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(H,G)- COINCIDENCE THEOREMS FOR MANIFOLDS

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Abstract

Let X be a paracompact space, let G be a finite group acting freely on X and let H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for $k \geq 1$, where V_k are the Wu classes of M. Suppose that $\operatorname{ind} X \geq n > (|G| - r)m$, where $r = \frac{|G|}{p}$. In this work, we estimate the cohomological dimension of the set $A(f, H, G)$ of (H, G)-coincidence points of f.

1 Introduction

Let G be a finite group which acts freely on a space X and let $f : X \to Y$ be a continuous map from X into another space Y. If H is a subgroup of G, then H acts on the right on each orbit Gx of G as follows: if $y \in Gx$ and $y = gx$, $g \in G$, then $hy = ghx$. A point $x \in X$ is said to be a (H, G)- *coincidence point* of f if f sends every orbit of the action of H on the G-orbit of x to a single point (See [5]). Of course, if H is the trivial subgroup, then every point of X is a (H, G)-coincidence. If $H = G$, this is the usual definition of G-coincidence, that is, $f(x) = f(gx)$, for all $g \in G$. If $G = \mathbb{Z}_p$ with p prime, then a nontrivial (H, G)-coincidence point is a G-coincidence point. Let us denote by $A(f, H, G)$ the set of all (H, G)-coincidence points. Borsuk-Ulam type theorems consists in estimating the cohomological dimension of the set $A(f, H, G)$. Two main directions considered of this problem are either when the target space Y is a manifold or Y is a CW complex. In the first direction are the papers of Borsuk [1] (the classical theorem of Borsuk-Ulam, for $H = G = \mathbb{Z}_2$, $X = S^n$ and $Y = R^n$), Conner and Floyd [3] (for $H = G = \mathbb{Z}_2$ and Y a n-manifold), Munkholm [9] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and $Y = R^m$), Nakaoka [10] (for $H = G = \mathbb{Z}_p$, X under certain (co)homological conditions and Y a m-manifold) and the following more general version proved by Volovikov [12] using the index of a free $\mathbb{Z}_p$-space X ($\operatorname{ind} X$, see Definition 2.2):

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Theorem A.[12, Theorem 1.2] *Let X be a paracompact free $\mathbb{Z}_p$-space of* $\operatorname{ind} X \geq n$, *and $f : X \to M$ a continuous mapping of X into an m-dimensional connected manifold M (orientable if $p > 2$). Assume that:*

(1) $f^*(V_i) = 0$ *for $i \geq 1$, where the V_i are the Wu classes of M; and*

(2) $n > m(p-1)$.

Then the $\operatorname{ind} A(f) \geq n - m(p-1) > 0$.

In the second direction are the papers of Izydorek and Jaworowski [7] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and Y a CW-complex), Gonçalves and Pergher [4] (for $H = G = \mathbb{Z}_p$, $X = S^n$ and Y a CW-complex) and for proper nontrivial subgroup H of G, Gonçalves, Jaworowski and Pergher [5] (for $H = \mathbb{Z}_p$ subgroup of a finite group G, X an homotopy sphere and Y a CW-complex) and Gonçalves, Jaworowski, Pergher and Volovikov [6](for $H = \mathbb{Z}_p$ subgroup of a finite group G, X under certain (co)homological assumptions and Y a CW-complex).

In this work, considering the target space $Y = M$ a manifold and H a proper nontrivial subgroup of G, we prove the following formulation of the Borsuk-Ulam theorem for manifolds in terms of (H, G)-coincidence.

Theorem 1.1. *Let X be a paracompact space of $indX \geq n$ and let G be a finite group acting freely on X and H a cyclic subgroup of G of prime order p. Let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M. Suppose that* $\operatorname{ind} X \geq n > (|G| - r)m$ *where* $r = \frac{|G|}{p}$, *then* $\operatorname{ind} A(f, H, G) \geq n - (|G| - r)m$. *Consequently,*

$$\operatorname{cohom.dim} A(f, H, G) \geq n - (|G| - r)m > 0.$$

Let us observe that if $H = G = \mathbb{Z}_p$, we have $(|G| - r)m = (p-1)m$ and therefore Theorem 1.1 generalizes Theorem A. obtained by Volovikov in [12] in terms of (H, G)-coincidence. For the case $n = (|G| - r)m$, p an odd prime, if we consider X a *modp* homology n-sphere in the Theorem 1.1(in this case, the continuous map f can be arbitrary), it follows from [10, Theorem 8] that $A(f, H, G) \neq \emptyset$. Also, Theorem 1.1 is a version for manifolds of the main result due to Gonçalves, Jaworowski and Pergher in [5].

