RT-MAT 99-15

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Maio 1999

Esta é uma publicação preliminar ("preprint").

A NORMAL COMPLEMENT FOR AN RA LOOP IN ITS INTEGRAL LOOP RING

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ABSTRACT. We show that an RA loop has a torsion-free normal complement in the loop of normalized units of its integral loop ring.

1. INTRODUCTION

Suppose L is a loop (perhaps a group) which has an alternative loop ring over the ring $\mathbb{Z}$ of rational integers. The loop of units (invertible elements) in $\mathbb{Z}L$ is a Moufang loop which contains L and it is of interest to see how L sits inside $\mathcal{U}(\mathbb{Z}L)$. If there exists a normal subloop $\mathcal{N}$ of $\mathcal{U}(\mathbb{Z}L)$ such that $L \cap \mathcal{U}(\mathbb{Z}L) = \{1\}$ and $\mathcal{U}(\mathbb{Z}L) = \pm L\mathcal{N}$, then $\mathcal{N}$ is called a *normal complement* of L. The search for a normal complement which is torsion-free (no nonidentity elements of finite order) is of great interest in group rings since the existence of a torsion-free normal complement implies a positive solution to the isomorphism problem: $\mathbb{Z}G \cong \mathbb{Z}H$ implies $G \cong H$. (See Theorem 2.5 in this paper.) On the other hand, the two problems are independent since, for example, the answer is "true" to the isomorphism problem for metabelian groups [Whi68] but "not true" to the existence of a torsion-free normal complement [RS83].

In this paper, we show that in the alternative loop ring of a Moufang loop L which is not associative, L indeed has a torsion-free normal complement. This has long been suspected and indeed has been established in the case that L/L' has exponent at most 6 [JL93], [GJM96, Proposition XII.4.1].

We now sketch briefly those facts about Moufang loops, alternative rings and loop rings which are required by this paper. Virtually all proofs can be found in the monograph [GJM96], but we endeavour also to cite original sources as much as possible.

An *alternative ring* is a ring which satisfies the identities $(yx)x = yx^2$ and $(xy)y = xy^2$ and a Moufang loop is a loop which satisfies the identity

$$(xy \cdot z)y = x(yz \cdot y). \tag{1.1}$$

1991 *Mathematics Subject Classification.* Primary 20N05; Secondary 17D05, 16S34, 16U60.

The first author extends most sincere thanks to the Instituto de Matemática e Estatística of the Universidade de São Paulo for its customary hospitality and congenial working environment and to FAPESP, Brasil, for its support.

This research was supported by the Natural Sciences and Engineering Research Council of Canada, Grant No. OGP0009087, and by FAPESP and CNPq of Brasil, Proc. 1998/13127-7 and Proc. 300243/79-O(RN) respectively.

March 26, 1999.

Note that any group is a Moufang loop and any associative ring is alternative and the converse is "nearly" true. Alternative rings and Moufang loops are *diassociative*: the subring (or subloop) generated by any pair of elements is associative. In fact, if three elements in an alternative ring (or Moufang loop) associate, then they generate an associative ring (or a group).

A *torsion element* in a loop is an element of finite order. In an RA loop, the set of torsion elements forms a subloop which is clearly normal. It is locally finite and, if L is finitely generated, actually finite [GJM96, Lemma VIII.4.1], [GM95, Lemma 2.1]. A loop is *torsion* if every element is a torsion element and *torsion-free* if it has no nonidentity elements of finite order.

The set of *units* (invertible elements) of an alternative ring with 1 is a Moufang loop. In an integral loop ring $\mathbf{Z}L$, the elements $\pm\ell$, $\ell \in L$, are clearly units. These are known as *trivial units*. A classical theorem of Higman says that if G is a torsion group, then the only units of $\mathbf{Z}G$ are trivial if and only if G is an abelian group of exponent 1, 2, 3, 4 or 6, or a hamiltonian 2-group [Hig40]. This theorem in fact can be generalized to torsion loops L for which $\mathbf{Z}L$ is an alternative ring [GJM96, Theorem VIII.3.2], [GP86].

