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Elliptic and hyperelliptic functions and complete minimal surfaces with handles.

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ELLIPTIC AND HYPERELLIPTIC FUNCTIONS AND COMPLETE MINIMAL SURFACES WITH HANDLES

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Abstract Using the classical theory of elliptic and hyperelliptic functions, we construct two complete orientable minimal surfaces in $\mathbb{R}^3$ with finite total curvature C and of genus p=1 and 2. The absolute value of C is as small as possible:

$$p = 1 \quad \text{and} \quad C = -8\pi$$

$$p = 2 \quad \text{and} \quad C = -12\pi$$

And both are of Enneper's type.

§ 1. Introduction

1.1. Properties of minimal surfaces

Let S be a complete orientable minimal surface in $\mathbb{R}^3$ with finite total curvature C. There associates a finitely connected Riemann surface R with boundary points P_ν:

$$S \longrightarrow R = \bar{R} \setminus \{P_1, \ldots, P_N\}$$

R is the parameter domain for isothermal parameters, $\bar{R}$ is a compact Riemann surface. Further S induces a meromorphic function g and a meromorphic differential fdz on $\bar{R}$, and, the induced metric is given by

$$(1) \qquad ds = \frac{1}{2}|f|(1 + |g|^2)|dz|$$

The coordinate differentials $\phi_j\,dz$ of S given by the Weierstrass formula:

$$(2) \qquad \begin{cases} \phi_1\,dz = \frac{1}{2}\,f(1 - g^2)dz, \\ \phi_2\,dz = \frac{i}{2}\,f(1 + g^2)dz, \\ \phi_3\,dz = fg\,dz, \end{cases}$$

and the coordinates are given by

$$(3) \qquad x_j = \mathrm{Re} \int \phi_j\,dz \;,\; j = 1,2,3.$$

Details can be found in Osserman [13, §8, §9].

To every boundary component P_ν there belongs an expanding tube of S in the sense of Cohn-Vossen [4] and we can define a number $k_\nu \in \mathbb{N}$, the order of the tube:

$$P_\nu \longrightarrow k_\nu \;,\; \nu = 1,\ldots,N,$$

which are geometrical quantities (See [5, p.160]). And it is known (See [5, Satz 2] or [10, Theorem 1.3]) that

Theorem 1 (Gauss-Bonnet equality for complete minimal surfaces)

Given a complete minimal surface of type S. Then the following equality holds:

$$(4) \qquad C = 2\pi(\chi - \sum_{\nu=1}^{N} k_\nu),$$

where χ is the Euler characteristic of S.

1.2. Problems and results

There are many complete minimal surfaces with genus p=0. There exist also many complete minimal surfaces of genus p>1, but their total curvatures are usually of large absolute values and the geometry of such surfaces is therefore quite difficult. Hence it's interesting to construct examples with $|C|$ as small as possible for fixed genus. Some discussions in this direction can be found in [2]. Formula (4) gives first informations:

$$C = 2\pi\left(2-2p-N-\sum_{\nu=1}^{N} k_\nu\right)$$

C	p			
-4π	0	$N=1$, $k=3$	Enneper's surface	no further surfaces
		$N=2$, $k_1=k_2=1$	Catenoid	
-8π	1	$N=1$, $k=3$	Enneper's type with 1 handle	if exist
		$N=2$, $k_1=k_2=1$	Catenoid's type with 1 handle	
⋮	⋮			
$-(p+1)4\pi$	p	$N=1$, $k=3$	Enneper's type with p handles	if exist
		$N=2$, $k_1=k_2=1$	Catenoid's type with p handles	

A surface S with N=1, k=1 does not exist [6]. And if G is the number of sheets of the g-function, then $C=-4\pi G$.

In this paper we construct two surfaces of Enneper's type for p=1 and p=2. We use the classical theory of elliptic and hyperelliptic functions and in both cases we construct with very symmetric Riemann surfaces $\bar{R}$.

