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THE F-FACTOR PROBLEM

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The $\mathcal{F}$-factor problem

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Abstract. Matchings in graphs have been investigated since the first days of graph theory. This paper surveys the $\mathcal{F}$-factor problem, which consists in deciding if a given graph has an $\mathcal{F}$-factor, that is a spanning subgraph whose components are isomorphic to members of a fixed family $\mathcal{F}$ of connected graphs.

If $\mathcal{F} = \{K_2\}$ this turns out to be the well-known perfect matching problem, which can be solved in polynomial time [8]. If $\mathcal{F} = \{F\}$ for any connected graph F non-isomorphic to K_1 or K_2 then the $\mathcal{F}$-factor problem is $\mathcal{NP}$-complete [13]. Surprisingly, for some large families of graphs the problem is polynomially solvable, for example those consisting of hypomatchable graphs plus K_2 [5, 14, 4], or yet some star graph families [1, 15]. For these families nice Tutte- and Berge-like characterizations of factorable graphs were obtained, together with some elegant structural results similar to the Gallai-Edmonds Theorem for maximum matchings [8]. All these results are summarized in this paper. For some of them we also give alternative simple proofs.

A recent conjecture of Loebl and Poljak [19] states that the $\mathcal{F}$-factor problem can be solved by a polynomial algorithm if and only if $\mathcal{F}$ is a matroidal family. We cover all the partial results available and also refer to a class of factor problems for which the Loebl and Poljak Conjecture has been settled as true: those defined by families of the form $\{K_2, F\}$, F connected [19, 20, 21].

Keywords. Packing, factor, matching, $\mathcal{NP}$-complete.

1 An overview of the problem

Consider $H = (V(H), E(H))$ a graph. A **matching** on H is a subset M of $E(H)$ which does not contain a pair of adjacent edges. We say that a vertex $v \in V(H)$ is **covered** by M if some edge of M is incident with v, and **free** otherwise. A matching M is **maximum** if no other matching of H covers more vertices of H than M, and **perfect** if every vertex of H is covered by M.

Matchings in graphs may also be understood as subgraphs with all components isomorphic to K_2. Going further, one may consider other kinds of components beyond K_2-like ones. This simple idea gives rise to a fascinating generalization of the traditional theory of matchings.

So let $\mathcal{F}$ be any fixed family of connected graphs. An $\mathcal{F}$-**packing** of H is a subgraph G of H whose components are isomorphic to members of $\mathcal{F}$. If G is a spanning subgraph of H then G is said to be an $\mathcal{F}$-**factor**, and H is an $\mathcal{F}$-**factorable** graph. The notions of covered and free vertices are similar to those for matchings. The number $|V(G)|$ is the **size** of G. The $\mathcal{F}$-packing G is **maximum** if there is no other packing of H which covers more vertices than G. The problem "Given a graph H, does H has an $\mathcal{F}$-factor?" is called $\mathcal{F}$-**factor problem**.

Factor problems were first stated as above in late seventies in [12], although some single such problems had already been investigated, e.g. the $\{K_3\}$-factor problem (Schaeffer, 1974, cf. [18]), the $\{P_2\}$-factor problem[1] (Johnson, 1977, cf. [13]) and the perfect fractional matching problem [26] (see page 3 for a

[1] P_2 is the path of length 2.

definition). Specifically, this survey focus on the computational complexity of factor problems, i.e. we want to summarize which of them are polynomial time solvable and which are $\mathcal{NP}$-complete.

All the polynomiality results we shall mention here were developed following two distinct approaches. The first searched for characterizations of **augmenting configurations** for each factor problem. These configurations play a similar role for factor problems as do alternating paths for the perfect matching problem. Necessary and sufficient conditions for a factor being maximum in terms of those configurations were found, which made possible to design new polynomial algorithms to solve the problems. This approach is represented by Hell and Kirkpatrick's work [14, 15].

The second line arose from a result of Uhry and Balas [26, 2], which found a relation between maximum fractional matchings with a minimum number of odd cycles and maximum ordinary matchings. This relation, known as the **Balas-Uhry property**, could also be proved to hold for several other factor problems than the perfect fractional matching problem [27, 4]. Cornuéjols, Hartvigsen and Pulleyblank introduced this approach in [5, 4]. In Loebl and Poljak's work [19, 20, 21] the Balas-Uhry property was regarded as a main tool for proving (in a non-suitable context for using augmenting configurations) the polynomial solvability of the most general known class of factor problems: those defined by **good graph families**.

A common feature of the polynomial time solvable problems is the so-called **matroidal property**. A graph family $\mathcal{F}$ is said to possess the matroidal property if for any graph H the sets of the form $T \subseteq V(H)$ such that there exists an F-packing G with $T \subseteq V(G)$ are the independent sets of a matroid on $V(H)$. Graph families with this property are **matroidal**. Loebl and Poljak in [19] posed the following conjecture for finite graph families: *the $\mathcal{F}$-factor problem is polinomially solvable if and only if $\mathcal{F}$ is matroidal.* All partial results about this conjecture relate matroidal to good families.

In this paper we shall give some insight on these features. We will first focus on problems defined by unitary graph families. Section 3 does a summary of two wide classes of factor problems: those defined either by complete bipartite graph families or by families into the form $\{K_2\} \cup \{F_1, \ldots, F_k\}$, where each F_i is hypomatchable, $1 \leq i \leq k$. Section 4 covers factor problems defined by hypomatchable graphs plus K_2 and sequential star set packings using the $\{K_2, K_3\}$-factor problem as an example. In Section 5 we will refer to the Loebl and Poljak Conjecture.

Alternative proofs of some theorems are presented in this paper. Proofs of the theorems just mentioned can be found in the references.

2 Uniform factorization

A factor problem is said to be **uniform** if it is defined by an unitary graph family. If $\mathcal{F} = \{F\}$ for some F we mostly say "F-factor problem" for $\{F\}$-factor problem. The K_3- and P_2-factor problems we mentioned in Section 1 are examples of uniform problems.

The result below, due to Hell and Kirkpatrick [12, 13], settles the complexity of all uniform factor problems.

