Existence results of positive solutions for Kirchhoff type equations via bifurcation methods

Willian Cintra[1] · João R. Santos Júnior[1] · Gaetano Siciliano[2] · Antonio Suárez[3]

Abstract

In this paper we address the following Kirchhoff type problem

$$\begin{cases} -\Delta(g(|\nabla u|_2^2)u + u^r) = au + bu^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in a bounded and smooth domain Ω in $\mathbb{R}^N$. By using change of variables and bifurcation methods, we show, under suitable conditions on the parameters a, b, p, r and the non-linearity g, the existence of positive solutions.

Keywords Kirchhoff type equation · Logistic non-linearity problem · Bifurcation method

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✉ Willian Cintra
willian_matematica@hotmail.com

João R. Santos Júnior
joaojunior@ufpa.br

Gaetano Siciliano
sicilian@ime.usp.br

Antonio Suárez
suarez@us.es

[1] Faculdade de Matemática, Instituto de Ciências Exatas e Naturais, Universidade Federal do Pará, Avenida Augusto corrêa 01, Belém, PA 66075-110, Brazil

[2] Departamento de Matemática, Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão 1010, São Paulo, SP 05508-090, Brazil

[3] Dpto. de Ecuaciones Diferenciales y Análisis Numérico, Fac. de Matemáticas, Univ. de Sevilla, Calle Tarfia s/n, Sevilla, Spain

1 Introduction

In the recent paper [12] two of the authors studied the following Kirchhoff type problem

$$\begin{cases} -\text{div}(m(u, |\nabla u|_2^2)\nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (1)$$

in the smooth and bounded domain $\Omega \subset \mathbb{R}^N$, $N \geq 1$, where f is a sublinear non-linearity and $m : \mathbb{R} \times [0, +\infty) \to \mathbb{R}$ is a function such that, setting

$$m_t : s \in \mathbb{R} \mapsto m(s, t) \in \mathbb{R},$$

the following hold:

(m0) $m : \mathbb{R} \times [0, +\infty) \to (0, +\infty)$ is continuous;
(m1) there is $\mathfrak{m} > 0$ such that $m(s, t) \geq \mathfrak{m}$ for all $s \in \mathbb{R}$ and $t \in [0, \infty)$;
(m2) for each $t \in [0, +\infty)$ the map $m_t : \mathbb{R} \to (0, +\infty)$ is strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, +\infty)$.

In particular the class of such admissible m is very huge. In [12] the problem was addressed by finding the fixed point of the map

$$S : t \in [0, +\infty) \mapsto \int_\Omega |\nabla u_t|^2 dx \in (0, +\infty)$$

where u_t is the unique solution of the auxiliary problem

$$\begin{cases} -\text{div}(m_t(u)\nabla u) = f(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (2)$$

Indeed, the main difficulty related to the method used was to guarantee existence, uniqueness and a priori bound with respect to t for the solutions of (2). In particular, the uniqueness was obtained in virtue of the sublinearity condition of the non-linearity f. However, as remarked in [12, Remark 5], the uniqueness of the solution to (2) was not necessary in order to employ the method: since the solution of (1) is obtained as a fixed point of S, all that is needed is the existence of a continuous branch of solutions $t \mapsto u_t$ to (2), rather than the uniqueness of the solution u_t.

We point out here that similar methods to that in [12] has been used recently in [13] to deal with a biharmonic Kirchhoff operator.

Motivated by the above remark, in this paper we study a Kirchhoff type problem where the uniqueness of the solution to the related auxiliary problem is not expected, due to the fact that the non-linearity is not sublinear. Indeed, in this paper bifurcation methods are used in order to circumvent the lack of uniqueness and obtain a continuum of positive solutions. We point out that bifurcation theory has been used by other authors to study Kirchhoff problems (see, for instance, [1,5,9] and references therein) and arguments combining bifurcation theory with Bolzano Theorem were used to analyze non-local elliptic problems in [2].

More specifically, we study here the existence of classical positive solutions for the following problem with a logistic type non-linearity:

$$\begin{cases} -\Delta(g(|\nabla u|_2^2)u + u^r) = au + bu^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous function satisfying minimal assumptions and a, b, r, p are constants such that $a > 0$, $p, r > 1$. Observe that problem (3) is of Kirchhoff type since the equation can be written as

$$-\text{div}\left(m(u, |\nabla u|_2^2)\nabla u\right) = au + bu^p$$

just defining $m(u, |\nabla u|_2^2) := g(|\nabla u|_2^2) + ru^{r-1}$. We point out here that, if we fix the value $|\nabla u|_2^2 = t$ and define the map for every $t \geq 0$

$$m_t : s \in \mathbb{R} \mapsto m(s, t) \in \mathbb{R}$$

the map m satisfies the above conditions (m0) and (m1); moreover for suitable values of r also (m2) holds, falling down into the class of m permitted in [12].

In this paper we give sufficient conditions, depending on the parameters a, b, r, p and on the function g in such a way that problem (3) admits at least a positive solution. We have to say, however, that many other cases remain open.

Here are our main results.

Theorem A *Assume that $g : [0, \infty) \to [0, \infty)$ is a continuous function with $\inf_{s>0} g(s) > 0$. If one of the following conditions is satisfied:*

(i) $b \leq 0$,
(ii) $b > 0$, $r = p$ and $b < \lambda_1$,
(iii) $b > 0$ and $r > p$,

then problem (3) admits at least one positive solution for each $a > g(0)\lambda_1$.

Hereafter λ_1 denotes the first eigenvalue of the Laplacian in Ω under homogeneous Dirichlet boundary conditions.

Theorem B *Assume that $g : [0, \infty) \to [0, \infty)$ is a bounded continuous function such that $g(0) > 0$.*

(i) *If $r = p$ and $b > \lambda_1$ then problem (3) admits at least one positive solution for each $a < g(0)\lambda_1$.*
(ii) *If $b > 0$, $1 < p/r < (N + 2)/(N - 2)$ and $g(0)\lambda_1 > \phi(s_0)$, then problem (3) admits at least one positive solution for each $a \in (\phi(s_0), g(0)\lambda_1)$.*
Here

$$s_0 := \left(\frac{b(p - 1)}{\lambda_1(r - 1)}\right)^{1/(r-p)}$$

is the maximum point of the function $\phi(s) := \lambda_1 s^{r-1} - bs^{p-1}$, $s \geq 0$.

Theorem C *Assume that $g : [0, \infty) \to [0, \infty)$ is a continuous and positive function. If $b = \lambda_1$ and $r = p < 2$, then problem (3) admits a positive solution if, and only if, $a \in \lambda_1 R[g] := \{\lambda_1 g(s); s \geq 0\} \subset (0, \infty)$.*

We emphasize that our approach consists in applying appropriate changes of variables, obtaining an associated semilinear problem. We carry out a detailed study of the solutions of this auxiliary problem via bifurcation results. See Proposition 2 and Corollary 1.

The paper is organized as follows. In Sect. 2 some preliminary results are given and a suitable parameter-dependent problem (see (P_λ)) is introduced in order to study the original problem (3).

Section 3 is devoted to study problem (P_λ). However to do that, by means of a change of variables, the problem is transformed into an equivalent one which is studied with bifurcation theory.

In Sect. 4 the proofs of the main results are given.

As a matter of notations, we set for $k \in \mathbb{N}$, $C_0^k(\overline{\Omega}) := C^k(\Omega) \cap C_0(\overline{\Omega})$ the set of functions defined on $\overline{\Omega}$ which are of class C^k in Ω and continuous up to the boundary satisfying the homogeneous Dirichlet boundary condition. Hereafter $|\cdot|_p$ stands for the $L^p(\Omega)-$norm and the uniform norm will be denoted by $\| \cdot \|$.