Finally, we prove the following nonsymmetric theorem for (H, G)-coincidences which is a version for manifolds of the main theorem in [8].

Theorem 1.2. *Let X be a compact Hausdorff space, let G be a finite group acting freely on S^n and let H be a cyclic subgroup of G of order prime p. Let $\varphi : X \to S^n$ be an essential map* [1] *and let $f : X \to M$ be a continuous map where M is a connected m-manifold (orientable if $p > 2$) and $f^*(V_k) = 0$, for*

[1] A map $\varphi : X \to S^n$ is said to be an essential map if φ induces nonzero homomorphism $\varphi^* : H^n(S^n; \mathbb{Z}_p) \to H^n(X; \mathbb{Z}_p)$.

all $k \geq 1$, *where* V_k *are the* Wu *classes of* M. *Suppose that* $n > (|G| - r)m$, *then*

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m,$$

where $r = \frac{|G|}{p}$ *and* $A_\varphi(f, H, G)$ *denotes the* (H, G)*-coincidence points of* f *relative to an essential map* $\varphi : X \to S^n$.

2 Preliminaries

We start by introducing some definitions as follows.

2.1 The $\mathbb{Z}_p$-index

We suppose that the cyclic group $\mathbb{Z}_p$ acts freely on a paracompact Hausdorff space X, where p is a prime number and we denote by $[X]^*$ the space of orbits of X by the action of $\mathbb{Z}_p$. Then, $X \to [X]^*$ is a principal $\mathbb{Z}_p$-bundle and we can consider a classifying map $c : [X]^* \to B\mathbb{Z}_p$.

Remark 2.1. It is well known that if $\hat{c}$ is another classifying map for the principal $\mathbb{Z}_p$-bundle $X \to X^*$, then there is a homotopy between c and $\hat{c}$.

Definition 2.2. *We say that the* $\mathbb{Z}_p$*-index of* X *is greater than or equal to* l *if the homomorphism*

$$c^* : H^l(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^l([X]^*; \mathbb{Z}_p)$$

is nontrivial. We say that the $\mathbb{Z}_p$*-index of* X *is equal to* l *if it is greater or equal than* l *and, furthermore,* $c^* : H^i(B\mathbb{Z}_p; \mathbb{Z}_p) \to H^i([X]^*; \mathbb{Z}_p)$ *is zero, for all* $i \geq l + 1$.

We denote the $\mathbb{Z}_p$*-index of* X *by* $\text{ind}\, X$.

2.2 The Wu classes

Let us consider the additives maps *reduced powers* and the *Steenrod squares* (see [11] and [2, Chapter II])

$$P^k : H^{m-2k(p-1)}(M; \mathbb{Z}_p) \to H^m(M; \mathbb{Z}_p)$$

e $Sq^k : H^{m-k}(M; \mathbb{Z}_p) \to H^m(M; \mathbb{Z}_p)$, respectively, for all $k \geq 1$. These maps are also defined in the cohomology of pairs of spaces, and therefore induce maps

$$P^k : H_c^{m-2k(p-1)}(M; \mathbb{Z}_p) \to H_c^m(M; \mathbb{Z}_p)$$

e $Sq^k : H_c^{m-k}(M; \mathbb{Z}_p) \to H_c^m(M; \mathbb{Z}_p)$ which satisfy the same properties that *reduced powers* and *Steenrod squares.*

The Wu classes are defined for $p > 2$, as follows. By formula of the universals coefficients,

$$H^{2k(p-1)}(M; \mathbb{Z}_p) \cong Hom(H_c^{m-2k(p-1)}(M; \mathbb{Z}_p), \mathbb{Z}_p),$$

this isomorphism carries $V \mapsto \overline{V}$, with $\overline{V}(x) = \langle V \smile x, [M] \rangle$, where $[M] \in H_m^c(M; \mathbb{Z}_p)$ is the fundamental class of M. Consider the homomorphism

$$h : H_c^{m-2k(p-1)}(M; \mathbb{Z}_p) \to \mathbb{Z}_p,$$

defined by $h(x) = \langle P^k(x), [M] \rangle$. Then, there is an unique element $V_k \in H^{2k(p-1)}(M; \mathbb{Z}_p)$ such that $h = \overline{V}_k$. The element V_k is the k-th Wu class of M, for $k \geq 0$. Similarly the Wu classes are defined for $p = 2$, in this case are used the maps *Steenrod squares*.

