THE HYPERCIRCLE INEQUALITY AND THE COLLOCATION METHOD OF SCHUMAKER

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1. Introduction

In this paper we prove a generalization of the hypercircle inequality found in [4] and apply it to the study of initial value systems of differential equations using collocation conditions. We arrive at the same method proposed and studied by Schumaker [7].

2. Generalized Hypercircle Inequality

Let H be an infinite dimensional Hilbert space.

2.1 Theorem: *Given* $L, L_1, \ldots, L_n : H \to \mathbb{R}^d$, *continuous linear maps, let* $a = (a_1, \ldots, a_n) \in \mathbb{R}^d \times \cdots \times \mathbb{R}^d$, $V_a = \{f \in H : L_i f = a_i, i = 1, 2, \ldots, n\}$, $V = V_0$ and $W = \{f \in V : Lf = 0\}$. **(i)** *If f is in V_a and f_0 is its orthogonal projection on $V^\perp$ then*

$$f_0 \in V_a \quad \text{and} \quad \| Lf - Lf_0 \| \leq \| L \|_V (\| f \|^2 - \| f_0 \|^2)^{1/2}$$

(ii) *Let $W^\perp$ be the orthogonal complement of W in V and denote its dimension by m. If $g_1, g_2, \ldots, g_m$ is any orthonormal basis of $W^\perp$ then*

$$\| L \|_V = \sup_{\|\lambda\|_2 \leq 1} \| K\lambda \|_{\mathbb{R}^d}$$

where K is the $d \times m$ matrix defined by $K_{ij} = (Lg_j)^i$.

2.2 Remark: If L_i^j is the j^{th} component of L_i for $1 \leq i \leq n$, $1 \leq j \leq d$ and $\{L_i^j : 1 \leq i \leq n, 1 \leq j \leq d\}$ is linearly independent then $V_a \neq \emptyset$ and $\dim V^\perp = n \times d$.

3. Application to ODE Systems

Set $E = \{f \in C^{m-1}([0,1], \mathbb{R}^d) : f^{(m)}$ is piecewise continuous$\}$, where $m \geq 2$, and let $L_0, L_1, \ldots, L_n : E \to \mathbb{R}^d$ be the continuous linear maps defined by $L_0 f = f(0)$ and $L_i f = f'(t_i) - A(t_i) f(t_i), 1 \leq i \leq n$, where A is a $d \times d$ matrix whose entries are continuous functions on $[0,1]$ and $0 = t_0 < t_1 < \cdots < t_n = 1$.

Moreover, define a positive symmetric bilinear form on $E \times E$ as

$$(f \mid g)_E = \sum_{i=0}^{n} (L_i \mid L_g) + \int_0^1 (f^{(m)}(t) \mid g^{(m)}(t)) dt,$$

where $(\ |\ )$ is the usual scalar product on $I\!R^d$.

3.1 Proposition: *There exists $\epsilon = \epsilon(A) > 0$ such that if $\max_{1\le i\le n} |t_i - t_{i-1}| < \epsilon$, then $(f \mid f)_E = 0$ implies $f = 0$.*

3.2 Remarks: (i) In the following we will assume that the points $t_0, t_1, \ldots, t_n$ satisfy the hypothesis of Proposition 3.1. Then, $(\ |\ )_E$ is a scalar product on E.
(ii) We will denote by $H = \hat{E}$, the completion of E relative to the above scalar product.

3.3 Proposition: (i) *The linear maps $L_i : E \to I\!R^d$, $0 \le i \le n$, are continuous.*
(ii) *The components L_i^j, $0 \le i \le n$, $1 \le j \le d$, form a linearly independent set.*

3.4 Proposition: $V^\perp = \{s \in S(P^d_{2m}, M, D) :$

$$\begin{aligned}
&s^{(m)}(0) = s^{(m+1)}(0) = \cdots = s^{(2m-2)}(0) = 0,\\
&s^{(m)}(1) = s^{(m+1)}(1) = \cdots = s^{(2m-3)}(1) = 0,\\
&jump(s^{(2m-1)} - A^t s^{(2m-2)})(t_i) = 0,\ 1 \le i \le n-1,\\
&s^{(2m-1)}(1) - A^t s^{(2m-2)}(1) = 0\}
\end{aligned}$$

where $jump(s)(t_i) = s(t_i+) - s(t_i-)$, $D = (t_1, \ldots, t_{n-1})$ and $M = (2, \ldots, 2)$. *

4. Schumaker's Collocation Method

In the previous section we laid the ground work for the application of Theorem 2.1 to the study of the problem

$$(4.1) \qquad \begin{cases} y'(t) = A(t)y(t) + r(t), & 0 < t \le 1 \\ y(0) = 0 \end{cases}$$

where A is a $d \times d$ matrix whose entries are continuous functions and $r \in C([0,1], I\!R^d)$.

In that section, we chose a suitable Hilbert's space H which contains the solution y of Problem (4.1). With the scalar product adopted in H, the mappings L_i turned out to be continuous. Moreover, the components L_i^j are linearly independent.

Though we don't know the exact solution of Problem (4.1), we do know how to compute the value of L_i on y:

$$\begin{aligned}
&L_i y = y'(t) - A(t_i)y(t_i) = r(t_i) \quad \text{and}\\
&L_0 y = y(0) = 0
\end{aligned}$$

Since the hypothesis of Theorem 2.1 were satisfied, we can use that theorem to obtain some information about the solution y from the following inequality

$$\| Lf - Lf_0 \| \le \| L \|_V (\| f \|^2 - \| f_0 \|^2)^{1/2}$$

Remark, according to Theorem 2.1, f_0 can be described in two equivalent ways:

$1^{\underline{st}})$ $\quad f_0 \in V_a$ *and* $f_0 \in V^\perp$
$2^{\underline{nd}})$ $\quad f_0 \in V_a$ *and* $\| f_0 \|_H = \inf_{f\in V_a} \| f \|_H$

* In Schumaker [6], the notation $S(P^d_{2m}, M, D)$ stands for the space of polinomial splines of degree less than or equal to $2m-1$, of class $C^{2m-3}([0,1], I\!R^d)$ with knots $0 < t_1 < t_2 < \cdots < t_{n-1} < 1$.

In section 3 we verify that $V^\perp$ is a space of splines. The first formulation says that there exists a unique spline $s \in V^\perp$ which satisfies the collocation conditions $s \in V_a$, that is

$$s'(t_i) = A(t_i)s(t_i) + r(t_i), \qquad i = 1, 2, \ldots, n$$
$$s(0) = 0$$

The second formulation says that the solution of the collocation problem is optimal in the sense that it minimizes the norm $\| \, \|_H$ among the elements of H satisfying the collocation conditions.

The collocation problem above, whose solution f_0 was shown to be unique, is the collocation problem proposed and studied by Schumaker in [7].

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