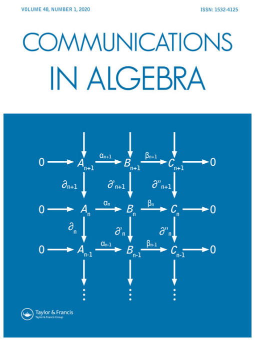

Communications in Algebra

ISSN: (Print) (Online) Journal homepage: https://www.tandfonline.com/loi/lagb20

Locally finite coalgebras and the locally nilpotent radical II

G. Santos Filho, L. Murakami & I. Shestakov

COMMUNICATIONS IN ALGEBRA®
https://doi.org/10.1080/00927872.2021.1947310

Locally finite coalgebras and the locally nilpotent radical II

G. Santos Filho, L. Murakami, and I. Shestakov

Department of Mathematics, IME, University of São Paulo, São Paulo, Brazil

ABSTRACT

In this article, we describe a criterion for an element of the dual space of an algebra to belong to the finite dual. This result is used to study when a certain subspace of the dual space is a subcoalgebra of the finite dual. We further apply it to find a right alternative coalgebra that is not locally finite. This work is motivated by a conjecture from I. Shestakov, which states that all coalgebras of a given variety are locally finite if, and only if, this variety admits locally nilpotent radical.

ARTICLE HISTORY

Received 19 January 2021
Communicated by Pavel Kolesnikov

KEYWORDS

Finite dual; locally finite coalgebras; right alternative algebras

2020 MATHEMATICS SUBJECT CLASSIFICATION

17A99; 17D15; 17A65

1. Introduction

Since the notion of Lie coalgebra was developed in [3], coalgebras in varieties of non-associative algebras have been studied. In particular, it is interesting to draw comparisons between results of the classic case of (co)associative coalgebras. One particular result that does not hold for non-associative coalgebras in general is the Fundamental Theorem of Coalgebras.

The Fundamental Theorem of Coalgebras states that every finitely generated associative coalgebra is finite dimensional. It was proved in [3] that an analogue of this result is not true for Lie coalgebras. W. Michaelis constructed a number of examples [4, 5] of infinite dimensional finitely generated Lie coalgebras.

It is still an open problem to find a necessary and sufficient condition in order to every finitely generated coalgebra of a given variety to be finite dimensional. The reader can refer to [7] for more information on varieties whose finitely generated coalgebras are known to be finite dimensional. On the other hand, examples of finitely generated coalgebras with infinite dimension are scarce and the Lie coalgebras constructed by W. Michaelis are still the only known ones to the authors.

Our first objective in this work is to study the finite dual of an algebra under certain circumstances in order to guarantee a well-behaved structure of coalgebra. Then, we use our results to construct an example of right alternative coalgebra that is finitely generated but infinite dimensional.

Proposition 1 and Theorem 2 gives us many examples of coalgebras in varieties, while Example 1 is an application of the latter to construct our desired example of right alternative coalgebra. As the variety of right alternative algebras does not admit locally nilpotent radical, the

CONTACT G. Santos Filho reis.g.filho@gmail.com Department of Mathematics, IME, University of São Paulo, São Paulo, Brazil.

existence of this example agrees with the following conjecture due to I. Shestakov: the finitely generated coalgebras of a given variety are finite dimensional if, and only if, this variety admits locally nilpotent radical (in the sense of Amitsur-Kurosh).

2. Bimodules and coalgebras

Let F be a field. In this article, algebras are assumed to be algebras over this field F and are not necessarily associative. If V is a vector space, then we denote by $\mathrm{End}\,(V)$ the space of all linear endomorphisms of V. The *dual space* of V, which is denoted by V^*, is the space of all linear functionals of the space V. If $\alpha \in V^*$ and $v \in V$, then we will denote the image of v through α by $\langle \alpha, v \rangle$.

If A is an algebra, then an *A-bimodule* is a triple (M, λ, ρ), in which M is a vector space and $\lambda : A \to \mathrm{End}\,(M), \rho : A \to \mathrm{End}\,(M)$ are linear maps called the *left action* and the *right action*, respectively. When there is no risk of mistake about the definition of the actions, we simply refer to the A-bimodule M. If $a \in A$, then we denote the image of a through the linear maps λ and ρ, respectively, by λ_a and ρ_a.

Let A be an algebra and $a \in A$. The linear maps $L_a, R_a \in \mathrm{End}\,A$ given by $L_a(x) = ax$ and $R_a(x) = xa$, for any $x \in A$, are called respectively the *left and right multiplication operators by a*. The triple (A, L, R) is an A-bimodule called the *regular A-bimodule*.

If $N \subseteq M$ is a subspace of the A-bimodule M, then we say that N is an *A-subbimodule of M* if the spaces

$$AN = \{\lambda_a(x) \,|\, x \in N, a \in A\} \text{ and } NA = \{\rho_a(x) \,|\, x \in N, a \in A\}$$

are subspaces of N.

