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AN OPEN SET OF INITIAL CONDITIONS CAN DRAW A HYPERBOLIC HORSESHOE

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An open set of initial conditions can draw a hyperbolic horseshoe

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Abstract

We prove that on any surface there is a C^∞ diffeomorphism exhibiting a wandering domain D with the following ergodic property: for any orbit starting at D the correspondent Birkhoff mean of Dirac measures converges to the invariant measure supported on a hyperbolic horseshoe Λ which is equivalent to the unique nontrivial Hausdorff measure in Λ. The construction is obtained by a perturbation of a diffeomorphism inside a Newhouse domain, where the unstable and stable foliations of Λ are relatively thick and in tangential position. We describe, in addition, the non–trivial topological properties of the wandering domain D.

1 Introduction.

Many aspects contribute to the richness and complexity of a dynamical system. One of them is the existence of *a wandering domain*, that is a non-empty connected open set whose forward orbit is a sequence of pairwise disjoint open sets. In general we might expect there were only trivial and unavoidable wandering domains (that is wandering domains attracted by some sort of well behaved attractor). Indeed, in the context of diffeomorphisms of the circle, if there exists enough differentiability, Denjoy [3] proved that non–trivial wandering domains do not exist. For endomorphisms of the circle or interval which have finitely many non-flat critical points the same type of result happens, see [6]. In the case of rational maps of the Riemann sphere, see [15]. For diffeomorphisms

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of a compact surface there are also results assuring that, under enough differentiability, wandering domains do not exist (see for example [1], [5], [11] and [12]). Another important aspect we shall consider is related to the structure of the sets which are most frequently visited by orbits of points. Again we might hope that these sets were well behaved attractors.

To be definite let us consider dynamical systems generated by iterations of a C^r $(r \geq 1)$ diffeomorphism f of a compact Riemannian surface M. Let $\mathrm{Diff}^r(M)$ denotes the set of such diffeomorphisms endowed with the C^r topology. We say that $A \subset M$ is an *attractor* for f if A is *compact*, *f–invariant*, i.e. $f(A) = A$, *f–transitive* (or simply *transitive*), i.e. there is $x \in A$ such that $\{f^n x\}_{n \geq 0}$ is dense in A, and the *basin of attraction*

$$B_f(A) = \{x \in M \; ; \; f^n x \to A \text{ as } n \to \infty\}$$

contains a neighbourhood of A.

A standing conjecture of Palis [13] claims, in its weaker version, that there is a dense set $\mathcal{P} \subset \mathrm{Diff}^r(M)$, $r \geq 1$, such that for every $f \in \mathcal{P}$ there are only finitely many attractors $A_1, \ldots, A_n$ for f, whose basins $B_f(A_1), \ldots, B_f(A_n)$ cover M up to Lebesgue measure zero. The conjecture may be improved by asking some good properties for the attractors, as for example *stochastic stability* and existence of an *SBR–measure*. In the latter case, there would be an invariant probability measure μ such that $\mathrm{supp}(\mu) = A$ and for (Lebesgue) almost every $x \in B_f(A)$ the Birkhoff sum of Dirac measures

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x}$$

converges, in the weak-* topology, to the measure μ. With respect to the abundance in $\mathrm{Diff}^r(M)$, one could also ask that generic k–parameter families intersect $\mathcal{P}$ for a total Lebesgue measure set of parameters. This would justify what is already expected in practical experiments: for almost all choices of parameters there are only finitely many invariant sets attracting almost all initial conditions.

However, it is a common sense that we are very far from proving or disproving this conjecture, mainly for $r \geq 2$. On the other hand some interesting and unexpected behaviours have been identified. For example, Newhouse showed in the 70's (see [8], [9], [10] and also [14], Chap. 6) that there are open sets $\mathcal{U} \subset \mathrm{Diff}^r(M)$, $r \geq 2$, and residual subsets $\mathcal{N} \subset \mathcal{U}$ such that every $f \in \mathcal{N}$ presents infinitely many cohexisting hyperbolic attracting periodic orbits (*sinks*). Much later, one of the authors [2] showed that in these open sets $\mathcal{U}$ there is a dense subset $\mathcal{C} \subset \mathcal{U}$ such that every $f \in \mathcal{C}$ presents infinitely many coexisting *Hénon–like strange attractors* (see [7] for a definition).

In this work we are concerned with another kind of behaviour: the existence of non-trivial wandering domains (for simplicity, just wandering domains). We prove that on any surface there exist C^∞ diffeomorphisms which exhibit wandering domains, and these diffeomorphisms we obtain are intimately related to the Newhouse phenomena. In addition, we can choose the diffeomorphism in such a way that for orbits starting at an open set, the Birkhoff sum as above converges to a measure whose support may be equal to a hyperbolic saddle or even a hyperbolic horseshoe.

2 Basic Concepts and Main Results

In dimension 2 a classical example which has (trivial) wandering domains was given by Bowen (see [16] and references therein): take for example a diffeomorphism with a source repelling all orbits to a saddle–connection (a separatrix of the unstable manifold coinciding with a separatrix of the stable manifold of the saddle) which bounds a topological disk containing the source. Then there are wandering domains whose forward orbits accumulate on the saddle connection. According to the definition we give in the sequel all of them are trivial.

Let us consider a C^r $(r \geq 1)$ diffeomorphism f of a compact Riemannian surface M. We say that a compact invariant set $A \subset M$ contained in the non–wandering set of f is *dynamically connected* if it is not the union of two nontrivial closed invariant disjoint sets. A dynamically connected set A is a *weak attractor* for f if its basin $B_f(A)$ contains an open set W which has only finitely many connected components and whose closure contains A. A set W like that is called an *immediate basin of attraction* for A. The saddle–connection of Bowen and attractors as defined in the Introduction are examples of weak attractors, as well as saddle–nodes. Inside the basin of a weak attractor there are trivial wandering domains. We define a *non-trivial wandering domain* (for simplicity, just wandering domain) to be a non-empty connected open set with disjoint orbit and with no forward iteration contained in an immediate basin of a weak attractor.

We state our main result.

Theorem 1. *Given any surface M there exists $f \in Diff^\infty(M)$ which has a (wandering) domain D such that:*

1. *$diam(f^n(D)) \to 0$ as $n \to \infty$ and ω_D, the union of the ω–limit sets of points in D, is contained in the union of a non-empty subset of a hyperbolic horseshoe Λ (possibly equal to Λ) with a non–empty subset of the tangency points between stable and unstable manifolds of Λ.*

2. *For any open connected set W such that $\overline{W}$ intersects ω_D there is $z \in W$ such that $f^n(z)$ converges to a hyperbolic sink.*

3. The accumulation points (in the weak- topology) of the sequence of measures*

$$\nu_n = \frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j x}$$

does not depend on x, for every $x \in D$, where δ_z is the Dirac measure supported on z, $z \in M$.

4. For every accumulation point ν of $(\nu_n)_n$, $supp(\nu) \subset \Lambda$. It may happen that the accumulation set of $(\nu_n)_n$ contains just one measure ν which satisfies one of the following properties:

(a) $\nu = \delta_p$, where p is a hiperbolic saddle fixed point of Λ.

(b) $supp(\nu) = \Lambda$ and ν is equivalent to the non–trivial Hausdorff measure of Λ.

It may also happen that the accumulation set of $(\nu_n)_n$ has more than one element (in other words, $(\nu_n)_n$ does not converge). We produce an example where this set is $\{t\delta_p + (1-t)\delta_{p'}; t \in [0,1]\}$, where p and p' are saddle points of Λ (this is analogous to what happens for the 'one–sided heteroclinic attractors' studied in [16]). In fact it will be clear that we have enough freedom to produce a great variety of examples with different accumulation sets.

The very last property in Theorem 1, as the title of this work suggests, can be theoretically used to 'draw' the hyperbolic horseshoe Λ. Divide the screen of the computer into pixels, each pixel corresponding to a square of a net. Pick a point $x \in D$ and for each pixel divide the number of visits to it by the total number of iterates already done. The pixel will be turned on if and only if its quotient does not tend to zero as the number of iterates increases.

In dimension one, we say that an interval I is wandering if its forward orbit is disjoint and there is no iterate of I contained inside the immediate basin of a periodic attractor. An equivalent definition is that no point of I has a periodic orbit as its ω–limit set. On the other hand, no point of the wandering domain D above has a periodic orbit as its ω–limit set, as an immediate consequence of the first statement of Theorem 1. However, this property seems to be not enough to define a wandering domain in dimension greater than one, since trivial examples could be produced. For instance, consider any open set inside a fundamental domain of a normally hyperbolic circle with an irrational rotation inside.

The diffeomorphism f of the statement of Theorem 1 is taken inside a Newhouse open set $\mathcal{U} \subset \mathrm{Diff}^\infty(M)$. This means that for some $f_0 \in \mathcal{U}$ there is a hyperbolic set Λ_0, which has a hyperbolic continuation $\Lambda = \Lambda_f$ defined for all $f \in \mathcal{U}$, and also for every $f \in \mathcal{U}$ there is a leaf of the stable set $W^s(\Lambda_f)$ in tangential position with a leaf of the unstable set $W^u(\Lambda_f)$. We strongly believe from our method of construction that the same properties stated in Theorem 1 are true for a dense set of diffeomorphisms inside any Newhouse open set $\mathcal{U}$. This result would depend on a technical realization of the ideas involved here, see for example [2] for a clue.

3 The starting diffeomorphism

The diffeomorphism f of Theorem 1 is constructed in two stages. First, we describe a starting diffeomorphism g of a disk $\mathcal{D} \subset R^2$ which exhibits a hyperbolic horseshoe Λ whose stable and unstable foliations are in tangential position. It can be realized (in any surface) as the restriction of a diffeomorphism to a local chart (for example by surgery around an attracting fixed point). Then we consider a class of perturbations for g and, among them, an arbitrarily small one (in the C^r topology, for any $r \geq 1$). These perturbations have support in $\mathcal{D}$, so that they can be trivially extended to M to give us the desired diffeomorphism.

We consider the square $Q = [-l, +l]^2$, for some $l > 0$, the disks

$$D_\pm = \{(x, y); \mp x \geq l\,,\ (x \pm l)^2 + y^2 \leq l^2\}$$

and the vertical strips

$$S_\pm = [\mp\frac{l}{2} - \sigma^{-1}l, \mp\frac{l}{2} + \sigma^{-1}l] \times [-l, +l]\,,$$

for some $\sigma > 2$ (see Figure 1).

Let $\mathcal{D} = D_+ \cup Q \cup D_-$ and $\mathcal{Z} \subset \text{Diff}^\infty(\mathcal{D})$ be the (nonempty) set of C^∞ diffeomorphisms $g : \mathcal{D} \to \mathcal{D}$ satisfying properties 1) to 7) below.

(1)$g(\mathcal{D}) \subset \text{int}(\mathcal{D})$, $g(D_+) \subset \text{int}(D_+)$ and $g(D_-) \cap D_- = \emptyset$.