Theorem 1 ([12, 13]). *Let F be any connected graph. Then the F-factor problem can be solved in polynomial time if F is isomorphic to K_1 or K_2 and is $\mathcal{NP}$-complete otherwise.*

The proof of Theorem 1 introduced a method which was used later in almost all $\mathcal{NP}$-completeness proofs for factor problems. We will briefly sketch this technique.

Hell and Kirkpatrick considered some special graphs, called **modules**, that are graphs with a non-empty set of distinguished vertices, said the **connectors** of the module. Non-connector vertices are **interior** vertices. Modules appear as induced subgraphs of larger graphs, denominated **modular extensions**. Only connector vertices of a module in a modular extension can be adjacent to vertices out of the module. Let M be a module, M' be a modular extension of M and G be an F-packing of M'. A vertex v of M is said to be bound to M by G if $v \in V(G') \Rightarrow V(G') \subseteq V(M)$, where G' is some component of G.

The interesting property of modules is its F-**coherence**. Let F be a connected graph and M a module. Then M is said to be F-**coherent** if

1) any F-factor of any F-factorable modular extension of M binds to M all its interior vertices and all or none of its connector vertices;

2) both M and $M \setminus \{$its connector vertices$\}$ are F-factorable.

Most of the proof of Theorem 1 is spent on building an F-coherent module with $|V(F)|$ connectors. This proves the $\mathcal{NP}$-completeness of the F-factor problem whenever F is a connected graph with at least three vertices, since such a module yields a quite straightforward reduction of the $|V(F)|$-dimensional matching problem[2] to the F-factor problem.

This result tells us that all uniform problems but essentially the K_1- and K_2-factor problems are unlikely to be solvable by polynomial time bounded algorithms, and therefore to be suitable to a generalization of the classical matching theory. As we shall see, non-uniform problems are more convenient for this purpose.

3 A summary of non-uniform problems

Two major results for non-uniform problems were proved in the early eighties, which established the complexity of a large class of such problems. For these results we need some definitions.

Let $\mathcal{F}$ be any connected graph family. If there is no family $\mathcal{I} \subseteq \mathcal{F}, \mathcal{I} \neq \emptyset$ of $(\mathcal{F} \setminus \mathcal{I})$-factorable graphs, then $\mathcal{F}$ is said to be **irreducible**. We regard the empty set as irreducible.

For all integers $m, n \geq 1$ we denote by $K_{m,n}$ the complete bipartite graph with parts of cardinality m and n, and by S_i the graph $K_{1,i}$, $i \geq 1$. A family of the form $\{S_i : 1 \leq i \leq k\}$ for any $k \geq 1$, or equal to $\{S_i : i \geq 1\}$, is a **sequential star family**.

A finite family of the form $\{K_2\} \cup \{F_1, \ldots, F_k\}$, where F_i is hypomatchable for any i, $1 \leq i \leq k$, is an **edge-and-hypomatchable family**.

Theorem 2 ([5, 14, 4]). *Let $\mathcal{F}$ be an edge-and-hypomatchable family. Then the $\mathcal{F}$-factor problem can be solved by a polynomial time-bounded algorithm.*

Theorem 3 ([15]). *Let $\mathcal{B}$ be an irreducible family of complete bipartite graphs. Then the $\mathcal{B}$-factor problem can be solved in polynomial time if $\mathcal{B}$ is a sequential star family and is $\mathcal{NP}$-hard otherwise.*

In what follows we mention some important factor problems whose complexity was settled by these results.

- The $\mathcal{K}$-factor problem, where $\mathcal{K}$ is any (non-necessarily finite) clique family which contains K_2;
- the $\{K_2\} \cup \mathcal{C}$-factor problem, defined by any finite family $\mathcal{C}$ of odd cycles;
- the $\mathcal{B}$-factor problem, where $\mathcal{B}$ is any complete bipartite graph family.

Other problems had their complexity previously established by the work of several authors, mainly on the perfect fractional matching problem and similar problems.

[2]See [11], pp. 46 for the 3-dimensional matching problem. We get the k-dimensional matching problem by replacing the 3 by k, for $k \geq 2$. This problem is $\mathcal{NP}$-complete for any $k \geq 3$ [17].

- The perfect fractional matching problem [26, 23, 2]. We denominate a fractional matching in a graph H any subgraph G of H whose components are induced by the support of some basic optimal solution of the following linear program:

$$\max \quad \sum_{e\in E(H)} x_e$$

$$\text{subject to} \quad \begin{cases} \sum_{e \text{ incident with } v} x_e \leq 1 & \forall v \in V(H), \\ x_e \geq 0 \quad \forall e \in E(H). \end{cases}$$

  It is a well-known fact that the components of G are either isolated edges or odd cycles [3]. Hence the perfect fractional matching problem and the $\{K_2\} \cup \{C_i : i \geq 3\}$-factor problem are equivalent in the sense that they have the same solutions.

- The $\{K_2\} \cup \{C_i : i > k\}$-factor problem for any $k \geq 3$ [6, 7].
- The sequential star family factor problem [1] (see also [27, 15]).

In [14] the following sufficient condition for a factor problem to be $\mathcal{NP}$-complete is introduced.

Theorem 4. *Let $\mathcal{F}$ be any finite family of cut-point-free graphs. If some graph F from $\mathcal{F}$ satisfies*

1) for some $v \in V(F)$ the graph $F' := (V(F'), E(F'))$ given by

$$V(F') := V(F) \setminus \{v\} \cup \{v_a, v_b\}, \quad v_a, v_b \notin V(F),$$
$$E(F') := E(F) \setminus \delta(v) \cup \{uv_a, uv_b : uv \in E(F) \text{ for all } u\}$$

is not $\mathcal{F}$-factorable, and

2) any $\mathcal{F} \setminus \{F\}$-factor of F leaves at least three uncovered vertices,

then the $\mathcal{F}$-factor problem is $\mathcal{NP}$-complete.

From this result it follows that the problems stated below are $\mathcal{NP}$-complete:

- The $\mathcal{K}$-factor problem, for any $\mathcal{K} \subseteq \{K_i : i \geq 3\}$ [14].
- The $\mathcal{C}$-factor problem defined by any (finite or not) cycle families except the following cases:
  * $\mathcal{C} = \{C_3, C_4, \ldots\}$,
  * $\mathcal{C} = \{C_4, C_5, \ldots\}$,
  * $\mathcal{C} = \{C_5, C_6, \ldots\}$,
  * $\mathcal{C} = \{C_3, C_5, C_6, \ldots\}$.