If (M, λ, ρ) is an A-bimodule and $S \subseteq M$ then we can define the subspaces

$$B^{(0)}\langle S \rangle = \mathrm{span}\,\langle S \rangle$$

$$B^{(k)}\langle S \rangle = \mathrm{span}\,\langle \tau^{(1)}_{a_1} \ldots \tau^{(k)}_{a_k}(m) \,|\, \tau^{(1)}, \ldots, \tau^{(k)} \in \{\lambda, \rho\},\ a_1, \ldots, a_n \in A,\ m \in S \rangle$$

for each positive integer k. The A-subbimodule of M generated by $S \subseteq M$, here denoted by $\mathrm{Bimod}_A\,\langle S \rangle$, is given the intersection of all A-subbimodules of M that contain the subset S. It is easy to check that

$$\mathrm{Bimod}_A\,\langle S \rangle = \sum_{k=0}^{\infty} B^{(k)}\langle S \rangle.$$

If (M, λ, ρ) is an A-bimodule, then we can define a multiplication over the vector space $A\dot{+}M$ given by

$$(a + m)(b + n) = ab + (\lambda_a(m) + \rho_b(n)),$$

for any $a, b \in A$ and any $m, n \in M$. With this product, $A\dot{+}M$ is an algebra called the *split-null extension* of the A-bimodule M. If $\mathcal{V}$ is a variety of algebras, then we say that the A-bimodule M is a *bimodule of* $\mathcal{V}$, if the split-null extension of M is an algebra of $\mathcal{V}$.

A *coalgebra* is a pair (C, Δ) where C is a vector space and $\Delta : C \to C \otimes C$ is a linear map called *comultiplication*. When there is no risk of mistake about the definition of Δ, we simply refer to the coalgebra C. We shall use the *Sweedler notation* for the comultiplication of elements of the coalgebra C, that is, if $x \in C$, then we will denote

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}.$$

A subspace $D \subseteq C$ is a *subcoalgebra* of C if $\Delta(D) \subseteq D \otimes D$. If $S \subseteq C$, then the subcoalgebra generated by S, denoted by $\text{Coalg}\langle S \rangle$, is the smallest subcoalgebra of C (in the sense of inclusion) that contains the subset S. We say that a coalgebra C is *locally finite* if every subcoalgebra generated by a finite subset of C is also finite dimensional.

Let C be a coalgebra with comultiplication $\Delta : C \to C \otimes C$. then a multiplication $\mathfrak{m} : C^* \otimes C^* \to C^*$ can be constructed on the dual space C^* in the following way

$$\langle \mathfrak{m}(\alpha \otimes \beta), x \rangle = \sum_{(x)} \langle \alpha, x_{(1)} \rangle \langle \beta, x_{(2)} \rangle$$

for each $\alpha, \beta \in C^*$ and $x \in C$. With this multiplication, C^* is called the *dual algebra of* C. For any subspace V, we can consider the linear transformation $\imath : V^* \otimes V^* \to (V \otimes V)^*$, given by $\langle \imath(\alpha \otimes \beta), v \otimes w \rangle = \langle \alpha, v \rangle \langle \beta, w \rangle$, for any $\alpha, \beta \in V^*$ and any $v, w \in V$. In the case where $V = C$, then the multiplication of the dual algebra of C can be written as $\mathfrak{m} = \imath \circ \Delta$, where Δ is the comultiplication of C.

The transformation $\imath$ is injective and it is a natural transformation in the sense that it does not depend on the choice of any basis on $C^* \otimes C^*$ or $(C \otimes C)^*$. For this reason we will use it to identify $C^* \otimes C^*$ as a subset of $(C \otimes C)^*$.

Let $\mathcal{V}$ be a variety of algebras. If C is a coalgebra such that its dual algebra belongs to $\mathcal{V}$, then we say that C is a coalgebra of $\mathcal{V}$. A classic result in the theory of associative coalgebras is the *Fundamental Theorem of Coalgebras*, which states that every associative coalgebra is locally finite. Analogues of the Fundamental Theorem of Coalgebra have been proved for a number of varieties (refer to [7]). It is still an open problem to find necessary and sufficient conditions in order to all coalgebras of a given variety to be locally finite.

If C is a coalgebra, then it has a structure of C^*-bimodule with left and right actions given, respectively, by

$$\alpha \rightharpoonup x = \sum_{(x)} \langle \alpha, x_{(2)} \rangle x_{(1)} \quad x \leftharpoonup \alpha = \sum_{(x)} \langle \alpha, x_{(1)} \rangle x_{(2)}, \tag{1}$$

where $x \in C$ and $\alpha \in C^*$.

3. Finite dual

As seen in the previous section, the dual space of a coalgebra has a natural structure of algebra. However, the same does not happen in the case of the dual space of an algebra (not even in the associative case). The finite dual of an algebra is, roughly speaking, the biggest subspace of the dual space in which the transpose of the multiplication defines a comultiplication. When looking for examples of coalgebras, it can be useful to study the finite dual of algebras.