2 Preliminaries

Our approach to deal with problem (3) is to consider the following auxiliary problem depending on the positive parameter λ:

$$\begin{cases} -\Delta(u/\lambda + u^r) = au + bu^p & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (P_\lambda)$$

By a solution we mean a pair $(\lambda, u_\lambda) \in (0, +\infty) \times C_0^2(\overline{\Omega})$ satisfying (P_λ) in the classical sense.

The idea is to find first an unbounded continuum, say Σ_0, of positive solution of (P_λ) via bifurcation theory. Then, using the classical Bolzano Theorem, we will search for zeros of the map,

$$h : (\lambda, u) \in \Sigma_0 \longmapsto \frac{1}{\lambda} - g(|\nabla u|_2^2) \in \mathbb{R}$$

which of course provides us a solution of (3).

In the following, given functions $A \in C^1(\overline{\Omega})$ and $B \in C(\overline{\Omega})$, satisfying $A(x) \geq A_0 > 0$ for $x \in \Omega$ and a suitable constant A_0, we will denote by

$$\sigma_1[-\text{div}(A(x)\nabla) + B(x)]$$

the principal eigenvalue of the problem

$$\begin{cases} -\text{div}(A(x)\nabla u) + B(x)u = \lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

It is well-known (see for instance [6,11]) that this eigenvalue is increasing with respect to A and B. When $A \equiv A_0$ is constant, we have $-\text{div}(A_0\nabla) = -A_0\Delta$, thus in this case we will write

$$\sigma_1[-A_0\Delta + B(x)].$$

By simplicity, we set

$$\lambda_1 := \sigma_1[-\Delta].$$

Moreover, we will denote by φ_1 the positive eigenfunction associated to λ_1 with $\|\varphi_1\| = 1$.

The first result gives necessary conditions on the parameters a, b, r, p in order to get solutions to (P_λ). Here the following function

$$\phi(s) := \lambda_1 s^{r-1} - bs^{p-1}, \quad s \geq 0, \tag{4}$$

plays an important role. It easy to see that if $r > p$ (resp. $r < p$) ϕ is bounded below (resp. above) and the minimum (resp. maximum) is attained at

$$s_0 = \left(\frac{b(p-1)}{\lambda_1(r-1)} \right)^{\frac{1}{r-p}}, \tag{5}$$

and

$$\phi(s_0) = \left(\frac{p-1}{\lambda_1} \right)^{\frac{p-1}{r-p}} \left(\frac{b}{r-1} \right)^{\frac{r-1}{r-p}} (p-r).$$

With these considerations, we can show the next result.

Lemma 1 *We have the following.*

(a) *Suppose $b \leq 0$. Then (P_λ) does not possess positive solution for $\lambda \leq \lambda_1/a$.*

(b) *Suppose $b > 0$ and $r > p$. Then (P_λ) does not possess positive solution for $\lambda < \lambda_1/(a - \phi(s_0))$. Moreover, if $\lambda > \lambda_1/a$, then (P_λ) does not possess positive solution (λ, u_λ) with $\|u_\lambda\| < (b/\lambda_1)^{1/(r-p)}$.*

(c) *Suppose $b > 0$ and $r = p$. If $b < \lambda_1$ (resp. $b > \lambda_1$), then (P_λ) does not posses positive solution for $\lambda < \lambda_1/a$ (resp. $\lambda > \lambda_1/a$). Moreover, if $b = \lambda_1$ then (P_λ) does not possess positive solution for $\lambda \neq \lambda_1/a$.*

(d) *Suppose $b > 0$, $r < p$ and $a > \phi(s_0)$. Then (P_λ) does not possess positive solution for $\lambda > \lambda_1/(a - \phi(s_0))$. Moreover, if $\lambda < \lambda_1/a$, then (P_λ) does not possess positive solution (λ, u_λ) with $\|u_\lambda\| < (b/\lambda_1)^{1/(r-p)}$.*

Proof To prove (a) and (b) we argue as follows. Multiplying (P_λ) by φ_1, integrating in Ω and applying the formula of integration by parts we get

$$\int_\Omega \nabla \left(\frac{u_\lambda}{\lambda} + u_\lambda^r \right) \nabla \varphi_1 = \int_\Omega (au_\lambda + bu_\lambda^p)\varphi_1$$

$$\int_\Omega \lambda_1 \left(\frac{u_\lambda}{\lambda} + u_\lambda^r \right) \varphi_1 = \int_\Omega (au_\lambda + bu_\lambda^p)\varphi_1.$$

Hence, (λ, u_λ) satisfies

$$\int_\Omega u_\lambda \varphi_1 \left[\lambda_1 u_\lambda^{r-1} - bu_\lambda^{p-1} - \left(a - \frac{\lambda_1}{\lambda} \right) \right] = 0. \tag{6}$$

If $b \leq 0$ the above equality implies that $a > \lambda_1/\lambda$. This implies (a).

On the other hand, assuming $b > 0$ since $r > p$, the function

$$\phi(s) = \lambda_1 s^{r-1} - bs^{p-1}, \quad s \geq 0.$$

is bounded below and by a direct calculation, $\min_{0 \leq s < \infty} \phi(s)$ is negative and it is attained at s_0. Therefore, for all $\lambda > 0$ such that

$$\lambda < \frac{\lambda_1}{a - \phi(s_0)},$$

we have

$$\phi(s_0) > a - \frac{\lambda_1}{\lambda} \quad \forall \lambda \in (0, \lambda_1/(a - \phi(s_0))].$$

Consequently,

$$\lambda_1 s^{r-1} - b s^{p-1} - \left(a - \frac{\lambda_1}{\lambda}\right) > 0 \quad \forall s \geq 0, \ \lambda \in (0, \lambda_1/(a - \phi(s_0))]$$

and (6) cannot be satisfied for any positive function u_λ, showing that $\lambda \geq \lambda_1/(a - \phi(s_0))$ is a necessary condition for the existence of positive solution of (P_λ) with $b > 0$ and $r > p$. Now, suppose that (λ, u_λ) is a positive solution of (P_λ) with $\lambda > \lambda_1/a$. Then, it verifies (6) and, hence,

$$\int_\Omega u_\lambda \varphi_1 (\lambda_1 u_\lambda^{r-1} - b u_\lambda^{p-1}) = \left(a - \frac{\lambda_1}{\lambda}\right) \int_\Omega u_\lambda \varphi_1 > 0. \tag{7}$$

If $\|u_\lambda\| < (b/\lambda_1)^{1/(r-p)}$, then

$$\phi(u_\lambda(x)) = \lambda_1 u_\lambda(x)^{r-1} - b u_\lambda(x)^{p-1} < 0 \quad \forall x \in \Omega.$$

and, therefore,

$$\int_\Omega u_\lambda \varphi_1 (\lambda_1 u_\lambda^{r-1} - b u_\lambda^{p-1}) < 0,$$

which is a contradiction with (7).

In the case (c), since $r = p$, (7) is equivalent to

$$(\lambda_1 - b) \int_\Omega u_\lambda^r \varphi_1 = \left(a - \frac{\lambda_1}{\lambda}\right) \int_\Omega u_\lambda \varphi_1$$

whence we easily infer the result.

The proof of paragraph (d) is similar to that of (b). □

In the next Section we will analyze the auxiliary problem (P_λ).

3 Study of the auxiliary problem (P_λ)

To study (P_λ) we introduce the following change of variables depending on the parameter λ:

$$w := \frac{u}{\lambda} + u^r. \tag{8}$$

Then, if we define the function $I_\lambda : s \in [0, +\infty) \mapsto \frac{s}{\lambda} + s^r \in [0, +\infty)$, which is a smooth diffeomorphism, and denote its inverse with q_λ, we have

$$w = I_\lambda(u) = \frac{u}{\lambda} + u^r \iff q_\lambda(w) = u.$$

The first result of this section collects some important properties related to the map q_λ.