(2)$g|D_+$ is a contraction; this means that there is a sink $q_+ \in D_+$ such that $\omega(z) = q_+$ for every $z \in D_+$.

(3) g is affine in a neighbourhood of S_+ and S_- and

$$\begin{array}{rccl} g|S_\pm : & S_\pm & \longrightarrow & Q \\ & (x, y) & \longmapsto & (\pm\sigma(x \pm \frac{1}{2}l), \mp\frac{l}{2} \pm \lambda y)\,, \end{array}$$

for some $0 < \lambda < \frac{1}{2}$ such that $\lambda\sigma < 1$. Define the horizontal strips $L_+ = g(S_+)$ and $L_- = g(S_-)$ and $\tilde{L}_+, \tilde{L}_0, \tilde{L}_-$ the connected components of $Q \setminus (L_+ \cup L_-)$, see Figure 1.

(4) If $\tilde{S}_+, \tilde{S}_0, \tilde{S}_-$ are the connected components of $Q \setminus (S_+ \cup S_-)$ (as indicated in Figure 1), then $g(\tilde{S}_+ \cup \tilde{S}_-) \subset D_+$ and $g(\tilde{S}_0) \subset D_-$.

A dynamically meaningful rectangle appears naturally under the conditions above. Let $R = J^u \times J^s$, where $J^u = [-a_u, +a_u]$, $J^s = [-a_s, +a_s]$, $a_u = \frac{l}{2(1-\sigma^{-1})}$, $a_s = \frac{l}{2(1-\lambda)}$.

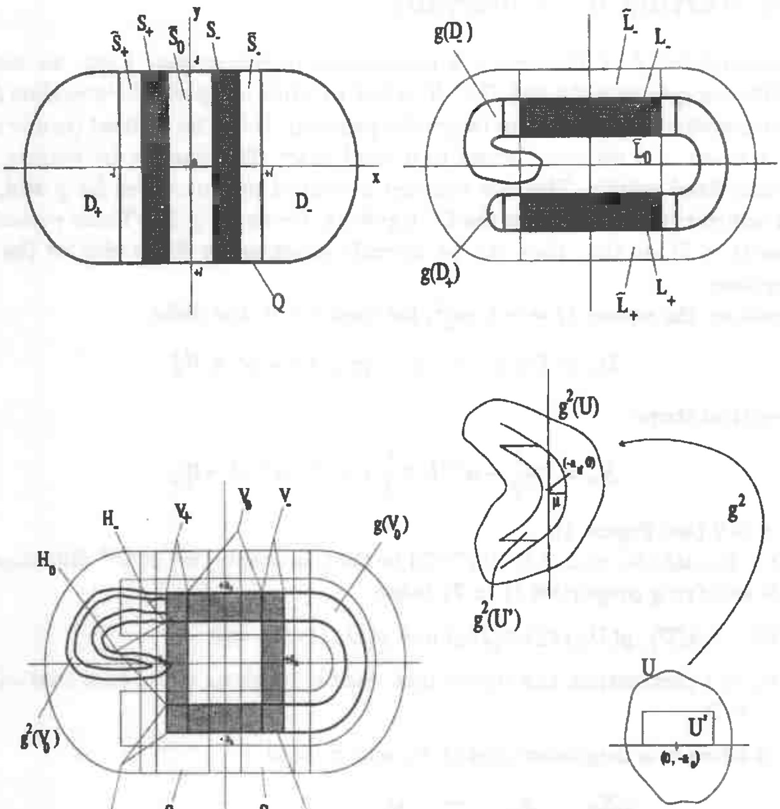

Figure 1: The starting diffeomorphism g.

This rectangle has the following properties: $J^u \times \{-a_s, +a_s\} \subset g(J^u \times \{-a_s\})$ and $g(\{-a_u, +a_u\} \times J^s) \subset \{-a_u\} \times J^s$, see Figure 1.

Observe that property 4) implies that $\omega(z) = q_+$ for any $z \in \tilde{S}_+ \cup \tilde{S}_-$. We claim that $\omega(z) = q^+$ for all $z = (x, y) \in Q$ such that $|x| > a_u$. Indeed, let $f(z) = (x', y')$. If $x < -a_u$ and $z \in S_+$ we have that $x' = \sigma(x + \frac{l}{2}) < -a_u$ and $-a_u - x' = \sigma(-a_u - x)$. Then, if $f(z) \in S_+$, we repeat this procedure. We get that, for some $k \geq 1$, $f^k(z) \in \tilde{S}_+$ and, because $\omega(z) = \omega(f^k(z)) = q_+$ the claim is proved. If $x > a_u$ and $z \in S_-$ we have that $x' = -\sigma(x - \frac{l}{2}) < -a_u$. Then, if $f(z) \in \tilde{S}_+$ it follows that $\omega(z) = \omega(f(z)) = q_+$ and the claim is proved. If $f(z) \in S_+$ we fall into the first case.

Let $\Lambda = \bigcap_{n=-\infty}^{\infty} g^n(Q)$ be the maximal invariant set in Q. It is not difficult to see that $\Lambda \subset R$ and Λ is the Cartesian product $K^u \times K^s$ of two Cantor sets, with J^u and J^s as convex hulls. The Cantor set K^u (resp. K^s) may be obtained by an inductive process where, starting from the interval J^u (resp. J^s), an open centred interval of proportion $1 - 2\sigma^{-1}$ (resp. $1 - 2\lambda$) is suppressed from each closed connected component remaining from the previous stage of the induction. To these Cantor sets we associate their *thickness*

$$\tau^u = \frac{\sigma^{-1}}{1 - 2\sigma^{-1}}, \quad \tau^s = \frac{\lambda}{1 - 2\lambda}.$$

We also associate to Λ its thickness $\tau(\Lambda) = \tau^u \tau^s$ (see [14], Chap. 4 for general definitions). The fith property is the following:

(5) $\tau(\Lambda) > 1$.

Let V_+ and V_- be the two vertical strips of R which are connected components of $R \cap g^{-1}(R)$, and let $H_\pm = g(V_\pm)$ be their images. Moreover, let $V_0 = R \setminus (V_+ \cup V_-)$ be the *vertical central gap* and $H_0 = R \setminus (H_- \cup H_+)$ be the *horizontal central gap*.

The last properties concern the return function from V_0 to H_0. Let U be a neighbourhood of $(0, -a_s)$, containing $[-2\delta, +2\delta] \times [-a_s, -a_s + 2a]$, $U' = [-\frac{\delta}{2}, +\frac{\delta}{2}] \times [-a_s, -a_s + a]$, for some $\delta > 0$ and $a > 0$ and suppose that

(6) $g^2(x, y) = (-a_u + \mu - \beta x^2 - \gamma(y + a_s), -\alpha x)$ for $(x, y) \in U$, where α, β, γ are positive constants and $\mu > 0$ will be chosen accordingly.

(7) If $(x, y) \in V_0$ and $(x', y') = g^2(x, y)$ is such that $x' > -a_u$ then $(x, y) \in U'$.

Property (6) is natural to impose and will give us a certain facility to apply our arguments, mainly in Section 8. Condition (7) implies that only one of the following possibilities occurs for $z \in \mathcal{D}$: $\omega(z) = q_+$, $z \in W^s(\Lambda)$ or the future orbit $\mathcal{O}_+(z) = \{z, g(z), \ldots, g^j(z), \ldots\}$ has marks

$$0 = N_0 \leq N_1 < N_1 + 2 < N_2 < \ldots N_{k-1} + 2 < N_k < \ldots$$

where $g^{N_k}(z) \in H_0 \cap g^2(U)$ for every $k \geq 1$ and $g^j(z) \notin V_+ \cup V_-$ only if $N_k - 2 \leq j \leq N_k - 1$ for some $k \geq 1$. In words, the iterates between H_0 and V_0 are done near the horseshoe, in the vertical strips V_+ and V_-, and if the orbit hits V_0 outside U' then it necessarily escapes to the sink q_+.

Observe also that in order to satisfy condition (7) for an already given U', we have to choose μ sufficiently small, in particular $\mu < \gamma a$ and $\mu < \beta(\frac{\delta}{2})^2$.

4 Gaps, bridges and foliations

Let $g \in \mathcal{Z}$ as in the previous Section. To each point $z \in \Lambda$ we associate an infinite sequence $\underline{z} = (\dots z_{-i} \dots z_{-1} z_0 z_1 \dots z_j \dots) \in \{-,+\}^{\mathbf{Z}}$, where $z_j = +$ if $g^j(z) \in V_+$ and $z_j = -$ if $g^j(z) \in V_-$, $j \in \mathbf{Z}$. Let

$$B^u(n; z_1 \dots z_n) = \{z \in R\, ;\ g^{j-1}(z) \in V_{z_j}\ ,\ j = 1, \dots, n\}\ ,$$

for $n \geq 1$, be the *unstable bridges* of Λ, where the natural number n is called the *generation* of the bridge. The *stable bridges* are defined as

$$B^s(n; z_1 \dots z_n) = \{z \in R\, ;\ g^{-j}(z) \in V_{z_j}\ ,\ j = 1, \dots, n\}\ .$$

We have already defined V_0 and H_0, the *unstable* and *stable central gaps.* Now let $G^u(n; z_1 \dots z_n)$ be the maximal vertical strip inside $B^u(n; z_1 \dots z_n)$ which lies between $B^u(n+1; z_1, \dots, z_n, +)$ and $B^u(n+1; z_1, \dots, z_n, -)$, and $G^s(n; z_1 \dots z_n)$ be the maximal horizontal strip inside $B^s(n; z_1 \dots z_n)$ which lies between $B^s(n+1; z_1, \dots, z_n, +)$ and $B^s(n+1; z_1, \dots, z_n, -)$, for every $n \geq 1$. These are, respectively, the *unstable* and *stable gaps.*

We are particularly concerned with the intersections of stable and unstable gaps. It is easy to verify that

$$g^j\,(G^u(n; z_1 \dots z_n) \cap H_0) = G^s(j; z_j \dots z_1) \cap G^u(n-j; z_{j+1} \dots z_n)\ ,$$

for $1 \leq j \leq n-1$ and, in particular,

$$g^n\,(G^u(n; z_1 \dots z_n) \cap H_0) = G^s(n; z_n \dots z_1) \cap V_0\ .$$

If we denote $\underline{z} = (z_1, \dots z_n)$ and $\underline{z}^{-1} = (z_n \dots z_1)$ then

$$g^n | G^u(n; \underline{z}) \cap H_0 : G^u(n; \underline{z}) \cap H_0 \to G^s(n; \underline{z}^{-1}) \cap V_0$$

is an affine diffeomorphism whose Jacobian is given by

$$\left(\prod_{j=1}^{n} z_j\right) \begin{pmatrix} \sigma^n & 0 \\ 0 & \lambda^n \end{pmatrix}\ ,$$

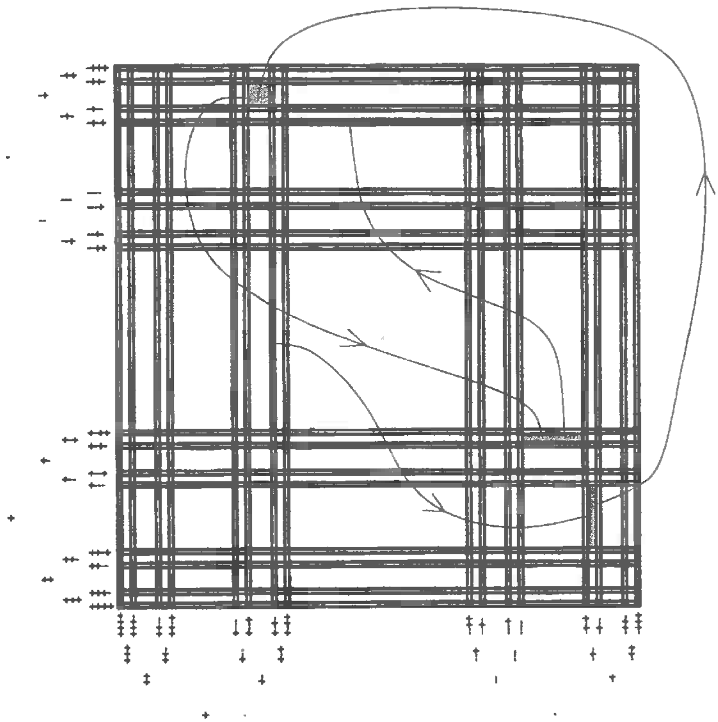

Figure 2: Iteration of gaps.