3 The Wu classes for product of manifolds

Let W and M be manifolds of dimensions w and m respectively, $w \geq m$, both orientables if $p > 2$. In the next lemma we obtain a characterization of the Wu classes of the product $W \times M$ in terms of the Wu classes of W and M.

Lemma 3.1. *Let W and M be connected manifolds of dimensions w and m respectively, with $w \geq m$ and W, M both orientables if $p > 2$. Then, the k-th Wu class of $W \times M$ is given by:*

$$v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s} \tag{3.1}$$

where U_s and V_{k-s} are the k-th and $(k-s)$-th Wu classes of W and M, respectively, and

$$\begin{cases} (-1)^{\Lambda(s)} = (-1)^{[w-s](k-s)} \equiv 1(\mathrm{mod}_2) & \text{if } p = 2, \\ (-1)^{\Lambda(s)} = (-1)^{[w-2s(p-1)](k-s)} & \text{if } p > 2. \end{cases}$$

First, we will need of the following result to prove Lemma 3.1:

Lemma 3.2. *Let W and M be manifolds and let R be a commutative ring with unity. If $u_1 \in H^p(W; R)$, $v_1 \in H^r(M; R)$, $u_2 \in H_c^q(W; R)$ and $v_2 \in H_c^s(M; R)$, then*

$$(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr} (u_1 \smile u_2) \times (v_1 \smile v_2).$$

Proof. We denote by $\pi : W \times M \to W$ and $p : W \times M \to M$ the natural projections, which are proper maps. We have

$$u_1 \times v_1 = \pi^*(u_1) \smile p^*(v_1) \in H^{p+r}(W \times M; R),$$

$$u_2 \times v_2 = \pi_c^*(u_2) \smile p_c^*(v_2) \in H_c^{q+s}(W \times M; R),$$

where π_c^* and p_c^* denote the induced maps in cohomology with compact support.

Thus, by [2, Chapter II Proposition 7.3 and Corollary 7.2], it follows that:

$$(u_1 \times v_1) \smile (u_2 \times v_2) = (-1)^{qr}(\pi^*(u_1) \smile \pi_c^*(u_2)) \smile (p^*(v_1) \smile p_c^*(v_2)).$$

By [2, Chapter II, Section 8.2] , we can conclude that

$$\begin{aligned}(u_1 \times v_1) \smile (u_2 \times v_2) &= (-1)^{qr}\pi_c^*(u_1 \smile u_2) \smile p_c^*(v_1 \smile v_2) \\ &= (-1)^{qr}(u_1 \smile u_2) \times (v_1 \smile v_2).\end{aligned}$$

■

Proof of Lemma 3.1. Firstly, we will show that for $p = 2$,

$$Sq^k(x) \quad = \quad \left[\sum_{s=0}^{k} U_s \times V_{k-s}\right] \smile x \quad \text{for all } x \in H_c^{(w+m)-k}(W \times M; \mathbb{Z}_2),$$

where $U_s \in H^s(W; \mathbb{Z}_2)$ and $V_{k-s} \in H^{k-s}(M; \mathbb{Z}_2)$ are the s–th and $(k-s)$–th Wu classes of W and M, respectively. Thus, by uniqueness $v_k = \sum_{s=0}^{k} U_s \times V_{k-s}$ is the k–th Wu class of $W \times M$.

From the Kunneth's formula [2, Chapter II Theorem 15.2],

$$H_c^{(w+m)-k}(W \times M; \mathbb{Z}_2) \cong \bigoplus_{\beta=0}^{(w+m)-k} H_c^{\beta}(W; \mathbb{Z}_2) \otimes H_c^{(w+m)-k-\beta}(M; \mathbb{Z}_2).$$