Of significance in this paper is the alternative *vector matrix algebra* $\mathfrak{Z}(F)$ over a field F of Max Zorn and its loop of units, the *general linear loop* $\mathrm{GLL}(2,F)$. The elements of $\mathfrak{Z}(F)$ are 2×2 matrices of the form $\left[\begin{smallmatrix} a & \mathrm{x} \\ \mathrm{y} & b \end{smallmatrix}\right]$, where $a, b \in F$, $\mathrm{x}, \mathrm{y} \in F^3$. Such matrices are added in the obvious way, but multiplied according to the following variation of the usual rule,

$$\begin{bmatrix} a_1 & \mathrm{x}_1 \\ \mathrm{y}_1 & b_1 \end{bmatrix} \begin{bmatrix} a_2 & \mathrm{x}_2 \\ \mathrm{y}_2 & b_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + \mathrm{x}_1 \cdot \mathrm{y}_2 & a_1\mathrm{x}_2 + b_2\mathrm{x}_1 - \mathrm{y}_1 \times \mathrm{y}_2 \\ a_2\mathrm{y}_1 + b_1\mathrm{y}_2 + \mathrm{x}_1 \times \mathrm{x}_2 & b_1b_2 + \mathrm{y}_1 \cdot \mathrm{x}_2 \end{bmatrix},$$

where $\cdot$ and $\times$ denote the dot and cross products respectively in F^3.

Zorn's algebra comes with a determinant function, $\left[\begin{smallmatrix} a & \mathrm{x} \\ \mathrm{y} & b \end{smallmatrix}\right] \mapsto ab - \mathrm{x}\cdot\mathrm{y}$, and the units of $\mathfrak{Z}(F)$ are precisely those matrices whose determinant is nonzero. These units form a loop which is denoted $\mathrm{GLL}(2,F)$ and called the *general linear loop*. It was first explored by Paige [Pai56] who showed that the centre of $\mathrm{GLL}(2,F)$ is $\pm\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and that, modulo its centre, $\mathrm{GLL}(2,F)$ is simple (and not associative). This result was proven again more recently, by different means, by Polcino Milies and Merlini Giuliani [MG].

Let A denote a quaternion algebra over a field F of characteristic different from 2. Let α be a nonzero element of F and u an element not in A. Let $C = A + Au$ with obvious addition, but multiplication defined by

$$(a + bu)(c + du) = (ad + \alpha d^* b) + (da + bc^*)u \tag{1.2}$$

for $a, b, c, d \in A$, $q \mapsto q^*$ denoting any involution in A. The algebra C is called a *Cayley-Dickson algebra*. Such an algebra is alternative, but not associative; moreover, it is known that every such algebra is either a division algebra or isomorphic to Zorn's vector matrix algebra over F [GJM96, Corollary I.4.17], [ZSSS82, Theorem 2.4.7].

Not all Moufang loops determine alternative loop rings. Those that do (independent of the characteristic of the coefficient ring) are called RA loops. The basic properties of RA loops themselves can be found in Section II.5.2 and Chapter IV of [GJM96]. See also [CG86] and [GP87].

Let G be a nonabelian group with commutator subgroup $G' = \{1, s\}$ of order 2, centre $\mathcal{Z}(G)$ and $G/\mathcal{Z}(G) \cong C_2 \times C_2$, the Klein 4-group. Defining

$$g^* = \begin{cases} g & g \in \mathcal{Z}(G) \\ sg & g \notin \mathcal{Z}(G) \end{cases} \tag{1.3}$$

one can show that $g \mapsto g^*$ is an involution of G (an antiautomorphism of order 2). Let $L = G \cup Gu$ for some indeterminate u and extend the multiplication from G to L by the rules

$$\begin{aligned} g(hu) &= (hg)u \\ (gu)h &= (gh^*)u \\ (gu)(hu) &= g_0 h^* g \end{aligned} \tag{1.4}$$

where $u^2 = g_0 \in \mathcal{Z}(G)$. Then L is an RA loop which we denote $M(G, *, g_0)$. Of fundamental importance is the fact that a loop is RA if and only if it is a loop of the form $M(G, *, g_0)$ [GJM96, Theorem IV.3.1, Theorem III.3.3, Proposition III.3.6]. See also [CG86] and the introduction to [GP87]. We also note that if $L = M(G, *, g_0)$ is RA, then $ZL = ZG$ and $\ell^2 \in \mathcal{Z}(L)$ for all $\ell \in \mathcal{Z}(L)$. Moreover, the unique nonidentity commutator, s, of G is a unique nonidentity commutator and a unique nonidentity associator of L.