Let A be an algebra. If S is a subspace of the dual space A^*, then S is called a *good subspace* of A^* if $\mathfrak{m}^*(S) \subseteq S \otimes S$ (by using the previously mentioned linear transformation $\imath$ to identify the sets $S \otimes S$ and $\imath(S \otimes S)$), in which $\mathfrak{m} : A \otimes A \to A$ is the multiplication of A and $\mathfrak{m}^* : A^* \to (A \otimes A)^*$ is the transpose of $\mathfrak{m}$. The *finite dual* of A, denoted by A°, is defined as the sum of the good subspaces of A^*. For more details on the construction the finite dual, the reader can check [1, p. 4699, 4670]. It is easy to show that A° is a coalgebra with comultiplication given by $\mathfrak{m}^*$.

It is worth mentioning two special cases of the finite dual.

First, let A be a finite dimensional algebra. The dual space of A is also finite dimensional and the finite dimensional spaces $A^* \otimes A^*$ and $(A \otimes A)^*$ are isomorphic through the linear transformation $\imath : A^* \otimes A^* \to (A \otimes A)^*$. In this case, we can define the comultiplication $\Delta : A^* \to A^* \otimes A^*$ given by the composition $\Delta = \mathfrak{m}^* \circ \imath^{-1}$, where $\mathfrak{m}$ is the multiplication of A and $\mathfrak{m}^*$ is the transpose transformation of $\mathfrak{m}$. Therefore, we have $A^\circ = A^*$.

Now let A be an associative algebra (of arbitrary dimension). It is well known [8] that the finite dual of A can be defined as

$$A^\circ = \{f \in A^* \mid \ker f \text{ contains a finite codimensional ideal of } A\}.$$

It was proved in [1] that this characterization is also true for any algebra A as long as A° is a locally finite coalgebra.

Let A be an algebra. The dual space A^* has a structure of A-bimodule with left and right actions given, respectively, by

$$\lambda_a(f) = f \circ R_a \, \rho_a(f) = f \circ L_a, \tag{2}$$

for any $a \in A$, $f \in A^*$, where L_a and R_a are respectively the left and right multiplication operators by a.

Note that if A is an algebra, then we can construct its finite dual A°, which is a coalgebra, and therefore we can also construct $(A^\circ)^*$, the dual algebra of A°. The elements of the coalgebra A° are subject to the actions of two bimodule structures: one is the structure of $(A^\circ)^*$-bimodule, given by the fact that A° is a coalgebra and, therefore, a bimodule over its dual algebra with actions given by (1); the second is due to the fact that A° is a subset of A^* and A^* is an A-bimodule with actions given by (2). The relation between these actions will be clarified in the next section.

For the rest of this work, we will often refer to the following result proved by [1, p. 4704] that offers a criterion for good subspaces of the dual space of an algebra.

Corollary 2.5 [1]. *Let A be an algebra and let S be a vector subspace of A^*. Then the following are equivalent:*

i. *The subspace S is an A-sub-bimodule of A^* such that, for any $f \in S$, the subspaces*

$$Af = \{\lambda_a(f) \mid a \in A\} \qquad fA = \{\rho_a(f) \mid a \in A\}$$

are finite dimensional.

ii. *S is good.*

Through the application of Corollary 2.5 from [1] to the split-null extension of an A-bimodule M, we can prove the following proposition, that may be useful to find examples of coalgebras in a specific variety.

Proposition 1. *Let A be a finite dimensional algebra and M be a bimodule over A. Then the finite dual of the split-null extension $A\dot{+}M$ is the dual space $(A\dot{+}M)^*$. Furthermore, if $\mathcal{V}$ is a variety of algebras and M is a bimodule of $\mathcal{V}$, then $(A\dot{+}M)^*$ is a coalgebra of $\mathcal{V}$.*

Proof. We shall show that, for any $f \in (A\dot{+}M)^*$, the space

$$B^{(1)}\langle f\rangle = \operatorname{span}\langle \lambda_x(f), \rho_x(f) \mid x \in A\dot{+}M\rangle$$

is finite dimensional which, in consequence, shows that the condition (i) of Corollary 2.5 from [1] is true, as Af and fA are subspaces of $B^{(1)}\langle f\rangle$. By consequence of the same corollary, $(A\dot{+}M)^*$ is a good subspace, that is, a coalgebra.

Let $\{a_1, \dots, a_n\}$ be a basis of A. The space A^* is finite dimensional and can be considered a subspace of $(A\dot{+}M)^*$. Therefore, we can define the following finite dimensional subspace of $(A\dot{+}M)^*$

$$V = A^* + \operatorname{span}\langle \lambda_{a_1}(f), \dots, \lambda_{a_n}(f), \rho_{a_1}(f), \dots, \rho_{a_n}(f)\rangle.$$

We shall prove that for any $x \in A\dot{+}M$, the elements $\lambda_x(f)$ and $\rho_x(f)$ belong to V. By linearity, it is sufficient to consider the cases in which $x \in A$ and $x \in M$.

If $x \in A$, then by linearity again, we can assume that $x \in \{a_1, ..., a_n\}$ and, trivially, we have $\lambda_x(f), \rho_x(f) \in V$. If $x \in M$, then for any $m \in M$,

$$\langle \lambda_x(f), m \rangle = \langle f, mx \rangle = \langle f, 0 \rangle = 0$$
$$\langle \rho_x(f), m \rangle = \langle f, xm \rangle = \langle f, 0 \rangle = 0$$

and we can conclude that $\lambda_x(f), \rho_x(f) \in A^* \subseteq V$. As desired, we proved that $B^{(1)}\langle f \rangle$ is a finite dimensional space and, by the discussion in the beginning of this proof, $(A\dot{+}M)^*$ is a coalgebra.