Therefore, for each $x \in H_c^{(w+m)-k}(M \times W; \mathbb{Z}_2)$ we can write

$$x = \sum_{\beta=0}^{(w+m)-k} x_\beta \times y_{(w+m)-k-\beta},$$

where $x_\beta \in H_c^{\beta}(W; \mathbb{Z}_2)$ and $y_{(w+m)-k-\beta} \in H_c^{(w+m)-k-\beta}(M; \mathbb{Z}_2)$. Thus, applying the k–th Steenrod square Sq^k on x and using the Cartan formula in [2], we obtain

$$\begin{aligned} Sq^k(x) &= \sum_{\beta=0}^{(w+m)-k} Sq^k[x_\beta \times y_{(w+m)-k-\beta}] \\ &= \sum_{\beta=0}^{(w+m)-k} \left(\sum_{s=0}^{k} Sq^s(x_\beta) \times Sq^{k-s}(y_{(w+m)-k-\beta})\right) \\ &= \sum_{s=0}^{k} \left(\sum_{\beta=0}^{(w+m)-k} Sq^s(x_\beta) \times Sq^{k-s}(y_{(w+m)-k-\beta})\right). \end{aligned} \tag{3.2}$$

Now, let us consider $\beta \neq w - s$ for each s fixed, $0 \leq s \leq k$.

If $\beta > w - s$, so $\beta + s > w$ and we have

$$Sq^s(x_\beta) \in H^{\beta+s}(W;\mathbb{Z}_2) = 0. \tag{3.3}$$

If $\beta < w - s$, so $m < w + m - \beta - s$ and we have

$$Sq^{k-s}(y_{(w+m)-k-\beta}) \in H^{(w+m)-\beta-s}(M;\mathbb{Z}_2) = 0. \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4), using that $Sq^s(x_{(w-s)}) = U_s \smile x_{w-s}$, $Sq^{k-s}(y_{m-(k-s)}) = V_{k-s} \smile y_{m-(k-s)}$, we conclude that

$$\begin{aligned} Sq^k(x) &= \sum_{s=0}^{k} Sq^s(x_{w-s}) \times Sq^{k-s}(y_{m-(k-s)}) \\ &= \sum_{s=0}^{k} [U_s \smile x_{w-s}] \times \left[V_{k-s} \smile y_{m-(k-s)}\right]. \end{aligned}$$

On the other hand, using Lemma 3.2 we have

$$\begin{aligned} v_k \smile x &= \left[\sum_{s=0}^{k} U_s \times V_{k-s}\right] \smile \left[\sum_{\beta=0}^{(w+m)-k} x_\beta \times y_{(w+m)-k-\beta}\right] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-k} [U_s \times V_{k-s}] \smile [x_\beta \times y_{(w+m)-k-\beta}] \\ &= \sum_{s=0}^{k} \sum_{\beta=0}^{(w+m)-k} [U_s \smile x_\beta] \times [V_{k-s} \smile y_{(w+m)-k-\beta}]. \end{aligned}$$

Again, let us consider $\beta \neq w - s$ for each s fixed, $0 \leq s \leq k$.

If $\beta > w - s$, then $U_s \smile x_\beta \in H_c^{\beta+s}(W;\mathbb{Z}_2) = \{0\}$.

If $\beta < w - s$, then $V_{k-s} \smile y_{(w+m)-k-\beta} \in H_c^{(w+m)-\beta-s}(M;\mathbb{Z}_2) = \{0\}$.

Therefore,

$$v_k \smile x = \sum_{s=0}^{k} [U_s \smile x_{w-s}] \times [V_{k-s} \smile y_{m-k-s}].$$

Now, in the analogous way, we will show that for $p > 2$,

$$P^k(x) = \left[\sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}\right] \smile x, \quad \text{for all } x \in H_c^{(w+m)-2k(p-1)}(W \times M;\mathbb{Z}_p),$$

where $U_s \in H^{2s(p-1)}(W;\mathbb{Z}_p)$ and $V_{k-s} \in H^{2(k-s)(p-1)}(M;\mathbb{Z}_p)$ are the s–th and $(k-s)$–th Wu classes of W and M, respectively. So, by uniqueness, we conclude that the class $v_k = \sum_{s=0}^{k} (-1)^{\Lambda(s)} U_s \times V_{k-s}$ is the k-th Wu class of $W \times M$.