2. Main Results

The notion of *augmentation* is important for us. Let R be a commutative and associative ring with 1. Let A be a normal subloop of an RA loop (or a group) L and let $\epsilon_A \colon RL \to R[L/A]$ denote the linear extension to RL of the natural homomorphism $L \to L/A$. This map is a ring homomorphism whose kernel is the ideal

$$\Delta(L, A) = \{\sum_{a \in A} \alpha_a (a - 1) \mid \alpha_a \in RL\}.$$

Since ϵ_A is surjective, we have $R[L/A] \cong RL/\Delta(L, A)$. In the special case $A = L$, we write $\epsilon = \epsilon_L$, calling this the *augmentation map* on RL and we call $\epsilon(\alpha)$ the *augmentation* of $\alpha \in RL$. For $\alpha = \sum \alpha_\ell \ell \in RL$, note that $\epsilon(\alpha) = \sum \alpha_\ell$. We define $\Delta(L) = \Delta(L, L)$ and call $\Delta(L)$ the *augmentation ideal* of L. The identity $\ell_1(\ell_2 - 1) = (\ell_1 \ell_2 - 1) - (\ell_1 - 1)$ shows that

$$\Delta(L) = \{\sum_{\ell \in L} \alpha_\ell (\ell - 1),\ \alpha_\ell \in R\}.$$

If A is a normal subloop of L, note that $\Delta(L, A) = (RL)\Delta(A)$. We refer the reader to Section VI.1 of [GJM96] where the ideas of this paragraph are explained in more detail.

Lemma 2.1. *Let A be a normal subloop of an RA loop L and let $\delta \in \Delta(L, A)$. Then there exists $a \in A$ such that $\delta \equiv a - 1 \pmod{\Delta(L)\Delta(A)}$.*

Proof. The identity

$$ab - 1 = (a - 1)(b - 1) + (a - 1) + (b - 1)$$

and its consequence

$$a^{-1} - 1 = -(a - 1) - (a - 1)(a^{-1} - 1)$$

imply that $a^n - 1 \equiv n(a-1) \pmod{\Delta(A)^2}$ for any $a \in A$ and $n \in \mathbb{Z}$. Thus, for any integers $n_1, n_2, \dots, n_k$ and any $a_1, a_2, \dots, a_k \in A$,

$$\sum_{i=1}^{k} n_i(a_i - 1) \equiv \Big(\prod_{i=1}^{k} a_i^{n_i}\Big) - 1 \pmod{\Delta(A)^2}. \tag{2.1}$$

Let $\delta \in \Delta(L, A)$ and write $\delta = \sum_{i,j} \delta_{ij}\ell_i(a_j - 1)$, $\delta_{ij} \in \mathbb{Z}$, $\ell_i \in L$, $a_j \in A$. Since

$$\ell(a-1) = (\ell-1)(a-1) + (a-1) \equiv a - 1 \pmod{\Delta(L)\Delta(A)},$$

we have $\delta \equiv \sum_j \delta_j(a_j - 1) \pmod{\Delta(L)\Delta(A)}$, $\delta_j = \sum_i \delta_{ij}$, and so $\delta \equiv a - 1 \pmod{\Delta(L)\Delta(A)}$, $a = \prod a_j^{\delta_j}$, by (2.1). □

At this point, we require the concept of the "support" of an element in a loop ring. If $\alpha = \sum_{\ell \in L} \alpha_\ell \ell$, $\alpha_\ell \in R$, is in a loop ring RL, the *support* of α is the set

$$\operatorname{supp}(\alpha) = \{\ell \in L \mid \alpha_\ell \neq 0\}.$$

Lemma 2.2. *Let L be a finite RA loop with centre $\mathcal{Z}$. Then $L \cap (1 + \Delta(L, \mathcal{Z})) = \mathcal{Z}$.*