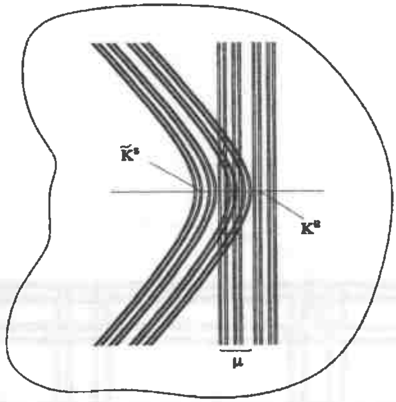

Figure 3: Foliations $\mathcal{F}^u_U$ and $\mathcal{F}^s_U$.

since all the iterates are done inside the linear region. In Figure 2 we show the iterations of $G^u(4;+-+-)\cap H_0$.

Let $p=(-a_u,-a_s)$ be the saddle fixed point of the boundary of Λ. Define the *unstable foliation*

$$\mathcal{F}^u(\Lambda)=\overline{W^u(p)\cap R}=[-a_u,+a_u]\times K^s$$

and the *stable foliation*

$$\mathcal{F}^s(\Lambda)=\overline{W^s(p)\cap R}=K^u\times[-a_s,+a_s]\,.$$

Moreover, let

$$\mathcal{F}^u_U=g^2(\mathcal{F}^u\cap U)$$

and

$$\mathcal{F}^s_U=\mathcal{F}^s\cap g^2(U)\,,$$

as depicted in Figure 3. The *locus* of tangencies between leaves of $\mathcal{F}^u_U$ and leaves of $\mathcal{F}^s_U$ is the horizontal line $\{y=0\}$, often called the *line of tangencies* (see [14], Chap. 6). The intersection of $\mathcal{F}^s_U$ with the line of tangency is a piece of the Cantor set K^u and the intersection of $\mathcal{F}^u_U$ with the same line is a piece of the Cantor set $\tilde{K}^s=-a_u+\mu-\gamma(K^s+a_s)$ (all possible intersections between K^u and $\tilde{K}^s$ must happen for these pieces). It is clear that the two Cantor sets intersect if and only if there is a tangency between leaves of the foliations. Since $\tau^s\tau^u > 1$ by hypothesis, we apply Newhouse's Gap Lemma (see [[14], Chap.4]) to ensure that in fact *there is* an intersection of the two Cantor sets, independently of the choice of μ.

The definition of stable and unstable bridges and gaps in R naturally induces a definition of (one–dimensional) bridges and gaps for K^u and $\tilde{K}^s$, via projection on the line of tangency. For this reason, we adopt the following notation. Let π_1 be the projection in the first coordinate and π_2 the projection in the second. Then the bridges and gaps of K^u are the sets $Br^u(n;\underline{z}) = \pi_1(B^u(n;\underline{z}))$ and $Ga^u(n;\underline{z}) = \pi_1(G^u(n;\underline{z}))$, while the bridges and gaps of $\tilde{K}^s$ are given by

$$Br^s(n;\underline{z}) = -a_u + \mu - \gamma(\pi_2(B^s(n;\underline{z})) + a_s)$$

and

$$Ga^s(n;\underline{z}) = -a_u + \mu - \gamma(\pi_2(G^s(n;\underline{z})) + a_s) \ ,$$

for all $n \geq 1$ and $\underline{z} \in \{+,-\}^n$.

5 Wandering intervals in a one–dimensional toy model

In this section we introduce a one–dimensional model that captures at least a fraction of the complexity of the diffeomorphisms considered here. For this model we discuss a sufficient condition for the existence of wandering intervals that inspires the remaining sections. At the end, the ideas developed for the two–dimensional case can be used *ipsis literis* to show that this condition is satisfied for a particular deformation of the original one–dimensional model.

Let Φ be the first return function to $g^2(U) \cap \{x \geq -a_u\}$ and $i(x) = (x, 0)$ be the inclusion of the x–axis. The one–dimensional 'toy model' is the function $\theta(x) = \pi_1 \circ \Phi \circ i(x)$.

The function θ has the following description (see Figure 4): its domain dom(θ) is a union of intervals, each one inside (and concentric with) a gap of K^u. Some gaps of K^u do not intersect dom(θ). A criterium to know whether a gap $Ga^u(n;\underline{z})$ contains a nontrivial component of dom(θ) is to determine the vertical position y_s of the centre of $G^s(n;\underline{z}^{-1})$ and verify

$$\mu - \gamma(y_s - (-a_s)) > 0 \ .$$

If I is a connected component of dom(θ), then $\theta|I$ is a quadratic function and $\theta(\partial I) = -a_u$. More precisely, if c is the critical point in the centre of I and $I \subset Ga^u(n;\underline{z})$ then $\theta(x) = \theta(c) - \beta\sigma^{2n}x^2$, $x \in I$. The critical value $\theta(c)$ coincides with the centre of the gap $Ga^s(n;\underline{z}^{-1})$ of the Cantor set $\tilde{K}^s$.

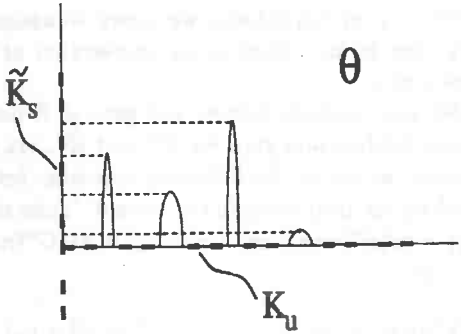

Figure 4: The one–dimensional toy model.

If μ is allowed to vary as a parameter we characterize a family $(\theta_\mu)_\mu$. As μ increases, the Cantor set $\tilde{K}^s = \tilde{K}^s_\mu = \mu + \tilde{K}^s_0$ is translated over K^u, and all defined critical values $\theta_\mu(c)$ go together, attached to the centres of the gaps (of course new connected components of the domain appear inside empty gaps of K^u).

A special situation arises when there is $c \in I'_\mu \subset I$ such that $\theta(I'_\mu) \subset I'_\mu$ and $\theta(\partial I'_\mu) \subset \partial I'_\mu$. This happens for values of μ such that $\theta_\mu(c)$ is near c. In other words, there is a parameter interval J such that for all $\mu \in J$ such an interval I'_μ does exist and for some $\mu_0 \in J$ we have $\theta_{\mu_0}(c) = c$. In particular, for some values $\mu \in J$ the function $\theta_\mu|I'_\mu$ presents a sink.

Newhouse ideas can be used in a simple way to show that for a residual set of parameters μ the function θ_μ presents infinitely many sinks (provided that $\tau^s \cdot \tau^u > 1$). The proof is based on two principles: i) whenever $K^u \cap (\mu_* + \tilde{K}^s_0) \neq \emptyset$ then there is a sequence of intervals $(\tilde{J}_k)_k$ accumulating on μ_* such that θ_μ presents a sink for all $\mu \in \tilde{J}_k$, $k \geq 1$ (and each sink is contained in a different gap of K^u); (ii) the thickness condition $\tau^s \cdot \tau^u > 1$ implies that $K^u \cap (\mu + \tilde{K}^s_0) \neq \emptyset$ for all μ (small). See for example [14] for more details.

Suppose θ has a *critical chain*, i.e. there is a sequence of critical points $(c_k)_{k\geq 1}$, $c_k \neq c_{k'}$ for $k \neq k'$, such that $\theta(c_k) = c_{k+1}$ for all $k \geq 1$ (although it seems extremely hard to satisfy this requirement, it is exactly the aim of Sections 6 and 7 to show that this is the case for some suitable perturbations of the original function).

Let $G^u_k = Ga^u(n_k; \underline{z}^{(k)})$ be the gap where c_k lies, for each $k \geq 1$. According to the description above, θ has a critical chain if and only if the centre of $Ga^s(n_k; [\underline{z}^{(k)}]^{-1})$ coincides with the centre of G^u_{k+1}, for all $k \geq 1$ (or equivalently the same assertion made with stable and unstable bridges, since the centre of each gap is the centre of the smaller bridge that contains it). In dimension two this will be the definition of a critical chain, since there are no critical points.

We claim, in our one–dimensional model, that if

$$\sum_{k=1}^{\infty} \frac{n_k}{2^k} < \infty$$

then there is a sequence of intervals $(T_k^*)_{k\geq 1}$, $c_k \in T_k^* \subset Ga^u(n_k; \underline{z}^{(k)})$ such that $\theta(T_k^*) \subset T_{k+1}^*$ and $\theta(\partial T_k^*) \subset \partial T_{k+1}^*$ for all $k \geq 1$. Therefore T_1^* is the wandering interval we were looking for.

Observe that $\theta(\partial T_k^*) \subset \partial T_{k+1}^*$ obliges T_k^* to be symmetric with respect to c_k. Let $l_k^* = \frac{1}{2}|T_k^*|$ be the radius of T_k^*. The problem of finding (T_k^*) now reduces to look for a sequence of positive numbers $(l_k^*)_{k\geq 1}$ such that

$$\beta \sigma^{2n_k} (l_k^*)^2 = l_{k+1}^* \, .$$

A solution to this problem is given by $l_k^* = b_k$, where

$$b_k = \beta^{-1} \sigma^{-\sum_{i=0}^{\infty} \frac{n_{k+i}}{2^i}} \, ,$$

which is positive for every $k \geq 1$ if and only if the summability condition above is satisfied (but we are not saying that if the sum diverges then there is no other positive solution).