Lemma 2 *The map $\lambda \in (0, \infty) \mapsto q_\lambda(s)$ is continuous and increasing, for all $s > 0$.*

Now let $\lambda > 0$ be fixed. Then the map $s \in (0, +\infty) \mapsto \dfrac{q_\lambda(s)}{s} \in (0, +\infty)$ has the following properties:

(q1) it is decreasing and $q_\lambda(s)/s \leq \lambda$,

(q2) $\lim\limits_{s \to 0^+} \dfrac{q_\lambda(s)}{s} = \lambda,$

(q3) $\lim\limits_{s \to +\infty} \dfrac{q_\lambda(s)}{s} = 0.$

On the other hand, the map $s \in (0, +\infty) \mapsto \dfrac{q_\lambda(s)^p}{s} \in (0, +\infty)$ *satisfies:*

(q4) $\lim\limits_{s \to 0^+} \dfrac{q_\lambda(s)^p}{s} = 0,$

(q5) it is increasing if $p \geq r$,

(q6) $\lim\limits_{s \to +\infty} \dfrac{q_\lambda(s)^p}{s} = \begin{cases} +\infty & \text{if } r < p, \\ 0 & \text{if } r > p, \\ 1 & \text{if } r = p. \end{cases}$

Proof The continuity of the map $\lambda \in (0, \infty) \mapsto q_\lambda(s)$, $s > 0$ follows by the definition of q_λ. To prove that it is increasing, that is,

$$\lambda' < \lambda'' \implies q_{\lambda'}(s) < q_{\lambda''}(s), \tag{9}$$

we proceed as follows. For every $s > 0$, we have:

$$\begin{aligned} I_{\lambda''}(q_{\lambda'}(s)) &= \frac{q_{\lambda'}(s)}{\lambda''} + q_{\lambda'}(s)^r \\ &< \frac{q_{\lambda'}(s)}{\lambda'} + q_{\lambda'}(s)^r \\ &= I_{\lambda'}(q_{\lambda'}(s)) \\ &= s \end{aligned}$$

from which (9) easily follows.

Now, let $\lambda > 0$ be fixed. Since q_λ is the inverse function of I_λ, it is increasing and verifies

$$s = I_\lambda(q_\lambda(s)) = \frac{q_\lambda(s)}{\lambda} + q_\lambda(s)^r$$

so that

$$\frac{q_\lambda(s)}{s} = \frac{1}{1/\lambda + q_\lambda(s)^{r-1}} \leq \lambda,$$

which proves (q1).

Furthermore, since $q_\lambda(0) = 0$ and $r > 1$ it follows that

$$\lim\limits_{s \to 0^+} \frac{q_\lambda(s)}{s} = \lambda \quad \text{and} \quad \lim\limits_{s \to +\infty} \frac{q_\lambda(s)}{s} = 0$$

which gives (q2) and (q3).

Moreover, being $p > 1$,

$$\lim\limits_{s \to 0^+} \frac{q_\lambda(s)^p}{s} = \lim\limits_{s \to 0^+} \frac{q_\lambda(s)}{s} q_\lambda(s)^{p-1} = 0,$$

proving (q4).

Finally, the identity

$$\frac{q_\lambda(s)^p}{s} = \frac{1}{q_\lambda(s)^{1-p}/\lambda + q_\lambda(s)^{r-p}}$$

gives (q5) and (q6). $\qquad\square$

Remark 1 We point out that, in the case $r > p > 1$, for each $\bar{\lambda} > 0$, the limit $\lim_{s \to \infty} q_\lambda(s)^p/s = 0$, is uniform in $\lambda \in [\bar{\lambda}, \infty)$. Indeed, for each $\delta > 0$, to obtain

$$\frac{q_\lambda(s)^p}{s} \leq \delta \left(\Longleftrightarrow s \leq I_\lambda((\delta s)^{1/p}) = \frac{(\delta s)^{1/p}}{\lambda} + (\delta s)^{r/p} \right)$$

it is sufficient to choose $s > 0$ (independent on λ) such that

$$1 \leq \delta^r s^{(r/p)-1} \left(\leq \frac{\delta^{1/p} s^{(1/p)-1}}{\lambda} + \delta^r s^{(r/p)-1} \right).$$

It should be noted that, by a similar argument, the limit $\lim_{s \to \infty} q_\lambda(s)/s = 0$ is also uniform in $\lambda \in [\bar{\lambda}, \infty)$.

We will use these properties later to get a priori bounds of the positive solutions of (10), uniform with respect to $\lambda \in [\bar{\lambda}, \infty)$.

Thus, under the above change of variable (8), problem (P_λ) is equivalent to

$$\begin{cases} -\Delta w = aq_\lambda(w) + bq_\lambda(w)^p & \text{in } \Omega, \\ w > 0 & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases} \tag{10}$$

in the sense that (λ, u_λ) is a solution of (P_λ) if, and only if, $(\lambda, w_\lambda) := (\lambda, u_\lambda/\lambda + u_\lambda^r)$ is a solution of (10).

We observe explicitly that the map

$$f : (\lambda, s) \in (0, +\infty) \times [0, +\infty) \longmapsto aq_\lambda(s) + bq_\lambda(s)^p \in \mathbb{R}$$

is locally Lipschitz. Note that for every $\lambda > 0$ there is always the trivial solution $w \equiv 0$ to (10).

Remark 2 In virtue of Lemma 1 we have the following necessary conditions for the existence of solutions to (10).

(a) Suppose $b \leq 0$. Then (10) does not possess positive solution for $\lambda \leq \lambda_1/a$.
(b) Suppose $b > 0$ and $r > p$. Then (10) does not possess positive solution for $\lambda < \lambda_1/(a - \phi(s_0))$. Moreover, If $\lambda > \lambda_1/a$, then (10) does not possess positive solution (λ, w_λ) with

$$\|w_\lambda\| < \frac{a}{\lambda_1} \left(\frac{b}{\lambda_1} \right)^{1/(r-p)} + \left(\frac{b}{\lambda_1} \right)^{r/(r-p)}.$$

(c) Suppose $b > 0$ and $r = p$. If $b < \lambda_1$ (resp. $b > \lambda_1$), then (10) does not posses positive solution for $\lambda < \lambda_1/a$ (resp. $\lambda > \lambda_1/a$). Moreover, if $b = \lambda_1$ then (10) does not possess positive solution for $\lambda \neq \lambda_1/a$.
(d) Suppose $b > 0$, $r < p$ and $a > \phi(s_0)$, then (10) does not possess positive solution for $\lambda > \lambda_1/(a - \phi(s_0))$. Moreover, if $\lambda < \lambda_1/a$, then (10) does not possess positive solution (λ, w_λ) with

$$\|w_\lambda\| < \frac{a}{\lambda_1} \left(\frac{b}{\lambda_1} \right)^{1/(r-p)} + \left(\frac{b}{\lambda_1} \right)^{r/(r-p)}.$$

Problem (10) will be studied with the help of bifurcation theory.

Definition 1 We say that $(\lambda_0, 0)$ (or simply λ_0) is a bifurcation point from the trivial solution of the equation in (10) if there exists a sequence (λ_n, w_n) of non-trivial solutions of (10) such that

$$(\lambda_n, w_n) \to (\lambda_0, 0) \quad \text{as } n \to \infty.$$

Now, we will obtain an unbounded continuum in $(0, \infty) \times C_0^1(\overline{\Omega})$ of positive solutions of (10) emanating from the trivial solution at $\lambda = \lambda_1/a$ and, hence, we prove a result of existence of positive solution of (10) and consequently of (P_λ).