Proof. Since $a = 1 + (a-1)$ and $a - 1 \in \Delta(L, \mathcal{Z})$, certainly $\mathcal{Z} \subseteq 1 + \Delta(L, \mathcal{Z})$. For the other inclusion, we first write $L = \bigcup_{q \in \mathcal{Q}} \mathcal{Z}q$ as the disjoint union of cosets of $\mathcal{Z}$. Without loss of generality, we may assume that 1 is in the *transversal* $\mathcal{Q}$. Suppose $\delta \in \Delta(L, \mathcal{Z})$. Then δ is an integral linear combination of terms of the form $\ell(a-1)$, $\ell \in L$, $a \in \mathcal{Z}$. Writing $\ell = qb$, with $q \in \mathcal{Q}$ and $b \in \mathcal{Z}$, we notice that

$$\ell(a-1) = qb(a-1) = q(ba-1) - q(b-1).$$

(In this calculation, we may associate freely since elements of $\mathcal{Z}$ associate with all other elements of $\mathbb{Z}L$.) It follows that $\delta = \sum_{i,j} \delta_{ij} q_i(a_j - 1)$, $\delta_{ij} \in \mathbb{Z}$, $q_i \in \mathcal{Q}$ and $a_j \in \mathcal{Z}$. Now suppose $\ell \in 1 + \Delta(L, \mathcal{Z})$. Then $\ell = 1 + \sum_{i,j} \delta_{ij} q_i(a_j - 1)$ and, with the understanding that certain δ_{ij} may be 0, we may assume in the summation that every $a_j \in \mathcal{Z}$ and every $q_i \in \mathcal{Q}$ appears. Moreover, we assume that $q_1 = a_1 = 1$. Thus

$$\ell = 1 + \sum_i \sum_{j \neq 1} \delta_{ij} q_i(a_j - 1) = 1 + \sum_i \sum_{j \neq 1} \delta_{ij} q_i a_j - \sum_i \big(\textstyle\sum_{j \neq 1} \delta_{ij}\big) q_i. \tag{2.2}$$

If $\ell = 1$, then $\ell \in \mathcal{Z}$ as desired. If $\ell \neq 1$, then 1 is not in the support of the right hand side of (2.2). Since $q_1 = 1$, no $q_i a_j = 1$ with $j \neq 1$. Thus $\sum_{j \neq 1} \delta_{1j} = 1$; in particular, $\delta_{1j_0} \neq 0$ for some j_0. It follows that a_{j_0} is in the support of the right hand side and hence in the support of the left. Thus $\ell = a_{j_0} \in \mathcal{Z}$. □

Corollary 2.3. *Let L be a finite RA loop with centre $\mathcal{Z}$. Then $L \cap (1 + \Delta(L)\Delta(\mathcal{Z})) = \{1\}$.*

Proof. As in the proof of Lemma 2.2, any $\delta \in \Delta(L, \mathcal{Z})$ can be written

$$\delta = \sum_{i,j} \delta_{ij} q_i(a_j - 1), \tag{2.3}$$

$a_j \in \mathcal{Z}$, $\delta_{ij} \in \mathbb{Z}$ and $q_i \in \mathcal{Q}$, a transversal of $\mathcal{Z}$ containing 1. Since $q_i a_j = q_r a_s$ implies $\mathcal{Z}q_i = \mathcal{Z}q_r$ and hence $q_i = q_r$, $a_j = a_s$, it is easy to see that the coefficients δ_{ij} in (2.3) are unique. Thus the map $\varphi\colon \Delta(L,\mathcal{Z}) \to \mathcal{Z}$ defined by

$$\sum_{i,j} \delta_{ij} q_i(a_j - 1) \mapsto \prod_j a_j^{\delta_j}, \quad \delta_j = \textstyle\sum_i \delta_{ij},$$

is well-defined and a homomorphism from the abelian group $(\Delta(L,\mathcal{Z}),+)$ to $\mathcal{Z}$. Under φ, the element

$$(qb-1)(a-1) = q(ba-1) - q(b-1) - (a-1) \tag{2.4}$$

maps to $(ba)b^{-1}a^{-1} = 1$; thus $\Delta(L)\Delta(\mathcal{Z}) \subseteq \ker\varphi$.