By Kunneth's formula,

$$H_c^{(w+m)-2k(p-1)}(W\times M;\mathbb{Z}_p)\cong\bigoplus_{\beta=0}^{(w+m)-2k(p-1)} H_c^{\beta}(W;\mathbb{Z}_p)\otimes H_c^{(w+m)-2k(p-1)-\beta}(M;\mathbb{Z}_p).$$

Therefore, for each $x\in H_c^{(w+m)-2k(p-1)}(W\times M;\mathbb{Z}_p)$ we can write

$$x=\sum_{\beta=0}^{(w+m)-2k(p-1)} x_\beta\times y_{(w+m)-2k(p-1)-\beta},$$

where $x_\beta\in H_c^{\beta}(W;\mathbb{Z}_p)$ and $y_{(w+m)-2k(p-1)-\beta}\in H_c^{(w+m)-2k(p-1)-\beta}(M;\mathbb{Z}_p)$. Thus, applying the k–th reduced power P^k on x and using the Cartan formula, we obtain

$$\begin{aligned}
P^k(x) &= \sum_{\beta=0}^{(w+m)-2k(p-1)} P^k[x_\beta\times y_{(w+m)-2k(p-1)-\beta}]\\
&= \sum_{\beta=0}^{(w+m)-2k(p-1)}\left(\sum_{s=0}^{k} P^s(x_\beta)\times P^{k-s}(y_{(w+m)-2k(p-1)-\beta})\right)\\
&= \sum_{s=0}^{k}\left(\sum_{\beta=0}^{(w+m)-2k(p-1)} P^s(x_\beta)\times P^{k-s}(y_{(w+m)-2k(p-1)-\beta})\right).
\end{aligned}\tag{3.5}$$

Now, let us consider $\beta\neq w-2s(p-1)$ for each s fixed, $0\le s\le k$.

If $\beta>w-2s(p-1)$, so $\beta+2s(p-1)>w$ and we have

$$P^s(x_\beta)\in H^{\beta+2s(p-1)}(W;\mathbb{Z}_p)=0. \tag{3.6}$$

If $\beta<w-2s(p-1)$, so $m<w+m-\beta-2s(p-1)$ and we have

$$P^{k-s}(y_{(w+m)-2k(p-1)-\beta})\in H^{(w+m)-\beta-2s(p-1)}(M;\mathbb{Z}_p)=0. \tag{3.7}$$

Therefore, from (3.5), (3.6) and (3.7), using that

$$P^s(x_{w-2s(p-1)})=U_s\smile x_{w-2s(p-1)},$$

$$P^{k-s}(y_{m-2(k-s)(p-1)})=V_{k-s}\smile y_{m-2(k-s)(p-1)},$$

we conclude

$$\begin{aligned}
P^k(x) &= \sum_{s=0}^{k} P^s(x_{w-2s(p-1)})\times P^{k-s}(y_{m-2(k-s)(p-1)})\\
&= \sum_{s=0}^{k}\left[U_s\smile x_{w-2s(p-1)}\right]\times\left[V_{k-s}\smile y_{m-2(k-s)(p-1)}\right].
\end{aligned}$$

On the other hand, using Lemma 3.2 we have

$$
\begin{aligned}
v_k \smile x &= \left[\sum_{s=0}^{k}(-1)^{\Lambda(s)}U_s \times V_{k-s}\right] \smile \left[\sum_{\beta=0}^{(w+m)-2k(p-1)} x_\beta \times y_{(w+m)-2k(p-1)-\beta}\right] \\
&= \sum_{s=0}^{k}\sum_{\beta=0}^{(w+m)-2k(p-1)} [(-1)^{\Lambda(s)}U_s \times V_{k-s}] \smile [x_\beta \times y_{(w+m)-2k(p-1)-\beta}] \\
&= \sum_{s=0}^{k}\sum_{\beta=0}^{(w+m)-2k(p-1)} (-1)^{(\Lambda(s)+\Gamma(s))}[U_s \smile x_\beta] \times [V_{k-s} \smile y_{(w+m)-2k(p-1)-\beta}],
\end{aligned}
$$

where $\Gamma(s) = (k-s)\beta$. Let us consider $\beta \neq w - 2s(p-1)$ for each s fixed, $0 \leq s \leq k$. If $\beta > w - 2s(p-1)$, then

$$U_s \smile x_\beta \in H_c^{\beta+2s(p-1)}(W;\mathbb{Z}_p) = \{0\}.$$

If $\beta < w - 2s(p-1)$, then

$$V_{k-s} \smile y_{(w+m)-2k(p-1)-\beta} \in H_c^{(w+m)-\beta-2s(p-1)}(M;\mathbb{Z}_p) = \{0\}.$$

