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Bifurcation structures in two-dimensional maps: The endoskeletons of shrimps

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ABSTRACT

Bifurcation structures for nonlinear dynamical systems in a space of two parameters often display geometric shapes resembling shrimps. For one-dimensional maps with two parameters and multiple extrema, the underlying structure of the shrimps can be elucidated by computing the locus of superstable cycles which form a "skeleton" that supports the shrimps. Here we use continuation methods to identify and compute structures in two-dimensional maps that play the same role as the skeleton in one-dimensional maps. This facilitates determining the complex geometries for situations in which there is multistability, and for which the regions of parameter space supporting stable orbits get vanishingly small.

1. Introduction

Complex dynamics are found as parameters are varied in dynamical systems. In one-parameter, one-dimensional maps with a quadratic maximum, universal sequences of stable periodic orbits and period-doubling bifurcations leading to chaos occur [1,2]. A next step up in complexity involves one-dimensional maps with two parameters such as the sine circle map that captures crucial features of periodically forced oscillators [3]. As the amplitude and frequency of the periodic forcing change, complex geometries, consisting of interlocking and overlapping zones emerge, Fig. 1 [4–6,8].

Superstable cycles, which are cycles passing through one or more extremal points in a one-dimensional map, provide insight into the organization of these zones. The "skeleton" is the locus of superstable cycles in a two-dimensional parameter space of one-dimensional maps. In the 1980s, several groups described the remarkable self-similar structure of the skeleton for several different one-dimensional maps with multiple extrema including circle maps of various degree, cubic maps, and quartic maps [7–12].

Related studies by Kapral and colleagues recognized the presence of "a rather odd 'fishhook' shape" in studies of bifurcations in the two-dimensional parameter space of the Rössler equation [8] and the two-dimensional Hénon map [13]. However, systematic analyses of two-dimensional parameter spaces were not carried out until a series of influential papers by Gallas and colleagues described "complicated pleopodic shrimplike structures" in the two-dimensional parameter space of the Hénon map [14,15].

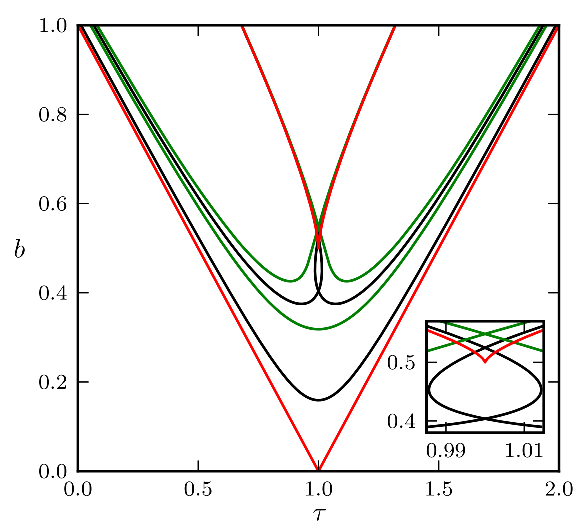

Fig. 1. Period-1 resonance zone for the sine circle map $x_{t+1} = x_t + \tau + b \sin 2\pi x_t$ showing the first period-doubling bifurcation and the period-1 and 2 superstable cycles. This geometry including all the adjacent 2×2^k, $k = 1, 2, \ldots,$ regions and associated chaotic regions is called a shrimp. In this and subsequent figures, the red curves represent lines of saddle-node bifurcations, the green curves represent period-doubling bifurcations, and the black curves represent the skeleton (where the trace of the associated Jacobian matrix is 0). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

A recent paper [16], which should be consulted for extensive references to papers by Gallas and others, proposes the following