An explicit solution to the sequence $(T_k^*)_k$ was found only because of the simplicity of the relation between the sizes of the intervals. This simplicity comes from the requirement that $\theta(c_k)$ coincides exactly with c_{k+1}, for all $k \geq 1$. Notwithstanding, the proof of the existence of the sequence of intervals could be performed even if it was done a weaker condition on the relative position of critical points and critical values. The method of proof is inspiring for Section 8, in the two–dimensional onset, where intervals become irregular rectangles and the width is not well defined. We point out however that the weaker condition is as difficult to satisfy as the original one, and there is in fact no gain in doing this for the one–dimensional onset.

Let $(b_k)_{k\geq 1}$ be defined as above, with the summability condition satisfied. This means that $b_k > 0$ for all $k \geq 1$. Suppose that

$$|\theta(c_k) - c_{k+1}| < \frac{1}{10} b_{k+1}$$

for all $k \geq 1$ (we say that θ has a *relaxed critical chain*). We claim that there is a sequence of intervals $(T_k^*)_{k\geq 1}$, each T_k^* centred at c_k such that $\theta(T_k^*) \subset T_{k+1}^*$ and $\theta(\partial T_k^*) \subset \partial T_{k+1}^*$.

To prove the claim, first consider the sequence $(T_k^0)_k$ where each T_k^0 is the connected component of $\mathrm{dom}(\theta)$ to which c_k belongs. Then $\theta(\partial T_k^0) \cap T_{k+1}^0 = \emptyset$, since $\theta(\partial T_k^0) = \{-a_u\}$

and all the T_k^0's are placed to the right of $-a_u$. Let l_k^0 be the radius of T_k^0, for all $k \geq 1$. We can give a lower bound for these radia. As

$$l_k^0 = \beta^{-1/2}\sigma^{-n_k}\sqrt{|\theta(c_k) - (-a_u)|}$$

and, for μ small (and hence n_1 big),

$$|\theta(c_k) - c_{k+1}| < \frac{1}{10}b_{k+1} << a_0\sigma^{-n_{k+1}} \leq |c_{k+1} - (-a_u)| ,$$

where a_0 is the half of the width of V_0, then

$$l_k^0 >> \frac{1}{10}b_k .$$

This implies that $\theta(c_k) \in T_{k+1}^0$ for all $k \geq 1$.

Once defined the starting sequence $(T_k^0)_k$ we inductively define the sequence $(T_k^{s+1})_k$ from the sequence $(T_k^s)_k$. The induction step is authorized if $\theta(c_k) \in T_{k+1}^s$ and $\theta(\partial T_k^s) \cap T_{k+1}^s = \emptyset$, for all $k \geq 1$. In that case,

$$T_k^{s+1} = \theta^{-1}(T_{k+1}^s)$$

for every $k \geq 1$.

Let l_k^s be the radius of T_k^s, for all $k \geq 1$ and $s \geq 0$. The procedure above implies that for a fixed $k \geq 1$ the sequence $(l_k^s)_s$ decreases as s increases, whenever the induction step is allowed. If the induction is defined for every $s \geq 0$ and $l_k^s \to l_k^* > 0$ as $s \to \infty$ then the intervals $T_k^* = [c_k - l_k^*, c_k + l_k^*]$ are the solution to our problem.

So it remains to show that the induction step is defined for any $s \geq 0$ and that the infimum of the sequence $(l_k^s)_s$ is positive for every $k \geq 1$. Suppose that the step s of the induction is done. Observe first that

$$l_k^{s+1} = \beta^{-1/2}\sigma^{-n_k}\sqrt{l_{k+1}^s + \theta(c_k) - c_{k+1}}.$$

So, if $l_{k+1}^s > \frac{1}{2}b_{k+1}$, we have that

$$l_k^{s+1} > \sqrt{\frac{3}{5}}b_k > \frac{1}{2}b_k .$$

As $l_k^0 > \frac{1}{2}b_k$ for all $k \geq 1$ then, by induction, $l_k^s > \frac{1}{2}b_k$ for every s. This will imply positiveness of the infimum and that $\theta(c_k) \in T_{k+1}^s$ for every s and k. The condition $\theta(\partial T_k^{s+1}) \cap T_{k+1}^{s+1} = \emptyset$ is clearly implied by the fact that $\theta(\partial T_k^{s+1}) \subset \partial T_{k+1}^s$ and $T_{k+1}^{s+1} \subset T_{k+1}^s$.

6 Linking Lemma

Up until now, we have given no serious restriction to the choice of μ, except that it must be small (and positive) according to the choice of U', just in order to satisfy Property (7) in Section 3. Suppose μ is inside this range. We have also seen that Property (5) and Newhouse's Gap Lemma guarantees that there is always at least one point of intersection between K^u and $\tilde{K}^s$. The content of the Linking Lemma we discuss in the sequel is that for some residual set of choices of μ near the origin there are in fact infinitely many intersections (or even a Cantor set of intersections) between K^u and $\tilde{K}^s$. However, we must be more precise about the way these intersections are obtained and then choose the suitable μ.

The Linking Lemma has been similarly applied in [2] to prove the cohexistence of infinitely many Hénon–like strange attractors, but it appeared before in [4].

Let $Br^u = Br^u(n;\underline{z})$ and $Br^s = Br^s(m;\underline{w})$, where $\underline{z} = (z_1, \dots, z_n)$ and $\underline{w} = (w_1, \dots, w_m)$, $n, m \geq 1$. We say that Br^u and Br^s are *linked* or, equivalently, (Br^u, Br^s) is a *linked pair* if $Br^u \cap Br^s \neq \emptyset$, Br^u is not contained inside a gap of $Br^s \cap \tilde{K}^s$ and Br^s is not contained in a gap of $Br^u \cap K^u$. As $Br^u \cap K^u$ and $Br^s \cap \tilde{K}^s$ are affine images of K^u and $\tilde{K}^s$, the product of their thickness is still greater than one, implying that if Br^u and Br^s are linked then $(Br^u \cap K^u) \cap (Br^s \cap \tilde{K}^s) \neq \emptyset$, by Newhouse's Gap Lemma.

A linked pair (Br^u, Br^s) is said to be *ξ–linked*, for $\xi \geq 0$, if

$$|Br^u \cap Br^s| \geq \xi \min\{|Br^u|, |Br^s|\} .$$

We say that the pair (Br^u, Br^s) is *proportional* if

$$\lambda |Br^u| < |Br^s| \leq |Br^u| .$$

In this case there is $A = A(\lambda, \sigma, a_s, a_u, \gamma) > 0$ such that

$$n - \frac{\log \lambda^{-1}}{\log \sigma} m \in [-A, A] .$$

Two bridges $Br_1^{u(s)}$ and $Br_2^{u(s)}$ are said to be *related* if they are the two maximal bridges properly contained in another bridge $\widehat{Br}^{u(s)}$.

Linking Lemma. *Let $\xi_0 = \frac{1}{2}\frac{\tau^u - 1}{\tau^u}$ and suppose that (Br^u, Br^s) is a linked pair. Then for any $\epsilon > 0$ there are $|\Delta| < \epsilon$, related unstable bridges Br_1^u, Br_2^u contained in Br^u and related stable bridges Br_1^s, Br_2^s contained in Br^s such that $(\Delta + Br_1^s, Br_1^u)$ and $(\Delta + Br_2^s, Br_2^u)$ are ξ_0–linked proportional pairs. Moreover*

$$|Br_i^u| \geq |Br_i^s| \geq \frac{\lambda^3 \epsilon}{2}$$

for $i = 1,2$ and whenever $\epsilon < |Br^u \cap Br^s|$.

Proof. If $\epsilon < |Br^u \cap Br^s|$ then let $\hat{b}_s$ and $\hat{b}_u$ be such that

$$\lambda^2\frac{\epsilon}{2} \leq \hat{b}_s < \lambda\frac{\epsilon}{2}\ ,\ \sigma^{-1}\hat{b}_u < \hat{b}_s \leq \hat{b}_u\ ,$$

and take the collection of all stable bridges of size $\hat{b}_s$ and all unstable bridges of size $\hat{b}_u$. As (Br^u, Br^s) is a linked pair, $(Br^u \cap K^u)$ and $(Br^s \cap \tilde{K}^s)$ have a common point, which implies that there is at least one linked pair $(\widehat{Br}^u, \widehat{Br}^s)$ with these sizes.

Note that $|\widehat{Br}^u| + |\widehat{Br}^s| < (\lambda\sigma)\frac{\epsilon}{2} + \lambda\frac{\epsilon}{2} < \epsilon$, so that there is an interval $\hat{I}$ of size smaller than ϵ for which $(\Delta + \widehat{Br}^s) \cap \widehat{Br}^u \neq \emptyset$ if and only if $\Delta \in \hat{I}$.

Take $\Delta \in \hat{I}$ such that the centre of $\Delta + \widehat{Br}^s$ coincides with the centre of $\widehat{Br}^u$. Denote by $\widehat{Ga}^u$ and $\widehat{Ga}^s$ the gaps in the centre of $\widehat{Br}^u$ and $\widehat{Br}^s$, respectively.

The thickness condition $\tau^u\tau^s > 1$ implies that $\max\{\tau^u, \tau^s\} > 1$. But $\tau^s < \tau^u$, since $\lambda < \sigma^{-1}$, hence $\tau^u > 1$. Therefore, each one of the two related bridges Br_1^u and Br_2^u contained in $\widehat{Br}^u$ of size $\sigma^{-1}|\widehat{Br}^u|$ must be bigger than $\widehat{Ga}^u$ (more precisely, $\sigma^{-1}|\widehat{Br}^u| = \tau^u|\widehat{Ga}^u|$). This implies, by the choice of $\hat{b}_u$ and $\hat{b}_s$, that $|\widehat{Br}^s| > |\widehat{Ga}^u|$, i.e. $\widehat{Ga}^u \subset \Delta + \widehat{Br}^s$.

Suppose that Br_1^u is to the left of Br_2^u and let Br_1^s, Br_2^s be the bridges contained in $\widehat{Br}^s$ of size $\lambda|\widehat{Br}^s|$, with Br_1^s to the left of Br_2^s.

The pairs (Br_i^u, Br_i^s), $i = 1,2$, are proportional, since

$$|Br_i^s| = \lambda|\widehat{Br}^s| \leq \lambda|\widehat{Br}^u| < \sigma^{-1}|\widehat{Br}^u| = |Br_i^u|$$

and

$$|Br_i^s| = \lambda|\widehat{Br}^s| > \lambda\sigma^{-1}|\widehat{Br}^u| = \lambda|Br_i^u|\ .$$

To prove that these pairs are ξ_0–linked, we consider two cases (see Figure 5): (a) $\Delta + \widehat{Ga}^s \subset \widehat{Ga}^u$; and (b) $\widehat{Ga}^u \subset \Delta + \widehat{Ga}^s$.