To this end, consider the map $\mathfrak{F} : (0, \infty) \times C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined by

$$\mathfrak{F}(\lambda, w) = w - (-\Delta)^{-1}[aq_\lambda(w) + bq_\lambda(w)^p],$$

where $(-\Delta)^{-1}$ is the inverse of the Laplacian operator under homogeneous Dirichlet boundary conditions. The operator $\mathfrak{F}$ is of class $\mathcal{C}^1$ and equation in (10) (including the boundary condition) can be written in the form

$$\mathfrak{F}(\lambda, w) = 0.$$

Moreover, since we are interested only in positive solutions with $\lambda > 0$, we can consider any $\mathcal{C}^1$-extension of $\mathfrak{F}$ on $\mathbb{R} \times C_0^1(\overline{\Omega}) \times C_0^1(\overline{\Omega})$. Thus, still denoting it by $\mathfrak{F}$, we have:

Proposition 1 *The value $\lambda = \lambda_1/a$ is the unique bifurcation point from the trivial solution to* (10) *with $\lambda > 0$. Moreover, from λ_1/a emanates an unbounded continuum in $(0, \infty) \times C_0^1(\overline{\Omega})$ of positive solutions $\widehat{\Sigma}_0$ of (10).*

Proof Thanks (q2) and (q4) of Lemma 2, we have

$$\lim_{s \to 0} \frac{aq_\lambda(s) + bq_\lambda(s)^p}{s} = a\lambda.$$

Thus, the linearization of $\mathfrak{F}$ at $(\lambda, 0)$, $\lambda > 0$, is given by

$$\partial_w \mathfrak{F}(\lambda, 0) = I_{\mathcal{C}_0^1(\overline{\Omega})} - a\lambda(-\Delta)^{-1}.$$

Consequently, $\partial_w \mathfrak{F}(\lambda, 0)$ is a Fredholm operator of index zero, analytic in λ. Moreover, its kernel verifies

$$N[\partial_w \mathfrak{F}(\lambda_1/a, 0)] = \text{span}[\varphi_1],$$

where $\varphi_1 > 0$ stands, as usual, for the principal eigenfunction associated to λ_1 and satisfying $\|\varphi_1\| = 1$. Furthermore, by a standard argument,

$$\partial_\lambda \partial_w \mathfrak{F}(\lambda_1/a, 0)\varphi_1 \notin R[\partial_w \mathfrak{F}(\lambda_1/a, 0)], \tag{11}$$

that is, $\lambda = \lambda_1/a$ is a 1-transversal eigenvalue of the family $\partial_w \mathfrak{F}(\lambda, 0)$, which is the transversality condition stated in [4]. Therefore, we can apply the unilateral bifurcation theorem (see [10, Theorem 6.4.3]) to conclude the existence of a continuum $\widehat{\Sigma}_0$ of positive solution of (10) satisfying one of the following non-excluding options: either

1. $\widehat{\Sigma}_0$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega})$.
2. There exists $\lambda_* \in \mathbb{R}$ such that $\lambda_* \neq \lambda_1/a$ and $(\lambda_*, 0) \in \Sigma_0$.
3. $\widehat{\Sigma}_0$ contains a point $(\lambda, w) \in \mathbb{R} \times (Y \backslash \{0\})$, where Y is the complement of $N[\partial_w \mathfrak{F}(\lambda_1/a, 0)]$ in $C_0^1(\overline{\Omega})$.

Let us prove that 2. and 3. cannot be satisfied.

If 2. occurs, then λ_* is a bifurcation point of (10) from the trivial solution. Since the bifurcation points of the trivial solution of (10) are, necessarily, simple eigenvalues of $-\Delta$ in Ω under homogeneous Dirichlet boundary conditions divided by a and the only simple eigenvalue is λ_1/a, we must have $\lambda_* = \lambda_1/a$, which is a contradiction.

Suppose now that 3. occurs. Note that we can take $Y = R[\partial_w \mathfrak{F}(\lambda_1/a, 0)]$. Indeed, since $\partial_w \mathfrak{F}(\lambda_1/a, 0)$ is a Fredholm operator of index zero, we have

$$\operatorname{codim} R[\partial_w \mathfrak{F}(\lambda_1/a, 0)] = \dim N[\partial_w \mathfrak{F}(\lambda_1/a, 0)] = 1$$

and, hence, the complement of $R[\partial_w \mathfrak{F}(\lambda_1/a, 0)]$ on $C_0^1(\overline{\Omega})$ is one dimensional. From the transversality condition (11), we obtain

$$R[\partial_w \mathfrak{F}(\lambda_1/a, 0)] \oplus N[\partial_w \mathfrak{F}(\lambda_1/a, 0)] = C_0^1(\overline{\Omega}).$$

Now, observe that the function w given in paragraph 3. is positive and

$$w \in Y = R[\partial_w \mathfrak{F}(\lambda_1/a, 0)] = R[I - \lambda_1(-\Delta)^{-1}]$$

which is impossible. Then $\widehat{\Sigma}_0$ is unbounded in $\mathbb{R} \times C_0^1(\overline{\Omega})$. □

With the aim of obtaining a priori estimates for the positive solutions of (10), we first recall the following result, which applies to the following general problem

$$\begin{cases} -\Delta u = f(\lambda, x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (12)$$

A solution for this problem is a pair (λ, u). We denote with $(12)_\lambda$ the above problem with λ fixed.

Theorem 1 (See [7, Theorem 2.2.]) *Assume that f is locally Lipschitz. Suppose that $I \subset \mathbb{R}$, is an interval and let $\Sigma \subset I \times C_0^2(\overline{\Omega})$ be a connected set of solutions of (12). Consider a continuous map $\overline{U} : I \to C_0^2(\overline{\Omega})$ such that $\overline{U}(\lambda)$ is a super-solution of $(12)_\lambda$ for every $\lambda \in I$, but not a solution. If $u_0 \leq (\not\equiv)\overline{U}(\lambda_0)$ for some $(\lambda_0, u_0) \in \Sigma$, then $u < \overline{U}(\lambda)$ in Ω, for all $(\lambda, u) \in \Sigma$.*

Then, coming back to our problem we have the following.

Lemma 3 *Let $(\lambda, w_\lambda) \in \widehat{\Sigma}_0$, where $\widehat{\Sigma}_0$ is given in Proposition 1 (and hence w_λ a positive solution of (10)).*

(a) If $b < 0$, then

$$\|w_\lambda\| \leq c/\lambda + c^r \quad \forall \lambda > 0,$$

where $c := (-a/b)^{1/(p-1)}$.

(b) If $b = 0$, then there exists $c_0 > 0$ such that

$$\|w_\lambda\| \leq c_0 \quad \forall \lambda > \lambda_1/a.$$

(c) If $b > 0$ and $p/r < 1$ then there exists $c_0 > 0$ such that

$$\|w_\lambda\| \leq c_0 \quad \forall \lambda \in Proj_{\mathbb{R}} \widehat{\Sigma}_0.$$

(d) If $b > 0$, $b \neq \lambda_1$ and $p/r = 1$ then for each compact subset $\Lambda \subset (0, \infty)$ there exists $c_0 > 0$ such that

$$\|w_\lambda\| \leq c_1 \quad \forall \lambda \in \Lambda.$$

(e) If $b > 0$ and $1 < p/r < (N + 2)/(N - 2)$, then for each compact subset $\Lambda \subset (0, \infty)$, there exists $c_2 > 0$ such that

$$\|w_\lambda\| \leq c_2 \quad \forall \lambda \in \Lambda.$$

Proof To prove (a), let $x_M \in \Omega$ be the point where w_λ attained its maximum on Ω. Then,

$$0 \leq -\Delta w_\lambda(x_M) = aq_\lambda(w_\lambda(x_M)) + bq_\lambda(w_\lambda(x_M))^p$$
$$0 \leq a + bq_\lambda(w_\lambda(x_M))^{p-1}.$$

Since $b < 0$, this inequality is equivalent to

$$q_\lambda(w_\lambda(x_M)) \leq (-a/b)^{1/(p-1)} =: c = q_\lambda(I_\lambda(c)),$$

and, hence,

$$w_\lambda(x_M) \leq c/\lambda + c^r = I_\lambda(c),$$

showing that $w_\lambda(x) \leq c/\lambda + c^r$ for all $x \in \Omega$.