  For finite cycle families Theorem 4 can be applied. For infinite families except the cases related above, [14] sketches a proof which uses Theorem 4 and a technique due to Papadimitriou and reported in [6]. If $\mathcal{C} = \{C_3, C_4, \ldots\}$ then we have the 2-factor problem which admits a polynomial time bounded algorithm [25]. The three remaining cases are open.

- The $\{K_2, F\}$-factor problem, where F is a 2-connected graph whose maximum matchings leave at least three free vertices.

The proofs of Theorems 1 and 4 and also the "$\mathcal{NP}$-completeness" part in the proof of Theorem 3 use a technique based on a straightforward extension of coherence for families. In all these cases some suitable multidimensional matching problem was reduced to the desired factor problem.

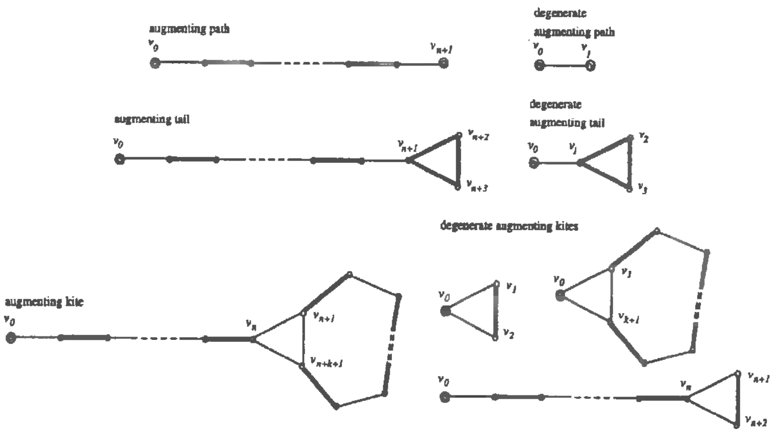


Figure 1: Augmenting configurations.

4 Edge-and-triangle packings

In order to give some insight on the polynomial non-uniform factor problems, we shall study in this chapter the $\{K_2, K_3\}$-factor problem.

Let H be any graph and G be a $\{K_2, K_3\}$-packing of H. We call **G-edges** the components of G isomorphic to K_2 and **G-triangles** all other components of G.

An **augmenting configuration** of G in H is a subgraph of H as any in Figure 1; that is, an **augmenting path, augmenting tail** or **augmenting kite**. In that picture n is any even non-negative integer and k is any odd positive integer. If $n = 0$ or $k = 1$ the configurations are said to be **degenerate**. Circled vertices are free of G and double edges are G-edges or belong to G-triangles. The subgraph in the augmenting kite induced by vertices $v_n, \ldots, v_{n+k+1}$ is the head of the kite, and v_0 is its end.

Clearly, a packing with an augmenting configuration can not be maximum. The converse is also true.

Theorem 5 ([14]). *A $\{K_2, K_3\}$-packing G of a graph H has maximum size if and only if there exists no augmenting path, tail or kite of G in H.*

The characterization of maximum packings stated above gives rise to a method for building such packings. Figure 2 describes the process.

This method successively builds $\{K_2, K_3\}$-packings $G_1, \ldots, G_t$, being G_i the packing G in the beginning of the i-th iteration ($1 \leq i \leq t$ for some finite t). Theorem 5 guarantees that the method stops, thus G_t is a maximum $\{K_2, K_3\}$-packing in H. At line (6) we rely on an oracle which, receiving as an argument a graph H and any $\{K_2, K_3\}$-packing G of H, tell us whether G admits an augmenting configuration in H. Further, this oracle returns an augmenting configuration if such one there exists.

We observe yet the use of a *maximum* matching at line (1). This has some quite interesting consequences, as the following lemma states.

Lemma 6. *The augmenting configuration at line (6) is always a kite.*

Proof. It can be easily verified that in every augmentation, say $G_i := G_{i-1} \Delta O$ for $i = 2, \ldots, t$, the number of components of G with at least one edge does not change if and only if O is a kite and that, if O is either

Instance: A graph H;
Output: a $\{K_2, K_3\}$-packing G in H of maximum size;

(1) Find a maximum matching M in H;
(2) $G := M$;
(3) maximum := false ;
(4) While not maximum
(5)   maximum := true ;
(6)   Try to find an augmenting configuration O of G in H.
(7)   If succeeded
(8) $G := G \triangle O$;
(9)   maximum := false ;
(10) Return G.

Figure 2: The First Method.

a path or a tail this number increases by 1. Let i be the first iteration in which O is not a kite. So G_{i+1} has one more component than G_i, and hence than G_1. But then any maximum matching of G_i is larger than G_1, a contradiction. □

A consequence of this fact is that in each iteration of the First Method a new triangle is set up in G. Further, there is no need of moving other triangles to make room for the new one.

Taking the Gallai-Edmonds Decomposition of H into account makes things even nicer. Some helpful schemes are seen at Figures 4 and 5. In these figures, round components of $H - A$ are components of $H[D_{t-1}]$ and rectangular components are components of $H[C_{t-1}]$.

The method at Figure 2 implements this idea. Henceforth the notation of [22] for the Gallai-Edmonds Decomposition will be assumed; that is let $D \subsetneq V(H)$ be the set of vertices free of some maximum matching in H, $A \subseteq V(H)$ be the set of vertices not in D which are adjacent to someone of D, and C be the remaining vertices of H. The Gallai-Edmonds Decomposition shall be denoted by the set triple (D, A, C). We will also refer to the subgraph $\bar{H}$ obtained from H by deleting C and the edges of $H[A]$ and by shrinking each component of $H[D]$ to a single vertex.

Theorem 7 below shows how the Second Method works iteration by iteration.