In case (a), they are automatically linked, since for each $i = 1,2$ only one of the boundary points of $\Delta + Br_i^s$ belongs to Br_i^u. Moreover,

$$\frac{|Br_i^s \cap Br_i^u|}{|Br_i^s|} = \frac{\frac{1}{2}(|\widehat{Br}^s| - |\widehat{Ga}^u|)}{\lambda|\widehat{Br}^s|} > \frac{1}{2}(\tau^u - 1)\frac{|\widehat{Ga}^u|}{\sigma^{-1}|\widehat{Br}^u|} = \xi_0\ ,$$

hence they are ξ_0–linked.

In case (b), $\Delta + Br_i^s \subset Br_i^u$, for $i = 1,2$. We have only to show that $\Delta + Br_i^s$ is not contained inside a gap of $Br_i^u \cap K^u$. Suppose by contradiction that this happens and take

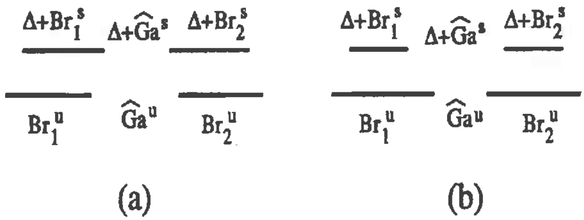

Figure 5: Two cases in the proof of the Linking Lemma.

$i = 1$, without loss of generality (or simply by simmetry). Let $\widetilde{Ga}^u$ be the referred gap, and $\widetilde{Br}^u$ be the greatest bridge adjacent to the right boundary point of $\widetilde{Ga}^u$. Clearly $\widetilde{Br}^u \subset \Delta + \widehat{Ga}^s$, hence

$$\frac{|\widetilde{Br}^u|}{|\widehat{Ga}^s|} \cdot \frac{|Br_1^s|}{|\widetilde{Ga}^u|} < 1 .$$

On the other hand,

$$\frac{|\widetilde{Br}^u|}{|\widetilde{Ga}^u|} \cdot \frac{|Br_1^s|}{|\widehat{Ga}^s|} > \tau^u \tau^s > 1 ,$$

a contradiction. As $|Br_i^s| = \lambda \hat{b}_s$ and $|Br_i^u| = \sigma^{-1}\hat{b}_u$, for $i = 1, 2$, the inferior bounds for their sizes follow. □

A consequence of the Lemma above is that for a residual set of translations of $\tilde{K}^s$ (each translation corresponding to a choice of μ) the intersection $K^u \cap \tilde{K}^s$ contains a Cantor set (see [4]). However, we use its precise statement to derive the following Lemma, adapted to the purposes of Section 8. It says that for a convenient small translation there are infinitely many linked pairs, with *generations growing linearly*. This will be much more than we need in order to satisfy the summability condition of the previous section.

Linear Growth Lemma. *Let $\xi_0 = \frac{1}{2}\frac{\tau^u - 1}{\tau^u}$ and $\mu > 0$ be chosen accordingly to the choice of U'. For any $\epsilon > 0$, there are $|\Delta| < \epsilon$ and collections of bridges $(Br_k^u)_{k \geq 1}$, $(Br_k^s)_{k \geq 1}$ such that $(\Delta + Br_k^s, Br_k^u)$ is a $\frac{\xi_0}{2}$–linked proportional pair for each $k \geq 1$. If n_k (respectively m_k) is the generation of Br_k^u (respectively Br_k^s) then*

$$m_k < m_{k+1} \leq m_k + N_s$$

$$n_k < n_{k+1} \leq n_k + N_u ,$$

where $N_s = N_s(\lambda, \xi_0)$ and $N_u = N_u(\lambda, \sigma, \xi_0)$.

Proof. Take the first n_0 such that $2a_u\sigma^{-(n_0+1)} \leq \mu$. This means that the bridge $Br_0^u = Br^u(n_0; ++\ldots+)$ has the left boundary point at $-a_u$ and the right one at the right of $-a_u+\mu$. On the other hand, take the first m_0 such that $2a_s\gamma\lambda^{m_0+1} \leq \mu$. Then the bridge $Br_0^s = Br^s(m_0; ++\ldots+)$ has its right boundary point at $-a_u+\mu$ and the left one at the left of $-a_u$. Therefore (Br_0^u, Br_0^s) is a linked pair and $|Br_0^u \cap Br_0^s| = \mu$.

By the Linking Lemma, for any $\epsilon < \mu$ there are $|\Delta_1| < \epsilon$, related unstable bridges Br_1^u, $\widetilde{Br}_1^u$ and related stable bridges Br_1^s, $\widetilde{Br}_1^s$ such that $(\Delta_1 + Br_1^s, Br_1^u)$ and $(\Delta_1 + \widetilde{Br}_1^s, \widetilde{Br}_1^u)$ are ξ_0–linked proportional pairs. Moreover, $|\widetilde{Br}_1^s| = |Br_1^s| \leq \frac{\lambda^2\epsilon}{2}$, $|\widetilde{Br}_1^u| = |Br_1^u| \leq \frac{\lambda\epsilon}{2}$ and $|Br_1^u| \geq |Br_1^s| > \frac{\lambda^3\epsilon}{2}$.

We proceed by induction. Let $\xi_j = \xi_0(1 - \sum_{i=1}^{j} 4^{-i})$ for all $j \geq 1$ and suppose that for any $k \geq 1$ we have:

1. real numbers $\Delta_1, \Delta_2, \ldots, \Delta_k$ such that
$$|\Delta_t| < \left(\frac{\lambda^2\xi_0}{8}\right)^{t-1} \epsilon$$
for all $t = 1, \ldots, k$;

2. bridges Br_t^u and Br_t^s, for $t = 1, \ldots, k$, such that $(\Delta_1 + \ldots + \Delta_k + Br_t^s, Br_t^u)$ is a ξ_{k-t}–linked proportional pair;

3.
$$\frac{\lambda^3\xi_0}{8}|Br_{t-1}^s| \leq |Br_t^s| \leq \frac{\lambda^2\xi_0}{8}|Br_{t-1}^s|$$
for all $t = 2, \ldots, k$;

4. a bridge $\widetilde{Br}_k^u$ related to Br_k^u and a bridge $\widetilde{Br}_k^s$ related to Br_k^s such that $(\Delta_1 + \ldots \Delta_k + \widetilde{Br}_k^s, \widetilde{Br}_k^u)$ is a ξ_0–linked proportional pair.

We will prove that the same is valid for $k+1$. At the end, the value $\Delta = \sum_{t=1}^{\infty} \Delta_t$ fits the statement of the Lemma.

Applying the Linking Lemma to the linked pair $(\Delta_1 + \ldots + \Delta_k + \widetilde{Br}_k^s, \widetilde{Br}_k^u)$ we obtain $|\Delta_{k+1}| < \epsilon_k \equiv \frac{\xi_0}{4}|\widetilde{Br}_k^s|$ and related bridges $Br_{k+1}^u, \widetilde{Br}_{k+1}^u \subset \widetilde{Br}_k^u$, $Br_{k+1}^s, \widetilde{Br}_{k+1}^s \subset \widetilde{Br}_k^s$ such that $(\Delta_1 + \ldots + \Delta_{k+1} + Br_{k+1}^s, Br_{k+1}^u)$ and $(\Delta_1 + \ldots + \Delta_{k+1} + \widetilde{Br}_{k+1}^s, \widetilde{Br}_{k+1}^u)$ are ξ_0–linked proportional pairs.

As $(\Delta_1 + \ldots + \Delta_k + Br_t^s, Br_t^u)$ is a ξ_{k-t}–linked proportional pair, for all $t = 1, \ldots, k$, then $(\Delta_1 + \ldots + \Delta_{k+1} + Br_t^s, Br_t^u)$ is a $(\xi_{k-t} - \frac{|\Delta_{k+1}|}{|Br_t^s|})$–linked proportional pair. But
$$\xi_{k-t} - \frac{|\Delta_{k+1}|}{|Br_t^s|} \geq \xi_0(1 - \sum_{i=1}^{k-t} 4^{-i}) - \frac{\xi_0}{4}\left(\frac{\lambda^2\xi_0}{8}\right)^{k-t} > \xi_{k+1-t}\,.$$

Finally, the Linking Lemma assures that

$$|Br^s_{k+1}| \geq \lambda^3 \frac{\epsilon_k}{2} = \frac{\lambda^3 \xi_0}{8} |Br^s_k| ,$$

proving the relation between m_{k+1} and m_k, for

$$N_s = \frac{\log \left(\frac{\lambda^3 \xi_0}{8}\right)^{-1}}{\log \lambda^{-1}} .$$

As the links are proportional,

$$\frac{\lambda^4 \xi_0}{8} |Br^u_k| < \frac{\lambda^3 \xi_0}{8} |Br^s_k| \leq |Br^s_{k+1}| \leq |Br^u_{k+1}| ,$$

proving the relation between n_{k+1} and n_k, for

$$N_u = \frac{\log \left(\frac{\lambda^4 \xi_0}{8}\right)^{-1}}{\log \sigma} .$$

□

7 Critical chains via perturbation

In Section 3 we have defined the starting diffeomorphism g, but we left μ as an arbitrary parameter. Now we fix μ as $\mu + \Delta$, according to the Linear Growth Lemma in Section 6, in order to have a sequence of $\frac{\xi_0}{2}$–linked proportional pairs $(\widetilde{Br}^u_k, \widetilde{Br}^s_k)$, $k \geq 1$, with generations $\tilde{n}_k$ and $\tilde{m}_k$, such that

$$\tilde{m}_k < \tilde{m}_{k+1} \leq \tilde{m}_k + N_s ,$$

$$\tilde{n}_k < \tilde{n}_{k+1} \leq \tilde{n}_k + N_u ,$$

for some $1 \leq N_s, N_u < \infty$.

We now describe the kind of C^∞ perturbations we aim at using to satisfy a Chain Property like that of Section 5. First, consider a non–negative and non–decreasing C^∞ function $\phi : \mathbf{R} \to \mathbf{R}$, null in $\{x \leq -1\}$ and equal to 1 in $\{x \geq 0\}$. Let $\rho > 0$, $I = [a, b]$ be an interval and define

$$\phi_{\rho,I}(x) = \left[\phi\left(\frac{x-a}{\rho|I|}\right) + \phi\left(-\frac{x-b}{\rho|I|}\right) - 1\right]$$

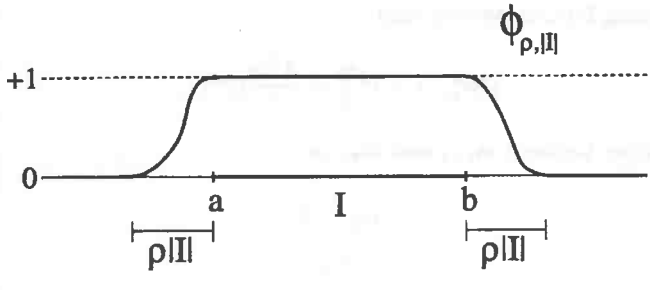

Figure 6: Bump function on an interval I.