In the case $b = 0$, we will build a family $\overline{W}(\lambda)$ of supersolutions of (10) for every $\lambda \in [\lambda_1/a, +\infty)$ and apply Theorem 1. To this aim, let e be the unique (positive) solution of

$$\begin{cases} -\Delta e = 1 & \text{in } \widehat{\Omega}, \\ e = 0 & \text{on } \partial\widehat{\Omega}, \end{cases}$$

for some regular domain $\Omega \subset\subset \widehat{\Omega}$; in particular $e_m := \min_{\overline{\Omega}} e > 0$. Let $K > 0$ be a constant big enough (independent on λ) such that

$$q_{\lambda_1/a}(Ke_m)^{r-1} > a\|e\| \quad \text{and} \quad Ke(x) \geq w_{\lambda_1/a}(x) \quad \forall x \in \Omega. \tag{13}$$

Then, we consider the map

$$\overline{W} : [\lambda_1/a, \infty) \to C_0^2(\overline{\Omega}) \quad \text{such that } \overline{W}(\lambda) = Ke.$$

We will show that $\overline{W}(\lambda) = Ke$ is a supersolution of (10) for every $\lambda \in [\lambda_1/a, +\infty)$, that is

$$\forall \lambda \geq \lambda_1/a : \quad K = -\Delta(Ke) \geq aq_\lambda(Ke) \quad \text{in } \Omega,$$

or equivalently

$$\forall \lambda \geq \lambda_1/a : \quad Ke \geq aq_\lambda(Ke)e \quad \text{in } \Omega.$$

Using that $I_\lambda(q_\lambda(s)) = q_\lambda(s)/\lambda + q_\lambda(s)^r = s$, for all $s \geq 0$, we are actually reduced to show that

$$\forall \lambda \geq \lambda_1/a : \quad \frac{q_\lambda(Ke)}{\lambda} + q_\lambda(Ke)^r \geq aq_\lambda(Ke)e \quad \text{in } \Omega,$$

and then, since $e(x) \geq e_m > 0$ in $\overline{\Omega}$, to prove that

$$\forall \lambda \geq \lambda_1/a : \quad \frac{1}{\lambda} + q_\lambda(Ke)^{r-1} \geq ae \quad \text{in } \Omega. \tag{14}$$

Now since $\lim_{s\to\infty} q_\lambda(s) = \infty$, $r > 1$ and $e_m > 0$, by the monotonicity of q_λ with respect to λ (see (9)) and the choice of K (see (13)) we obtain that, for every $\lambda \geq \lambda_1/a$:

$$\frac{1}{\lambda} + q_\lambda(Ke)^{r-1} \geq \frac{1}{\lambda} + q_\lambda(Ke_m)^{r-1} > q_{\lambda_1/a}(Ke_m)^{r-1} > a\|e\| \geq ae \quad \text{in } \overline{\Omega},$$

showing that (14) is satisfied and hence, $\overline{W}(\lambda) = Ke$ is a supersolution, but not a solution, of (10) for every $\lambda \in [\lambda_1/a, +\infty)$. Due to the choice of K satisfying (13), all the hypotheses of Theorem 1 are satisfied and then we have

$$w_\lambda < \overline{W}(\lambda) = Ke \leq K \max_{\overline{\Omega}} e := c_0,$$

completing the proof of (b).

To prove (c) we argue as above, building a family $\overline{W}(\lambda)$ of supersolutions of (10). We consider again the constant map

$$\overline{W} : [\lambda_1/a, \infty) \to C_0^2(\overline{\Omega}) \quad \text{such that } \overline{W}(\lambda) = Ke.$$

Then, $\overline{W}(\lambda) = Ke$ is a supersolution of (10) for every $\lambda \in \text{Proj}_{\mathbb{R}} \widehat{\Sigma}_0$ if

$$\forall \lambda \in \text{Proj}_{\mathbb{R}} \widehat{\Sigma}_0 : \quad K \geq aq_\lambda(Ke) + bq_\lambda(Ke)^p \quad \text{in } \Omega,$$

or equivalently

$$\forall \lambda \in \text{Proj}_{\mathbb{R}} \widehat{\Sigma}_0 : \quad 1 \geq a\frac{q_\lambda(Ke)}{Ke}e + b\frac{q_\lambda(Ke)^p}{Ke}e \quad \text{in } \Omega.$$

Since $r > p$, the limits $\lim_{s\to\infty} q_\lambda(s)/s = \lim_{s\to\infty} q_\lambda(s)^p/s = 0$ are uniform in $\lambda \in \text{Proj}_{\mathbb{R}} \widehat{\Sigma}_0$ (see Remark 1), we can obtain $K > 0$ large enough (independent on λ) such that $\overline{W}(\lambda) = Ke$ is a supersolution of (10), proving the result.

Now, let us prove (d). To this end, we consider two cases: $0 < b < \lambda_1$ and $b > \lambda_1$. For the first case we argue as above, building a family $\overline{W}(\lambda)$ of supersolutions of (10). Thus, let $0 < b < \lambda_1$ be fixed. By the monotonicity properties of principal eigenvalue with respect to the domain, we can get a regular domain $\widehat{\Omega}$ such that

$$\Omega \subset\subset \widehat{\Omega} \quad \text{and} \quad b < \widehat{\lambda}_1 < \lambda_1$$

where $\widehat{\lambda}_1$ stands for the principal eigenvalue of $-\Delta$ in $\widehat{\Omega}$ under homogeneous Dirichlet boundary conditions. Define

$$\overline{W} : [\lambda_1/a, \infty) \to C_0^2(\overline{\Omega}) \quad \text{such that } \overline{W}(\lambda) = K\widehat{\varphi}_1.$$

We will show that $\overline{W}(\lambda) = K\widehat{\varphi}_1$ is a supersolution of (10) for every $\lambda \in [\lambda_1/a, +\infty)$, that is

$$\forall \lambda \geq \lambda_1/a : \quad K\widehat{\lambda}_1\widehat{\varphi}_1 = -\Delta(K\widehat{\varphi}_1) \geq aq_\lambda(K\widehat{\varphi}_1) + bq(K\widehat{\varphi}_1)^p \quad \text{in } \Omega,$$

or equivalently

$$\forall \lambda \geq \lambda_1/a : \quad \widehat{\lambda}_1 \geq \frac{q_\lambda(K\widehat{\varphi}_1) + bq(K\widehat{\varphi}_1)^p}{K\widehat{\varphi}_1} \quad \text{in } \Omega. \tag{15}$$

Since $p = r$

$$\lim_{s\to\infty} \frac{q_\lambda(s) + bq(s)^p}{s} = b$$

[see (q6) in Lemma 2], we can chose $K > 0$ large enough such that (15) holds. By Theorem 1, we obtain the result.

