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On the aleatory variable $Y_n = 10^n X - [10^n X]$ for large n

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Abstract: In this paper, we consider the aleatory variable $Y_n = 10^n X - [10^n X]$ where X is an aleatory variable. Our goal, under nice conditions on X, is to prove that Y_n tends to an aleatory variable with uniform distribution on $[0,1]$.

1 Introduction

Let X be an aleatory variable and F the distribution function of variable X defined for all real numbers x by $F(x) = P\{X \leq x\}$. By the Probability Theory, [R₁], we know that F has the following properties: **i)** F is a nondecreasing function; **ii)** $\lim_{x \to \infty} F(x) = 1$; **iii)** $\lim_{x \to -\infty} F(x) = 0$; **iv)** F is right continuous, that is, for any x and any decreasing sequence x_n, $n \geq 1$, that converges to x, $\lim_{n \to \infty} F(x_n) = F(x)$.

For each natural n, consider the aleatory variable $Y_n = 10^n X - [10^n X]$, where $[\cdot]$ denotes the integer part.

Our goal is to show by putting nice conditions on F, that Y_n tends to an aleatory variable with uniform distribution on [0,1].

We denote by A^c the complement of the set A.

We will prove the following result:

Theorem 1 *Suppose that function F satisfies:*

a) F is absolutely continuous

b) There exists a closed set A of measure zero so that F is a C^1 function on A^c.

Then $\lim_{n\to\infty} G_n(y) \equiv y$, where G_n is the distribution function of Y_n for each n.

2 Some Initial Remarks

Remark 1 We note that if $0 \leq y \leq 1$ and $Y_n = 10^n X - [10^n X]$, then $X \in \bigcup_{i\in\mathbb{Z}} \left[\frac{i}{10^n}, \frac{i+y}{10^n}\right]$ if and only if $0 \leq Y_n \leq y$ for each n.

In fact, we have that if $\frac{i}{10^n} \leq X \leq \frac{i+y}{10^n}$ then $0 \leq X\,10^n - 1 \leq y \leq 1$ and then $0 \leq Y_n \leq y$.

Remark 2 Theorem 1 is easily proved if F is a C^1 function.

In fact, let f be a primitive of F. We have that $G_n(y) = \sum_{i\in\mathbb{Z}} \left[F\left(\frac{i+y}{10^n}\right) - F\left(\frac{i+y}{10^n}\right)\right]$ and by the Mean Value Theorem it follows that $G_n(y) = \sum_{i\in\mathbb{Z}} f(c_i)\frac{y}{10^n}$ where $\frac{i}{10^n} < c_i < \frac{i+y}{10^n} < \frac{i+1}{10^n}$.

Then, $\lim_{n\to\infty} G_n(y) = \lim_{n\to\infty} y \sum_{i\in\mathbb{Z}} f(c_i)\frac{i}{10^n} = y\int_{-\infty}^{\infty} f(x)dx = y \lim_{b\to\infty}\left[\lim_{a\to-\infty} F(b) - F(a)\right] = y$ by properties ii) and iii) of F.

Remark 3 We believe that Theorem 1 can be proved under weaker hypothesis.

3 Proof of Theorem 1

Take f so that $F' = f$ and take $\varepsilon > 0$. There exist numbers a and b, $a < b$ so that $F(a) < \varepsilon$ and $F(b) > 1 - \varepsilon$ by the properties of F.

Let ℓ_n and L_n be chosen in such a way that

$$\tilde{a} = \frac{\ell_n}{10^n} = \frac{\ell_m}{10^m} < a \quad \text{and}$$

$$\tilde{b} = \frac{L_n}{10^n} = \frac{L_m}{10^m} > b \quad \text{for all} \quad n, m \in \mathbb{N}\,.$$

Since F is absolutely continuous, there exists $\delta > 0$ so that if $\tilde{a} = x_0 < x_1 < \ldots < x_n = \tilde{b}$ and $\sum_{i=1}^{n} |x_i - x_{i-1}| < \delta$ then $\sum_{i=1}^{n} |F(x_i) - F(x_{i-1})| < \varepsilon$.

Let V be an open set of $[\tilde{a}, \tilde{b}]$ such that $A \subset V$, $mV < \frac{\delta}{2}$. Let $N_n' \subset \mathbb{N}$ be the set of index i such that $\left[\frac{i}{10^n}, \frac{i+1}{10^n}\right] \cap V \neq \phi$.

