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VERTEX-CONSTRAINED TRANSVERSALS IN A BIPARTITE GRAPH

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1. Introduction

Let G be a finite directed graph with directed bipartition (V_p, V_n), and u a positive integer function on V. McWhirter and Younger [McWY] proved that G has a u-constrained transversal of directed coboundaries if and only if for each nonnull subset M of V_p or V_n the number of components in the subgraph induced by $V-M$ is no greater than uM. This paper proves the following generalization:

Theorem: *For any finite directed bipartite graph G and positive integer function u on V, a subset r of e includes a u-constrained transversal if and only if $|\mathbf{X}| \leq uM$ for every r-supported nondegenerate structure* $(\mathbf{X}, M)$.

A *directed graph* G consists of a set V of *vertices* and a set e of *edges*, each edge having a *positive end* and a *negative end* in V. A *positive set* in G is a subset S of V such that no edge has its negative end in S and its positive end in $V-S$. A *negative set* is defined dually, that is, by interchanging 'positive' and 'negative' in the previous definition. A *signed set* is either a positive set or a negative set. A signed set is *nondegenerate* if it is distinct from V. A *positive partition* is a disjoint collection of nonnull positive sets. A *negative partition* is defined dually. A *signed partition* is either a positive or a negative partition; it is *nondegenerate* if distinct from $\{V\}$.

For a subset t of e and a signed set S, let tS denote the set of edges in t that have one end in S and the other in $V-S$. Sets of the form eS are *directed coboundaries*. For a signed partition $\mathbf{X}$, let $t\mathbf{X}$ abbreviate $t(\cup\mathbf{X})$ or, equivalently, $\cup\{tS : S \in \mathbf{X}\}$.

A *transversal (of directed coboundaries)* is a subset t of e such that $tS \neq \varnothing$ for every nonnull nondegenerate signed set S. (Hence, a disconnected graph has no transversal.) A transversal t is *u-constrained* if $|t\{v\}| \leq uv$ for each vertex v. Here, u is a positive integer function on

V, that is, a function from V to the set of strictly positive integers. The domain of u is extended to subsets of V by the definition $uM = \Sigma\{uv : v \in M\}$ for each subset M of V.

A *source* is a vertex v such that $\{v\}$ is a positive set. A *sink* is defined dually. The set of sources of the graph is denoted by V_p; the set of sinks, by V_n. The graph is *directed bipartite* if each vertex is a source or a sink, that is, if each edge has positive end in V_p and negative end in V_n.

In a directed bipartite graph, a *positive structure* is a pair $(\mathbf{X},M)$ where $\mathbf{X}$ is a positive partition and M is a subset of $V_n - \cup\mathbf{X}$. A *negative structure* is defined dually. The *null* structure is $(\emptyset,\emptyset)$. A structure $(\mathbf{X},M)$ is *nondegenerate* is $\mathbf{X}$ is nondegenerate; it is *marginal* if $|\mathbf{X}| = uM$; *submarginal* if $|\mathbf{X}| \le uM$; and *supermarginal* if $|\mathbf{X}| \ge uM$.

The McWhirter-Younger theorem gives necessary and sufficient conditions under which a directed bipartite graph has a u-constrained transversal. The conditions can be restated as follows: every nondegenerate structure $(\mathbf{X},M)$ such that $e\mathbf{X} \subset eM$ is submarginal. This paper shows that a recursive generalization of these conditions is necessary and sufficient for the existence of a u-constrained transversal within a specified subset r of e. The motivation for studying constrained transversals comes from the following conjecture by Woodall [W] and Edmonds-Giles [EG]:

> In any finite directed graph, a maximum disjoint collection of transversals of directed coboundaries is equal in cardinality to a minimum set of the form eS, S a nonnull nondegenerate signed set.

The equality has been proved for source-sink connected graphs [S] [FY]. The present Theorem may be relevant to the bipartite case of the conjecture.

We proceed to define a recursive generalization of the class of structures $(\mathbf{X},M)$ for which $r\mathbf{X} \subset eM$. The definition uses the following terminology. A signed partition $\mathbf{Y}$ is *internal* to a signed set S if $\cup\mathbf{Y} \subset S$. A structure $(\mathbf{Y},N)$ is *internal* to S if $\mathbf{Y}$ is internal to S and $N \subset S$.

Let r be a subset of e and $(\mathbf{X},M)$ a positive structure. An *r-support* of an element S of $\mathbf{X}$ is a negative structure $(\mathbf{X}_S,M_S)$, internal to S, such that $|\mathbf{X}_S| \ge uM_S$, $rS \subset eM \cup eM_S$, and each element of $\mathbf{X}_S$ has an r-support. The dual definition holds when $(\mathbf{X},M)$ is negative. To emphasize the dependence on M, we may say that $(\mathbf{X}_S,M_S)$ is an r-support of S *relative to M*. A structure $(\mathbf{X},M)$ is *r-supported* if each element S of $\mathbf{X}$ has an r-support.

Examples: The null structure is r-supported. Any structure $(\mathbf{X},M)$

such that $r\mathbf{X} \subset eM$ is r-supported.

The *size* of a signed partition $\mathbf{X}$ is the number $|\cup\mathbf{X}|$. If an element S of $\mathbf{X}$ has a support, then it has a support whose first coordinate is strictly smaller than $\mathbf{X}$, as the following argument shows. Let $(\mathbf{X}_S, M_S)$ be a support of S, and suppose that $\mathbf{X}_S$ is not strictly smaller than $\mathbf{X}$. Since the support is internal to S, therefore $\cup\mathbf{X}_S = S$, and so $M_S = \varnothing$. But then the null structure is a support of S.

We adopt the following convention. For an r-supported structure $(\mathbf{X}, M)$, if $(\mathbf{X}_S, M_S)$ is said to be an r-support of an element of $\mathbf{X}$, it is to be understood that $\mathbf{X}_S$ is strictly smaller that $\mathbf{X}$.

A subset r of e is *feasible* if every r-supported nondegenerate structure is submarginal. The Theorem can now be stated as follows: A subset r of e includes a u-constrained transversal if and only if r is feasible.

2. Reduction of the Theorem to the Fringe Lemma

An edge is *internal* to a signed set X if it has both ends in X. An edge is *external* to X if it is internal to $V-X$. These definitions extend naturally to a signed partition $\mathbf{X}$: an edge is *internal* or *external* to $\mathbf{X}$ if it is internal or external to $\cup\mathbf{X}$. For a subset t of e, let $t[X]$ denote the set of edges in t that are internal to X. Let $t[\mathbf{X}]$ and $t[V\sim\mathbf{X}]$ abbreviate $t[\cup\mathbf{X}]$ and $t[V-\cup\mathbf{X}]$.

The *fringe* of a structure $(\mathbf{X}, M)$ is the set $eM \cap e[V\sim\mathbf{X}]$, that is, the set of edges in eM that are external to $\mathbf{X}$. The fringe of $(\mathbf{X}, M)$ *in* a subset t of e is $tM \cap e[V\sim\mathbf{X}]$.

Property A: For a negative partition $\mathbf{Y}$ internal to a positive set S, each edge in $e\mathbf{Y}$ is internal to S and each edge in eS is external to $\mathbf{Y}$.

Corollary: For a negative structure $(\mathbf{Y}, N)$ internal to a positive set S, the intersection of eS and eN is a subset of the fringe of $(\mathbf{Y}, N)$.

The proofs are elementary.

Proof of the 'only if' assertion in the Theorem: Suppose r includes a u-contained transversal t. We proceed to show that t is feasible, that is,

(1) Every r-supported nondegenerate structure $(\mathbf{X}, M)$ is submarginal.

In order to carry out a proof by induction, we expand 1 by the addition of

(1') If $(\mathbf{X}, M)$ is marginal, then its fringe in t is null.