Now, let $b > \lambda_1$. We will proceed by contradiction. If (d) fails, there exists $(\lambda_n, w_n) \in \widehat{\Sigma}_0$ such that $\lambda_n \in \Lambda \subset (0, \infty)$ and

$$\|w_n\| \to \infty \quad \text{as } n \to \infty.$$

Moreover, up to a subsequence if necessary,

$$\lambda_n \to \lambda^* > 0.$$

Define

$$z_n := \frac{w_n}{\|w_n\|}, \quad n \geq 1.$$

Since (λ_n, w_n) is a positive solution of (10), we obtain that (λ_n, z_n) verifies

$$\begin{cases} -\Delta z_n = \dfrac{aq_{\lambda_n}(w_n) + bq_{\lambda_n}(w_n)^p}{\|w_n\|} & \text{in } \Omega, \\ z_n = 0 & \text{on } \partial\Omega. \end{cases} \tag{16}$$

Multiplying this equation by z_n, integrating in Ω and applying the formula of integration by parts gives

$$\|z_n\|_{H_0^1}^2 = \int_\Omega \frac{(aq_{\lambda_n}(w_n) + bq_{\lambda_n}(w_n)^p)z_n}{\|w_n\|}.$$

Since $r = p$, by Lemma 2, $q_\lambda(s)^p \leq s$ and $q_\lambda(s) \leq \lambda s$, for all $s \geq 0$ and $\lambda > 0$. Thus,

$$\|z_n\|_{H_0^1}^2 \leq \int_\Omega \frac{(a\lambda_n w_n + bw_n)z_n}{\|w_n\|} = \int_\Omega (a\lambda_n + b)z_n^2 \leq \left(a \max_{n\geq 1} \lambda_n + b \right) |\Omega|,$$

showing that z_n is bounded in $H_0^1(\Omega)$. By elliptic regularity, z_n is also bounded in $W^{2,m}(\Omega)$, $m > 1$. Thus, it follows from Morrey's compact embedding that, up to subsequence if necessary,

$$z_n \to z \quad \text{in } C(\overline{\Omega}).$$

Let $\psi \in C_0^\infty(\Omega)$. Multiplying (16) by ψ, integrating in Ω and applying the formula of integration by parts gives

$$-\int_\Omega z_n \Delta\psi = \int_\Omega \frac{(aq_{\lambda_n}(w_n) + bq_{\lambda_n}(w_n)^p)\psi}{\|w_n\|}$$
$$= \int_\Omega \frac{(aq_{\lambda_n}(z_n\|w_n\|) + bq_{\lambda_n}(z_n\|w_n\|)^p)z_n\psi}{z_n\|w_n\|}$$

In view of (q3) and (q6) of Lemma 2, letting $n \to \infty$ in the above equality yields

$$-\int_\Omega z\Delta\psi = \int_\Omega bz\psi.$$

Since $z \in H_0^1(\Omega)$, $z \geq 0$ in Ω and $\|z\| = 1$, it follows that $b = \lambda_1$, which is a contradiction with the inicial assumption $b > \lambda_1$.

Finally, to prove (e) observe that

$$\frac{aq_\lambda(s) + bq_\lambda(s)^p}{s^{p/r}} = \frac{aq_\lambda(s) + bq_\lambda(s)^p}{(q_\lambda(s)/\lambda + q_\lambda(s)^r)^{p/r}}$$
$$= \frac{a}{\left(q_\lambda(s)^{1-\frac{r}{p}}/\lambda + q_\lambda(s)^{r-r/p} \right)^{p/r}} + \frac{b}{(q_\lambda(s)^{1-r}/\lambda + 1)^{p/r}}$$

which implies

$$\lim_{s\to+\infty} \frac{aq_\lambda(s) + bq_\lambda(s)^p}{s^{p/r}} = b > 0.$$

By a classical result of Gidas and Spruck, see [8, Theorem 1.1], we obtain the conclusion. □

Remark 3 Note that when $b < 0$ we have a uniform bound with respect to λ of the positive solutions u_λ of problem (P_λ). Indeed from (a) of Lemma 3, since

$$\forall x \in \Omega : \ I_\lambda(u_\lambda(x)) = \frac{u_\lambda(x)}{\lambda} + u_\lambda^r(x) = w_\lambda(x) \le \frac{c}{\lambda} + c^r = I_\lambda(c),$$

we obtain $\|u_\lambda\| \le c$.

Now, we have a better information on the behaviour of the continuum of positive solutions emanating from the trivial solution at λ_1/a.

Proposition 2 *Let $\widehat{\Sigma}_0$ be the continuum of positive solutions of* (10) *given in Proposition 1.*

1. *Suppose that one of the following conditions is satisfied:*

   (i) $b \le 0$;
   (ii) $b > 0$, $r = p$ and $b < \lambda_1$;
   (iii) $b > 0$ and $r > p$.

   Then $(\lambda_1/a, \infty) \subset Proj_{\mathbb{R}} \widehat{\Sigma}_0$.
2. *Suppose that one of the following conditions is satisfied:*

   (i) $r = p$ and $b > \lambda_1$;
   (ii) $b > 0$, $1 < p/r < (N+2)/(N-2)$ and $a > \phi(s_0)$.

   Then $(0, \lambda_1/a) \subset Proj_{\mathbb{R}} \widehat{\Sigma}_0$ and (10) *does not possess positive solution for λ large.*
3. *Suppose that $r = p$ and $b = \lambda_1$, then $Proj_{\mathbb{R}} \widehat{\Sigma}_0 = \{\lambda_1/a\}$.*

Proof First, observe that in the cases (1) and (2), by Lemma 3, the positive solutions of (10) are bounded in $C(\overline{\Omega})$ and, by elliptic regularity, also in $C_0^1(\overline{\Omega})$. Thus, in view of Remark 2.

1. If (i), (ii) or (iii) occurs, then (10) does not possess positive solution for λ positive and small. Hence, $\widehat{\Sigma}_0$ has to be unbounded with respect to large values of λ, and this gives the conclusion.
2. If (i) or (ii) occurs, then there does not exist positive solution for λ large enough. Therefore, since there does not exist bifurcation point from infinity of positive solutions of (10) (by Lemma 3), the inclusion $(0, \lambda_1/a) \subset Proj_{\mathbb{R}} \widehat{\Sigma}_0$ must be satisfied.

Finally, the case $r = p$ and $b = \lambda_1$ is a direct consequence of Remark 2 (c). □

Figure 1 shows some admissible situations within the setting of Proposition 2.
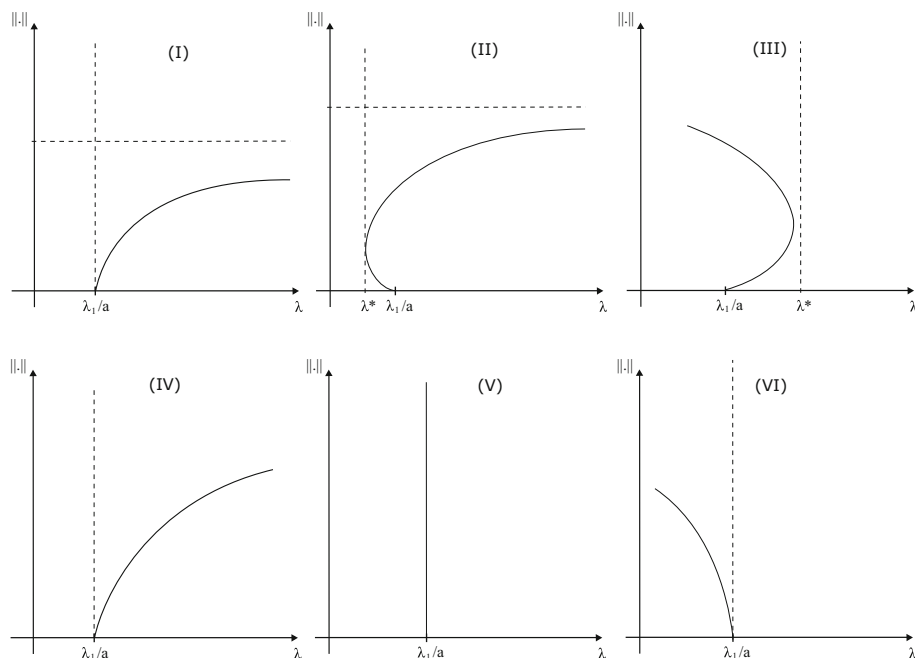
An immediate consequence of the above proposition is the following result of existence of positive solution of (10).