Furthermore, if M is a bimodule of the variety $\mathcal{V}$, that is, if the split-null extension $A\dot{+}M$ is an algebra of $\mathcal{V}$, then by the Corollary 4.3 from [1], the finite dual $(A\dot{+}M)^\circ$ is a coalgebra of $\mathcal{V}$. □

Although Proposition 1 gives us many examples of coalgebras (provided that we know examples of bimodules over finite dimensional algebras), we may still be difficult to determine the comultiplication of its elements. Later we shall prove Theorem 2, that gives us, under certain circumstances, examples of coalgebras with more well-behaved comultiplications.

4. Subcoalgebras of the finite dual

Let A be an algebra, A° be its finite dual and $f \in A^\circ$. Our objective is to determine the subcoalgebra of A° generated by f. As A° is a coalgebra, it has the structure of bimodule over its dual algebra. As a consequence of Corollary 1.5 from [1], the desired subcoalgebra is the $(A^\circ)^*$-subbimodule of A° generated by f. In other words,

$$\text{Coalg}\,\langle f \rangle = \text{Bimod}_{(A^\circ)^*}\,\langle f \rangle. \tag{3}$$

Equation (3) is not immediately useful from a practical point of view as we must determine all the elements of A° in order to understand the action of the algebra $(A^\circ)^*$. To simplify the task of determining $\text{Coalg}\,\langle f \rangle$, we shall use the structure of A-bimodule on A^* given by (2).

We must recall that a subspace of A^* is a good subspace if, and only if, it is a subcoalgebra of A°. Therefore, in view of (3), $\text{Bimod}_{(A^\circ)^*}\,\langle f \rangle$ is a good subspace of A^*. As a consequence of Corollary 2.5 from [1], $\text{Bimod}_{(A^\circ)^*}\,\langle f \rangle$ is an A-subbimodule of A^* and as $f \in \text{Bimod}_{(A^\circ)^*}\,\langle f \rangle$, we also have

$$\text{Bimod}_A\,\langle f \rangle \subseteq \text{Bimod}_{(A^\circ)^*}\,\langle f \rangle \tag{4}$$

Now, $\text{Bimod}_A\,\langle f \rangle$ is trivially an A-subbimodule of A^*. By the fact that $\text{Bimod}_{(A^\circ)^*}\,\langle f \rangle$ is a good subspace of A^*, by inclusion (4) and by Corollary 2.5 from [1], we can conclude that, for each $g \in \text{Bimod}_A\,\langle f \rangle$, the spaces gA and Ag are finite dimensional. Again, by Corollary 2.5 from [1] implies that $\text{Bimod}_A\,\langle f \rangle$ is a good subspace of A^* and a subcoalgebra of A°. In particular,

$$\text{Coalg}\,\langle f \rangle \subseteq \text{Bimod}_A\,\langle f \rangle \tag{5}$$

From (3, 4) and (5) we can conclude the following proposition.

Proposition 2. *Let A be an algebra and $f \in A^*$. If $f \in A^\circ$, then*

$$\text{Coalg}\,\langle f \rangle = \text{Bimod}_{(A^\circ)^*}\,\langle f \rangle = \text{Bimod}_A\,\langle f \rangle.$$

As an application of Proposition 2, we have Theorem 1, which can be considered a "local" version of the Corollary 2.5 from [1]. First we need to prove the following lemma.

Lemma 1. *Let A be an algebra and $f \in A^\circ$. Then the space*

$$B^{(1)}\langle f \rangle = \text{span}\,\langle \lambda_a(f),\ \rho_a(f) \mid a \in A \rangle$$

is finite dimensional.

Proof. Let $\Delta : A^\circ \to A^\circ \otimes A^\circ$ be the comultiplication of A° and $g_1, ..., g_n, h_1, ..., h_n \in A^\circ$ such that

$$\Delta(f) = \sum_{k=1}^{n} g_k \otimes h_k.$$

Then, for any $a \in A$,

$$\lambda_a(f) = \sum_{k=1}^{n} \langle h_k, a\rangle g_k \quad \text{and} \quad \rho_a(f) = \sum_{k=1}^{n} \langle g_k, a\rangle h_k$$

and therefore $\lambda_a(f), \rho_a(f) \in \text{span}\,\langle g_1, ..., g_n, h_1, ..., h_n\rangle$, for any $a \in A$. □

Theorem 1. *Let A be an algebra and $f \in A^*$. The following statements are equivalent:*

(i) $f \in A^\circ$;

(ii) *For every $\alpha \in \text{Bimod}_A\,\langle f\rangle$, one of the following spaces (and equivalently, both)*

$$\alpha A = \{\rho_a(\alpha)\,|\,a \in A\} \quad \textit{and} \quad A\alpha = \{\lambda_a(\alpha)\,|\,a \in A\}$$

is finite dimensional;

(iii) *For every positive integer k, the space*

$$B^{(k)}\langle f\rangle = \text{span}\,\langle \tau^{(1)}_{a_1} ... \tau^{(k)}_{a_k}(f)\,|\,a_1, ..., a_k \in A, \tau^{(1)}, ..., \tau^{(k)} \in \{\lambda, \rho\}\rangle$$

is finite dimensional.