§ 2. A complete minimal surface with p=1 and $C=-8\pi$

2.1. Every compact Riemann surface $\bar{R}$ of genus p=1 can be represented by a periodic parallelogram $(1,\tau)$ in the z-plane , denoted by $\bar{R}_\tau$. The most symmetric Riemann surface is the square $\bar{R}_i$. Now we construct a surface S which is conformally equivalent to $\bar{R}_i \setminus \{o\}$. We use the theory of elliptic functions. The Weierstrauss $\wp$-function plays an important role. For the square we have the invariants:

$$g_2 = 60 \sum_{k,k'}{}' \frac{1}{(k+k'i)^4} > 0 \ , \ g_3 = 0 \ ,$$

where g_2 and g_3 are the coefficients of the cubic differential equation associated to $\wp$. Now we prove

Theorem 2. There exists a complete minimal surface S with $p=1$, $C=-8\pi$, $N=1$, $k=3$, which is a surface of Enneper's type with one handle. We construct with

$$\text{(5)} \qquad \begin{cases} R = \bar{R}_i \setminus \{o\} \\ g(z) = A \dfrac{\wp'(z)}{\wp(z)} \ , \quad A = \sqrt{\dfrac{3\pi}{2g_2}} \\ f dz = 2\,\wp(z) dz \end{cases}$$

PROOF

a) Completeness, the boundary component and the order of the tube:

First we look at zeros and poles of g and f (instead of the differential fdz we take the function in the z-plane) and we find:

z	0	$\frac{1+i}{2}$
$f=2\wp$	∞,∞	0,0
g	∞	∞
S	tube k=3	regular point

∞,∞ means double pole, o,o means double zero. The $\wp$-function has a double zero at $z = \frac{1+i}{2}$ because we have the very symmetric square $\bar{R}_i$, hence $\frac{1+i}{2}$ is a regular point of S (See [13, Lemma 8.1]) and z=o is a tube of order 3 (See [5, p.160]). If S exists (i.e. the real parts of the residues and periods in (3) vanish), then S is complete and conformally equivalent to $\bar{R}_i \setminus \{o\}$. Therefore we have p=1, N=1, k=3 and $C=-8\pi$, because g is two-sheeted.

b) <u>The coordinate differentials</u>: By (2), (5) and the differential equation of the p-function we get

$$\phi_1 = \frac{1}{2} f(1-g^2) = \wp(1-A^2(\frac{\wp'}{\wp})^2) = \wp - A^2 \frac{4(\wp-e_1)(\wp-e_2)(\wp-e_3)}{\wp} .$$

Since the fundamental domain is a square lattice, we have

$$e_1 > 0,\ e_2 = 0,\ e_3 = -e_1 < 0 \quad \text{and} \quad e_1^2 = \frac{1}{4} g_2$$

and consequently

$$\phi_1 = \wp - 4A^2(\wp^2-e_1^2) = \wp - 4A^2(\frac{1}{6}\wp'' + \frac{1}{12} g_2 - \frac{1}{4} g_2) = -\frac{2}{3} A^2 \wp'' + \wp + \frac{2}{3}A^2 g_2$$

Similarly, we find

$$\phi_2 = \frac{2i}{3} A^2 \wp'' + i\wp - \frac{2i}{3} A^2 g_2 ,$$

and obviously

$$\phi_3 = 2 A\wp'$$

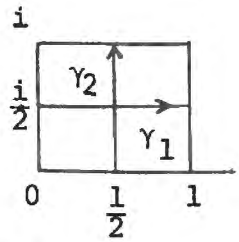
c) <u>Residues and periods</u>: Since our differentials $\phi_j dz$ have only one pole, the residues are zero:

$$\operatorname*{Res}_{z=0} \phi_j \, dz = 0 \quad \text{for} \quad j = 1,2,3.$$

To get the periods we must look at the six integrals:

$$\int_{\gamma_\nu} \phi_j \, dz \quad , \quad j = 1,2,3; \quad \nu = 1,2$$

with



For ϕ_3 dz, we see easily

(6) $$x_3 = 2A \operatorname{Re} \wp(z),$$

which is a nice formula for the third coordinate. The value distribution of the Weirstrass $\wp$-function related to the square lattice is well-known (See [12, p.303]).

The first period of ϕ_1 dz:

$$\int_{\gamma_1} \phi_1 \, dz = \eta_1 + \frac{2}{3} A^2 g_2$$

Using $\wp(\frac{1}{2} + it) = - \wp(\frac{i}{2} + t) \geq 0$ and the Legendre's relation we find

$$\eta_1 = - \pi \quad \text{and} \quad \eta_2 = i\pi$$

The first period vanishes if $A = \overset{+}{(-)} \sqrt{\frac{3\pi}{2g_2}}$. The "—" sign gives only a reflected surface $\tilde{S}$, see [7]. The second period of $\phi_1$dz:

$$\int_{\gamma_2} \phi_1 dz = \int_{\gamma_2} (\wp + \frac{2}{3} A^2 g_2) dz = \eta_2 + \frac{2}{3} A^2 g_2 i = 2\pi i$$