Theorem 7. *Consider the i-th iteration of the Second Method, $i \geq 1$. Let C_i and D_i be respectively C and D at the beginning of this iteration, P_i be P at line (7) and $\hat{D}_i$ be $\hat{D}$ at line (10). Let $\bar{H}_i$ be the bipartite graph obtained from H by deleting the vertices of C_i, the edges of $H[A]$ and shrinking each component of $H[D_i]$ to a single vertex. Then*

i) either P_i is totally embedded in some component of $H[D_i]$, or P_i has its end in some component of $H[D_i]$, flows through the edges of $\bar{H}_i$ and $H[D_i]$ and its head lies into a second component of $H[D_i]$;

ii) all the components of $H[D_i]$ are hypomatchable;

iii) all the components of $H[C_i]$ are $\{K_2, K_3\}$-factorable;

iv) G_i induces on each component of $H[D_i]$ a matching which leaves exactly one free vertex in this component;

v) G_i induces a $\{K_2, K_3\}$-factor in each component of $H[C_i]$;

vi) each vertex of A is matched by G_i with a vertex lying into a distinct component of $H[D_i]$.

Instance: A graph H;
Output: a $\{K_2, K_3\}$-packing G in H of maximum size.

(1) Find a maximum matching M in H by Edmonds' Algorithm;
(2) Build the Gallai-Edmonds Decomposition (D, A, C) of H;
(3) $G := M$;
(4) maximum := false ;
(5) While not maximum
(6)   maximum := true ;
(7)   Try to find an augmenting kite P of G in H.
(8)   If succeeded
(9) $G := G \,\Delta\, P$;
(10) $\hat{D} :=$ $H[D]$-component which contains the head of P;
(11) $D := D \setminus V(\hat{D})$;
(12) $C := C \cup V(\hat{D})$;
(13)   maximum := false ;
(14) Return G.

Figure 3: The Second Method.

Proof. We shall demonstrate this theorem by induction on i.

Basis: $i = 1$

Itens ii) through vi) are true by the Gallai-Edmonds Theorem. Vertex v_0 of P_1 is free of G_1 and so it is in some component of D_1. Suppose that P_1 is not totally contained in this component. If $n > 0$ by items iv) through vi) we have that the path from v_0 through v_n in P_1 satisfies i). So v_n is in a second component of $H[D_1]$, say $\hat{D}_1$. We claim that the vertices $v_{n+1}, \ldots, v_{n+k+1}$ lie also into $\hat{D}_1$. Suppose not, that is let j be the smallest index between 0 and k such that $v_{n+j+1} \notin \hat{D}_1$. Clearly $v_j \in A$ and is even. Because of iv), v) and vi), as the path from v_{n+j+1} to v_{n+k+1} in P_1 is G_1-alternating and has odd length, then v_{n+k+1} lies into a third component of $\hat{D}_1$, a contradiction. As the claim's proof works also if $n = 0$, we get a contradiction also in this case.

Step: $i > 1$

Suppose as induction hypothesis that items i) through vi) are valid at the $(i-1)$-th iteration. From lines (9) through (12) of the method we have $G_i := G_{i-1} \,\Delta\, P_i$, $D_i := D_{i-1} \setminus V(\hat{D}_i)$, and $C_i := C_{i-1} \cup V(\hat{D}_1)$. As $G_i[\hat{D}_i]$ is a factor in $H[\hat{D}_i]$, items ii) through vi) are straightforward. Proceeding as in the basis, we get i) also proved. □

Figures 4 and 5 shows the state of D, A, C and $\bar{H}$ at the last iteration. If G_t is a $\{K_2, K_3\}$-factor, let $\bar{C}$ be $V(H)$, and $\bar{D} = \bar{A} = \emptyset$. Otherwise let $\bar{A}$ be the set of all vertices in A which are reachable from some G_t-free vertex by an alternating path through the G_t-edges of $\bar{H}_t$, $\bar{D}$ be the set of vertices lying in those components of D_t which are reachable in the same way, and $\bar{C}$ be $V(H) \setminus (\bar{D} \cup \bar{A})$. With this definition, for each component D' of D_t, either $V(D') \cap \bar{D} = \emptyset$ or $V(D') \subsetneq \bar{D}$.

By Theorem 7 the components of $H[\bar{D}]$ are hypomatchable. It is not difficult to see that they also are not $\{K_2, K_3\}$-factorable, since otherwise G_t could still be enlarged. It turns out that these components form a barrier against further enlargements of G_t, in the same way hypomatchable components of the Gallai-Edmonds Decomposition do for maximum matchings. This motivates the following definition.

Let $\mathcal{F}$ be a connected graph family, and H be a graph. Then H is said to be an $\mathcal{F}$-critical graph if H is not $\mathcal{F}$-factorable and $H - v$ is $\mathcal{F}$-factorable for any $v \in V(H)$. The next result, proved in [5] and partially

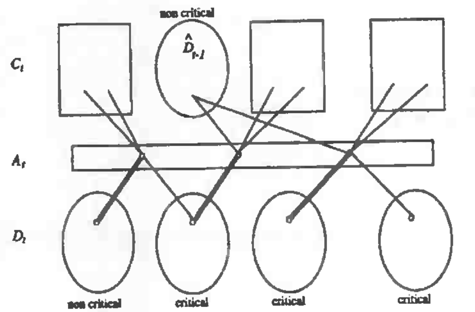


Figure 4: Beginning of the last iteration of the Second Method.

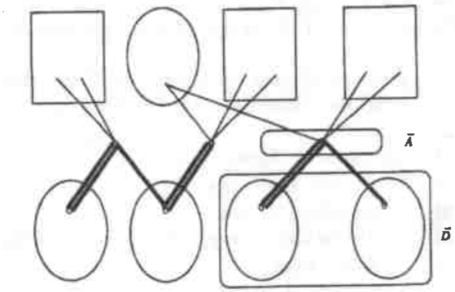


Figure 5: The graph $\bar{H}_t$ is indicated by thick lines. Sets $\bar{D}$ and $\bar{A}$ are also indicated.

in [14], provides a quite convenient characterization of critical graphs.