Proof. $(i) \iff (ii)$: Simply apply Corollary 2.5 from [1] for $S = \text{Bimod}_A\,\langle f\rangle$ and note that $f \in A^\circ$ if, and only if, $\text{Bimod}_A\,\langle f\rangle \subseteq A^\circ$.

$(i) \Rightarrow (iii)$: If $f \in A^\circ$, then by Lemma 1, $B^{(1)}\langle f\rangle = \text{span}\,\langle g_1, ..., g_n\rangle$ is finite dimensional. The desired conclusion follows by induction after noting that, for any positive integer k, if $B^{(k)}\langle f\rangle = \text{span}\,\langle g_1, ..., g_n\rangle$, for some $g_1, ..., g_n \in A^\circ$, then

$$B^{(k+1)}\langle f\rangle = B^{(1)}\langle g_1\rangle + ... + B^{(1)}\langle g_n\rangle.$$

$(iii) \Rightarrow (ii)$: Let $f \in A^*$. Note that

$$\text{Bimod}_A\,\langle f\rangle = \sum_{k=0}^{\infty} B^{(k)}\langle f\rangle,$$

let $\alpha \in \text{Bimod}_A\,\langle f\rangle$ and suppose, without loss of generality, that $\alpha \in B^{(t)}\langle f\rangle$ for some positive integer t. Because of linearity, we can also consider that $\alpha = \tau^{(1)}_{a_1} ... \tau^{(t)}_{a_t}(f)$, for some $a_1, ..., a_t \in A$ and $\tau^{(1)}, ..., \tau^{(t)} \in \{\lambda, \rho\}$. Then,

$$\begin{aligned} \alpha A &= \text{span}\,\langle \lambda_a(\alpha)\,|\,a \in A\rangle \\ &= \text{span}\,\langle \lambda_a \tau^{(1)}_{a_1} ... \tau^{(t)}_{a_t}(f)\,|\,a \in A\rangle \\ &\subseteq B^{(t+1)}\langle f\rangle \end{aligned}$$

and we can conclude that αA (similarly, $A\alpha$) is a finite dimensional subspace. □

5. Applications

Now we shall apply Theorem 1 to study when a certain subspace of A^* is a subcoalgebra of the finite dual. This will give us another source of examples of coalgebras and we will use it to obtain a nonlocally finite right alternative coalgebra.

Let A be an algebra, let $\mathcal{B}$ be a basis of A and let $\mathcal{B}^* = \{f_b \in A^* \mid b \in \mathcal{B}\}$ be the dual set of $\mathcal{B}$, in which for every $b \in \mathcal{B}$, the linear map $f_b : A \to F$ is given by

$$\langle f_b, x \rangle = \begin{cases} 1, & \text{if } x = b \\ 0, & \text{if } x \neq b \end{cases}$$

for $x \in \mathcal{B}$. If A is a finite dimensional algebra, then for every basis $\mathcal{B}$, it is easy to check (using Theorem 1, for instance) that $f_b \in A^\circ$, for each $b \in \mathcal{B}$. In this case, $\mathcal{B}^*$ is a basis of A^* called the dual basis (with respect to $\mathcal{B}$) and we have $A^\circ = A^*$.

On the other hand, if A is an infinite dimensional algebra with basis $\mathcal{B}$, then the subspace

$$V_{\mathcal{B}} = \operatorname{span}\langle f_b \mid b \in \mathcal{B} \rangle$$

is a proper subspace of A^*. Also, the inclusion $V_{\mathcal{B}} \subseteq A^\circ$ is no longer necessarily true. As an application of Theorem 1 we shall present a sufficient condition over a basis $\mathcal{B}$ of an infinite dimensional algebra A in order to guarantee this inclusion.

Definition 1. Let A be an algebra, $\mathcal{B}$ a basis of A and $x \in A$. If $x = \sum_{i=1}^n \alpha_i b_i$, where $\alpha_i \in F$ is nonzero and $b_i \in \mathcal{B}$ for each $i = 1, \dots, n$, then the *support of* x (with respect to the basis $\mathcal{B}$) is the set $\operatorname{supp} x = \{b_1, \dots, b_n\}$.

Lemma 2. *Let A be an algebra and $\mathcal{B}$ be a basis of A. Suppose that $x \in \mathcal{B}$ is an element such that the set*

$$P(x) = \{(b_1, b_2) \in \mathcal{B} \times \mathcal{B} \mid x \in \operatorname{supp}(b_1 b_2)\}$$

is finite. Let (λ, ρ) be the action of the A-bimodule A^*. Then

$$\lambda_b(f_x) = \sum_{(a,b) \in P(x)} \langle f_x, ab \rangle f_a$$
$$\rho_b(f_x) = \sum_{(b,a) \in P(x)} \langle f_x, ba \rangle f_a$$

and the space

$$B^{(1)}\langle f_x \rangle = \operatorname{span}\langle \lambda_a(f_x),\ \rho_a(f_x) \mid a \in A \rangle = Af_x + f_x A$$

is finite dimensional.