(see Figure 6). If $\|\cdot\|_r$ is the norm given by the derivatives until order r then

$$\|\phi_{\rho,I}\|_r \leq \frac{1}{(\rho|I|)^r}\|\phi\|_r \, .$$

Recall that $U' \subset [-\delta,\delta]\times[-a_s,-a_s+a]$, and define

$$\psi = \phi_{\frac{1}{4},[-\delta,+\delta]} \, .$$

Let $\widetilde{Br}^s_k$ be written as $\widetilde{Br}^s_k = Br^s(\tilde{m}_k;\underline{\tilde{w}}^{(k)})$ and take the interval

$$I_k = \pi_2\left(B^s(\tilde{m}_k;\underline{\tilde{w}}^{(k)})\right) \, ,$$

i.e. the y–projection of the stable bridge $B^s(\tilde{m}_k;\underline{\tilde{w}}^{(k)})$. As the projections of the adjacent stable gaps to this bridges have length bigger than $\frac{1}{\tau^s}|I_k|$, the functions

$$\phi_k = \phi_{\frac{1}{3\tau^s},I_k} \, , \; k \geq 1 \, ,$$

have disjoint supports.

Let $\underline{t} = (t_1,t_2,\ldots,t_k,\ldots)$ be a sequence of real numbers and let

$$h_{\underline{t}}(x,y) = \left(x, y - \gamma^{-1}\psi(x)\sum_{k=1}^{\infty} t_k\phi_k(y)\right) \, .$$

Then

$$\|h_{\underline{t}} - \mathrm{Id}\|_r \leq C\gamma^{-1}(3\tau^s)^r \sum_{k=1}^{\infty}\frac{t_k}{|I_k|^r}\|\phi\|_r \, .$$

We call $\underline{t}$ a *perturbation vector*. If $(\underline{t}^{(M)})_{M\geq 1}$ is a sequence of perturbation vectors, with $\underline{t}^{(M)} = (t_1^M, \ldots, t_k^M, \ldots)$, satisfying

$$|t_k^M| < \lambda^{Mk} ,$$

then $h_{\underline{t}^{(M)}} \to \mathrm{Id}$ as $M \to \infty$ in the C^r topology, for any $r \geq 0$ (since $|I_k| \geq 2a_s\lambda^{\tilde{m}_k}$ and $\tilde{m}_k \leq \tilde{m}_1 + N_s k$).

For a sufficiently large M we define the diffeomorphism

$$f = f_M = g \circ h_{\underline{t}^{(M)}} .$$

Let us examine the effect of this perturbation to the dynamics. Since $f = g$ outside U, the horseshoe is preserved and the dynamics near it remains the same. The only change is in the image of the stable bridges and gaps that cross U. More precisely, if $B^s(m;\underline{w}) \subset B^s(\tilde{m}_k;\underline{\tilde{w}}^{(k)})$ then

$$f^2(x,y) = (t_k^M, 0) + g(x,y)$$

for every $(x,y) \in B^s(m;\underline{w}) \cap \{-\delta \leq x \leq \delta\}$. In particular, the correspondent bridge $Br^s(m;\underline{w})$ is translated by t_k^M in the line of tangency.

Critical Chain Lemma. *For every $M >> N_s$ there is a sequence $\underline{t} = (t_1, t_2, \ldots, t_k, \ldots)$ with $t_1 = 0$, $|t_k| < \lambda^{Mk}$, $k \geq 1$, and a bridge $Br_k^u = Br^u(n_k;\underline{z}^{(k)}) \subset \widetilde{Br}_k^u$, such that the centre of $t_{k+1} + Br^s(n_k;[\underline{z}^{(k)}]^{-1})$ coincides with the centre of $Br^u(n_{k+1};\underline{z}^{(k+1)})$, for every $k \geq 1$. Moreover, we can write*

$$\underline{z}^{(k)} = \underline{\hat{z}}^{(k)}\underline{z}_0^{(k)}[\underline{\hat{w}}^{(k+1)}]^{-1}$$

where $\underline{\hat{z}}^{(k)}$, $\underline{z}_0^{(k)}$, $\underline{\hat{w}}^{(k+1)}$ have sizes $\hat{n}_k$, n_k^0, $\hat{m}_{k+1}$, respectively, with

$$\hat{n}_k + \hat{m}_{k+1} \leq 5\frac{\log \lambda^{-1}}{\log \sigma}Mk$$

and n_k^0 arbitrary.

Proof. As in the Linking Lemma, let $\widehat{Br}_k^s \subset \widetilde{Br}_k^s$ and $\widehat{Br}_k^u \subset \widetilde{Br}_k^u$ form a linked pair, with sizes

$$\lambda^2 \cdot \frac{\lambda^{Mk}}{2} \leq |\widehat{Br}_k^s| < \lambda\frac{\lambda^{Mk}}{2} ,$$

$$\sigma^{-1}|\widehat{Br}_k^u| < |\widehat{Br}_k^s| \leq |\widehat{Br}_k^u| .$$

Let J_k be the interval such that $t_k \in J_k$ if and only if $(t_k + \widehat{Br}^s_k) \cap \widehat{Br}^u_k \neq \emptyset$. By the choice of sizes, $|J_k| < \lambda^{Mk}$.

Let $\widehat{Br}^u_k$ and $\widehat{Br}^s_k$ be written as

$$\widehat{Br}^u_k = Br^u(\hat{n}_k; \hat{\underline{z}}^{(k)}) \ , \ \widehat{Br}^s_k = Br^s(\hat{m}_k; \hat{\underline{w}}^{(k)}) \ ,$$

and define $Br^u_k = Br^u(n_k; \underline{z}^{(k)})$, where $n_k = \hat{n}_k + n^0_k + \hat{m}_{k+1}$ and

$$\underline{z}^{(k)} = \hat{\underline{z}}^{(k)} \underline{z}^{(k)}_0 [\hat{\underline{w}}^{(k+1)}]^{-1}$$

and $\underline{z}^{(k)}_0$ is an arbitrary sequence of size n^0_k. With this definition, $Br^u_k \subset \widehat{Br}^u_k$ and $Br^s_{k+1} \equiv Br^s(n_k; [\underline{z}^{(k)}]^{-1}) \subset \widehat{Br}^s_{k+1}$, for all $k \geq 1$. Now it is enough to take $t_{k+1} \in J_{k+1}$ such that the centre of Br^s_{k+1} coincides with the centre of Br^u_{k+1} and the proof is complete. □

An important remark concerning the last Lemma is the freedom to choose the intermediate sequence $\underline{z}^0_k$, whose combinatorics is the responsible for the alternative ergodic properties deduced in Section 9. We fix from now on its size (and in Section 9 its code) and derive some quantitative information.

Take $n^0_k = k^2$, in such a way that

$$\frac{\hat{n}_k + \hat{m}_{k+1}}{n^0_k} \to 0$$

as $k \to \infty$. At the same time, for every $\eta > 0$ there is $k_0 = k_0(\eta)$ such that

$$n_{k+1} < (1 + \eta) n_k$$

for every $k \geq k_0$.

Let η_0 be such that if one defines $\eta_1 = \frac{3\eta_0}{1-\eta_0}$ then $\lambda \sigma^{1+\eta_1} < 1$ and take $k_0 = k_0(\eta_0)$ as above. Without loss of generality we can suppose that $k_0 = 1$, by simply choosing the first gap of the critical chain as $Ga^u(n_{k_0}; \underline{z}^{(k_0)})$. By the same argument we can suppose n_1 sufficiently big with respect to some constants that depend only on the definition parameters of g.

If $\eta < 1$ then the numbers

$$b_k = \beta^{-1} \sigma^{-\sum_{i=0}^{\infty} \frac{n_{k+i}}{2^i}} \ , \ k \geq 1 \ ,$$

are positive. A lower bound is given by

$$b_k \geq \beta^{-1} \sigma^{-2n_k(1-\eta_0)^{-1}} \ ,$$

for all $k \geq 1$. On the other hand, as $n_{k+1} > n_k$,

$$b_k \leq \beta^{-1}\sigma^{-2}\sigma^{-2n_k}$$

for all $k \geq 1$. For the calculations of the next section, remember that

$$\beta\sigma^{2n_k}b_k^2 = b_{k+1} ,$$

for all $k \geq 1$.

Conclusion. If we take $f = g \circ h_{\underline{t}}$, where $\underline{t}$ is chosen accordingly to the Critical Chain Lemma, then we have a sequence of rectangles $(R_k)_{k\geq 1}$ with the following properties:

1. $R_k \subset G^u(n_k; \underline{z}^{(k)})$, where $\underline{z}^{(k)}$ is specified as above;
2. the centre of R_k is the centre $(x_k, 0)$ of $G^u(n_k; \underline{z}^{(k)})$; the height of R_k equals the height of $G^u(n_k; \underline{z}^{(k)}) \cap H_0$ and its width is equal to $2\delta\sigma^{-n_k}$; in other words,

$$R_k = f^{-n_k}\left(\{-\delta \leq x \leq +\delta\} \cap G^s(n_k; [\underline{z}^{(k)}]^{-1})\right) ;$$

3. let $F_k = f^{n_k+2}|R_k$; then

$$F_k : (x_k + x, y) \mapsto (x_{k+1} - \beta\sigma^{2n_k}x^2 \pm \gamma\lambda^{n_k}y, \pm\alpha\sigma^{n_k}x) ;$$

   in particular $F_k(x_k, 0) = (x_{k+1}, 0)$.

8 Construction of a restrictive chain of rectangles

We will say that R is a C^r*–rectangle*, $r \geq 0$, if

$$R = \{(x, y); -y_R \leq y \leq y_R, h_R^l(y) \leq x \leq h_R^r(y)\} ,$$

where h_R^l, h_R^r are C^r functions and $h_R^l(y) < h_R^r(y)$ for every $y \in [-y_R, y_R]$.

The boundary of R is the union of the four segments

$$\begin{aligned}
\partial_u R &= \{y = y_R, h_R^l(y_R) \leq x \leq h_R^r(y_R)\} , \\
\partial_b R &= \{y = -y_R, h_R^l(-y_R) \leq x \leq h_R^r(-y_R)\} , \\
\partial_l R &= \{-y_R \leq y \leq y_R, x = h_R^l(y)\} , \\
\partial_r R &= \{-y_R \leq y \leq y_R, x = h_R^r(y)\} ,
\end{aligned}$$

where the indices u, b, r and l come from *up*, *bottom*, *right* and *left*.

Let $(R_k)_{k\geq 1}$ be the sequence of rectangles in the Conclusion of Section 7, each R_k inside an unstable gap of generation n_k. We say that a sequence $(R_k^*)_{k\geq 1}$ where $R_k^* \subset R_k$

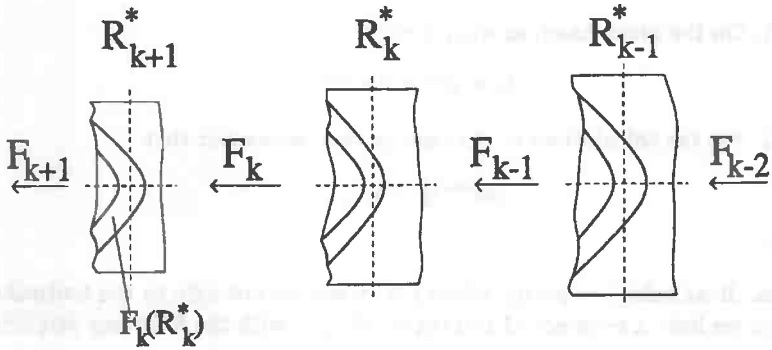

Figure 7: Restrictive critical chain of rectangles.

is a *restrictive critical chain of rectangles* if $F_k(R_k^*) \subset R_{k+1}^*$ and $F_k(\partial_l R_k^* \cup \partial_r R_k^*) \subset \partial_l R_{k+1}^*$ (see Figure 7).