Now suppose $\ell \in 1 + \Delta(L)\Delta(\mathcal{Z})$ and write $\ell = 1 + \delta$, $\delta \in \Delta(L)\Delta(\mathcal{Z})$. We have $\varphi(\delta) = 1$. On the other hand, since $\Delta(L)\Delta(\mathcal{Z}) \subseteq \Delta(L,\mathcal{Z})$, $\ell \in \mathcal{Z}$ by Lemma 2.2 so, by definition of φ, $\varphi(\delta) = \varphi(\ell - 1) = \ell$. It follows that $\ell = 1$. □

Since the augmentation map $\epsilon\colon \mathbf{Z}L \to \mathbf{Z}$ is a homomorphism, if $\mu \in \mathbf{Z}L$ is a unit, $\epsilon(\mu) = \pm 1$. We call μ *normalized* if $\epsilon(\mu) = +1$. The set $\mathcal{U}_1$ of normalized units in $\mathbf{Z}L$ is a loop (containing L) and that $\mathcal{U}(\mathbf{Z}L) = \pm\mathcal{U}_1$. It follows that if $\mathcal{N}$ is a normal complement for L in $\mathcal{U}_1$, then $\mathcal{N}$ is a normal complement for $\pm L$ in the full unit loop.

Theorem 2.4. *Let L be a finite RA loop with centre $\mathcal{Z}$. Then $\mathcal{N} = (1+\Delta(L)\Delta(\mathcal{Z}))\cap\mathcal{U}(\mathbf{Z}L)$ is a torsion-free normal complement for L in the loop $\mathcal{U}_1$ of normalized units in $\mathbf{Z}L$.*

Proof. The set $\Delta(L)\Delta(\mathcal{Z})$ is an ideal of $\mathbf{Z}L$ because $\Delta(L)$ is an ideal and $\Delta(\mathcal{Z})$ is central. This implies that $\mathcal{N}$ is a normal subloop of $\mathcal{U}_1$. By Corollary 2.3, it remains only to prove that $L\mathcal{N} = \mathcal{U}_1$ and that $\mathcal{N}$ is torsion-free.

Let $\epsilon\colon \mathbf{Z}L \to \mathbf{Z}[L/\mathcal{Z}]$ denote the ring homomorphism which is the linear extension to $\mathbf{Z}L$ of the natural loop homomorphism $L \to L/\mathcal{Z}$ which maps ℓ to $\mathcal{Z}\ell = \overline{\ell}$. Let $\overline{\alpha}$ denote the image in $\mathbf{Z}[L/\mathcal{Z}]$ of $\alpha \in \mathbf{Z}L$ under the homomorphism $\epsilon_{\mathcal{Z}}$. Let $\mu \in \mathcal{U}_1$. Since $L/\mathcal{Z} \cong C_2 \times C_2$, the units in $\mathbf{Z}[L/\mathcal{Z}]$ are trivial by Higman's Theorem. Thus the image $\overline{\mu}$ of μ in $\mathbf{Z}L/\Delta(L,\mathcal{Z}) \cong \mathbf{Z}[L/\mathcal{Z}]$ is an element of the form $\pm\overline{\ell}$, $\ell \in L$, and, in fact, $+\overline{\ell}$ since the augmentation of $\overline{\mu}$ is $+1$. So $\mu - \ell \in \ker\epsilon_{\mathcal{Z}} = \Delta(L,\mathcal{Z})$ and we have $\mu = \ell(1+\delta)$ for some $\delta \in \Delta(L,\mathcal{Z})$. By Lemma 2.1, $\delta \equiv a - 1 \pmod{\Delta(L)\Delta(\mathcal{Z})}$ for some $a \in \mathcal{Z}$, so, for some $\delta_1, \delta_2 \in \Delta(L)\Delta(\mathcal{Z})$, we have $\mu = \ell(1 + a - 1 + \delta_1) = \ell(a + \delta_1) = \ell a^{-1}(1+\delta_2) \in L\mathcal{N}$. Thus $L\mathcal{N} = \mathcal{U}_1$.