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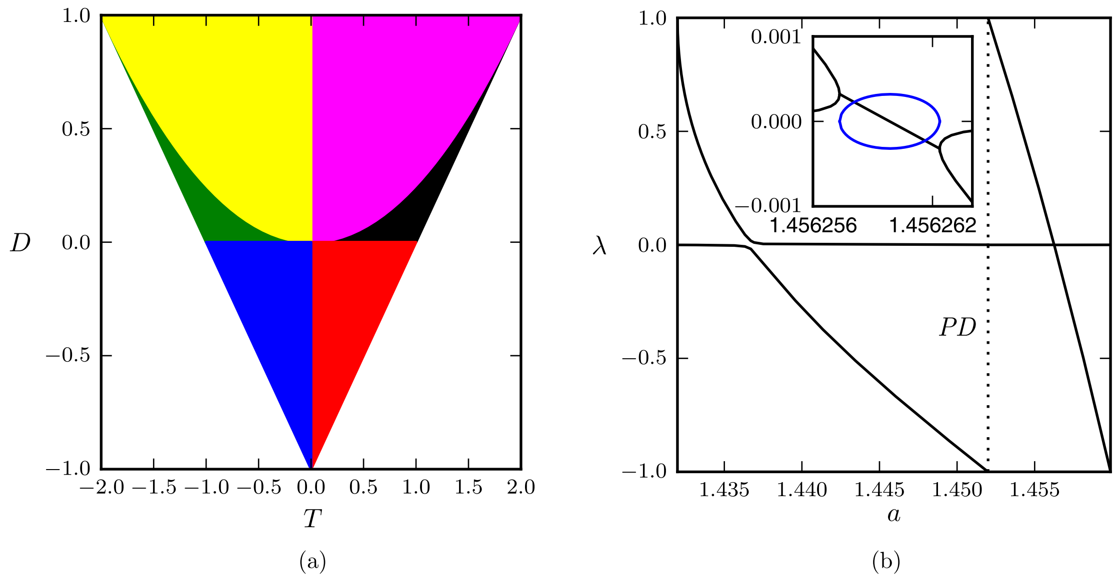

Fig. 2. (a) The stability triangle [23]. For values of the determinant D and trace T falling in the colored regions, the eigenvalues lie in the unit circle leading to local stability. Blue, red, green and black regions represent D and T obtained by real eigenvalues, while yellow and magenta arise from complex eigenvalues. (b) Typical trajectories of eigenvalues for two-dimensional maps as a parameter is varied from a boundary that gives a saddle-node bifurcation to a period-doubling bifurcation. This case is for the Hénon map with $b = 0.2$, showing the eigenvalues of period-5 and period-10 cycles. To the left of the period-doubling PD, $D < 0$, the eigenvalues are always real and the trajectories as a increases do not cross. To the right of PD, $D > 0$ and there exists a small region where the eigenvalues are complex (the blue curve in the inset gives the imaginary part of the eigenvalues). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

definition: "Shrimps are formed by a regular set of adjacent windows centered around the main pair of intersecting superstable parabolic arcs. A shrimp is a doubly infinite mosaic of stability domains composed by an innermost main domain plus all the adjacent stability domains arising from two period-doubling cascades together with their corresponding domains of chaos". Shrimps, as these geometric structures are now known, have been widely observed in electrical circuits, delay differential equations, and chemical oscillators [17,16,18].

Although cycles containing an extremum are guaranteed to be stable in one-dimensional maps, and thus are suitable for defining the skeleton of bifurcation structures, for two-dimensional maps, we need a new method to determine the skeleton. In the following, we combine continuation techniques [19,20] with earlier results on the stability of cycles in higher dimensional maps [21] to show how to determine the skeleton of two-dimensional maps. We illustrate the method on the Hénon map [22,13,14].

2. Theory

Consider the n-dimensional map, where $n = 1, 2$

$$\mathbf{x}' = f(\mathbf{x}). \tag{1}$$

The Jacobian, $\mathbf{J}$, is an $n \times n$ matrix composed of the elements $J_{i,j} = \partial f_i / \partial x_j$. To determine the stability of a cycle of period p, we compute the product

$$\mathbf{M} = \mathbf{J}(\mathbf{x}^p) \cdot \ldots \cdot \mathbf{J}(\mathbf{x}^1), \tag{2}$$

where $\mathbf{x}^k$, $k = 1, \ldots, p$, are the points $\mathbf{x}, \ldots, f^{(p-1)}(\mathbf{x})$ on the cycle. The eigenvalues λ_k, $k = 1, \ldots, n$, determine the stability of the cycle. If all eigenvalues lie in the unit circle, then the cycle is stable. The largest Lyapunov exponent is $(1/p)\ln|\lambda_1|$, where $|\lambda_1|$ is the absolute value (or modulus) of the leading eigenvalue. Thus, in one-dimensional maps for a cycle containing an extremal point, we find that $M = 0$, the Lyapunov exponent is $-\infty$, and it is straightforward to determine the locus of period-p superstable cycles in a two-dimensional parameter space, by taking the extremal point as the initial condition and looking for fixed points on the period-p map [4]. A saddle-node bifurcation occurs when the leading eigenvalue crosses 1, and a period-doubling bifurcation occurs when the leading eigenvalue crosses -1. For two-dimensional parameter spaces the boundaries of the saddle-node and period-doubling bifurcations are typically lines in parameter space that bound a set of parameter values in which a stable periodic orbit exists for some set of initial conditions. In one-dimensional maps, the locus of the superstable cycle of the same period is a line in between both boundaries – called a "skeleton" [4] or a "spine" [23].