Theorem 8 **([5, 14]).** *Let H be a graph. Then H is $\{K_2, K_3\}$-critical if and only if H is hypomatchable and does not have a $\{K_2, K_3\}$-factor with exactly one triangle if and only if H is hypomatchable and is not $\{K_2, K_3\}$-factorable.*

From what we have said before, the components of $H[\bar{D}]$ are $\{K_2, K_3\}$-critical. The Second Method and Theorem 8 thus give us an effective way of finding maximum $\{K_2, K_3\}$-packings. The search for augmenting kites at line (7) can be replaced by the search for reachable non-$\{K_2, K_3\}$-critical hypomatchable components. Theorem 8 guarantees that this search can be performed in polynomial time in the order of the component we are working on.

The following results generalize Tutte's Theorem, Berge's Formula and the Gallai-Edmonds Theorem. Their proofs make use of sets $\bar{D}$ and $\bar{A}$.

Theorem 9 **([5, 14]).** *Let H be a graph. Let $b(A)$ be the number of $\{K_2, K_3\}$-critical components of the graph $H - A$, where $A \subseteq V(H)$. Then H is $\{K_2, K_3\}$-factorable if and only if $b(A) \leq |A|$ for all $A \subseteq V(H)$.*

Theorem 10 **([14]).** *Let H be any graph and $def_{\{K_2,K_3\}}(H)$ the number of G-free vertices of H, where G is a maximum $\{K_2, K_3\}$-packing of H. Then*

$$def_{\{K_2,K_3\}}(H) = \max\{b(A) - |A| : A \subseteq V(H)\}.$$

Theorem 11 **([4]).** *Let H be a graph. Let $\tilde{D}$, $\tilde{A}$ and $\tilde{C}$ be the following subsets of $V(H)$:*

- *$\tilde{D}$ is the set of vertices from H which are free of some maximum $\{K_2, K_3\}$-packing of H.*
- *$\tilde{A}$ is the set of neighbors of $\tilde{D}$ not belonging to D.*
- *$\tilde{C}$ is the set $V(H) \setminus (\tilde{A} \cup \tilde{D})$.*

Then

i) the components of $H[\tilde{D}]$ are $\{K_2, K_3\}$-critical;

ii) $H[\tilde{C}]$ is $\{K_2, K_3\}$-factorable;

iii) a $\{K_2, K_3\}$-packing G of H is maximum if and only if:

- *G induces on each component of $H[\tilde{D}]$ a $\{K_2, K_3\}$-packing which leaves exactly one free vertex;*
- *G induces a $\{K_2, K_3\}$-factor in $H[\tilde{C}]$;*
- *each vertex from $\tilde{A}$ is matched by a G-edge with some distinct component of $H[\tilde{D}]$.*

We shall give here an alternative proof for the self-contained proof of Cornuéjols and Hartvigsen in [4]. Our proof relies on the Second Method and Theorem 7.

Proof. If H is $\{K_2, K_3\}$-factorable the result is obvious. So assume the contrary. Then $D_t \neq \emptyset$ and there is a G_t-free vertex in each component of $H[D_t]$, according to Theorem 7. Let the sets $\bar{D}$, $\bar{A}$ and $\bar{C}$ be defined as before. We claim that $\tilde{D} = \bar{D}$.

Let $v \in \tilde{D}$ and $\bar{G}$ be a maximum $\{K_2, K_3\}$-packing missing v. Let P be a maximal G_t-alternating, $\bar{G}$-alternating path with all internal vertices having degree 1 in $\bar{G}$, beginning at v. Let w be the other end of P. There are four cases to investigate.

If P has odd length and w is $\bar{G}$-free then $\bar{G}$ can be enlarged by an augmenting path, a contradiction.

If P has odd length and w is covered by some $\bar{G}$-triangle then $\bar{G}$ can be enlarged by an augmenting tail, contradiction.

If P has even length and w is G_t-free, then it lies in a $\{K_2, K_3\}$-critical component of $H[D_t]$. It turns out that P flows through G_t-edges and then, by alternating on P, we get $v \in \bar{D}$.

Finally, if P has even length and w is covered by a G_t-triangle then by the maximality of P the other two vertices of this triangle belong to P. That yields an augmenting kite of $\bar{G}$, again a contradiction.

So we get $\tilde{D} \subseteq \bar{D}$. Now, for any $v \in \bar{D}$ we can easily modify G_t to get another maximum $\{K_2, K_3\}$-packing which misses v, thus proving the converse. Let D' be the component of $H[\bar{D}]$ which contains v. As $H[D']$ is hypomatchable, there is a K_2-packing T which misses v. If D' has a G_t-free vertex then $G_t[V(H) - D'] \cup T$ is a maximum $\{K_2, K_3\}$-packing which misses v. If not, let P be an even G_t-alternating path beginning at some G_t-free vertex w, and ending in D'. Then $G_t \triangle P$ is a maximum $\{K_2, K_3\}$-packing which misses some vertex of D', and thus $(G_t \triangle P)[V(H) - D'] \cup T$ is a maximum $\{K_2, K_3\}$-packing which misses v. This proves $\tilde{D} = \bar{D}$.

Let v e an vertex of $\bar{A}$. Then by definition $\bar{A} \subseteq A$ and v is reachable from some G_t-free vertex by a G_t-alternating path P through $\bar{H}_t$. As A is independent in $\bar{H}_t$ and has no G_t-free vertex, P has length at least one and the vertex which precedes v in P belongs to $\bar{D}$. As $\bar{D} = \tilde{D}$ we have $v \in Adj(\tilde{D}) \setminus \tilde{D} = \tilde{A}$.

Conversely, if $v \in \tilde{A}$ then v is adjacent to some vertex $w \in \tilde{D} = \bar{D}$. By definition, the component of $H[\bar{D}]$ which contains w is reachable, therefore v is also reachable.

Thus from our claim we have $\tilde{A} = \bar{A}$ and $\tilde{C} = \bar{C}$. Items i) and ii) now follow from Theorem 7, so only item iii) remains to be proved.

Let $\bar{G}$ be some $\{K_2, K_3\}$-packing satisfying the conditions of item iii). We shall prove that $\bar{G}$ and G_t have the same size. Indeed,

$$\begin{aligned}\text{size of } \bar{G} = |V(H)| - (\#\text{ of components of } H[\tilde{D}]) + |\tilde{A}| = \\ |V(H)| - (\#\text{ of components of } H[\bar{D}]) + |\bar{A}| + |A \setminus \bar{A}| - (\#\text{ of components of } H[D_t \setminus \bar{D}])^{a} = \\ |V(H)| - (\#\text{ of components of } H[D_t]) + |A| = \text{size of } G_t.\end{aligned}$$

[a] $|A \setminus \bar{A}| = (\#$ of components of $H[D_t \setminus \bar{D}])$ by Theorem 7.