Note that for $\beta = w - 2s(p-1)$, $\Gamma(s) = \Lambda(s)$ and therefore,

$$
\begin{aligned}
v_k \smile x &= \sum_{s=0}^{k}(-1)^{2\Lambda(s)}[U_s \smile x_{w-2s(p-1)}] \times [V_{k-s} \smile y_{m-2(k-s)(p-1)}] \\
&= \sum_{s=0}^{k}[U_s \smile x_{w-2s(p-1)}] \times [V_{k-s} \smile y_{m-2(k-s)(p-1)}].
\end{aligned}
$$

■

4 Proofs of the main theorems

Now, we denote by $a_1, \ldots, a_r$ a set of representatives of the left lateral classes of G/H, where $r = \frac{|G|}{p}$. We define, for each $i = 1, \ldots, r$, $g_i : X \to X$ by $g_i(x) = a_i x$. Consider the map $F : X \to M^r$ defined by

$$F = (f_1 \times \ldots \times f_r) \circ d,$$

where $d : X \to X^r$ is the diagonal map and $f_i = f \circ g_i$. We prove the following

Lemma 4.1. *If $f^*(V_k) = 0$, for all $k \geq 1$ where the V_k are the Wu classes of M then the homomorphism F^* induced by F is such that $F^*(v_k) = 0$ for all $k \geqslant 1$, where v_k are the Wu classes of M^r,* *Proof.* It suffices to show that $(f_1 \times \cdots \times f_r)^*(v_k) = 0$, for $k \geqslant 1$. The proof will be done by induction on r. If $r = 1$, then $F = f_1$ and $f_1{}^*(V_k) = g_1{}^* \circ f^*(V_k) = 0$. Now, let us denote by

$$p_1 : M^{r-1} \times M \to M^{r-1},\ p_2 : M^{r-1} \times M \to M,$$

$$q_1 : X^{r-1} \times X \to X^{r-1},\ q_2 : X^{r-1} \times X \to X$$

the natural projections.

If $r = 2$, we have $v_k = \sum_{s=0}^{k}(-1)^{\Lambda(s)} V_s \times V_{k-s}$, then

$$(f_1 \times f_2)^*(v_k) = \sum_{s=0}^{k}(-1)^{\Lambda(s)}(f_1 \times f_2)^*(V_s \times V_{k-s}).$$

Since $V_s \times V_{k-s} = p_1{}^*(V_s) \smile p_2{}^*(V_{k-s})$, it follows that

$$\begin{aligned}(f_1 \times f_2)^*(v_k) &= \sum_{s=0}^{k}(-1)^{\Lambda(s)}(f_1 \times f_2)^*(p_1{}^*(V_s)) \smile (f_1 \times f_2)^*(p_2{}^*(V_{k-s})) \\ &= \sum_{s=0}^{k}(-1)^{\Lambda(s)} q_1{}^* \circ f_1{}^*(V_s) \smile q_2{}^* \circ f_2{}^*(V_{k-s}) = 0.\end{aligned}$$

If $r > 2$, we have

$$\begin{aligned}(f_1 \times \cdots \times f_{r-1}) \circ q_1 &= p_1 \circ (f_1 \times \cdots \times f_r) \\ f_r \circ q_2 &= p_2 \circ (f_1 \times \cdots \times f_r).\end{aligned}$$

Since, by Lemma 3.1, $v_k = \sum_{s=0}^{k}(-1)^{\Lambda(s)} U_s \times V_{k-s}$ and assuming inductively that $(f_1 \times \cdots \times f_{r-1})^*(U_s) = 0$, we conclude that

$$(f_1 \times \cdots \times f_r)^*(v_k) = (f_1 \times \cdots \times f_r)^*(\sum_{s=0}^{k}(-1)^{\Lambda(s)} U_s \times V_{k-s}) =$$

$$= \sum_{s=0}^{k}(-1)^{\Lambda(s)}(f_1 \times \cdots \times f_r)^*(p_1{}^*(U_s)) \smile (f_1 \times \cdots \times f_r)^*(p_2{}^*(V_{k-s})) =$$

$$= \sum_{s=0}^{k}(-1)^{\Lambda(s)} q_1{}^* \circ (f_1 \times \cdots \times f_{r-1})^*(U_s) \smile q_2{}^* \circ f_r{}^*(V_{k-s}) = 0.$$

■

4.1 Proof of Theorem 1.1 and its consequences

In this section, we present the proofs of Theorems 1.1 and its consequences, as follows.