Rectangles Lemma. *Let f be a diffeomorphism obtained from g as in Section 7. Then there is a restrictive critical chain of rectangles $(R_k^*)_{k\geq 1}$.*

In Section 9 we define $D = R_1^*$ and explore the outgoing consequences. The remaining of this Section is devoted to prove the Lemma above.

Let $y_k^* = 20\alpha\beta^{-1/2}\sqrt{b_k}$ and

$$\begin{aligned} R_k^{\text{int}} &= [x_k - \frac{1}{2}b_k, x_k + \frac{1}{2}b_k] \times [-y_k^*, y_k^*]\,, \\ R_k^{\text{ext}} &= [x_k - 10b_k, x_k + 10b_k] \times [-y_k^*, y_k^*]\,. \end{aligned}$$

We will inductively construct a sequence of C^1–rectangle sequences

$$((R_k^s)_{k\geq 1})_{s\geq 0}$$

such that for every $s \geq 0$ and $k \geq 1$ we have:

1. $y_{R_k^s} = y_k^*$;
2. $R_k^{s+1} \subset R_k^s$;
3. $R_k^1 = R_k^{\text{ext}}$;
4. $R_k^s \supset R_k^{\text{int}}$;

5. let $h^l_{k,s} \equiv h^l_{R^s_k}$ and $h^r_{k,s} \equiv h^r_{R^s_k}$; there is d small such that

$$|Dh^l_{k,s}|, |Dh^r_{k,s}| \le dy^*_k ;$$

6. $F_k(\partial_l R^s_k \cup \partial_r R^s_k) \cap R^s_{k+1} = \emptyset$;

7. $F_k(R^s_k) \cap \partial R^s_{k+1} = F_k(R^s_k) \cap \partial_l R^s_{k+1} \neq \emptyset$;

8. $\partial_l R^{s+1}_k \cup \partial_r R^{s+1}_k = F_k^{-1}(F_k(R^s_k) \cap \partial_l R^s_{k+1})$.

By (1)–(5) above, $h^{l(r)}_{k,s}$ converges, uniformly in the C^0 topology, to a C^0 function $h^{l(r)}_{k,*}$ in $y \in [-y^*_k, y^*_k]$, and this will define the left and right boundaries of R^*_k. From (6)–(8) it follows that $F_k(R^*_k) \subset R^*_{k+1}$ and $F_k(\partial_l R^*_k \cup \partial_r R^*_k) \subset \partial_l R^*_{k+1}$.

Width comparison. Given the above definition of y^*_k, we claim that the horizontal semi–width of $F_k(R^s_k)$ (equal to $\gamma\lambda^{n_k} y^*_k$) is much smaller than the semi–width of R^{int}_{k+1} (equal to $\frac{1}{2}b_{k+1}$). We compare the sizes by the quotient

$$\frac{\gamma\lambda^{n_k} y^*_k}{\frac{1}{2}b_{k+1}} = 40\alpha\gamma\lambda^{n_k}\sigma^{-n_k/2}\sigma^{\frac{3}{4}\sum_{i=0}^{\infty}\frac{n_{k+1+i}}{2^i}} .$$

By the remarks just after the Critical Chain Lemma, in Section 7, the quotient above is bounded by $40\gamma\alpha(\lambda\sigma^{1+\eta_1})^{n_k}$, which is small if n_1 is chosen sufficiently big. This estimate will be used in the proof of the Pre–images Lemma below.

We will show by induction in $s \ge 0$ that for every $k \ge 1$ and $y_0 \in [-y^*_k, y^*_k]$ the image under F_k of the segment $\{y = y_0\} \cap R^s_k$ (which we call a *parabolic segment*) intersects $\partial_l R^s_{k+1}$ transversally exactly two times. Moreover, if the pre–images of the two points are written as $(h^l_{k,s+1}(y_0), y_0)$ and $(h^r_{k,s+1}(y_0), y_0)$, then

$$h^l_{k,s}(y_0) < h^l_{k,s+1}(y_0) < x_k - \frac{1}{2}b_k < x_k + \frac{1}{2}b_k < h^r_{k,s+1}(y_0) < h^r_{k,s}(y_0) ,$$

where $h^l_{k,0} \equiv x_k - 10b_k$ and $h^r_{k,0} \equiv x_k + 10b_k$. This will give the next sequence $(R^{s+1}_k)_{k\ge1}$ in the induction process.

First step. If $x = \pm 10b_k$ and $|y| \le y^*_k$ and $(x_{k+1} + x', y') = F_k(x_k + x, y)$ then $|x'| > 90b_{k+1}$ and $|y'| \le y^*_{k+1}$. This can be shown with the expression of F_k and the Width Comparison above. The consequence is that the parabolic segments $F_k([x_k - 10b_k, x_k + 10b_k] \times y_0)$, for $|y_0| \le y^*_k$, cross $\{x - 10b_{k+1}\} \times \{|y| \le y^*_{k+1}\}$ exactly two times.

Induction. We have only to show that (i) if $|h^{l(r)}_{k,s} - x_k| > \frac{1}{2}b_k$ for all $k \ge 1$ then $|h^{l(r)}_{k,s+1} - x_k| > \frac{1}{2}b_k$, for all $k \ge 1$; (ii) if $|Dh^{l(r)}_{k,s}| < dy^*_k$ for all $k \ge 1$ then $|Dh^{l(r)}_{k,s+1}| < dy^*_k$;

(iii) the slope of the parabolic images in R_{k+1} with respect to the vertical is greater than dy^*_{k+1} if $|x - x_k| > \frac{1}{2}b_k$, i.e. transversal to the lateral boundaries of R^s_{k+1}, for any $s \geq 0$. These requirements will be proved by the two Lemmas below.

Let $d = \frac{1}{100} \cdot \frac{\beta}{100\alpha^2}$ and let C_k be a uniform cone field in R_k given by

$$C_k = C_k(x,y) = \{v = (v_x, v_y); |v_x| \leq dy^*_k |v_y|\} ,$$

for every $k \geq 1$.

Cone Lemma. *With the definitions above, if n_1 is big enough we have that*

$$DF_k^{-1}(x_{k+1} + x', y') \cdot C_{k+1}(x_{k+1} + x', y') \subset C_k(F_k^{-1}(x_{k+1} + x', y'))$$

whenever $|y'| \geq \frac{1}{10}\alpha\beta^{-1/2}\sqrt{b_{k+1}} = \frac{1}{200}y^*_{k+1}$.

Proof. Take

$$F_k^{-1}(x_{k+1} + x', y') = \left(x_k \pm \alpha^{-1}\sigma^{-n_k}y', \pm\gamma^{-1}\lambda^{-n_k}(x' + \beta\alpha^{-2}(y')^2)\right)$$

and

$$\begin{aligned} (v_x, v_y) &= DF_k^{-1}(x_{k+1} + x', y') \cdot (u_x, u_y) \\ &= \left(\pm\alpha^{-1}\sigma^{-n_k}u_y, \pm\gamma^{-1}\lambda^{-n_k}(u_x + 2\beta\alpha^{-2}y'u_y)\right) . \end{aligned}$$

If y' is as above and $(u_x, u_y) \in C_{k+1}$ then

$$\frac{|u_x|}{2\beta\alpha^{-2}|y'| \cdot |u_y|} \leq \frac{100\alpha^2 d}{\beta} < \frac{1}{100} ,$$

hence

$$\frac{|v_x|}{|v_y|} < \frac{\alpha\gamma}{\beta} \cdot \frac{\sigma^{-n_k}\lambda^{n_k}}{|y'|} < \frac{10\gamma}{\beta^{1/2}} \cdot \frac{\sigma^{-n_k}\lambda^{n_k}}{\sqrt{b_{k+1}}} .$$

Comparing with $dy^*_k = 20d\alpha\beta^{-1/2}\sqrt{b_k}$ we get

$$\frac{|v_x|/|v_y|}{dy^*_k} < \frac{\gamma\beta}{2d\alpha}(\lambda\sigma^{1+m})^{n_k} ,$$

which is smaller than one for a sufficiently big n_1, proving the cone invariance.

□

Pre–images Lemma. *Let $(x_{k+1} + x', y') = F_k(x_k + x, y)$ and $(u_x, u_y) = DF_k(x_k + x, y) \cdot (1, 0)$. If $|x'| > \frac{1}{2}b_{k+1}$ and $|y| \leq y^*_k$ then*

1. $|y'| > \frac{1}{40} y^*_{k+1}$;

2. $\frac{|u_x|}{|u_y|} > d y^*_{k+1}$;

3. $|x| > \frac{1}{2} b_k$.

Proof. As $x' = -\beta\sigma^{2n_k} x^2 \pm \gamma\lambda^{n_k} y$, then

$$\beta\sigma^{2n_k} x^2 \geq \frac{1}{2} b_{k+1} - \gamma\lambda^{n_k} y^*_k > \frac{1}{3} b_{k+1} ,$$

by the estimate in the Width Comparison. Hence

$$|x| \geq \frac{1}{\sqrt{3}} \beta^{-1/2} \sigma^{-n_k} \sqrt{b_{k+1}} = \frac{1}{\sqrt{3}} \sqrt{b_k} > \frac{1}{2} b_k ,$$

proving (3). On the other hand,

$$|y'| = \alpha\sigma^{n_k} |x| > \frac{1}{2} \alpha\beta^{-1/2} \sqrt{b_{k+1}} = \frac{1}{40} y^*_{k+1} ,$$

proving (1). Finally

$$(u_x, u_y) = (-2\beta\sigma^{2n_k} x, \pm\alpha\sigma^{n_k}) ,$$

and therefore

$$\frac{|u_x|}{|u_y|} \geq \frac{\beta}{\alpha} \sigma^{n_k} b_k = \frac{\beta}{20\alpha^2} y^*_{k+1} = 500 d y^*_{k+1} ,$$

proving (2). □

9 Properties of the wandering domain

In this section we derive the properties stated in Theorem 1 about D, which is defined as the rectangle R^*_1 in Section 8.

Diameter. Each rectangle R^*_k has height $2y^*_k$ proportional to $\sqrt{b_k}$ and width smaller than $20 b_k$, and b_k goes to zero as k goes to ∞. In the n_k subsequent iterates, the height is shrinked by a factor of λ^{n_k}, although the width is enlarged by a factor of σ^{n_k}. But at the end, the width of $f^{n_k}(R^*_k)$ is of the order of the height of R^*_{k+1}. This proves that the diameter of $f^n(R^*_1)$ goes to zero with n.