To prove the converse, let G' be a maximum $\{K_2, K_3\}$-packing. Then all G'-free vertices are in $\tilde{D} = \bar{D}$.

If G' does not match each vertex of $\tilde{A} = \bar{A}$ with a vertex in a distinct component of $H[\tilde{D}] = H[\bar{D}]$, then G' is smaller than G_t, since there is more unmatched $H[\bar{D}]$-components in G' than in G_t and these components are not $\{K_2, K_3\}$-factorable.

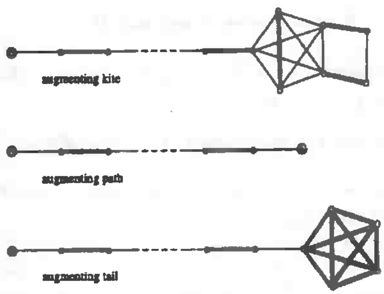


Figure 6: Augmenting configurations for the $\{K_2, K_5\}$-factor problem.

As $Adj(\tilde{C}) \subseteq \tilde{C} \cup \tilde{A}$, then G' induces a $\{K_2, K_3\}$-factor in $H[\tilde{C}]$. Moreover, as $Adj(\tilde{D}) \subseteq \tilde{D} \cup \tilde{A}$ and the components of $H[\tilde{D}]$ are $\{K_2, K_3\}$-critical, then G' induces in these components a $\{K_2, K_3\}$-packing which leaves exactly one free vertex. □

Cornuéjols and Hartvigsen's proof takes the Gallai-Edmonds Decomposition (D, A, C) of H and a maximum matching M of H which covers a maximum number of critical components of D, and uses M-alternating forests to define a new three-partition of $V(H)$ which turns out to be the same given by the sets $\tilde{D}$, $\tilde{A}$ and $\tilde{C}$ introduced here. By taking proper packings in each block of their partition, they exhibit a maximum $\{K_2, K_3\}$-packing in H. It follows afterwards that this partition satisfies items i) to iii) and is the unique partition given by Theorem 11.

We remark that this last proof, together with a suitable characterization of critical graphs (as in Theorem 8) and a subroutine for building a proper M, yields an algorithm for maximum $\{K_2, K_3\}$-packings whose polynomiality depends of this subroutine's. The Loebl and Poljak's algorithm for good graph family packing, which shall be covered in Chapter 5, exploits this idea in a more general context.

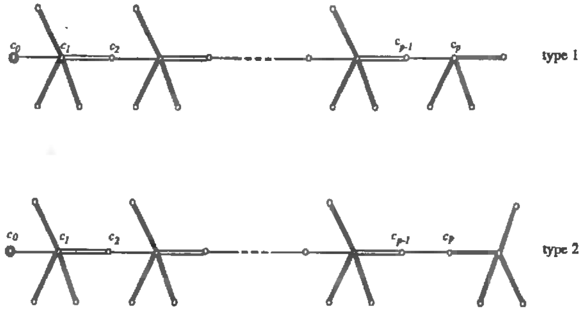
All these results generalize to factor problems defined by edge-and-hypomatchable families. Figure 6 shows some examples of augmenting configurations for $\mathcal{F} = \{K_2, K_5\}$. We ask for a finite number of hypomatchable graphs in $\mathcal{F}$ to be able to do the criticality test in polynomial time according to a generalized version of Theorem 8. For the $\mathcal{K}$-factor problem, $\mathcal{K} \subseteq \{K_i : i \geq 1\}$, $K_2 \in \mathcal{K}$, however, there is no need of this, for if one takes the maximal irreducible sub-family of $\mathcal{K}$ then an equivalent and two-element-family factor problem is obtained.

Sequential finite star set factor problems also can be handled this way. The augmenting configurations are presented in Figure 7. Although Hell and Kirkpatrick [15] have proposed three kinds of augmenting configurations, only two of them are really needed, as we show in [24]. Further, only type 1 ones are required when we start augmenting from a maximum matching. It turns out that the same approach we've seen for the $\{K_2, K_3\}$-factor problem works in the task of giving a polynomial algorithm for sequential star finite set problems. Finally, the sequential infinite star set factor problem is trivial since every graph without isolated vertices is factorable by this family [27, 15].

5 A conjecture for polynomial factor problems

Let $\mathcal{F}$ be any connected graph family, and H be a graph. Consider the following set family defined on $V(H)$.

$$\mathcal{I} := \{T \subseteq V(H) \colon \text{there is some } F\text{-packing } G \text{ in } H \text{ such that } T \subseteq V(G)\}.$$



In type 1 and type 2 configurations the vertices c_i: $0 < i < p$ have degree k when i is odd and 1 when i is even. Vertex c_0 is free and vertex c_p has degree less than k. Type 1 configurations have c_p of degree 1 if and only if it is covered by an isolated edge of the packing.

Figure 7: Augmenting configurations for the $\{S_1, \ldots, S_k\}$-factor problem.

Family $\mathcal{I}$ is clearly hereditary, that is, it contains the empty set and all subsets of S whenever S is itself a member of $\mathcal{I}$. It was proved for the perfect matching problem in [8], for the edge-and-hypomatchable factor problem in [4] and for the sequential star factor problem in [27] that the sets in $\mathcal{I}$ are just the independent sets of a matroid on $V(H)$. If this is true for all H then $\mathcal{F}$ is said to possess the **matroidal property**.

The matroidal property seems to be a common feature of the polynomial time solvable factor problems. This remark motivated the following conjecture on factor problems to be posed:

Conjecture (Loebl and Poljak, 1988 [19]). *The following statements about a finite graph family $\mathcal{F}$ are equivalent:*

- *$\mathcal{F}$ is matroidal;*
- *the $\mathcal{F}$-factor problem is polynomial.*

To strengthen the evidences of the Loebl-Poljak Conjecture to be true, Loebl e Poljak introduced a new class of graph families, the **good families**, which contains both sequential star and edge-and-hypomatchable families, and established the complexity of $\{K_2, F\}$-factor problem in a way which agrees with the conjecture (under the assumption that $\mathcal{P} \neq \mathcal{NP}$).