We can find n_0 so that if $n \geq n_0$ then $\sum_{i \in N_n'} \left|\frac{i+1}{10^n} - \frac{i}{10^n}\right| < \delta$ by the choice of V.

By using the definition of $G_n(y)$ and the properties of F we have that

$$\begin{aligned}
G_n(y) &= \sum_{i \in \mathbb{Z}} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] \leq \sum_{i \leq \ell_n} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] \\
&+ \sum_{i=\ell_n}^{L_{n-1}} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] + \sum_{i \geq L_n} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] \\
&= F\left(\frac{\ell_n}{10^n}\right) + \sum_{i=\ell_n}^{L_{n-1}} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] \\
&+ 1 - F\left(\frac{L_n}{10^n}\right) < F(a) + \sum_{i=\ell_n}^{L_{n-1}} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right] \\
&+ 1 - 1 + \varepsilon < 2\varepsilon + \sum_{i=\ell_n}^{L_{n-1}} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right)\right].
\end{aligned}$$

Then, $0 \leq \left| G_n(y) - \sum_{i=\ell_n}^{L_n-1} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right] \right| < 2\varepsilon$.

But $\sum_{i=\ell_n}^{L_n-1} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right] = \sum_{i\in N'_n} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right] +$ $\sum_{i\in(N'_n)^c} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right]$ and $\sum_{i\in N'_n} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right] < \varepsilon$ because F is absolutely continuous on N'_n .

Because F is C^1 on A^c and $\left[\frac{i}{10^n}, \frac{i+1}{10^n}\right] \subset \left[\tilde{a}, \tilde{b}\right] - V = \tilde{V}$ if $i \notin N'_n$, we can use the Mean Value Theorem on A^c, that is, there exists $\frac{i}{10^n} < c_i < \frac{i+1}{10^n}$ such that $F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) = f(c_i) \cdot \frac{y}{10^n}$ if $i \notin N'_n$.

Let us show that $\sum_{i=\ell_n}^{L_n-1} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right]$ converges to $\left(\int_{\tilde{a}}^{\tilde{b}} f(x)dx\right) \cdot y$ when n goes to infinity.

We have that

$$\left| \sum_{i\in N'_n} \left[F\left(\frac{i}{10^n} + \frac{y}{10^n}\right) - F\left(\frac{i}{10^n}\right) \right] + \sum_{i\in(N'_n)^c} \frac{f(c_i)}{10^n} \cdot y \right.$$

$$\left. - \left(\sum_{i\in N'_n} \int_{\frac{i}{10^n}}^{\frac{i+1}{10^n}} f(x)dx + \sum_{i\in(N'_n)^c} \int_{\frac{i}{10^n}}^{\frac{i+1}{10^n}} f(x)dx \right) \cdot y \right|$$

$$< 2\varepsilon + \left| \sum_{i\in(N'_n)^c} \frac{f(c_i) - f(d_i)}{10^n} \cdot y \right| < 2\varepsilon + \frac{\varepsilon}{\tilde{b} - \tilde{a}} \cdot y \cdot (\tilde{b} - \tilde{a})$$

and this last inequality holds because if $i \notin N'_n$, the set $\left[\frac{i}{10^n}, \frac{i+1}{10^n}\right] \subset \left[\tilde{a}, \tilde{b}\right] = V = \tilde{V}$ and $\tilde{V}$ is a closed set in $\left[\tilde{a}, \tilde{b}\right]$. Then $\tilde{V}$ is a compact set and f is a uniformily continuous function on $\tilde{V}$. So, given $\varepsilon > 0$, there is a $\delta_* > 0$ such that if $|x - y| < \delta_*$ then $|f(x) - f(y)| < \frac{\varepsilon}{\tilde{b} - \tilde{a}}$. Thus, we can take n so large that $\frac{1}{10^n} < \delta_*$.

By observing that $y = \left(\int_{-\infty}^{\infty} f(x)dx\right) \cdot y = \left(\int_{-\infty}^{\bar{a}} f(x)dx + \int_{\bar{a}}^{\bar{b}} f(x)dx + \int_{\bar{b}}^{\infty} f(x)dx\right) \cdot y < \left(\varepsilon + \int_{\bar{a}}^{\bar{b}} f(x)dx + \varepsilon\right) \cdot y$, we have for n sufficiently large that $|G_n(y) - y| < 7\varepsilon$ and the theorem is proved.

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