The proof is by induction on the size of $\mathbf{X}$. If the size is 0, then $\mathbf{X} = \varnothing$ and so the assertion 1,1' is trivially true. Now suppose the size of $\mathbf{X}$ is nonzero. Assume $|\mathbf{X}| \geq uM$, for otherwise 1,1' is trivially true. Let S be an arbitrary element of $\mathbf{X}$ and $(\mathbf{X}_S, M_S)$ an r-support of S. Then, by the Corollary of Property A, $tS \subset eM \cup f_S$, where f_S is the fringe of $(\mathbf{X}_S, M_S)$ in t. By induction hypothesis, $(\mathbf{X}_S, M_S)$ is marginal, and therefore $f_S = \varnothing$. So, $tS \subset eM$. Since this holds for each S in $\mathbf{X}$, therefore $t\mathbf{X} \subset eM$. Hence, $|t\mathbf{X}| \leq |tM|$. But $|t\mathbf{X}| \geq |\mathbf{X}|$ since $\mathbf{X}$ is nondegenerate and t is a transversal; $|\mathbf{X}| \geq uM$ by hypothesis; and $uM \geq |tM|$ since t is u-constrained. Hence $|t\mathbf{X}| = |tM|$ and equalities hold throughout. In particular, $|\mathbf{X}| = uM$ and $t\mathbf{X} = tM$. The first equality verifies 1; the second, 1'. This completes the induction and so proves the 'only if' assertion. □

The proof of the 'if' assertion proceeds by induction on r: given a feasible subset t of r, we search for an edge α in t such that t-α is not feasible. Here t-α is an abbreviation of $t - \{\alpha\}$. An edge α is *critical* if t-α is not feasible. Critical edges can be characterized in terms of the following concept.

For subset t of e and edge α in t, a structure $(\mathbf{X}, M)$ is $t$modα-*supported* if each element S of $\mathbf{X}$ has a $(t$-$\alpha)$-support $(\mathbf{X}_S, M_S)$ that is t-supported. Our convention about supports is naturally extended: for any $t$modα-supported structure $(\mathbf{X}, M)$, if a structure $(\mathbf{X}_S, M_S)$ is said to be a $t$modα-support of an element of $\mathbf{X}$, it is to be understood that $\mathbf{X}_S$ is strictly smaller than $\mathbf{X}$.

Critical Edge Lemma: *Let t be a feasible subset of e. For any critical edge α in t, there exists a $t$modα-supported nondegenerate structure $(\mathbf{X}, M)$ such that $|\mathbf{X}| > uM$.*

Proof. Observe that

(1) For every $(t$-$\alpha)$-supported structure $(\mathbf{Y}, N)$, if α is external to $\mathbf{Y}$, then $(\mathbf{Y}, N)$ is t-supported.

The proof of this observation is a simple exercise.

Since t-α is not feasible, there exists a $(t$-$\alpha)$-supported nondegenerate structure $(\mathbf{X}, M)$ such that $|\mathbf{X}| > uM$. Let Ω denote the collection of all such structures. We will show that some structure in Ω is $t$modα-supported.

For each $(\mathbf{X}, M)$ in Ω, α is nonexternal to some element S of $\mathbf{X}$. Otherwise, by 1, $(\mathbf{X}, M)$ would be t-supported, contrary to the feasibility of t. We call S the *α-element* of $\mathbf{X}$.

Choose $(\mathbf{X}, M)$ in Ω so that the α-element of $\mathbf{X}$ is minimal. Then

(2) α is noninternal to **X**,

as we proceed to show. Suppose the contrary. Then α is internal to the α-element, S, of **X**. Let $(\mathbf{Y},N)$ be a $(t\text{-}\alpha)$-support of S. Since $(\mathbf{Y},N)$ is internal to S, therefore $\mathbf{Y}^+ = \mathbf{Y} \cup \{V-S\}$ is a signed partition and $(\mathbf{Y}^+,N)$ a structure. In this structure, each element of $\mathbf{Y}^+$ other than $V-S$ has a $(t\text{-}\alpha)$-support. Now consider $V-S$. Let $\mathbf{X}^- = \mathbf{X} - \{S\}$ and observe that the structure $(\mathbf{X}^-,M)$ is internal to $V-S$, supermarginal, and $(t\text{-}\alpha)$-supported. Since $(t\text{-}\alpha)(V-S) = (t\text{-}\alpha)S \subset eM \cup eN$, therefore $(\mathbf{X}^-,M)$ is a $(t\text{-}\alpha)$-support of $V-S$. So, the structure $(\mathbf{Y}^+,N)$ is $(t\text{-}\alpha)$-supported. Moreover, $|\mathbf{Y}^+| = |\mathbf{Y}| + 1 \geq uN + 1 > uN$. Hence, $(\mathbf{Y}^+,N)$ lies in Ω. The α-element, T, of $\mathbf{Y}^+$ is in **Y**. Since **Y** is internal to S and strictly smaller than **X**, therefore T is a proper subset of S. This is contrary to the minimality of S. The contradiction establishes 2.

We can show now that

(3) $(\mathbf{X},M)$ is $t \bmod \alpha$-supported.

Let S be an arbitrary element of **X** and $(\mathbf{X}_S,M_S)$ a $(t\text{-}\alpha)$-support of S. Since $\mathbf{X}_S$ is internal and α is noninternal to **X**, Property A implies that α is external to $\mathbf{X}_S$. Now, by 1, $(\mathbf{X}_S,M_S)$ is t-supported. This holds for each S in **X**, whence $(\mathbf{X},M)$ is $t \bmod \alpha$-supported. This completes the proof of the Lemma. □

Reduction of the 'If' assertion in the Theorem to the Fringe Lemma: Suppose r is feasible and let t be a minimal feasible subset of r. The following argument shows that t is a transversal. Let S be a nonnull nondegenerate positive set and M a minimal subset of $V_n - S$ such that $tS \subset eM$. Then the structure $(\{S\},M)$ is t-supported. Since t is feasible, $uM \geq |\{S\}|$, whence $M \neq \varnothing$. By minimality of M, $tS \neq \varnothing$. The same argument shows that $tS \neq \varnothing$ for every nonnull nondegenerate negative set S. Hence, t is a transversal.

The proof that t is u-constrained is based upon the following

Fringe Lemma: Let $(\mathbf{Y},N)$ be a supermarginal nondegenerate structure such that $t\mathbf{Y} \subset eN$. For any edge α in the fringe of $(\mathbf{Y},N)$, every $t \bmod \alpha$-supported nondegenerate structure is submarginal.

The proof of the Lemma is given in section 5. To show that t is u-constrained we use the following corollary of the Lemma:

(1) The fringe in t of every t-supported supermarginal nondegenerate structure $(\mathbf{Y},N)$ is null.

The proof of 1 is by induction on the size of **Y**. If the size is 0, then $\mathbf{Y} = N = \varnothing$ and so the assertion is trivially true. Now suppose the size of **Y** is nonzero. Let S be an arbitrary element of **Y** and $(\mathbf{Y}_S,N_S)$ a t-

support of S. By the Corollary of Property A, $tS \subset eN \cup f_S$, where f_S is the fringe of (Y_S, N_S) in t. By induction hypothesis, $f_S = \emptyset$. So, $tS \subset eN$. This holds for each S in Y, whence $tY \subset eN$. Now, by the Fringe Lemma and the Critical Edge Lemma, no edge in the fringe of (Y,N) is critical. Since t is minimal feasible, the fringe of (Y,N) in t must be null.

This corollary has the following consequence:

(2) For each α in t, there exists a signed set S such that $tS = \{\alpha\}$.

Here is a proof of this statement. By the Critical Edge Lemma, there exists a $t$modα-supported nondegenerate structure (X,M) such that $|X| > uM$. Let S be an arbitrary element of X and (X_S, M_S) a $t$modα-support of S. By the Corollary of Property A, $tS \subset eM \cup f_S \cup \{\alpha\}$, where f_S is the fringe in t of (X_S, M_S). By 1, $f_S = \emptyset$. Hence $tS \subset eM \cup \{\alpha\}$. This holds for each S in X, whence $tX \subset eM \cup \{\alpha\}$. Observe that $\alpha \in tS$ for some S in X, for otherwise (X,M) is t-supported, contrary to the feasibility of t. Let $X^- = X - \{S\}$ and let f be the fringe in t of the structure (X^-, M). Then $tS \subset \{\alpha\} \cup f$. Since $tX^- \subset eM$ and $|X^-| \geq uM$, therefore, by 1, $f = \emptyset$. So, $tS \subset \{\alpha\}$. This establishes 2.