Proof of Theorem 1.1. We consider the map F defined previously. We have

$$A(f,H,G) \supset A_F = \{x \in X : F(x) = F(hx), \forall h \in H\}.$$

In fact, let x be a point in the set A_F, then

$$(f(a_1x),\ldots,f(a_rx)) = (f(a_1hx),\ldots,f(a_rhx)),$$

for all $h \in H$. Thus, $f(a_ix) = f(a_ihx)$, for all $h \in H$ and $i = 1,\ldots,r$. According to the definition of the action of H on the orbit Gx, $ha_ix := a_ihx$, for $i = 1,\ldots,r$. Thus, f collapses each orbit determined by the action of H on a_ix, for $i = 1,\ldots,r$, therefore $x \in A(f,H,G)$.

Now we observe that $H \cong \mathbb{Z}_p$ acts freely on X by restriction and by hypothesis $\operatorname{ind} X \geq n > n-(p-1)rm$. By Lemma 4.1, $F^*(v_k) = 0$, for all $k \geq 1$, where v_k are the Wu classes of M^r. Thus, according to [12, Theorem 1.2]

$$\operatorname{ind} A_F \geq n-(p-1)rm = n-(|G|-r)m.$$

Let us consider the inclusion $i : A_F \to A(f,H,G)$, which is an equivariant map, and so it induces $\bar{i} : [A_F]^* \to [A(f,H,G)]^*$ a map between the orbit spaces. Therefore, if $c : [A(f,H,G)]^* \to B\mathbb{Z}_p$ is any classifying map, we have that $c \circ \bar{i} : [A_F]^* \to B\mathbb{Z}_p$ is a classifying map. Thus,

$$\operatorname{ind} A(f,H,G) \geq \operatorname{ind} A_F \geq n-(|G|-r)m,$$

■

Corollary 4.2. *Let X be a paracompact space and let G be a finite group acting freely on X. Let M be a orientable m-manifold, and p a prime number that divide $|G|$. Suppose that $indX \geq n > (|G|-r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$ such that $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that*

$$\text{cohom.dim}\, A(f,H,G) \geq n-(|G|-r)m.$$

Proof. Let p be a prime number such that divide $|G|$. By Cauchy Theorem, there is a cyclic of order p subgroup H of G . Then, we apply Theorem 1.1.

■

Remark 4.3. Let us observe that, if $f^* : H^i(M;\mathbb{Z}_p) \to H^i(X;\mathbb{Z}_p)$ is trivial, for $i \geq 1$, and p is the smallest prime number dividing $|G|$, then $r = \frac{|G|}{p} \geq \frac{|G|}{q}$, where q can be any other prime number dividing $|G|$. Thus, $n > (|G| - \frac{|G|}{q})m$, therefore for each prime number q dividing $|G|$, there exists a cyclic subgroup of order q, H_q of G such that $\operatorname{ind} A(f,H_q,G) \geq n-(|G|-r)m$.

The following theorem is a version for manifolds of the main result in [5]

Theorem 4.4. *Let G be a finite group which acts freely on n-sphere S^n and let H be a cyclic subgroup of G of prime order p. Let $f : S^n \to M$ be a continous map where M be a m-manifold (orientable if $p > 2$). If $n > (|G| - r)m$ where $r = \frac{|G|}{p}$, then*

$$\text{cohom.dim}(A(f, H, G)) \geq n - (|G| - r)m.$$

Proof. Since $n > (|G| - r)m \geq m$, $f^*(V_k) = 0$, for all $k \geq 1$. Moreover, ind $S^n = n$ and thus we apply the Theorem 1.1. ■

4.2 Proof of Theorem 1.2

Now, let us consider X a compact Hausdorff space and an essential map $\varphi : X \to S^n$. Suppose G be a finite group de order s which acts freely on S^n and H be a subgroup of order p of G. Let $G = \{g_1, ..., g_s\}$ be a fixed enumeration of elements of G, where g_1 is the identity of G. A nonempty space X_φ can be associated with the essential map $\varphi : X \to S^n$ as follows:

$$X_\varphi = \{(x_1, ..., x_s) \in X^s : g_i\varphi(x_1) = \varphi(x_i), i = 1, ..., s\},$$

where X^s denotes the s-fold cartesian product of X. The set X_φ is a closed subset of X^s and so it is compact. We define a G-action on X_φ as follows: for each $g_i \in G$ and for each $(x_1, ..., x_s) \in X_\varphi$,

$$g_i(x_1, ..., x_s) = (x_{\sigma_{g_i}(1)}, ..., x_{\sigma_{g_i}(s)}),$$

where the permutation σ_{g_i}, is defined by $\sigma_{g_i}(k) = j$, $g_k g_i = g_j$. We observe that if $x = (x_1, ..., x_s) \in X_\varphi$ then $x_i \neq x_j$, for any $i \neq j$ and therefore G acts freely on X_φ.

