{65}$$

Consequently, by ODE theory in [19], we can find sufficiently large $T \gg 1$ satisfying

$$\begin{aligned}
\widetilde{v}_\Sigma(\tau) &= A_1 e^{\lambda_1\tau} + A_2 e^{\lambda_2\tau} + A_3 e^{\lambda_3\tau} + A_4 e^{\lambda_4\tau} \\
&\quad + B_1 \int_T^\tau e^{\lambda_1(\tau-t)} \widetilde{f}(\widetilde{\mathcal{V}}(t))\, \mathrm{d}t + B_2 \int_T^\tau e^{\lambda_2(\tau-t)} \widetilde{f}(\widetilde{\mathcal{V}}(t))\, \mathrm{d}t \\
&\quad - B_3 \int_\tau^\infty e^{\lambda_3(\tau-t)} \widetilde{f}(\widetilde{\mathcal{V}}(t))\, \mathrm{d}t - B_4 \int_\tau^\infty e^{\lambda_4(\tau-t)} \widetilde{f}(\widetilde{\mathcal{V}}(t))\, \mathrm{d}t,
\end{aligned}$$

where A_1, A_2, A_3, A_4 are constants depending on T, B_1, B_2, B_3, B_4 are constants not depending on T, and

$$\lambda_1 = -\frac{n}{2},\ \lambda_2 = -\frac{n-4}{2},\ \lambda_3 = \frac{n}{2} \text{ and } \lambda_4 = \frac{n-4}{2}$$

are the solutions to the characteristic equation $\lambda^4 - K_2\lambda^2 + K_0\lambda = 0$. In addition, by (65) we obtain that $A_3 = A_4 = 0$. Hence, we use the same ideas in [21, Theorem 3.1] to arrive at

$$\widetilde{v}_\Sigma(\tau) = \mathcal{O}(e^{-\frac{n-4}{2}\tau}) \quad \text{as} \quad \tau \to \infty \quad \text{or} \quad v_\Sigma(t) = \mathcal{O}(e^{\frac{n-4}{2}t}) \quad \text{as} \quad t \to -\infty.$$

Eventually, undoing the cylindrical transformation, we have that $u_\Sigma(r) = \mathcal{O}(1)$ as $r \to 0$, which finishes the proof of the claim.

Therefore, using the last claim, we get u_Σ is uniformly bounded, which implies $u_i \in C^0(\mathbb{R}^n)$ for all $i \in I$. Finally, standard elliptic regularity theory provides that $\mathcal{U} \in C^{4,\zeta}(\mathbb{R}^n, \mathbb{R}^p)$ for some $\zeta \in (0,1)$ and for all $i \in I$; this concludes the proof of the proposition. □

Proposition 4.20. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_p$). If $\mathcal{P}_{\mathrm{sph}}(\mathcal{U}) < 0$, then $\mathcal{U}$ is fully-singular.*

Proof. Suppose by contradiction $\mathcal{U}$ is semi-singular, that is, there exists some $i_0 \in I \setminus I_\infty$. We may suppose without loss of generality $\{i_0\} = I \setminus I_\infty$, which yields

$$\lim_{\substack{r\to 0\\ i\neq i_0}} u_i(r) = \infty \quad \text{and} \quad \liminf_{r\to 0} u_{i_0}(r) = C_{i_0} < \infty. \tag{66}$$

Claim 1.$\lim_{t\to\infty} v_{i_0}(t) = \infty$.

Indeed, using Lemma 4.16, we have that $\gamma^{-1}|v_{i_0}^{(1)}(t)| \le v_{i_0}(t) \le C_i e^{-\gamma t}$ for all $i \in I \setminus \{i_0\}$, which provides $\varphi_{i_0}(t) \to 0$ as $t \to \infty$. Hence, since $P < 0$, we get that $H < 0$, which combined with Remark 4.18 yields $\sum_{\substack{i=1\\ i\neq i_0}}^{p} \varphi_i(t) = -H$. Let us divide the rest of the proof into two steps:

Step 1: For each $i \in I \setminus \{i_0\}$, there exists $C_i > 0$ such that $u_i(r) \ge C_i r^{-\gamma}$ for all $r \in (0, 1]$.

First, it is equivalent to $\inf_{t\ge 0} v_i(t) \ge C_i$ in cylindrical coordinates. Assume by contradiction that it does not hold. Then, there exists $\{t_k\}_{k\in\mathbb{N}} \subset (0,\infty)$ such that $t_k \to \infty$ and $v_i(t_k) \to 0$ as $k \to \infty$. Moreover, using Lemma 4.16 for all $i \in I$ one obtains $0 \le \gamma^{-1}|v_i^{(1)}(t_k)| \le v_i(t_k) \to 0$, which yields that $\varphi_i(t_k) \to 0$. This is a contradiction, and the proof of Step 1 is finished.