Let v be any vertex in V_n. We deduce the inequality $|t\{v\}| \leq uv$ from 2. For each α in $t\{v\}$, let S_α be a signed set such that $tS_\alpha = \{\alpha\}$. Adjust notation, by substituting $V - S_\alpha$ for S_α if necessary, so that S_α is positive. Let X denote the collection $\{S_\alpha : \alpha \in t\{v\}\}$. For distinct α and β, the set $S_\alpha \cap S_\beta$ is null since $t(S_\alpha \cap S_\beta) = \emptyset$ and t is a transversal. So, X is a partition. Since $tX \subset e\{v\}$, the structure $(X, \{v\})$ is t-supported. Since X is nondegenerate and t is feasible, $|X| \leq u\{v\}$. Hence, $|t\{v\}| \leq uv$. The same argument shows that $|t\{v\}| \leq uv$ for each v in V_p. So, t is u-constrained. □

The proof of the Theorem has now been reduced to the Fringe Lemma.

3. Properties of Supported Structures

Positive structures (X,M) and (Y,N) are *disjoint* if $\cup X \cap \cup Y = \emptyset$, $M \cap \cup Y = \emptyset$, and $N \cap \cup X = \emptyset$. If (X,M) and (Y,N) are disjoint, let $(X,M) \cup (Y,N)$ denote the structure $(X \cup Y, M \cup N)$.

Disjoint Structures Property: *Let t be a subset of e. If structures (X,M) and (Y,N) are disjoint and t-supported, then the structure $(X,M) \cup (Y,N)$ is t-supported.*

The proof is elementary.

Let $\mathbf{X}$ and $\mathbf{Y}$ be signed partitions. The *part of* $\mathbf{X}$ *internal to* $\mathbf{Y}$, denoted $\mathbf{X} \times \mathbf{Y}$, is the partition $\{X \in \mathbf{X} : X \subset Y \text{ for some } Y \text{ in } \mathbf{Y}\}$. The *part of* $\mathbf{X}$ *noninternal to* $\mathbf{Y}$ is $\mathbf{X} - (\mathbf{X} \times \mathbf{Y})$. The partition $\mathbf{X}$ is *internal to* $\mathbf{Y}$ if $\mathbf{X} = \mathbf{X} \times \mathbf{Y}$. For a subset M of V, $M \times \mathbf{Y}$ is an alternative notation for $M \cap \cup\mathbf{Y}$. For a negative set Y, $M \times Y$ and $\mathbf{X} \times Y$ are abbreviations of $M \times \{Y\}$ and $\mathbf{X} \times \{Y\}$. For a structure $(\mathbf{X},M)$, let $(\mathbf{X},M) \times Y$ denote the structure $(\mathbf{X} \times Y, M \times Y)$.

Internal Part Property: *Let t be a subset of e, Y a negative set, and $(\mathbf{X},M)$ a positive structure. If $(\mathbf{X},M)$ is t-supported, then so is $(\mathbf{X},M) \times Y$. For any edge α external to Y, if $(\mathbf{X},M)$ is t modα-supported, then $(\mathbf{X},M) \times Y$ is t-supported.*

Proof. Let S be an arbitrary element of $\mathbf{X} \times Y$, and $(\mathbf{X}_S,M_S)$ a t-support or t modα-support of S. Then $tS \subset eM \cup eM_S$ or $tS \subset eM \cup eM_S \cup \{\alpha\}$. By Property A, $tS \subset e[Y]$. Since $\{\alpha\} \cap e[Y] = \emptyset$ and $eM \cap e[Y] \subset e(M \times Y)$, therefore $tS \subset e(M \times Y) \cup eM_S$. The same argument can be made for each S in $\mathbf{X} \times Y$. Hence, $(\mathbf{X},M) \times Y$ is t-supported. □

A *refinement* of a positive partition $\mathbf{X}$ is any positive partition $\dot{\mathbf{X}}$ such that $\cup\dot{\mathbf{X}} = \mathbf{X}$ and each element of $\dot{\mathbf{X}}$ is a subset of some element of $\mathbf{X}$. A refinement $\dot{\mathbf{X}}$ of $\mathbf{X}$ is *proper* if $\dot{\mathbf{X}} \neq \mathbf{X}$; it is *finest* if $\mathbf{X}$ has no proper refinement. The relevant properties of any refinement $\dot{\mathbf{X}}$ of $\mathbf{X}$ are: $e\dot{\mathbf{X}} = e\mathbf{X}$; $|\dot{\mathbf{X}}| \geq |\mathbf{X}|$; $|\dot{\mathbf{X}}| = |\mathbf{X}|$ iff $\dot{\mathbf{X}} = \mathbf{X}$.

For a positive partition $\mathbf{X}$ and a negative partition $\mathbf{Y}$, let $\mathbf{X} \cdot \mathbf{Y}$ denote the collection of all nonnull sets of the form $X \cap Y$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$.

Property R: For a positive partition $\mathbf{X}$ and a negative partition $\mathbf{Y}$, if $\mathbf{X}$ is internal to $\cup\mathbf{Y}$, then $\mathbf{X} \cdot \mathbf{Y}$ is a refinement of $\mathbf{X}$.

The proof is elementary.

Refinement Property: *Let t be a feasible subset of e, $(\mathbf{X},M)$ a structure, and $\dot{\mathbf{X}}$ a refinement of $\mathbf{X}$. If $(\mathbf{X},M)$ is t-supported, then so is $(\dot{\mathbf{X}},M)$. For any α in t, if $(\mathbf{X},M)$ is t modα-supported then so is $(\dot{\mathbf{X}},M)$.*

Proof. The proof of the first assertion process by induction on the size of $\mathbf{X}$. If the size is 0, then $\mathbf{X} = \emptyset$ and so the assertion is trivially true. Now suppose the size of $\mathbf{X}$ is not 0. We give a proof of the case $|\dot{\mathbf{X}}| = |\mathbf{X}| + 1$; the general case will follow by induction. Specifically, we assume that $\dot{\mathbf{X}}$ can be obtained from $\mathbf{X}$ by partitioning an element S of $\mathbf{X}$ into two signed sets, S_1 and S_2. To prove that $(\dot{\mathbf{X}},M)$ is t-supported, we need only show that S_1 and S_2 have t-supports.

Let $(\mathbf{Y},N)$ be a t-support of S, and let $\dot{\mathbf{Y}}$ denote $\mathbf{Y} \cdot \{S_1,S_2\}$. By Property R, $\dot{\mathbf{Y}}$ is a refinement of $\mathbf{Y}$. By induction hypothesis, $(\dot{\mathbf{Y}},N)$ is t-supported. For $i = 1,2$, by the Internal Part Property,

(1) the structure $(\dot{\mathbf{Y}},N) \times S_i$ is t-supported.

Since $tS_i \subset tS \subset eM \cup eN$ and one end of each edge in $eN \cap tS_i$ is in $N \times S_i$,

(2) $tS_i \subset eM \cup e(N \times S_i)$.

From 1, since $\dot{\mathbf{Y}}$ is nondegenerate and t is feasible,

(3) $|\dot{\mathbf{Y}} \times S_i| \leq u(N \times S_i)$ for $i = 1,2$.

Since $\dot{\mathbf{Y}}$ is internal to $\{S_1,S_2\}$, therefore $|\dot{\mathbf{Y}} \times S_1| + |\dot{\mathbf{Y}} \times S_2| \geq |\dot{\mathbf{Y}}|$. But $|\dot{\mathbf{Y}}| \geq |\mathbf{Y}| \geq uN$ and N is the disjoint union of $N \times S_1$ and $N \times S_2$. Hence

(4) $|\dot{\mathbf{Y}} \times S_1| + |\dot{\mathbf{Y}} \times S_2| \geq u(N \times S_1) + u(N \times S_2)$.