Corollary 1 *We have the following.*

(a) *Suppose that either $b \le 0$ or $0 < b < \lambda_1$ and $r = p$. Then* (10) *admits (at least) a positive solution if, and only if, $\lambda > \lambda_1/a$. Moreover, assuming $\lambda > \lambda_1/a$, if*

   (i) $b = 0$, *or*
   (ii) $b < 0$ *and $p \ge r$*

   the solution is unique.

Fig. 1 Possible bifurcation diagrams of solution of (10) for the case (I) $b \leq 0$; (II) $b > 0$ and $r > p$; (III) $b > 0$, $1 < p/r < (N+2)(N-2)$ and $a > \phi(s_0)$; (IV) $0 < b < \lambda_1$ and $r = p$; (V) $b = \lambda_1$ and $r = p$; (VI) $b > \lambda_1$ and $r = p$

(b) Suppose $b > 0$ and $r > p$. Then (10) admits (at least) a positive solution if $\lambda > \lambda_1/a$. Moreover, there exists $\lambda^* \in (0, \lambda_1/a)$ such that (10) admits (at least) two positive solutions for each $\lambda \in (\lambda^*, \lambda_1/a)$.

(c) Suppose that $r = p$ and $b > \lambda_1$. Then (10) admits (at least) a positive solution if $0 < \lambda < \lambda_1/a$.

(d) Suppose that $b > 0$, $r < p$, $p/r < (N+2)/(N-2)$ and $a > \phi(s_0)$. Then (10) admits (at least) a positive solution if $0 < \lambda < \lambda_1/a$. Moreover, there exists $\lambda^* > \lambda_1/a$ such that (10) admits (at least) two positive solutions for each $\lambda \in (\lambda_1/a, \lambda^*)$.

(e) Suppose $r = p$ and $b = \lambda_1$. Then (10) admits a positive solution if, and only if, $\lambda = \lambda_1/a$. Moreover, all positive solutions of (10) are given by

$$ w = c\varphi_1 \quad c \in \mathbb{R}, c > 0. $$

Proof (a) The existence is a consequence of Proposition 1 and Remark 2 (a). The uniqueness follows by the same arguments as in [3, Section 2], once that, in case (i) $s \mapsto aq(\lambda, s)/s$ is decreasing by Lemma 2 item (q1), and in case (ii) $s \mapsto (aq(\lambda, s) + bq(\lambda, s)^p)/s$ is decreasing if $p \geq r$ by Lemma 2 item (q5).

(b) It follows again by Proposition 1 combined with Remark 2 (b) that ensures that the bifurcation at $\lambda = \lambda_1/a$ is subcritical. Indeed, for $\lambda > \lambda_1/a$ there does not exist positive solution of (10) with small norm. Hence, the bifurcation is subcritical.

The proofs of (c) and (d) are similar.

In the case (e), Remark 2 (c) ensures that $\lambda = \lambda_1/a$ is a necessary condition for the existence of positive solution of (10). Moreover, for all $c > 0$,

$$
\begin{aligned}
-\Delta(c\varphi_1) = \lambda_1 c\varphi_1 &= \lambda_1 I(q_{\lambda_1/a}(c\varphi_1)) \\
&= \lambda_1 \left(\frac{q_{\lambda_1/a}(c\varphi_1)}{\lambda_1/a} + q_{\lambda_1/a}(c\varphi_1)^p \right) \\
&= a q_{\lambda_1/a}(c\varphi_1) + \lambda_1 q_{\lambda_1/a}(c\varphi_1)^p,
\end{aligned}
$$

showing that $w = c\varphi_1$, $c > 0$ is the positive solution of (10). This ends the proof. □

Remark 4 Another consequence of Propositions 1 and 2 is the existence of an unbounded continuum Σ_0 of positive solutions of (P_λ), namely,

$$
\Sigma_0 := \{(\lambda, u) \in \mathbb{R} \times C_0^1(\overline{\Omega}); \ (\lambda, u/\lambda + u^r) \in \widehat{\Sigma}_0\}.
$$

Moreover, Proposition 2 still remains valid if replace $\widehat{\Sigma}_0$ by Σ_0.

4 Proofs of the main theorems

In this section we will prove results of existence of positive solution of (3), under suitable assumptions on g. For the reader convenience we rewrite here the theorems we are going to prove.

The analysis will be done in three cases and in the first two we will use the following classical Bolzano Theorem (see for instance [2]).

Theorem 2 *Let X be a Banach space, $I \subset \mathbb{R}$ be an interval and $\Sigma_0 \subset I \times X$ be a continuum. Assume that $h : \Sigma_0 \to \mathbb{R}$ is a continuous function such that for some $(\mu_1, u_1), (\mu_2, u_2) \in \Sigma_0$ it holds $h(\mu_1, u_1)h(\mu_2, u_2) < 0$. Then there exists $(\lambda_*, u_*) \in \Sigma_0$ such that $h(\lambda_*, u_*) = 0$.*

Roughly speaking, our results depend on the position of $\mathrm{Proj}_{\mathbb{R}} \Sigma_0$ with respect to λ_1/a.

4.1 The case $(\lambda_1/a, \infty) \subset \mathrm{Proj}_{\mathbb{R}} \Sigma_0$

Our first result of existence of positive solution of (3) deals with the case in which $(\lambda_1/a, \infty) \subset \mathrm{Proj}_{\mathbb{R}} \Sigma_0$. To this end, we will assume that g satisfies the next hypothesis:

(g1) $g : [0, \infty) \to [0, \infty)$ is a continuous function and there exists a constant $g_0 > 0$ such that

$$
g(s) > g_0 \quad \forall s \in (0, +\infty),
$$

The first one is.

Theorem A *Assume that g satisfies (g1). If one of the assumptions of Proposition 2 (1) occurs, then (3) admits at least one solution for each $a > g(0)\lambda_1$.*

Proof If one of the assumptions of Proposition 2 (1) occurs, by Corollary 1 and Remark 4, $(\lambda_1/a, \infty) \subset \mathrm{Proj}_{\mathbb{R}} \Sigma_0$, where Σ_0 is the unbounded continuum of positive solutions of (P_λ).

Consider the continuous map $h : \Sigma_0 \subset (0, +\infty) \times C_0^1(\overline{\Omega}) \to C_0^1(\overline{\Omega})$ defined by

$$
h(\lambda, u) := \frac{1}{\lambda} - g(|\nabla u|_2^2).
$$

Then the zeros of h are positive solutions of (3). Let us apply the Bolzano Theorem to h. In $(\mu_1, u_1) := (\lambda_1/a, 0) \in \overline{\Sigma}_0$ we have

$$h(\mu_1, u_1) = h(\lambda_1/a, 0) = \frac{a}{\lambda_1} - g(0) > 0$$

thanks to $a > \lambda_1 g(0)$.