Let us consider a continuous map $f : X \to M$, where M is a topological space and $\tilde{f} : X_\varphi \to M$ given by $\tilde{f}(x_1, ..., x_s) = f(x_1)$,

Definition 4.5. *The set $A_\varphi(f, H, G)$ of (H, G)-coincidence points of f relative to φ is defined by*

$$A_\varphi(f, H, G) = A(\tilde{f}, H, G).$$

Proof of Theorem 1.2. Let $\tilde{f} : X_\varphi \to M$ given by $\tilde{f}(x_1, ..., x_r) = f(x_1)$, that is, $\tilde{f} = f \circ \pi_1$, where π_1 is the natural projection on the 1-th coordinate. By hypothesis, $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, then we have $\tilde{f}^*(V_k) = 0$, for all $k \geq 1$. Moreover, the $\mathbb{Z}_p$-index of X_φ is equal to n by [8] Theorem 3.1. In this way, X_φ and $\tilde{f}$ satisfy the hypothesis of Theorem 1.1 which implies that the $\mathbb{Z}_p$-index of the set $A(\tilde{f}, H, G)$ is greater than or equal to $n - (|G| - r)m$. By definition, $A_\varphi(f, H, G) = A(\tilde{f}, H, G)$, and then

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m.$$

■

By a similar argument to that used in the proof of Corollary 4.2 we have the following corollary of Theorem 1.2

Corollary 4.6. *Let X be a compact Hausdorff space and let G be a finite group acting freely on S^n. Let M be a orientable m-manifold and p a prime number dividing $|G|$. Suppose that $n > (|G| - r)m$, where $r = \frac{|G|}{p}$. Then, for a continuous map $f : X \to M$, with $f^*(V_k) = 0$, for all $k \geq 1$, where V_k are the Wu classes of M, there exists a non-trivial subgroup H of G, such that*

$$\text{cohom.dim}\, A_\varphi(f, H, G) \geq n - (|G| - r)m.$$

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References

[1] Borsuk, K., *Drei Sätze über die n-dimensionale euklidische Sphäre*, Fund. Math. **20** 177–190, 1933.

[2] Bredon, G., *Sheaf Theory*, Graduate Texts in Mathematics, 170, Springer, New York, (1997).

[3] Conner, P.E.; Floyd E.E. *Differentiable Periodic Maps*. Ergebnisse der Math. Springer-Verlag Berlin **33**, 1964.

[4] Gonçalves, D.L.; Pergher, Pedro L.Q., *$\mathbb{Z}_p$-coincidence for maps of spheres into CW-complexes*, Kobe J. Math. **15**, (1998), 191–195.

[5] Gonçalves, D.L.; Jaworowski, J.; Pergher, Pedro L.Q., *Measuring the size of the coincidence set*, Topology Appl. **125** (2002), 465–470.

[6] Gonçalves, D.L.; Jaworowski, J.; Pergher, Pedro L.Q., A.Yu. Volovikov, *Coincidences for maps of spaces with finite group actions*. Topology Appl. 145 (2004), no. 1-3, 61–68.

[7] Izydorek, M.,; Jaworowski, J., *Antipodal coincidence for maps of spheres into complexes*, Proc. Amer. Math. Soc. 123 (6) (1995) 1947-1950.

[8] De Mattos, Denise; Dos Santos, Edivaldo L., *On nonsymmetric theorems for (H, G)-coincidences*. Topological Methods in Nonlinear Analysis, v. 33, (2009) 105-120.

[9] Munkholm, H.J., *Borsuk-Ulam theorem for proper $\mathbb{Z}_p$-actions on (mod p homology) n-spheres*. Math. Scan. **24** (1969) 167-185.

[10] Nakaoka, M., *Generalization of Borsuk-Ulam theorem*. Osaka J. Math. **7** (1970) 423-441.

[11] Steenrod, N. E., *Cohomology operations* , Princeton Univ. Press, Princeton,N.J. (1962).

[12] Volovikov, A. Yu., *Mappings of free Zp-spaces into manifolds*, Izv. Akad. Nauk SSRR Ser. Math., (46) n.1, (1992) 36–55.

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