By virtue of 4, equality holds in 3. So, by 1 to 4, the structure $(\dot{\mathbf{Y}},N) \times S_i$ is a t-support of S_i. The proof of the first assertion is complete. For a proof of the second assertion, replace 2 by $tS_i \subset eM \cup e(N \times S_i) \cup \{\alpha\}$. □

4. Meet, Join, and Difference

This section defines the meet, join, and difference of two structures. The context in which these operations will be used, the proof of the Fringe Lemma, contains the following elements: a feasible subset t of e, a supermarginal structure $(\mathbf{Y},N)$ such that $t\mathbf{Y} \subset eN$, an edge α in the fringe of $(\mathbf{Y},N)$, and a $t\,\mathrm{mod}\alpha$-supported structure $(\mathbf{X},M)$. The object of the proof is to show that $(\mathbf{X},M)$ is submarginal. There are two cases: either both structures are positive, or one is negative and the other positive. In each case, $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are combined into two new structures. In the first case, the new structures are called the meet and join of $(\mathbf{X},M)$ and $(\mathbf{Y},N)$; they are analogous to the intersection, $X \cap Y$, and union, $X \cup Y$, of positive sets X and Y. In the second case, the resulting structures are called differences; these are analogous to the differences $X - Y$ and $Y - X$ of a negative set X and a positive set Y. Each new structure is submarginal; this is because it, or some related structure, is t-supported. The submarginality of $(\mathbf{X},M)$ is then deduced by a cardinality relation. In the case where $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are positive, this cardinality relation, described in the Lemma on Meet and Join, is analogous to the identity $|X \cap Y| + |X \cup Y| = |X| + |Y|$. In the second case, this relation, given in the Lemma on Differences, is analogous to the set relation $|X - Y| + |X \cap Y| + |Y - X| + |Y \cap X| = |X| + |Y|$.

The first half of this section considers the meet and the join; the second half studies the difference.

The *meet* of positive partitions $\mathbf{X}$ and $\mathbf{Y}$, denoted $\mathbf{X} \wedge \mathbf{Y}$, is the collection of all nonnull sets of the form $X \cap Y$, $X \in \mathbf{X}$, $Y \in \mathbf{Y}$. The *join*, $\mathbf{X} \vee \mathbf{Y}$, is the collection of all sets of the form $\cup \mathbf{J}$, where $\mathbf{J}$ is a minimal nonnull subcollection of $\mathbf{X} \cup \mathbf{Y}$ such that $\cup \mathbf{J}$ is disjoint from $\cup(\mathbf{X} \cup \mathbf{Y} - \mathbf{J})$. Since the intersection and the union of positive sets are positive sets, $\mathbf{X} \wedge \mathbf{Y}$ and $\mathbf{X} \vee \mathbf{Y}$ are positive partitions.

Let $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ be positive structures. The *meet* of these structures, denoted $(\mathbf{X},M) \wedge (\mathbf{Y},N)$, is the structure $(\mathbf{X} \wedge \mathbf{Y}, (M \times \mathbf{Y}) \cup (N \times \mathbf{X}) \cup (M \cap N))$. The *join*, denoted $(\mathbf{X},M) \vee (\mathbf{Y},N)$, is the structure $(\mathbf{X} \vee \mathbf{Y}, (M \sim \mathbf{Y}) \cup (N \sim \mathbf{X}))$, where $M \sim \mathbf{Y}$ is an abbreviation of $M - \cup\mathbf{Y}$. When $\mathbf{X}$ and $\mathbf{Y}$ are understood, the second coordinates of the meet and the join are denoted by $M \wedge N$ and $M \vee N$ respectively.

Meet and join have the following property: for any subset t of e, if $t\mathbf{X} \subset eM$ and $t\mathbf{Y} \subset eN$, then $t(\mathbf{X} \wedge \mathbf{Y}) \subset e(M \wedge N)$ and $t(\mathbf{X} \vee \mathbf{Y}) \subset e(M \vee N)$. The generalization of the second part of this property to the context of the Fringe Lemma is true: if $(\mathbf{X},M)$ is $t$moda-supported and $t\mathbf{Y} \subset eN$, then the join $(\mathbf{X},M) \vee (\mathbf{Y},N)$ is t-supported. This is the main assertion of Lemma J below. The generalization of the first part of the property is not true: the meet may not be t-supported. Despite that, if t is feasible, the meet is submarginal. The proof of this submarginality is one of the objectives of the Main Lemma in the next section.

Lemma J: *Let t be a subset of e and $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ positive structures. If $t\mathbf{Y} \subset eN$ and $(\mathbf{X},M)$ is t-supported or $t$moda-supported for some edge α in the fringe of $(\mathbf{Y},N)$, then $(\mathbf{X},M) \vee (\mathbf{Y},N)$ is t-supported.*

Proof. For each S in $\mathbf{X}$, let $(\mathbf{X}_S,M_S)$ be a t-support or $t$moda-support of S. Let J be an arbitrary element of $\mathbf{X} \vee \mathbf{Y}$ and $\mathbf{X}'$ and $\mathbf{Y}'$ the subcollections of $\mathbf{X}$ and $\mathbf{Y}$ such that $J = \cup(\mathbf{X}' \cup \mathbf{Y}')$. Let $(\mathbf{X}_J,M_J)$ denote the structure $\cup\{(\mathbf{X}_S,M_S) : S \in \mathbf{X}'\}$. We wish to show that $(\mathbf{X}_J,M_J)$ is a t-support of J relative to $M \vee N$.

By construction, $(\mathbf{X}_J,M_J)$ is supermarginal and internal to J. By the Disjoint Structures Property, $(\mathbf{X}_J,M_J)$ is t-supported. There remains the proof that $tJ \subset e(M \vee N) \cup eM_J$. An elementary argument shows that $tJ \subseteq t\mathbf{X}' \cup t\mathbf{Y}' - e[\mathbf{X}] \cup e[\mathbf{Y}]$. By hypothesis, $t\mathbf{Y}' \subset eN$ and $t\mathbf{X}' \subset eM \cup eM_J \cup eN$, since $\alpha \in eN$. Hence, $tJ \subset (eN - e[\mathbf{X}]) \cup$

$(eM - e[\mathbf{Y}]) \cup eM_J \subset e(N \sim \mathbf{X}) \cup e(M \sim \mathbf{Y}) \cup eM_J \subset e(M \vee N) \cup eM_J$.

So, $(\mathbf{X}_J, M_J)$ is a t-support of J. This holds for each J in $\mathbf{X} \vee \mathbf{Y}$. Hence, $(\mathbf{X} \vee \mathbf{Y}, M \vee N)$ is t-supported. □

Lemma on Meet and Join: *Let* $(\mathbf{X},M)$ *and* $(\mathbf{Y},N)$ *be positive structures. If* $(\mathbf{X},M) \wedge (\mathbf{Y},N)$ *and* $(\mathbf{X},M) \vee (\mathbf{Y},N)$ *are submarginal, then* $|\mathbf{X}| + |\mathbf{Y}| \leq uM + uN$. *If* $(\mathbf{X},M)$ *and* $(\mathbf{Y},N)$ *are supermarginal, then* $|\mathbf{X} \wedge \mathbf{Y}| + |\mathbf{X} \vee \mathbf{Y}| \geq u(M \wedge N) + u(M \vee N)$.

Proof. We begin by showing the supermodular inequality $|\mathbf{X} \wedge \mathbf{Y}| + |\mathbf{X} \vee \mathbf{Y}| \geq |\mathbf{X}| + |\mathbf{Y}|$. Let B be a bipartite graph whose vertex-set is the disjoint union of $\mathbf{X}$ and $\mathbf{Y}$ and whose edges are the pairs (X,Y) such that $X \in \mathbf{X}$, $Y \in \mathbf{Y}$, and $X \cap Y \neq \emptyset$. Then the number of edges of B is $|\mathbf{X} \wedge \mathbf{Y}|$ and the number of components of B is $|\mathbf{X} \vee \mathbf{Y}|$. For any graph, the number of edges plus the number of components is no smaller than the number of vertices. The supermodular inequality follows.

The intersection and union of $M \wedge N$ and $M \vee N$ are equal to $M \cap N$ and $M \cup N$ respectively. Hence, $u(M \wedge N) + u(M \vee N) = uM + uN$. This equality and the supermodular inequality in the previous paragraph prove the Lemma. □

Now consider the difference between structures.

For negative set X and positive partition $\mathbf{Y}$, let $X \sim \mathbf{Y}$ abbreviate $X - \cup\mathbf{Y}$. For negative partition $\mathbf{X}$, let $\mathbf{X} \sim \mathbf{Y}$ denote the collection of all nonnull sets of the form $X \sim \mathbf{Y}$, $X \in \mathbf{X}$. Note that $\mathbf{X} \sim \mathbf{Y}$ is a negative partition.