We now show that $\mathcal{N}$ is torsion-free. Because a torsion unit with a central element in its support is necessarily an element of L [GJM96, Corollary VIII.1.2], [GM89, Corollary 2.2] and in view of Corollary 2.3, it suffices to show that each element of $1 + \Delta(L)\Delta(\mathcal{Z})$ has a central element in its support. Let then $\alpha \in 1 + \Delta(L)\Delta(\mathcal{Z})$. Using (2.4), we may write

$$\alpha = 1 + \sum_{q\in\mathcal{Q},a\in\mathcal{Z}} \alpha_{qa} q(a-1) + \sum_{b\in\mathcal{Z}} \beta_b(b-1), \tag{2.5}$$

$\alpha_{qa}, \beta_b \in \mathbf{Z}$, $a, b \in \mathcal{Z}$ and $\mathcal{Q}$ a transversal of $\mathcal{Z}$ in L containing 1. In the right hand side of (2.5), a fixed $a_0 \in \mathcal{Z}$, $a_0 \neq 1$, has coefficient $\alpha_{1a_0} + \beta_{a_0}$. Thus, if $\alpha_{1a_0} + \beta_{a_0} \neq 0$, then a_0 is in the support of α and we have the desired result. On the other hand, if $\alpha_{1a} + \beta_a = 0$ for all $a \in \mathcal{Z}$, then the coefficient of 1 on the right side of (2.5) is

$$1 - \sum_{a\in\mathcal{Z}} \alpha_{1a} - \sum_k \beta_{a\in\mathcal{Z}} = 1 \neq 0$$

so 1 is in the support of α. □

The original proof of the following "isomorphism theorem" appeared in [GM89]. See also [GJM96, Theorem IX.1.1].

Theorem 2.5. *Let L and L_1 be finite RA loops with L_1 torsion and suppose that $\mathbb{Z}L_1 \cong \mathbb{Z}L$. Then $L_1 \cong L$.*

Proof. Note first that L and L_1 have the same order, since each is the rank of the same free $\mathbb{Z}$-module. Suppose $\varphi\colon \mathbb{Z}L_1 \to \mathbb{Z}L$ is the given isomorphism and let $\mathcal{N}$ be a torsion-free normal complement for L_1 in $\mathcal{U}_1(\mathbb{Z}L_1)$. Then $\varphi(\mathcal{N})$ is torsion-free in $\mathcal{U}_1(\mathbb{Z}L)$ and so $L \cap \varphi(\mathcal{N}) = \{1\}$. Since $[\mathcal{U}_1(\mathbb{Z}L)\colon \varphi(\mathcal{N})] = |L_1| = |L| = [L\varphi(\mathcal{N})\colon \varphi(\mathcal{N})]$, we have $\mathcal{U}_1(\mathbb{Z}L) = L\varphi(\mathcal{N})$. Thus

$$L_1 \cong \frac{\mathcal{U}_1(\mathbb{Z}L_1)}{\mathcal{N}} \cong \frac{\mathcal{U}_1(\mathbb{Z}L)}{\varphi(\mathcal{N}} \cong \frac{L\varphi N}{\varphi N} \cong \frac{L}{L \cap \varphi N} = L. \quad \square$$

3. Related Questions

In view of Theorem 2.4, it is natural to ask if L can ever be a direct factor of $\mathcal{U}_1$. As we shall see, with L finite, this happens only when $\mathcal{U}_1 = L$.

Theorem 3.1. *Let L be a finite RA loop. Then L is normal in $\mathcal{U}(\mathbb{Z}L)$ if and only if $\mathcal{U}(\mathbb{Z}L)$ is itself an RA loop and this occurs if and only if $\mathcal{U}(\mathbb{Z}L) = \pm L$.*

Proof. If $\mathcal{U}(\mathbb{Z}L)$ is an RA loop, then the torsion units form a subloop of $\mathbb{Z}L$, so the loop T of torsion elements in L is either abelian group or a hamiltonian Moufang 2-loop [GJM96, Corollary XII.2.14], [GM95, Theorem 3.1]. Here, $T = L$ is a hamiltonian Moufang 2-loop. In this case, the generalization of Higman's Theorem to alternative loop rings says $\mathcal{U}(\mathbb{Z}L) = \pm L$, so L is certainly normal in $\mathcal{U}(\mathbb{Z}L)$.