On the other hand,

$$\limsup_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to+\infty}} h(\lambda, u) = \limsup_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to+\infty}} \left(\frac{1}{\lambda} - g(|\nabla u|_2^2)\right) = -\liminf_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to+\infty}} g(|\nabla u|_2^2)$$

and since $g(s) \geq g_0 > 0$, it follows that

$$\limsup_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to+\infty}} h(\lambda, u) \leq -g_0 < 0.$$

Therefore, there exists some $(\mu_2, u_2) \in \Sigma_0$ such that $h(\mu_2, u_2) < 0$. By the Bolzano Theorem 2 we find $(\lambda_*, u_*) \in \Sigma_0$ satisfying

$$h(\lambda_*, u_*) = \frac{1}{\lambda_*} - g(|\nabla u_*|_2^2) = 0.$$

In particular, (λ_*, u_*) is a positive solution of (P_λ) with $1/\lambda_* = g(|\nabla u_*|_2^2)$, that is,

$$\begin{cases} -\Delta(g(|\nabla u_*|_2^2)u_* + u_*{}^r) = au_* + bu_*{}^p & \text{in } \Omega, \\ u_* = 0 & \text{on } \partial\Omega. \end{cases}$$

Thus, u_* is a positive solution of (3). □

4.2 The case $(0, \lambda_1/a) \subset \text{Proj}_{\mathbb{R}}\Sigma_0$.

Now, we will prove our second result of existence of solutions to (3) that deals with the case in which $(0, \lambda_1/a) \subset \text{Proj}_{\mathbb{R}}\Sigma_0$. For this, we will assume that:

(g2) $g : [0, \infty) \to [0, \infty)$ is a bounded continuous function such that $g(0) > 0$.

Then we have

Theorem B *Assume that g satisfies (g2).*

(i) If $r = p$ and $b > \lambda_1$ then problem (3) admits (at least) one solution for each $a < g(0)\lambda_1$.
(ii) If $b > 0$, $r < p$, $p/r < (N+2)/(N-2)$ and $g(0)\lambda_1 > \phi(s_0)$ then problem (3) admits (at least) one solution for each $a \in (\phi(s_0), g(0)\lambda_1)$.
Recall that ϕ and s_0 are defined in (4) and (5).

Proof In both cases, by Corollary 1 and Remark 4, $(0, \lambda_1/a) \subset \text{Proj}_{\mathbb{R}}\Sigma_0$. Thus:
If (i) occurs then in $(\mu_1, u_1) := (\lambda_1/a, 0) \in \overline{\Sigma}_0$ we have

$$h(\mu_1, u_1) = h(\lambda_1/a, 0) = \frac{a}{\lambda_1} - g(0) < 0,$$

thanks to $a < g(0)\lambda_1$. On the other hand, since g is bound,

$$\liminf_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to0^+}} h(\lambda, u) = \liminf_{\substack{(\lambda,u)\in\Sigma_0 \\ \lambda\to0^+}} \left(\frac{1}{\lambda} - g(|\nabla u|_2^2)\right) = +\infty.$$

Therefore, there exists some $(\mu_2, u_2) \in \Sigma_0$ such that $h(\mu_2, u_2) > 0$. By Bolzano Theorem 2 we find $(\lambda_*, u_*) \in \Sigma_0$ satisfying $h(\lambda_*, u_*) = 0$, thus u_* is a positive solution of (3). Similarly, if (ii) occurs we have

$$h(\lambda_1/a, 0) = \frac{a}{\lambda_1} - g(0) < 0 \quad \text{and} \quad \liminf_{\substack{(\lambda, u) \in \Sigma_0 \\ \lambda \to 0^+}} h(\lambda, u) = +\infty.$$

By Bolzano Theorem 2 we find $(\lambda_*, u_*) \in \Sigma_0$ satisfying $h(\lambda_*, u_*) = 0$, thus u_* is a positive solution of (3). □

4.3 The case $\Sigma_0 = \{\lambda_1/a\}$

To finish, we will show a result of existence of positive solution of (3) when $\Sigma_0 = \{\lambda_1/a\}$. In this case we will not use the Bolzano Theorem 2 and the only assumption on g is that it is a continuous positive function.

Theorem C *Assume that $g : [0, \infty) \to [0, \infty)$ is continuous and positive. If $b = \lambda_1$ and $r = p < 2$, then problem (3) admits a positive solution if, and only if, $a \in \lambda_1 R[g] := \{\lambda_1 g(s); s \geq 0\} \subset (0, \infty)$.*

Proof Since $b = \lambda_1$ and $r = p$, it follows from Corollary 1 and Remark 4 that $\Sigma_0 = \{\lambda_1/a\}$. Moreover, all positive solutions of (3) are given by

$$u_c := q_{\lambda_1/a}(c\varphi_1) \iff \frac{a}{\lambda_1} u_c + u_c^r = c\varphi_1 \quad c \in \mathbb{R}, \ c > 0.$$

Thus,

$$\nabla(au_c + \lambda_1 u_c^r) = (a + \lambda_1 r u_c^{r-1})\nabla u_c = c\lambda_1 \nabla\varphi_1$$

$$\nabla u_c = \frac{c\lambda_1}{a + \lambda_1 r u_c^{r-1}}\nabla\varphi_1$$

$$|\nabla u_c|^2 = \left(\frac{c\lambda_1}{a + \lambda_1 r u_c^{r-1}}\right)^2 |\nabla\varphi_1|^2$$

and

$$|\nabla u_c|_2^2 = \int_\Omega \left(\frac{c\lambda_1}{a + \lambda_1 r q_{\lambda_1/a}(c\varphi_1)^{r-1}}\right)^2 |\nabla\varphi_1|^2$$

$$= \int_\Omega \left(\frac{\lambda_1}{a/c + \lambda_1 r c^{r-2}(q_{\lambda_1/a}(c\varphi_1)/c)^{r-1}}\right)^2 |\nabla\varphi_1|^2.$$

Consequently, since $r < 2$, the map $c \in [0, \infty) \mapsto |\nabla u_c|_2 \in [0, \infty)$ is continuous and increasing. Let us prove that it is one-to-one. Indeed, by monotonicity, it follows that $c \mapsto |\nabla u_c|_2$ is an injection. To prove that it is a surjection, it is sufficient to show that

$$\lim_{c \to +\infty} |\nabla u_c|_2 = +\infty.$$

Indeed, it follows from Lemma 2,

$$q_{\lambda_1/a}(c\varphi_1)^{r-1} \leq \lambda_1(c\varphi_1)^{r-1}/a \leq \lambda_1 c^{r-1}/a.$$

Thus,

$$
\begin{aligned}
|\nabla u_c|_2^2 &= \int_\Omega \left(\frac{c\lambda_1}{a + \lambda_1 r q_{\lambda_1/a}(c\varphi_1)^{r-1}} \right)^2 |\nabla\varphi_1|^2 \\
&\geq \int_\Omega \left(\frac{c\lambda_1}{a + r\lambda_1^2 c^{r-1}/a} \right)^2 |\nabla\varphi_1|^2 = \left(\frac{\lambda_1}{ac^{-1} + r\lambda_1^2 c^{r-2}/a} \right)^2.
\end{aligned}
$$

Once that $r < 2$, one can infer

$$
\lim_{c \to +\infty} |\nabla u_c|_2 = +\infty.
$$

Subsequently, for each $a \in \{\lambda_1 g(s); s \geq 0\}$, there exists $s' \geq 0$ such that

$$
a = \lambda_1 g(s').
$$

By the previous discussion, we can choose $c > 0$ satisfying

$$
|\nabla u_c|_2 = s'.
$$

Therefore

$$
a = \lambda_1 g(|\nabla u_c|_2) \Longleftrightarrow \frac{1}{\lambda_1/a} = g(|\nabla u_c|_2)
$$

and u_c is a solution of (3). □

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