For a negative partition $\mathbf{X}$ and a positive structure $(\mathbf{Y},N)$, the *part of* $\mathbf{X}$ *with supermarginal* $(\mathbf{Y},N)$*-filling* is the collection of all X in $\mathbf{X}$ for which $(\mathbf{Y},N) \times X$ is supermarginal.

Let $(\mathbf{X},M)$ be a negative structure and $(\mathbf{Y},N)$ a positive structure. The *difference* between these structures, denoted $(\mathbf{X},M) \sim (\mathbf{Y},N)$, is the structure $(\mathbf{X}^* \sim \mathbf{Y}^\bullet, M \sim \mathbf{Y}^\bullet)$, where $\mathbf{X}^*$ is the part of $\mathbf{X}$ with supermarginal $(\mathbf{Y},N)$-filling; $\mathbf{Y}^\bullet$ is $\mathbf{Y} - (\mathbf{Y} \times \mathbf{X}^*)$, that is, the part of $\mathbf{Y}$ noninternal to $\mathbf{X}^*$; and $M \sim \mathbf{Y}^\bullet$ is $M - \cup\mathbf{Y}^\bullet$.

The difference has the following property: for any subset t of e, if $t\mathbf{X} \subset eM$ and $t\mathbf{Y} \subset eN$, then $(\mathbf{X},M) \sim (\mathbf{Y},N)$ is t-supported; specifically, each element $S \sim \mathbf{Y}^\bullet$ of $\mathbf{X}^* \sim \mathbf{Y}^\bullet$ has $(\mathbf{Y},N) \times S$ for support. The generalization of this property to the context of the Fringe Lemma is true: for feasible t, if $(\mathbf{X},M)$ is t modα-supported and $t\mathbf{Y} \subset eN$, then $(\mathbf{X},M) \sim (\mathbf{Y},N)$ is t-supported. This will be shown within the proof of Main Lemma in the next section. Under the same conditions,

$(\mathbf{Y},N) \sim (\mathbf{X},M)$ may not be t-supported. But a t-supported structure is obtained by adjoining to $(\mathbf{Y},N) \sim (\mathbf{X},M)$ the supports of some of the elements of $\mathbf{X}$. This is shown within the proof of the following lemma.

Lemma D: *Let t be a feasible subset of e, $(\mathbf{Y},N)$ a nondegenerate positive structure, and $(\mathbf{X},M)$ a negative structure. If $t\mathbf{Y} \subset eN \cup eM$ and $(\mathbf{X},M)$ is t-supported or $t$modα-supported for some edge α in the fringe of $(\mathbf{Y},N)$, then $(\mathbf{Y},N) \sim (\mathbf{X},M)$ is submarginal.*

Proof. Let $\mathbf{Y}^{*}$ be the part of $\mathbf{Y}$ with supermarginal $(\mathbf{X},M)$-filling, and $\mathbf{X}^{\bullet}$ the part of $\mathbf{X}$ noninternal to $\mathbf{Y}^{*}$. For each S in $\mathbf{X}^{\bullet}$, let $(\mathbf{X}_S,M_S)$ be a t-support or $t$modα-support of S. Let $(\mathbf{X}_1,M_1)$ denote the structure $\cup\{(\mathbf{X}_S,M_S) : S \in \mathbf{X}^{\bullet}\}$. Since $(\mathbf{X}_1,M_1)$ is internal to $\mathbf{X}^{\bullet}$, it is disjoint from the structure $(\mathbf{Y}^{*} \sim \mathbf{X}^{\bullet},\ N \sim \mathbf{X}^{\bullet})$. So, the pair $((\mathbf{Y}^{*} \sim \mathbf{X}^{\bullet}) \cup \mathbf{X}_1,$ $(N \sim \mathbf{X}^{\bullet}) \cup M_1)$ is a structure. We will show that it is t-supported. The submarginality of $(\mathbf{Y},N) \sim (\mathbf{X},M)$ will then follow directly.

By the Disjoint Structures Property, $(\mathbf{X}_1,M_1)$ is t-supported. In particular, each element of $\mathbf{X}_1$ is t-supported relative to $(N \sim \mathbf{X}^{\bullet}) \cup M_1$. There remains the proof that each element of $\mathbf{Y}^{*} \sim \mathbf{X}^{\bullet}$ is t-supported relative to $(N \sim \mathbf{X}^{\bullet}) \cup M_1$.

Let Y be an arbitrary element of $\mathbf{Y}^{*}$. By the Internal Part Property, since α is external to Y, therefore $(\mathbf{X},M) \times Y$ is t-supported. We proceed to verify that this structure is a support of $Y \sim \mathbf{X}^{\bullet}$ relative to $(N \sim \mathbf{X}^{\bullet}) \cup M_1$. By definition of $\mathbf{Y}^{*}$, $(\mathbf{X},M) \times Y$ is supermarginal; since $\mathbf{X} \times Y$ is parallel to $\mathbf{X}^{\bullet}$ and M is disjoint from $\cup\mathbf{X}$, therefore $(\mathbf{X},M) \times Y$ is internal to $Y \sim \mathbf{X}^{\bullet}$. There remains the proof of the inclusion $t(Y \sim \mathbf{X}^{\bullet}) \subset e(N \sim \mathbf{X}^{\bullet}) \cup eM_1 \cup e(M \times Y)$. An elementary argument shows that $t(Y \sim \mathbf{X}^{\bullet}) \subset tY \cap e[V \sim \mathbf{X}^{\bullet}] \cup t\mathbf{X}^{\bullet} \cap e[Y] \cup t\mathbf{X}^{\bullet} \cap tY$. By hypothesis, $tY \subset eN \cup eM$ and $t\mathbf{X}^{\bullet} \subset eM \cup\ eM_1 \cup e[V \sim \mathbf{Y}]$ since $\alpha \in e[V \sim \mathbf{Y}]$. Hence, $t(Y \sim \mathbf{X}^{\bullet}) \subset e(N \sim \mathbf{X}^{\bullet}) \cup eM_1 \cup eM$. Since the positive end of each edge in $eM \cup t(Y \sim \mathbf{X}^{\bullet})$ is in $M \times Y$, therefore $t(Y \sim \mathbf{X}^{\bullet}) \subset e(N \sim \mathbf{X}^{\bullet}) \cup eM_1 \cup e(M \times Y)$.

Since $\mathbf{Y}$ and $\mathbf{X}_1$ are nondegenerate and t is feasible, $((\mathbf{Y}^{*} \sim \mathbf{X}^{\bullet}) \cup \mathbf{X}_1, (N \sim \mathbf{X}^{\bullet}) \cup M_1)$ is submarginal. But $|\mathbf{X}_1| = \Sigma|\mathbf{X}_S| \geq \Sigma uM_S = uM_1$, with summations over all S in $\mathbf{X}^{\bullet}$, whence $(\mathbf{X}_1,M_1)$ is supermarginal. So, $(\mathbf{Y}^{*} \sim \mathbf{X}^{\bullet}, N \sim \mathbf{X}^{\bullet})$ is submarginal. This completes the proof of the Lemma. □

Lemma on Differences: *Let $(\mathbf{X},M)$ be a negative structure and $(\mathbf{Y},N)$ a positive structure such that each of $\mathbf{X}$ and $\mathbf{Y}$ has no proper refinement. If (a) $(\mathbf{X},M) \times Y$ is submarginal for each Y in $\mathbf{Y}$, (b) $(\mathbf{Y},N) \times X$ is submarginal for each X in $\mathbf{X}$, and (c) the differences*

$(\mathbf{X},M) \sim (\mathbf{Y},N)$ and $(\mathbf{Y},N) \sim (\mathbf{X},M)$ are submarginal, then $|\mathbf{X}| + |\mathbf{Y}| \leq uM + uN$.