For two-dimensional maps the trace $T = M_{1,1} + M_{2,2} = \lambda_1 + \lambda_2$, and the determinant $D = M_{1,1}M_{2,2} - M_{1,2}M_{2,1} = \lambda_1\lambda_2$, where also $\lambda_1, \lambda_2 = T/2 \pm \sqrt{T^2 - 4D}/2$, help us understand the structure of the bifurcations in parameter space. For a given cycle, the stability triangle [23,21] in Fig. 2(a) determined for $\mathbf{M}$ in Eq. (2) defines the values of D and T for which the periodic orbit is stable, and can be used to code the regions of parameter space. Provided $|D| < 1$, the locus of points in parameter space for which $T = 0$ defines a structure that plays the same role as the skeleton in one-dimensional maps since it lies between the boundaries at which there is a saddle-node bifurcation (right-hand boundary) and a period-doubling boundary (left-hand boundary). In this Letter we consider situations in which D is either always positive or always negative. Fig. 2(b) shows schematic trajectories of the eigenvalues as a parameter is varied from a boundary that gives a saddle-node bifurcation to a period-doubling bifurcation. There are two cases. If $-1 < D < 0$ there is necessarily a point where $T = 0$. For $1 > D > 0$ there will be a region in which the eigenvalues become complex. In this case, there are typically two values where the eigenvalues are real and equal.

Continuation methods can be used to trace the loci of points where $T = 0$ or where the eigenvalues are real and equal in parameter space for a given periodic orbit and its period-doubled descendants. To do this, after doing a one-parameter continuation for $f^{(p)}(\mathbf{x}) - \mathbf{x} = 0$ detecting zeros of $T = M_{1,1} + M_{2,2}$, we consider the following extended algebraic system:

$$f^{(p)}(\mathbf{x}) - \mathbf{x} = 0,$$
$$T = M_{1,1} + M_{2,2} = 0. \tag{3}$$

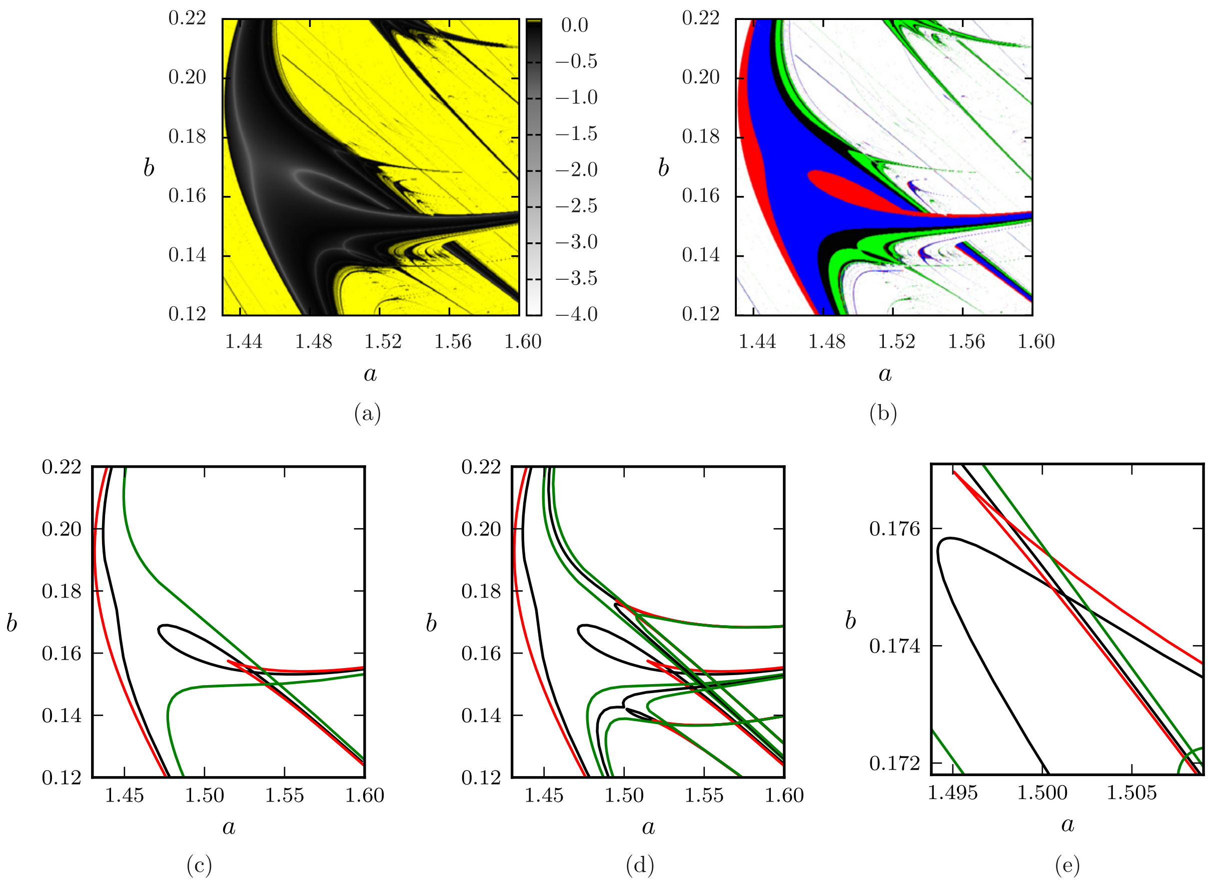

Fig. 3. Bifurcation structure of a shrimp in the Hénon map. (a) The largest Lyapunov exponent. Yellow represents chaotic regions, with a positive largest Lyapunov exponent. (b) Coloring based on the region of D and T in the stability triangle. (c) Bifurcation diagram for period-5 cycles in the Hénon map. (d) The same diagram as (c), but also including the same curves for period-10 cycles. (e) Magnification of (d) showing the Type B unfolding. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this Letter.)

A continuation package like AUTO [20,19] considers this a three-dimensional system with one parameter. Therefore, one of the two parameters needs to be the third state vector coordinate in the continuation system, and the other parameter can then be the continuation parameter.

Similarly the points where the eigenvalues are real and equal can be continued. To do this we simply replace the second condition in (3) as follows:

$$f^{(p)}(\mathbf{x}) - \mathbf{x} = 0,$$
$$T^2 - 4D = (M_{1,1} + M_{2,2})^2 - 4(M_{1,1}M_{2,2} - M_{1,2}M_{2,1}) = 0. \quad (4)$$

3. Results

We now apply these results to compute the skeleton of shrimps in the Hénon map [22,13,14,24] $x' = a - x^2 + by$, $y' = x$. For this map we investigate period-5 orbits that, using numerical continuation, can be traced back to an eigenvalue resonance on the Neimark–Sacker bifurcation at $b = -1$ for $a \approx 2.276$ where the eigenvalues are equal to $\exp(\pm(2/5) \times 2\pi i)$. The Hénon map has the special property that its determinant $D = -b^p$. Therefore, for period-5 $D = -b^5 < 0$ and for all doubled cycles $D > 0$. If $|b| < 1$, for periodic orbits of high period, D can be quite small.

We now investigate the structure of its period-5 and doubled period-10 cycles for $b \approx 0.18$ in a region already identified as having a shrimp [14,15]. We display three views of the same region of parameter space: the largest Lyapunov exponent; the color based on the values of D and T in the stability triangle; and the skeleton determined using the continuation techniques.

In Fig. 3(a) we depict the largest Lyapunov exponent and in Fig. 3(b) we plot the associated color from the color triangle. In Fig. 3(b) $-b^5 \approx -1.9 \times 10^{-4}$, and $b^{10} \approx 3.6 \times 10^{-8}$. This means that for the period-5 cycle we always have real eigenvalues, with blue and red regions separated by curves where $T = 0$. For the period-10 cycle the eigenvalues may be complex, but for such small values of the determinant the complex regions vanish numerically and thus a green–black border can be seen. For such small values of the determinant, the green–black and red–blue borders provide good approximations for the loci of points in two-dimensional maps that play the same role as the skeleton in one-dimensional maps [21]. Thus it is sufficient to compute only the $T = 0$ curves by continuation. The continuation analysis was supplemented by standard numerical continuation of the red saddle-node and green period-doubling curves. The results are shown in Fig. 3. For the period-5 orbit the stable region is delimited by a red saddle-node curve on the left, and a period-doubling bifurcation on the right. A second saddle-node bifurcation generates a cusp configuration, much like in the one-dimensional map in Fig. 1. Similar structures exist for the period-10 cycle.