Theorem 12 ([20]). *Let $\mathcal{F}$ be a finite good graph family. Then the $\mathcal{F}$-factor problem can be solved by a polynomial time bounded algorithm.*

Theorem 13 ([19]). *Let $\mathcal{F}$ be a good graph family. Then $\mathcal{F}$ is matroidal.*

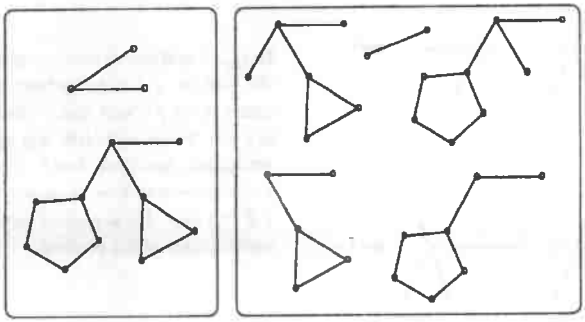
Theorem 14 ([19, 20, 21]). *Let $\mathcal{F} = \{K_2, F\}$ where F is any connected graph. Then*

1) if $\mathcal{F}$ is good or reducible then the $\mathcal{F}$-factor problem is polynomially solvable, otherwise it is $\mathcal{NP}$-complete;

2) $\mathcal{F}$ is matroidal if and only if $\mathcal{F}$ is good or reducible.

Into the good families, besides K_2, hypomatchable graphs and stars, there is another kind of graph: the **propeller**. A graph H is said to be a **k-propeller** ($k \geq 0$) if it possesses a vertex c, called the **center** of H, such that $H - c$ contains $k+1$ hypomatchable components with at least one isomorphic to K_1. The unique vertex in this component is the **root** of H, denoted by r, the edge cr is the **stick**, and the remaining k components are called the **blades** of H. In these notes we shall consider K_2 as a propeller with no blades, that is as a 0-propeller. The set of blades will be denoted by $B(H)$. A propeller H' with the same root and center as H and such that $B(H') \subseteq B(H)$ is a **subpropeller** of H.

A **closed family of propellers** is a family $\mathcal{P}$ of propellers which satisfies:



The seven graphs in this figure form a closed family of propellers. If one wants the two graphs on the left bubble in the family then the graphs on the right bubble must also be present.

Figure 8: A closed family of propellers.

- Heredity: if $F \in \mathcal{P}$ and F' is isomorphic to any subpropeller of F, then $F' \in \mathcal{P}$.
- Blade Exchange: If $F, F' \in \mathcal{P}$ and F' is not isomorphic to a subpropeller of F, then for each blade $D \in B(F)$ such that the the number of blades isomorphic to D is greater in F than in F' there is a blade $D' \in B(F')$ such that F' has more isomorphic-to-D' blades than F, and the blade $F - D + D'$ is in $\mathcal{P}$.

An example of a closed family of propellers can be seen at Figure 8.

A family of graphs is said to be **good** if it can be written as $\{K_2\} \cup \mathcal{H} \cup \mathcal{P}$ where $\mathcal{H}$ is a hypomatchable graph family and $\mathcal{P}$ is a propeller closed family.

The Balas-Uhry property is a basic tool for Loebl and Poljak's proof of the polynomial time solvability of the good family factor problem [20]. This property was first proved for the fractional matching problem in [26, 2] and is stated in the next theorem.

Theorem 15. *Let G be a fractional matching of a graph H having a minimum number of odd cycles. Then a maximum matching in G is also a maximum matching in H.*

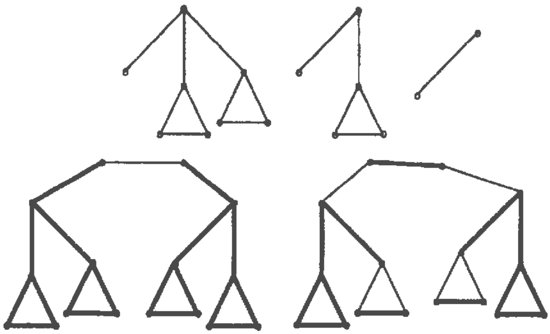
This result motivated the following definition [16]. Let $\mathcal{F}$ be some graph family which contains K_2. A family $\mathcal{F}$ of graphs is said to possess the **Balas-Uhry property** if for any graph H and any $\mathcal{F}$-packing G of H with a minimum number of components non isomorphic to K_2, a maximum matching in G is also a maximum matching in H.

This property was proved to hold for edge-and-hypomatchable families in [4] and also for a larger class of graph families in [16]. However, for good families the original property does not hold, and Figure 9 shows a counterexample. We introduce now the following generalization of the Balas-Uhry property. For this let the **deficiency** of a graph be the number of its vertices free of a maximum matching. We note that the deficiency of a good family packing is equal to its number of blades and hypomatchable graphs.

Definition . *Let $\mathcal{F}$ be a graph family. If for any graph H and any maximum $\mathcal{F}$-packing G of H whose deficiency is minimum over all maximum $\mathcal{F}$-packings of H, a maximum matching in G is also maximum in H, then $\mathcal{F}$ has the **generalized Balas-Uhry property**.*

Theorem 16. *Let $\mathcal{F}$ be a good family of graphs. Then for any graph H and any maximum $\mathcal{F}$-packing G of H of minimum deficiency, a maximum matching in G is also maximum in H.*

This result was partially proved by Kano and Poljak [16], together with a partial converse. We give now a simple proof of this.



In the top our graph family is shown. Below, we have two different factors of the same graph, both with the same number of components non isomorphic to K_2. A maximum matching of the left packing leaves four free vertices and in the right one only two free vertices are left.

Figure 9: A counterexample for the Balas-Uhry property.