Proof. The proof consists of the following chain of inequalities:

$$\begin{aligned}
|\mathbf{X}| + |\mathbf{Y}| &= |\mathbf{X}^* \sim \mathbf{Y}^\bullet| + |\mathbf{X}^* \times \cup\mathbf{Y}^\bullet| + |\mathbf{X} - \mathbf{X}^*| \\
&\quad + |\mathbf{Y}^* \sim \mathbf{X}^\bullet| + |\mathbf{Y}^* \times \cup\mathbf{X}^\bullet| + |\mathbf{Y} - \mathbf{Y}^*| && (1) \\
&\leq |\mathbf{X}^* \sim \mathbf{Y}^\bullet| + |\mathbf{X} \times \mathbf{Y}^\bullet| + |\mathbf{X}^\bullet - \mathbf{X}^*| \\
&\quad + |\mathbf{Y}^* \sim \mathbf{X}^\bullet| + |\mathbf{Y} \times \mathbf{X}^\bullet| + |\mathbf{Y}^\bullet - \mathbf{Y}^*| && (2) \\
&\leq u(M \sim \mathbf{Y}^\bullet) + u(M \times \mathbf{Y}^\bullet) \\
&\quad + u(N \sim \mathbf{X}^\bullet) + u(N \times \mathbf{X}^\bullet) && (3) \\
&= uM + uN
\end{aligned}$$

To verify 1, observe that one part of $\mathbf{X}^*$ is internal to $\cup\mathbf{Y}^\bullet$, the other is noninternal to $\cup\mathbf{Y}^\bullet$. And the latter has the same number of elements as $\mathbf{X}^* \sim \mathbf{Y}^\bullet$. Symmetric considerations hold for $\mathbf{Y}$.

The part of $\mathbf{X}^*$ internal to $\cup\mathbf{Y}^\bullet$ is a subcollection of $\mathbf{X} \times \cup\mathbf{Y}^\bullet$. Since $\mathbf{X}$ has no proper refinement, Property R implies that $\mathbf{X} \times \cup\mathbf{Y}^\bullet = \mathbf{X} \times \mathbf{Y}^\bullet$. This verifies the first part of 2. For the second part, observe that ech element of $\mathbf{X} - \mathbf{X}^*$ intersects N and is therefore noninternal to $\mathbf{Y}$, in particular noninternal to $\mathbf{Y}^*$. Hence $\mathbf{X} - \mathbf{X}^* = \mathbf{X}^\bullet - \mathbf{X}^*$.

By hypothesis, $|\mathbf{X}^* \sim \mathbf{Y}^\bullet| \leq u(M \sim \mathbf{Y}^\bullet)$ and $|\mathbf{Y}^* \sim \mathbf{X}^\bullet| \leq u(N \sim \mathbf{X}^\bullet)$. To complete the verification of 3 let $\mathbf{Y}^{\bullet *}$ abbreviate $\mathbf{Y}^\bullet \cap \mathbf{Y}^*$ and observe that

$$\begin{aligned}
&|\mathbf{X} \times \mathbf{Y}^\bullet| + |\mathbf{Y}^\bullet - \mathbf{Y}^*| = \\
&\quad = |\mathbf{X} \times \mathbf{Y}^{\bullet *}| + \Sigma\{|\mathbf{X} \times Y| : Y \in \mathbf{Y}^\bullet - \mathbf{Y}^*\} + |\mathbf{Y}^\bullet - \mathbf{Y}^*| \\
&\quad \leq u(M \times \mathbf{Y}^{\bullet *}) + \Sigma\{u(M \times Y) - 1 : Y \in \mathbf{Y}^\bullet - \mathbf{Y}^*\} + |\mathbf{Y}^\bullet - \mathbf{Y}^*| \\
&\quad = u(M \times \mathbf{Y}^{\bullet *}) + u(M \times (\mathbf{Y}^\bullet - \mathbf{Y}^*)) \\
&\quad = u(M \times \mathbf{Y}^\bullet).
\end{aligned}$$

Similarly, $|\mathbf{Y} \times \mathbf{X}^\bullet| + |\mathbf{X}^\bullet - \mathbf{X}^*| \leq u(N \times \mathbf{X}^\bullet)$. This completes the proof of the Lemma. □

5. Proof of the Fringe Lemma

Fringe Lemma: *Let t be a feasible subset of e and $(\mathbf{Y},N)$ a supermarginal nondegenerate structure such that $t\mathbf{Y} \subset eN$. For any edge α of*

t in the fringe of $(\mathbf{Y},N)$, *every* $t$modα*-supported nondegenerate structure is submarginal.*

Proof. Let $(\mathbf{X},M)$ be a $t$modα-supported nondegenerate structure. Adjust notation so that $\mathbf{X}$ has no proper refinement. The adjustment consists of replacing $\mathbf{X}$ by its finest refinement $\dot{\mathbf{X}}$, and is justified as follows: by the Refinement Property, $(\dot{\mathbf{X}},M)$ is $t$modα-supported; since $|\mathbf{X}| \leq |\dot{\mathbf{X}}|$, the submarginality of $(\dot{\mathbf{X}},M)$ will imply the submarginality of $(\mathbf{X},M)$.

Let $\dot{\mathbf{Y}}$ be a refinement of $\mathbf{Y}$. Since $t\dot{\mathbf{Y}} = t\mathbf{Y} \subset eN$, therefore $(\dot{\mathbf{Y}},N)$ is t-supported, and consequently submarginal. By hypothesis, $(\mathbf{Y},N)$ is supermarginal. Hence $\dot{\mathbf{Y}} = \mathbf{Y}$. This argument shows that $\mathbf{Y}$ has no proper refinement.

Structures $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ satisfy the hypothesis of the Main Lemma below with $(\mathbf{Y},N)$ in place of $(\mathbf{Y}^\circ,N^\circ)$. So, if $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are positive, then

(1) $(\mathbf{X},M) \wedge (\mathbf{Y},N)$ is submarginal;

if $(\mathbf{X},M)$ is negative and $(\mathbf{Y},N)$ positive, then

(2) $(\mathbf{X},M) \sim (\mathbf{Y},N)$ is submarginal.

Now the proof follows up these alternatives.

Case 1: $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are positive.

If α is not in $e\mathbf{X}$, then $(\mathbf{X},M)$ is t-supported; since t is feasible and $\mathbf{X}$ is nondegenerate, $(\mathbf{X},M)$ is submarginal. Now suppose α is in $e\mathbf{X}$. Then the negative end of α is in $V \sim \mathbf{X}$ as well as $V \sim \mathbf{Y}$, whence $\mathbf{X} \vee \mathbf{Y}$ is nondegenerate. By Lemma J, $(\mathbf{X},M) \vee (\mathbf{Y},N)$ is t-supported. Since t is feasible,

(3) $(\mathbf{X},M) \vee (\mathbf{Y},N)$ is submarginal.

From 1 and 3, by the Lemma on Meet and Join, $|\mathbf{X}| + |\mathbf{Y}| \leq uM + uN$. Now, since $(\mathbf{Y},N)$ is supermarginal, $(\mathbf{X},M)$ is submarginal.

Case 2: $(\mathbf{X},M)$ is negative and $(\mathbf{Y},N)$ positive.

Let Y be an arbitrary element of $\mathbf{Y}$. Since α is external to Y, the Internal Part Property shows that $(\mathbf{X},M) \times Y$ is t-supported. Since $\mathbf{X}$ is nondegenerate and t feasible,

(4) $(\mathbf{X},M) \times Y$ is submarginal.

By a similar argument, for each X in $\mathbf{X}$,

(5) $(\mathbf{Y},N) \times X$ is submarginal.

By Lemma D,

(6) $(\mathbf{Y},N) \sim (\mathbf{X},M)$ is submarginal.

Recall that neither $\mathbf{X}$ nor $\mathbf{Y}$ has proper refinements. Then, by the Lemma on Differences, assertions 2 and 4-6 imply that $|\mathbf{X}| + |\mathbf{Y}| \leq uM + uN$. Now, since $(\mathbf{Y},N)$ is supermarginal, $(\mathbf{X},M)$ is submarginal.

The proof is complete except for the Main Lemma. □

Main Lemma: *Let t be a feasible subset of e and $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ nondegenerate structures. Suppose $t\mathbf{Y} \subset eN \cup e[V \sim \mathbf{X}]$ and $(\mathbf{X},M)$ is t-supported or t modα-supported for some edge α in the fringe of $(\mathbf{Y},N)$.*
1. If $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are positive, then $(\mathbf{X},M) \wedge (\mathbf{Y}^\circ,N^\circ)$ is submarginal for every $(\mathbf{Y}^\circ,N^\circ)$ such that $\mathbf{Y}^\circ \subset \mathbf{Y}$, $N^\circ \subset N$, $t(\mathbf{Y} - \mathbf{Y}^\circ) \subset e(N - N^\circ)$, and $|\mathbf{Y} - \mathbf{Y}^\circ| \geq u(N - N^\circ)$.
2. If $\mathbf{X},M$ is negative and $(\mathbf{Y},N)$ is positive, then $(\mathbf{X},M) \sim (\mathbf{Y},N)$ is submarginal.