Step 2: There exists $\varrho \in C^\infty(\mathbb{R} \setminus \{0\})$ such that $\lim_{r\to 0} \varrho(r) = \infty$ and

$$u_{i_0}(r) \ge \varrho(r) \quad \text{for all} \quad r \in (0, 1].$$

First, it is easy to check that there exists $C_0 > 0$ such that $u_{i_0}(r) \ge C_0$ for all $r \in (0, 1]$. Second, writing the Laplacian in spherical coordinates, we have

$$r^{1-n}\partial_r \left[r^{n-1}\partial_r \Delta u_{i_0}(r)\right] = c(n)|\mathcal{U}|^{2^{**}-2} u_{i_0}.$$

Now use the estimates in Step 1 to obtain,

$$\partial_r \left[r^{n-1}\partial_r \Delta u_{i_0}(r)\right] \ge c_0 r^{n-5},$$

which, by integrating, implies

$$r^{n-1}\partial_r \Delta u_{i_0}(r) \ge c_1 r^{n-4} + c_2.$$

By proceeding as before, we can find $c_1, c_2, c_3, c_4 \in \mathbb{R}$ satisfying

$$u_{i_0}(r) \ge c_1 r^{-1} + c_2 r^{1-n} + c_3 r^{-n} + c_4,$$

which concludes the proof of Step 2.

Eventually, passing to the limit as $r \to 0$ in Step 2, we obtain that u_{i_0} blows-up at the origin. Hence, Claim 1 holds.

This is a contradiction with (66). Therefore, semi-singular solutions cannot exist, and the proposition is proved. □

4.7. **Proof of Theorem 1.2.** Finally, we have conditions to connect the information we have obtained to prove our classification result. Our idea is to apply the analysis of the Pohozaev invariant and ODE methods together with Theorem 1.1, and Propositions 4.19 and 4.20, which can be summarized as follows

Theorem 4.21. *Let $\mathcal{U}$ be a strongly positive solution to ($\mathcal{S}_p$). There exist only two possibilities for the sign of the Pohozaev invariant:*
(i) *If $\mathcal{P}_{\rm sph}(\mathcal{U}) = 0$, then $\mathcal{U} = \Lambda^* u_{x_0,\mu}$, where $u_{x_0,\mu}$ is given by* (1) (*Spherical solution);*
(ii) *If $\mathcal{P}_{\rm sph}(\mathcal{U}) < 0$, then $\mathcal{U} = \Lambda^* u_{a,T}$, where $u_{a,T}$ is given by* (3) (*Emden–Fowler solution).*

Proof. (i) It follows directly by Proposition 4.19 and Theorem 1.1.
(ii) First, since solutions are strongly positive and fully-singular, it follows that $I_+ = I_\infty = I$, which makes the quotient functions $q_{ij} = v_i/v_j$ well-defined for all $i, j \in I$. Moreover, we show that they are constants. Notice that v_i and v_j satisfy,

$$\begin{cases} v_i^{(4)} - K_2 v_i^{(2)} + K_0 v_i = c(n)|\mathcal{V}|^{2^{**}-2} v_i \\ v_j^{(4)} - K_2 v_j^{(2)} + K_0 v_j = c(n)|\mathcal{V}|^{2^{**}-2} v_j, \end{cases}$$

which provides

$$\left(v_i^{(4)} v_j - v_j^{(4)} v_i\right) = -K_2 \left(v_i^{(2)} v_j - v_i v_j^{(2)}\right). \tag{67}$$

Furthermore, a standard computation yields

$$q_{ij}^{(4)} = \frac{v_i^{(4)} v_j - v_i v_j^{(4)}}{v_j^2} - 4 v_j^{(1)} v_j^{-1} q_{ij}^{(3)} - 6 v_j^{(2)} v_j^{-1} q_{ij}^{(2)} - 4 v_j^{(3)} v_j^{-1} q_{ij}^{(1)},$$

which, combined with (67), implies that the quotient satisfy the following fourth order homogeneous Cauchy problem,

$$\begin{cases} q_{ij}^{(4)} + 4 v_j^{(1)} v_j^{-1} q_{ij}^{(3)} + 6 v_j^{(2)} v_j^{-1} q_{ij}^{(2)} + (4 v_j^{(3)} v_j^{-1} + K_2) q_{ij}^{(1)} = 0 \quad \text{in} \quad \mathbb{R} \\ q_{ij}(0) = a_i/a_j, \quad q_{ij}^{(1)}(0) = q_{ij}^{(2)}(0) = q_{ij}^{(3)}(0) = 0. \end{cases}$$

Hence, using Lemma 4.14 the Picard–Lindelöf uniqueness theorem, it follows that $q_{ij} \equiv a_i/a_j$. Thus, by the same argument in the proof of Theorem 1.1, one can find $\Lambda^* \in \mathbb{S}^{p-1}_{+,*}$ such that $\mathcal{V}(t) = \Lambda^* v_{a,T}(t)$, where $v_{a,T}$ is given by (3). By undoing the cylindrical transformation, the proof is concluded. □

As a consequence, we provide a sharp global estimate for the blow-up rate of singular solutions to ($\mathcal{S}_p$) near the origin.

Corollary 4.22. *Let $\mathcal{U}$ be a strongly positive singular solution to ($\mathcal{S}_p$). Then, there exist $C_1, C_2 > 0$ such that*

$$C_1 |x|^{\frac{4-n}{2}} \leq |\mathcal{U}(x)| \leq C_2 |x|^{\frac{4-n}{2}} \quad \text{for all} \quad x \in \mathbb{R}^n \setminus \{0\}.$$

In other terms, one has

$$|\mathcal{U}(x)| = \mathcal{O}(|x|^{\frac{4-n}{2}}) \quad \text{as} \quad x \to 0.$$

Proof. It is direct application of Theorem 4.21 and Lemma 4.15. □

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