Proof. Consider G a maximum $\mathcal{F}$-packing of minimum deficiency in H. The set of all subgraphs of G which are either hypomatchable components or blades shall be denoted by $T(G)$. From we've said before, $|T(G)|$ is minimum over all maximum $\mathcal{F}$-packings of H. Let M be a maximum matching of G. Clearly M induces a maximum matching in each component of G. In particular for the propellers there is an edge of M incident with each center. We shall suppose that these edges are the sticks of their respective propellers.

Let $C := (c_0, \ldots, c_p)$ be an M-alternating path of odd length, with c_0 free of M. We claim that C satisfies

i) C is G-alternating;

ii) the vertex c_i is the center of some propeller of G for each odd i, $0 < i < p$;

iii) the edge $c_{i-1}c_i$ is the stick of this propeller for each even i, $0 < i < p$

and that i), ii) and iii) imply

iv) c_p is the center of some propeller of G.

We remark that this claim implies that M is a maximum matching of H.

This shall be proved by induction on p. If $p = 1$, then i), ii) and iii) are vacuously valid. For contradiction, suppose that c_1 is not the center of a propeller of G. Then c_1 is either free of G or covered by some graph in $T(G)$ or by a root of some k-propeller, $k \geq 1$. As c_0 is free of M, then it is either free of G or covered by someone in $T(G)$. If c_0 or c_1 are free of G then G can be enlarged, otherwise $|T(G)|$ can be decreased. In any case we get a contradiction.

For the step consider $C' := (c_0, \ldots, c_{p-2})$. By the induction hypothesis, C' satisfies i) through iii) and this implies that c_{p-2} is the center of some propeller of G. So the edge $c_{p-2}c_{p-1}$ must be the stick of this propeller because C' is M-alternating and M contains the stick of every propeller in G. Further, $c_{p-1}c_p$ does not belong to G. This proves i) through iii).

Suppose iv) false. So again we have c_0 free of G or covered by someone in $T(G)$, and c_p either free of G or covered by some element of $T(G)$ or as a root of some k-propeller, $k \geq 1$. As we've just seen in the basis this yields an $\mathcal{F}$-packing G' which either is larger than G or has the same size and a smaller deficiency. Figure 10 summarizes all the cases. □

As in the case of the $\{K_2, K_3\}$-factor problem, recognition of critical graphs plays a main role in building a polynomial algorithm for the good family factor problem from the Gallai-Edmonds Decomposition. Proving a characterization for these graphs as in Theorem 8 is straightforward using the generalized Balas-Uhry property. Loebl and Poljak presented such an algorithm in [20]. However this algorithm is more

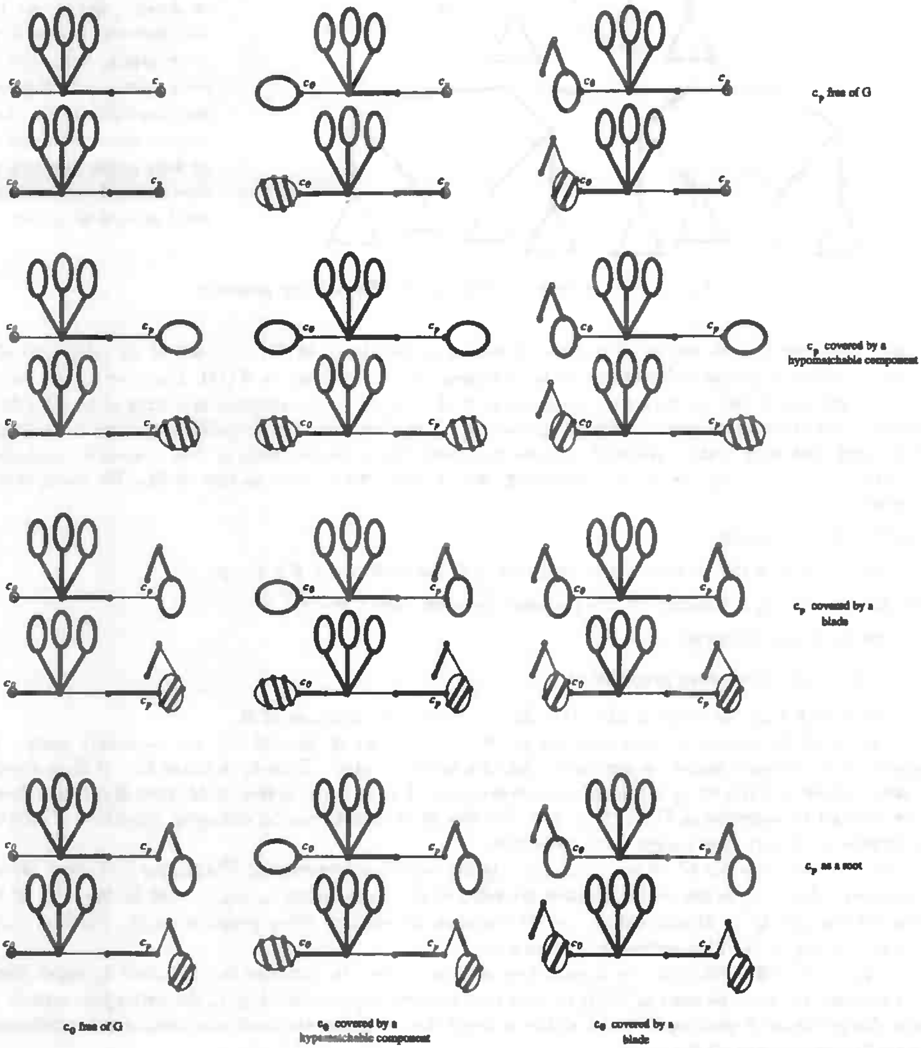


Figure 10: Proof of Theorem 16. For each case we exhibit G and G' in thick edges. Blades and hypomatchable components are represented by round bubbles.

complicated than those we've seen so far, because an instance of the matroid partition problem [10, 9] must be solved. (For this we need the heredity and blade exchange properties of good families.) We have no notice of an algorithm which does not require that, so it would be of interest finding one.

We remark that the family $\{C_3, C_4, \ldots\}$ is not matroidal, even though it defines a polynomial factor problem, as we've mentioned in Section 3. So the equivalence between families which define polinomial factor problems and matroidal graph families is not valid for infinite graph families.

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