Proof. The proof is by induction on the size of $\mathbf{X}$. If the size is 0, then $\mathbf{X} = \varnothing$ and the assertions of the Lemma are trivially true. Now suppose the size of $\mathbf{X}$ is nonzero.

Case 1. $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ are positive.

For each S in $\mathbf{X}$, let $(\mathbf{X}_S,M_S)$ be a t-support of S. Let $(\mathbf{X}_1,M_1)$ denote the structure $\cup\{(\mathbf{X}_S,M_S) : S \in \mathbf{X}\}$. By the Disjoint Structures Property,

(1.1) $(\mathbf{X}_1,M_1)$ is t-supported.

Since $(\mathbf{X}_S,M_S)$ is supermarginal for each S in $\mathbf{X}$,

(1.2) $(\mathbf{X}_1,M_1)$ is supermarginal.

Since $|\mathbf{X}_1| \geq uM_1$ and $|\mathbf{Y} - \mathbf{Y}^\circ| \geq u(N - N^\circ)$, the desired inequality $|\mathbf{X} \wedge \mathbf{Y}^\circ| \leq u(M \wedge N^\circ)$ is equivalent to

(1.3) $|\mathbf{X}_1| + |\mathbf{Y} - \mathbf{Y}^\circ| + |\mathbf{X} \wedge \mathbf{Y}^\circ| \leq uM_1 + u(N - N^\circ) + u(M \wedge N^\circ)$.

Let $\mathbf{Y}_1$ be a finest refinement of the partition $(\mathbf{Y} - \mathbf{Y}^\circ) \cup (\mathbf{X} \wedge \mathbf{Y}^\circ)$, and let N_1 denote $(N - N^\circ) \cup (M \wedge N^\circ)$. Observe that $(\mathbf{Y}_1,N_1)$ is a positive structure. Now, 1.3 is implied by

(1.4) $|\mathbf{X}_1| + |\mathbf{Y}_1| \leq uM_1 + uN_1$.

By the Lemma on Differences, 1.4 follows from

(1.5) $\mathbf{X}_1$ has no proper refinement,

(1.6) $(\mathbf{X}_1,M_1) \times Y_1$ is submarginal for each Y_1 in $\mathbf{Y}_1$,

(1.7) $(\mathbf{Y}_1,N_1) \times X_1$ is submarginal for each X_1 in $\mathbf{X}_1$,

(1.8) $(\mathbf{X}_1,M_1) \sim (\mathbf{Y}_1,N_1)$ is submarginal, and

(1.9) $(\mathbf{Y}_1,N_1) \sim (\mathbf{X}_1,M_1)$ is submarginal.

So, the proof of this case is reduced to the proofs of 1.5 to 1.9.

This paragraph proves 1.5. Let $\dot{\mathbf{X}}_1$ be a refinement of $\mathbf{X}_1$. By 1.1 and the refinement Property, $(\dot{\mathbf{X}}_1,M_1)$ is t-supported, and consequently submarginal. By 1.2, $(\mathbf{X}_1,M_1)$ is supermarginal. Hence $\dot{\mathbf{X}}_1 = \mathbf{X}_1$ and so $\mathbf{X}_1$ has no proper refinement.

Let Y_1 be an element of $\mathbf{Y}_1$. By 1.1 and the Internal Part Property, $(\mathbf{X}_1,M_1) \times Y_1$ is t-supported. Since t is feasible and $\mathbf{X}_1$ is nondegenerate, 1.6 holds.

The proofs of 1.7 to 1.9 depend on the inclusion

(1.10) $t\mathbf{Y}_1 \subset eN_1 \cup f_1$,where f_1 is the fringe of $(\mathbf{X}_1,M_1)$ in t.

An elementary argument shows that $t(\mathbf{X} \wedge \mathbf{Y}^\circ) = t\mathbf{X} \cap e[\mathbf{Y}^\circ] \cup t\mathbf{Y}^\circ \cap e[\mathbf{X}] \cup t\mathbf{X} \cap t\mathbf{Y}^\circ$. By hypothesis, $t\mathbf{Y}^\circ \subset eN^\circ \cup e(N - N^\circ) \cup e[V \sim \mathbf{X}]$ and $t\mathbf{X} \subset eM \cup eM_1 \cup e[V \sim \mathbf{Y}]$, since $\alpha \in e[V \sim \mathbf{Y}]$. In fact, $t\mathbf{X} \subset eM \cup f_1 \cup e[V \sim \mathbf{Y}]$ by the Corollary of Property A. Hence, $t(\mathbf{X} \wedge \mathbf{Y}^\circ) \subset e(M \times \mathbf{Y}^\circ) \cup e(N^\circ \times \mathbf{X}) \cup e(M \cap N^\circ) \cup e(N - N^\circ) \cup f_1 = e(M \wedge N^\circ) \cup e(N - N^\circ) \cup f_1$. Now, $t\mathbf{Y}_1 = t(\mathbf{Y} - \mathbf{Y}^\circ) \cup t(\mathbf{X} \wedge \mathbf{Y}^\circ) \subset e(N - N^\circ) \cup e(M \wedge N^\circ) \cup f_1 = eN_1 \cup f_1$.

Let X_1 be an element of $\mathbf{X}_1$. By 1.10, $t\mathbf{Y}_1 \subset eN_1 \cup e[V - X_1]$. Hence, by Property A, $t(\mathbf{Y}_1 \times X_1) \subset e(N_1 \times X_1)$. So, the structure $(\mathbf{Y}_1,N_1) \times X_1$ is t-supported. Since t is feasible and $\mathbf{Y}_1$ is nondegenerate, 1.7 holds.

Claim 1.8 is given by assertion 2 of the induction hypothesis with $(\mathbf{X}_1,M_1)$ and $(\mathbf{Y}_1,N_1)$ in place of $(\mathbf{X},M)$ and $(\mathbf{Y},N)$. Claim 1.9 is given by Lemma D with $(\mathbf{Y}_1,N_1)$ and $(\mathbf{X}_1,M_1)$ in place of $(\mathbf{Y},N)$ and $(\mathbf{X},M)$. The conditions for the appeal to the induction hypothesis and Lemma D are satisfied: $\mathbf{X}_1$ is smaller than $\mathbf{X}$; $\mathbf{X}_1$ and $\mathbf{Y}_1$ are nondegenerate; by 1.1, $(\mathbf{X}_1,M_1)$ is t-supported; by 1.10, $t\mathbf{Y}_1 \subset eN_1 \cup e[V \sim \mathbf{X}_1]$ as well as $t\mathbf{Y}_1 \subset eN_1 \cup eM_1$.

The proof of case 1 is complete.

Case 2: $(\mathbf{X},M)$ is negative and $(\mathbf{Y},N)$ positive.

Recall that $(\mathbf{X},M) \sim (\mathbf{Y},N) = (\mathbf{X}^{\mathbf{z}} \sim \mathbf{Y}^{\bullet}, M \sim \mathbf{Y}^{\bullet})$, where $\mathbf{X}^{\mathbf{z}}$ is the part of $\mathbf{X}$ with supermarginal $(\mathbf{Y},N)$-filling and $\mathbf{Y}^{\bullet}$ is $\mathbf{Y} - (\mathbf{Y} \times \mathbf{X}^{\mathbf{z}})$. Let S be an element of $\mathbf{X}^{\mathbf{z}}$ and $(\mathbf{X}_S,M_S)$ a structure that t-supports or t modα-supports S. Let $(\mathbf{X}'_S,M'_S)$ abbreviate $(\mathbf{X}_S,M_S) \times (S \sim \mathbf{Y}^{\bullet})$ and $(\mathbf{Y}'_S,N'_S)$ abbreviate $(\mathbf{Y},N) \times (S \sim \mathbf{Y}^{\bullet})$. Let $j = (\mathbf{X}'_S,M'_S) \vee (\mathbf{Y}'_S,N'_S)$. Then

(2.1) j is t-supported,

(2.2) j is internal to $S \sim \mathbf{Y}^*$,

(2.3) j is supermarginal, and

(2.4) $t(S \sim \mathbf{Y}^*) \subset e(M \sim \mathbf{Y}^*) \cup e(M'_S \vee N'_S)$.