Proof. If $x \in \mathcal{B}$ is such that $P(x)$ is a finite set, then

$$V = \operatorname{span}\langle f_a \mid (a, b) \in P(x), \text{ for some } b \in \mathcal{B} \rangle$$

is a finite dimensional space. We will show that $B^{(1)}\langle f_x \rangle$ is a subspace of V.

If $a, b \in \mathcal{B}$, then

$$\langle \lambda_b(f_x), a \rangle = \langle f_x, R_b(a) \rangle = \langle f_x, ab \rangle = \begin{cases} \alpha \neq 0 & \text{se } x \in \operatorname{supp}(ab) \\ 0 & \text{otherwise.} \end{cases}$$

Let $b \in \mathcal{B}$. If there exists no element $a \in \mathcal{B}$ such that $x \in \operatorname{supp}(ab)$, then $\lambda_b(f_x) = 0$ belongs to V. Otherwise, as $P(x)$ is finite, there exists a finite number of elements $a_1, \dots, a_n \in \mathcal{B}$ such that $x \in \operatorname{supp}(a_1 b), \dots, \operatorname{supp}(a_n b)$. Then, the functional $\lambda_b(f_x)$ coincides with

$$\langle f_x, a_1 b \rangle f_{a_1} + \dots + \langle f_x, a_n b \rangle f_{a_n}$$

when evaluated on elements of the basis $\mathcal{B}$. Therefore,

$$\lambda_b(f_x) = \sum_{(a,b)\in P(x)} \langle f_x, ab\rangle f_a. \tag{6}$$

In particular, $\lambda_b(f_x)$ belongs to V.

In conclusion,

$$Af_x = \text{span}\,\langle \lambda_b(f_x)\,|\,b\in B\rangle \subseteq V.$$

Similarly, we prove that

$$\rho_b(f_x) = \sum_{(b,a)\in P(x)} \langle f_x, ba\rangle f_a, \tag{7}$$

and that f_xA is a finite dimensional space. Therefore, $B^{(1)}\langle f_x\rangle = Af_x + f_xA$ is a finite dimensional space. □

Theorem 2. *Let A be an algebra and $\mathcal{B}$ be a basis of A. Suppose that for every $x\in\mathcal{B}$ the set*

$$P(x) = \{(b_1,b_2)\in\mathcal{B}\times\mathcal{B}\,|\,x\in \text{supp}(b_1b_2)\}$$

is finite. Then the space $V_{\mathcal{B}} = span\,\langle f_b\,|\,b\in\mathcal{B}\rangle$ is a subcoalgebra of the finite dual A°.

Proof. Let $b\in\mathcal{B}$ and $g=f_b$. We shall prove by induction over k that for every positive integer there exists a finite subset $\{b_1,\dots,b_n\}\subseteq\mathcal{B}$ such that

$$B^{(k)}\langle g\rangle = \text{span}\,\langle \tau^{(1)}_{a_1}\dots\tau^{(k)}_{a_k}(g)\,|\,a_1,\dots,a_k\in A, \tau^{(1)},\dots,\tau^{(k)}\in\{\lambda,\rho\}\rangle$$

is a subspace of span $\langle f_{b_1},\dots,f_{b_n}\rangle$. The base of induction follows from Lemma 2.

Let t be a positive integer such that $B^t\langle g\rangle$ is a subspace of span $\langle f_{b_1},\dots,f_{b_n}\rangle$, for some $b_1,\dots,b_n\in\mathcal{B}$. As $B^{(t+1)}\langle f\rangle = A\;B^{(t)}\langle g\rangle + B^{(t)}\langle g\rangle\;A$, it follows that

$$\begin{aligned} B^{(t+1)}\langle g\rangle &\subseteq Af_{b_1}+\dots+Af_{b_m}+f_{b_1}A+\dots+f_{b_m}A \\ &= (Af_{b_1}+f_{b_1}A)+\dots+(Af_{b_m}+f_{b_m}A) \\ &= B^{(1)}\langle f_{b_1}\rangle+\dots+B^{(1)}\langle f_{b_m}\rangle \end{aligned}$$

By Lemma 2, the spaces $B^{(1)}\langle f_{b_1}\rangle,\dots,B^{(1)}\langle f_{b_m}\rangle$ are finite dimensional subspaces of $V_{\mathcal{B}}$ and, therefore, so is $B^{(t+1)}\langle f\rangle$.

By induction, $B^{(k)}\langle f_x\rangle\subseteq V_{\mathcal{B}}$ is a finite dimensional subspace of $V_{\mathcal{B}}$ for each nonnegative integer k. By Theorem 1, f_x belongs to the coalgebra A° for each $x\in\mathcal{B}$ and the space $V_{\mathcal{B}}$ is a subspace of A°. By (6) and (7), $V_{\mathcal{B}}$ is an A-subbimodule of A° and, by Proposition 2, $V_{\mathcal{B}}$ is a subcoalgebra of A°, concluding our proof. □

As an application of Theorem 2, we will construct an example of a right alternative coalgebra that is not locally finite.