Description of ω_D. Let E be the intersection $G^u \cap G^s$ between an unstable and a stable gap, eventually including the central gaps. Then the future orbit of R^*_1 intersects E at most one time. This is because each R_k^* lies inside a different unstable gap (intersected with H_0) and different gaps have disjoint orbits until hitting V_0. The consequence is that E contains no accumulation point of the orbit of D, i.e.

$$\omega_D \cap R \subset \mathcal{F}^s(\Lambda) \cup \mathcal{F}^u(\Lambda) .$$

Let us see who are the accumulation points inside $f^2(U)$. They are exactly the accumulation points of the sequence $(R_k^*)_k$. As each R_k^* is inside a different unstable gap, the accumulation points must lie in $\mathcal{F}^s(\Lambda)$. Moreover, as the rectangles are centred in $(x_k, 0)$ and the diameters are shrinking, the accumulation points are on the line of tangencies ($\{y = 0\}$ in our example). On the other hand, for each $k \geq 2$, R_k^* intersects the image under f^2 of a stable gap, hence the accumulation points are contained in $f^2(\mathcal{F}^u(\Lambda))$. Therefore the accumulation points in $f^2(U)$ are contained in the set of tangencies between the two foliations. The same is clearly true for U and $f(U)$. The set $f(U)$ is also the only place where ω_D can intersect the complement of R, otherwise the orbits escape to the sink q_+, according to the Property (7) of Section 3.

By the same reason, every point of ω_D in $\mathcal{F}^s(\Lambda) \setminus \Lambda$ has a point in the pre–orbit which is in $f^2(U)$, hence, by the invariance of ω_D and the above, it belongs to $W^u(\Lambda) \cap W^s(\Lambda)$. At the same time, every point of ω_D in $\mathcal{F}^u(\Lambda) \setminus \Lambda$ has a point in the future orbit in U, hence in $W^u(\Lambda) \cap W^s(\Lambda)$.

Therefore we have proved that $\omega_D \subset W^s(\Lambda) \cap W^u(\Lambda)$, with a nonempty intersection outside Λ. It remains to show that there is a nonempty intersection with Λ.

But a point in $W^s(\Lambda) \setminus \Lambda$ accumulates in Λ, hence by the invariance of ω_D it intersects Λ.

Escaping points. Every point of $z \in \Lambda$ is accumulated by unstable gaps, by the two sides of the stable leaf of $\mathcal{F}^s(\Lambda)$ to which it belongs. Every unstable gap eventually hits V_0 after an appropriate number of iterations, and the corresponding image crosses V_0 horizontally. Hence there are points of this image outside U, in particular every point outside a vertical strip containing U, and these points escape to the sink q_+. Pushing backwards, this means that z is surrounded (and accumulated) by open vertical strips made of points that escape to the sink q_+. Therefore every open set W such that ∂W intersects z must intersect these vertical strips and hence contains points that escape to the sink.

Accumulation measures. Let $\nu(x)$, $x \in D$, be an accumulation measure of the sequence $\nu_l(x)$ given in Section 1. This means that there is a sequence l_j such that

$$\frac{1}{l_j} \sum_{i=0}^{l_j - 1} \delta_{f^i(x)} \to \nu(x)$$

as $j \to \infty$.

Clearly $\text{supp}(\nu(x)) \subset \omega_D$. We will show that for every $z \in \omega_D \setminus \Lambda$ there is $\epsilon > 0$ such that $\nu(x)(B_\epsilon(z)) = 0$. As such a point z has a point of the orbit inside $f^2(U)$, it suffices to show it for $z \in \omega_D \cap f^2(U)$. In this case, take $\epsilon > 0$ such that $B_\epsilon(z) \subset f^2(U)$ and observe that

$$\nu(x)(f^2(U)) = \lim_{j\to\infty} \frac{\sharp\{0 \le i < l_j; f^i(x) \in f^2(U)\}}{l_j} .$$

Let $k(j)$ be the biggest integer such that

$$n_1 + n_2 + \ldots + n_{k(j)} < l_j ,$$

where the n_k's give the number of iterations to send R_k^* from $f^2(U)$ to U. Then

$$\frac{\sharp\{0 \le i < l_j; f^i x \in f^2(U)\}}{l_j} \le \frac{k(j)}{n_1 + n_2 + \ldots + n_{k(j)}} .$$

But $n_k \ge k^2$, by the choices made in Section 7, so that the quotient is bounded by $1/k(j)$. As l_j goes to infinity with j, also $k(j)$ does it, and the claim is proved.

The accumulation measures do not depend on the initial condition. Two measures ν and $\tilde{\nu}$ with support in Λ are equal if and only if they have the same pound on every cylinder $\Lambda \cap B^u \cap B^s$, where $B^u = B^u(n; \underline{z})$, $B^s(m; \underline{w})$, $\underline{z} = (z_1 \ldots z_n)$, $\underline{w} = (w_1 \ldots w_m)$, $n, m \ge 1$.

If $\nu(x)$ is an accumulation measure of the forward orbit of x, $x \in D$, then $\nu(x)(\Lambda \cap B^u \cap B^s) = \nu(x)(B^u \cap B^s)$ is an accumulation point for the mean of passages of the forward orbit of x inside $B^u \cap B^s$. But by the construction of D, $f^i(x) \in B^u \cap B^s$ if and only if $f^i(D) \subset B^u \cap B^s$. Hence $\nu(x) = \nu(y)$ for any $x, y \in D$.

As the accumulation measures do not depend on the initial condition taken inside D, we will denote, for simplicity, $\nu_l = \nu_l(x)$, an equality which also makes sense when measuring intersections of bridges.

Convergence to the Dirac measure on a saddle point. Suppose that for each k we choose $\underline{z}_k^0 = (+ + \ldots +)$ (with size k^2), according to Section 7. We will prove that

$$\lim_{l\to\infty} \nu_l \left(B^u(n; + \ldots +) \cap B^s(m; + \ldots +)\right) = 1 ,$$

for every $n, m \ge 1$. This clearly implies that $\nu = \delta_{p_+}$, where $p_+ = (-a_s, -a_u)$ is the saddle fixed point corresponding to the code $(\ldots + + + + \ldots)$.

Recall that for every $k \ge 1$,

$$f^{2k + \sum_{j=1}^{k} n_j}(D) \subset G^u(n_{k+1}; \underline{z}_{k+1}) \cap H_0 ,$$

where $\underline{z}_k = \underline{z}'_k \underline{z}^0_k \underline{z}''_k$, and the sizes of $\underline{z}'_k$, $\underline{z}^0_k$ and $\underline{z}''_k$ are, respectively, n'_k, n^0_k and n''_k, with

$$n'_k + n''_k \leq Ck \, , \; n^0_k = k^2 \, ,$$

for some constant $C > 0$. So the number of visits of the orbit of D to $B^u \cap B^s = B^u(n; + \dots +) \cap B^s(m; + \dots +)$ can be counted by

$$\sum_{k \geq 1} \sharp\{1 \leq i < n_k; f^i\left(G^u(n_k; \underline{z}_k) \cap H_0\right) \subset B^u \cap B^s\} \, .$$

Let k_1 be such that $k_1^2 > n + m$ and $k \geq k_1$. Denoting $\underline{z}_k = (z_1 z_2 \dots z_{n_k})$, recall that (Section 4)

$$f^i\left(G^u(n_k; \underline{z}_k) \cap H_0\right) = G^u(n_k - i; z_{i+1} \dots z_{n_k}) \cap G^s(i; z_i \dots z_1) \, ,$$

for all $i = 1, \dots, n_k - 1$. Therefore we conclude that among the first $n_k - 1$ iterates of $G^u(n_k; \underline{z}_k) \cap H_0$ at least $k^2 - (n + m)$ iterates fall inside $B^u \cap B^s$, that is a proportion greater than

$$1 - \frac{n + m + Ck}{k^2} \, .$$

This proves that the frequency of visits in $B^u \cap B^s$ of the orbit of D tends to 1 or, in other words, $\nu(B^s \cap B^u) = 1$.

An example for the lack of convergence for the frequency measures. Let $(k_s)_{s \geq 1}$ be such that

$$\sum_{k=k_s+1}^{k_{s+1}} k^2 > s \sum_{k=1}^{k_s} k^2 \, ,$$

for all $s \geq 1$. For every $k_s < k \leq k_{s+1}$, s odd, define $\underline{z}^0_k = (- - \dots -)$, and for s even, $\underline{z}^0_k - (+ + \dots +)$.

It is easy to show that the accumulation measures are all the convex combinations of the Dirac measures supported on the saddles p_+ and p_-, where p_- corresponds to the code $(\dots - - - - \dots)$ (in fact, the set of accumulation measures is always convex).

Convergence to a measure equivalent to the nontrivial Hausdorff measure on Λ. It is enough to find suitable sequences $\underline{z}^0_k$ such that the limit measure satisfies

$$\nu(B^u \cap B^s) = 2^{-n} 2^{-m}$$

for all $B^u = B^u(n; \underline{z})$ and $B^s = B^s(m; \underline{w})$, $n, m \geq 1$. In this case ν is exactly the nontrivial Hausdorff measure $\mathcal{H}$ in Λ, which is also an invariant measure for f.

Let $\hat{z} \in \Lambda$ be a generic point for $\mathcal{H}$, i.e. such that

$$\lim_{l\to\infty} \sum_{i=0}^{l-1} \delta_{f^i(\hat{z})} = \mathcal{H}(B^u \cap B^s) ,$$

for all pairs of bridges B^u and B^s. For fixed B^u and B^s and given any $\epsilon > 0$, there is $l_0 = l_0(\epsilon)$ such that

$$\frac{\sharp\{0 \leq i < l; f^i(\hat{z}) \in B^s \cap B^u\}}{l} \in [1-\epsilon, 1+\epsilon]2^{-n}2^{-m}$$

for any $l \geq l_0$ (probably $l_0 >> 2^n 2^m$). If we denote the code of $\hat{z}$ by

$$\underline{\hat{z}} = (\ldots \hat{z}_{-2}\hat{z}_{-1}\hat{z}_0\hat{z}_1\hat{z}_2 \ldots) ,$$

then we see approximately $2^{-n}2^{-m}l$ times the block $[\underline{w}]^{-1}\underline{z}$ inside $(\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_l)$.

Now take k_0 such that $k_0^2 \geq l_0$, and for each $k \geq k_0$ the sequence

$$\underline{z}_k^0 = (\hat{z}_0, \hat{z}_1, \ldots, \hat{z}_{k^2}) .$$

Then it is easy to show that for this choice the sequence $\nu_l(B^u \cap B^s) \in [1-2\epsilon, 1+2\epsilon]\mathcal{H}(B^u \cap B^s)$, for large l. As ϵ is arbitrary, $\nu(B^u \cap B^s) = \mathcal{H}(B^u \cap B^s)$, and since B^u and B^s are arbitrary, $\nu = \mathcal{H}$.

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