Hence, j t-supports $S \sim \mathbf{Y}^*$ relative to $M \sim \mathbf{Y}^*$. The same argument can be made for each S in $\mathbf{X}^{\mathbf{z}}$. Hence, the structure $(\mathbf{X}^{\mathbf{z}} \sim \mathbf{Y}^*, M \sim \mathbf{Y}^*)$ is t-supported. Since t is feasible and $\mathbf{X}$ is nondegenerate, the structure is submarginal. This reduces case 2 to the proofs of 2.1 to 2.4.

This paragraph proves 2.1. By Property A, $t\mathbf{Y}'_S \subset t\mathbf{Y} \cap e[S \sim \mathbf{Y}^*] \subset eN \cap e[S \sim \mathbf{Y}^*] \subset eN'_S$. By the Internal Part Property with $(\mathbf{X}_S,M_S)$ and $S \sim \mathbf{Y}^*$ in place of $(\mathbf{X},M)$ and Y, the structure $(\mathbf{X}'_S,M'_S)$ is t-supported. Now, by Lemma J, j is t-supported.

Since $(\mathbf{X}'_S,M'_S)$ and $(\mathbf{Y}'_S,N'_S)$ are internal to $S \sim \mathbf{Y}^*$, so is their join. This verifies 2.2.

Claim 2.3 follows, by the Lemma on Meet and Join, from

(2.3a) $(\mathbf{X}'_S,M'_S) \wedge (\mathbf{Y}'_S,N'_S)$ is submarginal,

(2.3b) $(\mathbf{Y}'_S,N'_S)$ is supermarginal, and

(2.3c) $(\mathbf{X}'_S,M'_S)$ is supermarginal.

Claim 2.3a follows from assertion 1 of the induction hypothesis with $(\mathbf{X}'_S,M'_S)$ and $(\mathbf{Y}'_S,N'_S)$ and $(\mathbf{Y}'_S,N'_S)$ in place of $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ and $(\mathbf{Y}^\circ,N^\circ)$. The conditions for this appeal to the induction hypothesis are satisfied: $\mathbf{X}'_S$ is smaller that $\mathbf{X}$; $\mathbf{X}'_S$ and $\mathbf{Y}'_S$ are nondegenerate; $t\mathbf{Y}'_S \subset eN'_S$ and $(\mathbf{X}'_S,M'_S)$ is t-supported, as shown in the proof of 2.1.

Claim 2.3b is verified as follows. Since $\mathbf{Y} \times (S \sim \mathbf{Y}^*) = \mathbf{Y} \times S$, a consequence of $\cup(\mathbf{Y} \times S)$ disjoint from $\cup\mathbf{Y}^*$, therefore $|\mathbf{Y}'_S| = |\mathbf{Y} \times S|$. Since $S \in \mathbf{X}^{\mathbf{z}}$, therefore $|\mathbf{Y} \times S| \geq u(N \times S)$. Since $N \times S = N \times (S \sim \mathbf{Y}^*)$, therefore $u(N \times S) = uN'_S$. So, $|\mathbf{Y}'_S| \geq uN'_S$.

Claim 2.3c is given by the inequalities

(2.3c') $\quad |\mathbf{X}'_S| \geq |\mathbf{X}_S| - |\mathbf{X}_S \wedge \mathbf{Y}^*|$

(2.3c") $\quad \geq uM_S - u(M_S \wedge N^*)$, where $N^* = N - (N \times \mathbf{X}^{\mathbf{z}})$

(2.3c"') $\quad \geq uM'_S$.

This paragraph verifies 2.3c'. By definition of $\mathbf{X}'_S$, each element of $\mathbf{X}_S - \mathbf{X}'_S$ intersects $\cup\mathbf{Y}^*$. So, each element of $\mathbf{X}_S - \mathbf{X}'_S$ includes an element of $\mathbf{X}_S \wedge \mathbf{Y}^*$; hence $|\mathbf{X}_S| - |\mathbf{X}'_S| \leq |\mathbf{X}_S \wedge \mathbf{Y}^*|$.

Inequality 2.3c" follows from $|\mathbf{X}_S| \geq uM_S$ and $|\mathbf{X}_S \wedge \mathbf{Y}^*| \leq u(M_S \wedge N^*)$. The former is true by definition of $(\mathbf{X}_S,M_S)$. The latter is given by assertion 1 of the induction hypothesis with $(\mathbf{X}_S,M_S)$

and $(\mathbf{Y},N)$ and $(\mathbf{Y}^*,N^*)$ in place of $(\mathbf{X},M)$ and $(\mathbf{Y},N)$ and $(\mathbf{Y}^*,N^*)$. The conditions for this appeal to the induction hypothesis are satisfied, as the following analysis shows. First, $\mathbf{X}_S$ is smaller than $\mathbf{X}$. Second, $t\mathbf{Y} \subset eN \cup e[V \sim \mathbf{X}_S]$ since $\mathbf{X}_S$ is internal to $\mathbf{X}$. Third, $(\mathbf{X}_S,M_S)$ is t-supported. Fourth, $t(\mathbf{Y} - \mathbf{Y}^*) = t(\mathbf{Y} \times \mathbf{X}^*) \subset e(N \times \mathbf{X}^*) = e(N - N^*)$, where the inclusion follows from Property A. Fifth, $|\mathbf{Y} - \mathbf{Y}^*| = |\mathbf{Y} \times \mathbf{X}^*| = \Sigma|\mathbf{Y} \times X| \geq \Sigma u(N \times X) = u(N \times \mathbf{X}^*) = u(N - N^*)$, with summations over all X in $\mathbf{X}^*$, where the inequality is given by the definition of $\mathbf{X}^*$.

Since $(\mathbf{X}_S,M_S)$ is internal to $\mathbf{X}^*$, therefore $M_S \wedge N^* = (M_S \times \mathbf{Y}^*) \cup (N^* \times \mathbf{X}_S) \cup (M_S \cap N^*) \subset (M_S \times \mathbf{Y}^*) \cup (N^* \times \mathbf{X}^*)$. By definition of N^*, $N^* \times \mathbf{X}^* = \emptyset$. Hence, $M_S - (M_S \wedge N^*) = M_S - (M_S \times \mathbf{Y}^*) = M'_S$. Inequality 2.3c''' follows.

There remains the proof of the inclusion 2.4. An elementary argument shows that $t(S \sim \mathbf{Y}^*) = tS \cap e[V \sim \mathbf{Y}^*] \cup t\mathbf{Y}^* \cap e[S] \cup t\mathbf{Y}^* \cap tS$. By hypothesis $t\mathbf{Y}^* \subset eN \cup e[V - S]$ and $tS \subset eM \cup eM_S \cup eN$, since $\alpha \in eN$. So, $t(S \sim \mathbf{Y}^*) \subset e(M \sim \mathbf{Y}^*) \cup eM_S \cup eN$. In fact, since the positive end of each edge in $eM_S \cap t(S \sim \mathbf{Y}^*)$ is in $M_S \times (S \sim \mathbf{Y}^*)$, and the positive end of each edge in $eN \cap t(S \sim \mathbf{Y}^*)$ is in $N \times (S \sim \mathbf{Y}^*)$, therefore $t(S \sim \mathbf{Y}^*) \subseteq e(M \sim \mathbf{Y}^*) \cup eM'_S \cup eN'_S$. Since $\mathbf{Y}'_S$ and $\mathbf{X}'_S$ are internal to $S \sim \mathbf{Y}^*$, therefore, by Property A, $t(S \sim \mathbf{Y}^*)$ is a subset of $e[V \sim \mathbf{Y}'_S]$ and of $e[V \sim \mathbf{X}'_S]$. So, $t(S \sim \mathbf{Y}^*) \subset e(M \sim \mathbf{Y}^*) \cup e(M'_S \sim \mathbf{Y}'_S) \cup e(N'_S \sim \mathbf{X}'_S) = e(M \sim \mathbf{Y}^*) \cup e(M'_S \vee N'_S)$.

The proof of the Main Lemma is complete. □

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