Although the shrimp structures for the Hénon map and the sine circle map in Fig. 1 appear to be the same, there are differences revealed by the current analysis. Consider the schematic diagram in Fig. 4. The top diagram represents the "classic" structure that forms the building block of the self-similar bifurcation structures in bimodal maps [4–6,8–12]. The upper crossing of the superstable orbits represents a point in the diamond shaped region of parameter space bounded by the saddle-node bifurcation curve with the cusp and two period-doubling bifurcation curves, where

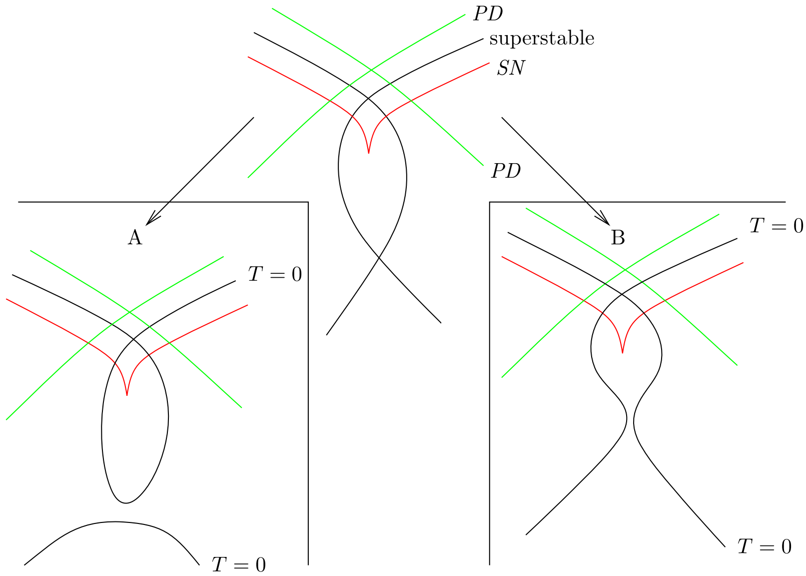

Fig. 4. Unfolding of the skeleton. The top panel represents the "classic" geometry in a bimodal one-dimensional map. In two-dimensional maps the skeleton unfolds in two different ways – Type A has a loop geometry and Type B a crossing without a loop.

there is bistability, whereas the lower crossing represents a point in parameter space where there is a periodic orbit for which the derivative of two of the iterates equals zero.

In two-dimensional maps, the lower crossing represents a non-generic situation, so generically it will not exist. The reason for this is that in the continuation system such a point represents a branch point, which is generically of codimension-three and can only be of codimension-two if certain symmetries are present [25]. In one-dimensional maps, the trace of the scalar matrix (that is, the value of the matrix itself) $\mathbf{M}$ in Eq. (2) is the product of the traces of the matrices $\mathbf{J}(\mathbf{x}^k)$, but for higher dimensions this symmetry is lost. Rather the skeletal structure unfolds in two different ways which we show as Type A and Type B. The period-5 structure in Fig. 3(c) reflects the Type A geometry, whereas the period-10 structure in Fig. 3(e) represents the Type B geometry.

4. Conclusions

Although the concept of resonance tongues is well established experimentally, the observation of the fine structures theoretically predicted from analysis remains an important challenge.

As parameters are changed in nonlinear systems, there are complex bifurcations with possibility of multistability. Although extremely delicate structures that are observed over minute regions of parameter space and/or over minute ranges of initial conditions may be difficult to observe experimentally, it is nevertheless of interest to have numerical and analytical methods that can be used to dissect bifurcation structures as parameters change. For one-dimensional bimodal maps, symbolic dynamics and kneading theory can be used to analyze the structure of the superstable cycles in parameter space [11,12]. However for nonlinear differential equations and for maps of dimension 2, analyses of bifurcation structures have largely relied on shooting methods in which the dynamics are determined for a specific initial condition over a range of parameter values. The current work enables direct computation of structures that are very difficult to identify and analyze using shooting methods. Although the term "shrimps" captures the delicate geometries observed in these numerical studies, the detailed topological structure of the shrimps for two-dimensional maps and other types of dynamical systems still needs to be analyzed. For example, for models of periodically stimulated nonlinear oscillators with finite relaxation time to the limit cycle, we anticipate that the shrimp structures will display complex evolution as the relaxation time to the limit cycle varies [26].

The current work shows that continuation methods can be used to analyze delicate bifurcation structures in the two-dimensional Hénon map, that are difficult to analyze using shooting methods. Since the loci at which $T = 0$ play a similar role to the superstable cycle in two-dimensional maps continuation methods that identify where $T = 0$ should help elucidate the fine details of bifurcations in a wide range of different examples. For higher n-dimensional maps there are additional quantities that matter, namely the coefficients c_i of the characteristic polynomial, that comprise all possible distinct product combinations of eigenvalues taken i at a time for $i = 1, 2, \ldots, n$ [23]. Those quantities can all be restricted in a similar fashion as T and D using continuation methods.

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