Conversely, assume that L is normal in its unit loop. If $\nu \in \mathcal{U}(\mathbb{Z}L)$ and $\ell \in L$, then $\nu^{-1}\ell\nu$ is an element of the finite set L. It follows that each $\mu = \sum_{\ell \in L} \mu_\ell \ell \in \mathcal{U}(\mathbb{Z}L)$ has just finitely many conjugates of the form $\nu^{-1}\mu\nu$. Such a loop is called *FC* and it is known that if $\mathcal{U}(\mathbb{Z}L)$ is FC, then $\mathcal{U}(\mathbb{Z}L)$ is RA [GJM96, Corollary 2.14], [GM95, Theorem 3.3]. □

Remark 3.2. The condition that $\mathcal{U}(\mathbb{Z}L)$ be RA is equivalent to many other conditions on this unit loop, including nilpotency and the n-Engel and FC properties [GJM96, Corollary XII.2.14].

Example 3.3. There do indeed exist RA loops L with $\mathcal{U}(\mathbb{Z}L) \neq \pm L$ also RA. By Corollary XII.2.14 of [GJM96] (see also [GM95, Theorem 3.3], it is sufficient to construct an RA loop with a torsion subloop T which is an abelian group such that if $x \in L$ does not centralize T, then $x^{-1}tx = t^{-1}$ for all $t \in T$. To construct such an L, let $A = \langle s \rangle \times \langle b \rangle$, $s^2 = 1$ be the direct product of a cyclic group of order 2 and an infinite cyclic group $\langle b \rangle$. Let G be the group generated by A and elements x, y subject to the relations

$$ax = xa,\ ay = ya \text{ for } a \in A,\ x^2 = s,\ y^2 = b, (x, y) = s.$$

Then $G' = \{1, s\}$, $\mathcal{Z} = A$ and $G/\mathcal{Z} = \langle x \rangle \times \langle y \rangle \cong C_2 \times C_2$. The loop $M(G, *, b) = G \cup Gu$, where $u^2 = b$, is RA with torsion subloop $T = \langle s, x \rangle$ and $y^{-1}x^{-1}y = (y, x)x^{-1} = sx^{-1} = x$.

As we now show (Theorems 3.4 and 3.6), in rather dramatic contrast to Theorem 3.1, L is never normal in the unit loop of a loop algebra over a field F and $\mathcal{U}(FL)$ is never RA.

Theorem 3.4. *Let L be a finite RA loop and let F be a field. Then L is not normal in $\mathcal{U}(FL)$.*

Proof. Assume that L is indeed normal in $\mathcal{U}(FL)$. As in the proof of Theorem 3.1, $\mathcal{U}(FL)$ must be an FC loop, so F is finite [GJM96, Corollary XII.3.5], [GM96a], say of characteristic $p > 0$. Since L is finite, we can write $L = L_2 \times L_{2'}$ as the direct product of an RA 2-loop L_2 and an abelian group $L_{2'}$ [GJM96, Proposition V.1.1], [CG86, Theorem 6]. If $\ell \in L_2$ and $\mu \in \mathcal{U}(FL)$, then $\mu^{-1}\ell\mu$ is a 2-element of L and hence in L_2. It follows that we may assume that L is a 2-loop.

Let F_p denote the field of p elements and suppose first that $p \neq 2$. Then FL is semisimple and hence the direct sum of fields F_i and Cayley-Dickson algebras A_i [GJM96, Corollary VI.4.8] [GM96b, Theroem 2.8], so $\mathcal{U}(FL)$ is the product of the unit loops of these F_i and A_i. Since each A_i is finite, it is not a division algebra. (A finite alternative division ring is commutative by Wedderburn's Theorem, since the subring generated by any two elements is a finite division ring, and hence associative [ZSSS82, Theorem 3, p. 143].) Thus each A_i is a Zorn's vector matrix algebra over a field K and the projection of $\pm L$ in A_i is a subloop of the unit loop $\mathcal{U}(A_i)$ of A_i. Since L is normal in $\mathcal{U}(FL)$, the projection of $\pm L$ is normal in $\mathcal{U}(A_i)$ which, as explained in the introduction is the general linear loop $\mathrm{GLL}(2,K)$. Since $\pm\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ is the only one nontrivial normal subloop of $\mathrm{GLL}(2,K)$, either the image of $\pm L$ is $\mathrm{GLL}(2,K)$ or it is central (perhaps trivial). The first possibility cannot occur, however, since G is a nonabelian normal subloop in L and there are no such subloops of $\mathrm{GLL}(2,K)$. The second possibility cannot occur either since the image of L in $\mathcal{U}(FL)$ is not central. Thus $p = 2$.