Example 1. Let $A = M_2(F) = \text{span}\,\langle e_{11},e_{12},e_{21},e_{22}\rangle$ be the algebra of 2×2 matrices with entries in F. For every $h\in\mathbb{Z}$, let $M^{(h)} = \text{span}\,\langle m_1^{(h)},m_2^{(h)},m_3^{(h)},m_4^{(h)}\rangle$, where $\{m_1^{(h)},m_2^{(h)},m_3^{(h)},m_4^{(h)}\}$ are linearly independent elements, and consider $M=\bigoplus_{h\in\mathbb{Z}}M^{(h)}$. The space M is a A-bimodule with action given by

$$\begin{array}{ll}
m_1^{(h)}e_{11}=m_1^{(h)} & m_2^{(h)}e_{11}=0\\
m_1^{(h)}e_{12}=0 & m_2^{(h)}e_{12}=m_3^{(h)}\\
m_1^{(h)}e_{21}=m_4^{(h)} & m_2^{(h)}e_{21}=0\\
m_1^{(h)}e_{22}=0 & m_2^{(h)}e_{22}=m_2^{(h)}\\
 & \\
m_3^{(h)}e_{11}=m_3^{(h)} & m_4^{(h)}e_{11}=0\\
m_3^{(h)}e_{12}=0 & m_4^{(h)}e_{12}=m_1^{(h)}\\
m_3^{(h)}e_{21}=m_2^{(h)} & m_4^{(h)}e_{21}=0\\
m_3^{(h)}e_{22}=0 & m_4^{(h)}e_{22}=m_4^{(h)}\\
 & \\
e_{11}m_1^{(h)}=m_1^{(h)} & e_{11}m_2^{(h)}=0\\
e_{12}m_1^{(h)}=0 & e_{12}m_2^{(h)}=-m_4^{(h)}\\
e_{21}m_1^{(h)}=-m_3^{(h)} & e_{21}m_2^{(h)}=0\\
e_{22}m_1^{(h)}=0 & e_{22}m_2^{(h)}=m_2^{(h)}\\
 & \\
e_{11}m_3^{(h)}=-m_1^{(h+1)} & e_{11}m_4^{(h)}=m_4^{(h)}-m_1^{(h+1)}\\
e_{12}m_3^{(h)}=0 & e_{12}m_4^{(h)}=m_1^{(h)}\\
e_{21}m_3^{(h)}=-m_2^{(h)}+m_3^{(h+1)}-m_4^{(h+1)} & e_{21}m_4^{(h)}=-2m_2^{(h)}+m_3^{(h+1)}-m_4^{(h+1)}\\
e_{22}m_3^{(h)}=m_3^{(h)}+m_1^{(h+1)} & e_{22}m_4^{(h)}=m_1^{(h+1)}
\end{array}$$

for any $h \in \mathbb{Z}$. With the previously described action, we have the following results:

- M is a right alternative A-bimodule;
- the set

$$\mathcal{B}=\{e_{11},e_{12},e_{21},e_{22}\}\cup\{m_1^{(h)},m_2^{(h)},m_3^{(h)},m_4^{(h)}\,|\,h\in\mathbb{Z}\}$$

  is a basis of the split-null extension $A\dot{+}M$ that satisfies the hypothesis of Theorem 2;
- the coalgebra $V_{\mathcal{B}}=\operatorname{span}\langle f_b\,|\,b\in\mathcal{B}\rangle$ contains a right alternative coalgebra that is not locally finite.

Proof. One can easily conclude that the A-bimodule described is right alternative by observing the definition of a similar $M_2(F)$-bimodule in [6, p. 911]. In other words, the split-null extension $A\dot{+}M$ is a right alternative algebra and, by Corollary 4.3 from [1], $(A\dot{+}M)^\circ$ is a right alternative coalgebra.

Observing the action of A over the elements of M, we can conclude that any element of the basis $\mathcal{B}$ occurs a finitely many times in the table of multiplication of $A\dot{+}M$ (when written in terms of the basis $\mathcal{B}$). In particular, for any $x\in\mathcal{B}$, the set $P(x)=\{(b_1,b_2)\,|\,b_1,b_2\in\mathcal{B},\ x\in \operatorname{supp}(b_1b_2)\}$ is finite. By Theorem 2, the space $V_{\mathcal{B}}=\operatorname{span}\langle f_b\,|\,b\in\mathcal{B}\rangle$ is a subcoalgebra of $(A\dot{+}M)^\circ$.