Let F_2 denote the field of 2 elements. Clearly $F_2L \subseteq FL$ and L is normal in $\mathcal{U}(F_2L)$. Let $\epsilon\colon F_2L \to F_2$ be the augmentation map. Since ϵ is a ring homomorphism, if $\alpha \in F_2L$ is a unit, necessarily $\epsilon(\alpha) = 1$. On the other hand, if $\alpha \in F_2L$ has augmentation 0, then α lies in the augmentation ideal $\Delta(L)$ which is known to be nilpotent [Goo95], [MZ, Theorem 3.4]. In particular, α is not a unit. Thus $\alpha \in F_2L$ is a unit if and only if $\epsilon(\alpha) = 1$. Let $g, h \in L$ be two elements which do not commute. Then $\mu = 1 + g + h$ is a unit in F_2L which does not commute with g, so $\mu^{-1}g\mu = t \in L$. We have $g\mu = \mu t$, implying $g + g^2 + gh = t + gt + ht$. Now $g^2 \neq g$, $g^2 \neq gh$ and $g^2 \neq gt$; thus $g^2 = t$ or $g^2 = ht$. If $g^2 = t$, then t is central (the square of any element of L is central) and so is $g = \mu t \mu^{-1} = t$. a contradiction. So $g^2 = ht$ which, after cancellation, gives $g + gh = t + gt$ and $g \in \{gh, t, gt\}$, which is not true. □

Remark 3.5. Let R be any commutative associative ring with 1 and of characteristic $p > 0$. Since R contains F_p, the proof of Theorem 3.4 shows that L is never normal in $\mathcal{U}(RL)$. Such is not the case with group rings over finite rings; for example, the symmetric group S_3 is normal in $\mathcal{U}(F_2S_3)$ [Seh78, §6.2, p. 215].

Theorem 3.6. *Let L be a finite RA loop and F a field. Then $\mathcal{U}(FL)$ is not RA.*

Proof. Suppose $\mathcal{U}(FL)$ is an RA loop. Since L contains an RA 2-loop, we may assume that L itself is a 2-loop.

Suppose that the characteristic of F is different from 2. Again, $\mathcal{U}(FL)$ is the product of the unit groups of fields and Cayley-Dickson algebras. Let A be one of the Cayley-Dickson algebras and $\mathcal{U} = \mathcal{U}(A)$ its unit loop. As a subloop of the RA loop $\mathcal{U}(FL)$, clearly $\mathcal{U}$ is also RA. If A has zero divisors, then A is a Zorn's vector matrix algebra and $\mathcal{U}$ is a general linear loop. This loop is not RA for a variety of reasons; for instance, it contains the general linear group and hence does not possess a unique nonidentity commutator. On the other hand, if A is a division algebra, then $\mathcal{U} = A \setminus \{0\}$ contains Q_8, the quaternion group of order 8. In Q_8, the unique nonidentity commutator is -1, so this is the unique nonidentity commutator in $\mathcal{U}$. Since $i + j$ and i do not commute, we would have $(i+j)i = -i(i+j)$, giving $2i^2 = 0$, a contradiction. Thus $\operatorname{char} F = 2$.

Let F_2 denote the field of 2 elements. Since $F_2L \subseteq FL$, $\mathcal{U}(F_2L)$ is an RA loop. In particular, this unit loop has a unique nonidentity commutator-associator which is necessarily the unique nonidentity commutator-associator, s, of L. As noted in the proof of Theorem 3.4, $\alpha \in F_2L$ is a unit if and only if $\epsilon(\alpha) = 1$, there $\epsilon\colon F_2L \to F_2$ is the augmentation map. Let g and h be two elements of L which do not commute. Thus $hg = sgh$. Then $1 + g + h$ is a unit which does not commute with g, so the commutator of these two elements is also s. The equation $(1+g+h)g = sg(1+g+h)$ implies $g + g^2 = sg + sg^2$ and hence $1+g = s+sg$, an impossibility since $g \neq 1$, $g \neq s$ and $g \neq sg$. □

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