Let k be a integer. By observing the action of the algebra A over M described above, we can note that

$$\begin{aligned}P(m_1^{(k)})=\{&(m_1^{(k)},e_{11}),(m_4^{(k)},e_{12}),(e_{11},m_1^{(k)}),(e_{11},m_3^{(k-1)}),(e_{11},m_4^{(k-1)}),\\&(e_{12},m_4^{(k)}),(e_{21},m_4^{(k-1)}),(e_{22},m_3^{(k-1)}),(e_{22},m_4^{(k-1)})\}\end{aligned}$$

from which we can conclude using formula (6) that

$$\rho_{e_{11}}f_{m_1^{(k)}}=f_{m_1^{(k)}}-f_{m_3^{(k-1)}}-f_{m_4^{(k-1)}}$$

Similarly,

$$\rho_{e_{21}} f_{m_1^{(k)}} = 0$$
$$\rho_{e_{21}} f_{m_3^{(k-1)}} = -f_{m_1^{(k-1)}} + f_{m_3^{(k-2)}} + f_{m_4^{(k-2)}}$$
$$\rho_{e_{21}} f_{m_4^{(k-1)}} = -f_{m_3^{(k-2)}} - f_{m_4^{(k-2)}}$$

and thus

$$\rho_{e_{21}} \rho_{e_{11}} f_{m_1^{(k)}} = f_{m_1^{(k-1)}}.$$

In particular, if D is subcoalgebra of $(A\dot{+}M)^\circ$ and $f_{m_1^{(k)}} \in D$, then $f_{m_1^{(k-1)}} \in D$. Thus, the linearly independent set $\{f_{m_1^{(k)}} \mid k \in \mathbb{Z}, k \leq 1\}$ is a linearly independent subset of the A-bimodule generated (i.e. the coalgebra generated) by $f_{m_1^{(1)}}$.

In conclusion, the coalgebra of $(A\dot{+}M)^\circ$ generated by $f_{m_1^{(k)}}$ is an infinite dimensional right alternative coalgebra. The infinite dimensional coalgebra Coalg $\langle f_{m_1^{(k)}} \rangle$ is evidently finitely generated and, thus, it is not a locally finite coalgebra. □

As far as the authors know, Example 1 is the only example of a non-locally finite coalgebra that is not a Lie coalgebra. In view of Shestakov's conjecture, we can highlight the following corollary of Example 1.

Corollary. *Not every right alternative coalgebra is locally finite. In other words, an analogue of the Fundamental Theorem of Coalgebras is not true for right alternative coalgebras.*

The corollary gives us another example of variety for which its finitely generated coalgebras are not necessarily finite dimensional. As the variety of right alternative algebras does not admit locally nilpotent radical, Shestakov's conjecture remains open.

Considering other varieties that do not admit locally nilpotent radical and determining if its coalgebras are locally finite may help us understand whether Shestakov's conjecture is true or not.

Acknowledgements

When the paper was already submitted, V. N. Zhelyabin has informed us on the paper [2] by D. Kh. Kozybaev, where a non-locally finite Novikov coalgebra was constructed, showing that an analogue for the Fundamental Theorem of Coalgebras is not true for Novikov coalgebras. The same result was claimed for the class of right alternative coalgebras. However, the example of non-locally finite right alternative coalgebra constructed contained a mistake, which was corrected by U. Umirbaev in a recent personal communication. For the previously described reasons, the authors would like to thank V. N. Zhelyabin and U. Umirbaev respectively.

Funding

G. Santos Filho is supported by grant CAPES 88882.386845/2019-01. L. Murakami is supported by grant FAPESP 2018/23690-6. I. Shestakov is supported by grants FAPESP 2018/23690-6 and CNPq 303916/2014-1. Conselho Nacional de Desenvolvimento Científico e Tecnológico; Coordenação de Aperfeiçoamento de Pessoal de Nível Superior; Fundação de Amparo à Pesquisa do Estado de São Paulo.

References

[1] Anquela, J. A., Cortés, T., Montaner, F. (1994). Nonassociative coalgebras. *Commun. Algebra* 22(12): 4693–4716. DOI: 10.1080/00927879408825096.

[2] Kh. Kozybaev, D. (2000). Right-alternative and right-symmetric coalgebras. *Sci. Educ. South Kazakhstan*. 19:155–163.

[3] Michaelis, W. (1974). Lie Coalgebras (with a proof of an analogue of the Poincaré-Birkhoff-Witt Theorem). PhD thesis. ProQuest LLC, Ann Arbor, MI. –University of Washington.
[4] Michaelis, W. (1985). The dual Poincaré-Birkhoff-Witt theorem. *Adv. Math*. 57(2):93–162. DOI: 10.1016/0001-8708(85)90051-9.
[5] Michaelis, W. (1990). An example of a nonzero Lie coalgebra M for which $\mathrm{Loc}(M) = 0$. *J. Pure Appl. Algebra* 68(3):341–348.
[6] Murakami, L. S. I., Shestakov, I. P. (2001). Irreducible unital right alternative bimodules. *J. Algebra* 246(2): 897–914. DOI: 10.1006/jabr.2001.8990.
[7] Santos Filho, G., Murakami, L., Shestakov, I. (2021). Locally finite coalgebras and the locally nilpotent radical i. *Linear Algebra Appl*. 621:235–253.
[8] Sweedler, M. E. (1969). *Hopf Algebras*. *Mathematics Lecture Note Series*. New York: W. A. Benjamin, Inc.