Similarly, we find the two periods of $\phi_2$dz:

$$\int_{\gamma_1} \phi_2 dz = - 2\pi i \quad \text{and} \quad \int_{\gamma_2} \phi_2 dz = 0$$

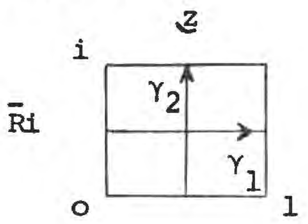
So all residues and periods are zero or purely imaginary and theorem 2 is proved

q.e.d.

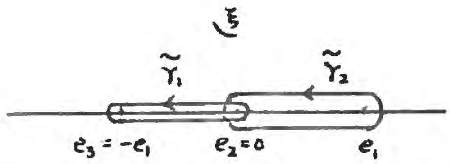
REMARK. This theorem is a positive answer to the question posed in [1, § 4].

2.2. Translation into hyperelliptic language

To find minimal surfaces with genus $p>1$, it is natural to start with a hyperelliptic Riemann surface $\bar{R}$. To get some idea we translate theorem 2 into hyperelliptic language.



translation ⟷



this concrete Riemann surface is produced by $w = \sqrt{\xi(\xi^2-e_1^2)}$

$$g(z) = A\frac{\wp'(z)}{\wp(z)} \longleftrightarrow \tilde{g}(\xi) = 2A\frac{w}{\xi}$$

$$fdz = 2\wp(z)dz \longleftrightarrow \tilde{f}d\xi = \frac{\xi}{w}d\xi$$

On hyperelliptic functions see [11, p. 188]

§ 3. A complete minimal surface with $p=2$ and $C = -12\pi$.

Analogical to theorem 2 we now prove

Theorem 3. There exists a complete minimal surface S with $p=2$, $C = -12\pi$, $N=1$, $k=3$. This is a surface of Enneper's type with two handles. We construct with

$\bar{R}$ = the concrete Riemann surface produced by

$$w = \sqrt{z(z^2-a^2)(z^2-b^2)}$$

$$(7)\qquad g(z) = B\frac{w}{z^2-a^2}$$

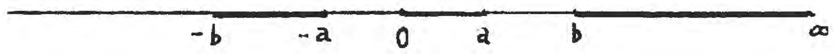
$$fdz = \frac{z^2-a^2}{w}dz$$

where a,b and B are proper real constants with $0<a<b$ and $0<B$.

PROOF

a) Completeness, the boundary component and the order of the tube:

$\bar{R}$:



Looking at the zeros and poles of g and fdz and by local uniformization, we find:

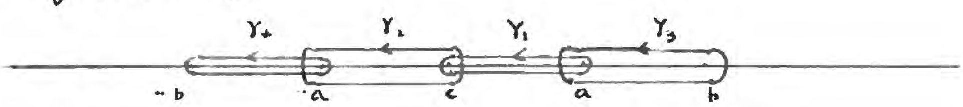
z	-b	-a	o	a	b	∞
g	o	∞	o	∞	o	∞
fdz		o,o		o,o		∞,∞
S		reg. point		reg. point		tube k=3

If S exists (i.e the real parts of the residues and periods in (3) vanish), then S is complete and conformally equivalent to $\bar{R}\setminus\{\infty\}$ with $p=2$, $N=1$, $k=3$ and $C = -12\pi$, because g is three-sheeted.

b) Residues and periods. Since the differentials $\phi_j dz$ have only one pole, the residues are zero. To compute the periods, we look at the following twelve integrals:

$$\int_{\gamma_\nu} \phi_j dz \ , \quad j = 1,2,3; \quad \nu = 1,2,3,4,$$

with γ_ν given by



There is obviously no problem with $\phi_3 dz = Bdz$:

$$x_3 = Bx \quad , \quad z = x+iy.$$

c) The last eight periods. From (2) and (7), we get

$$\phi_1 dz = \frac{1}{2}\,\frac{z^2-a^2-B^2z(z^2-b^2)}{w}\,dz$$

and the first period is

$$\int_{\gamma_1} \phi_1 dz = \frac{1}{2}\left\{\int_0^a \frac{x^2-a^2-B^2x(x^2-b^2)}{\sqrt{x(x^2-a^2)(x^2-b^2)}}dx - \int_0^a \frac{x^2-a^2-B^2x(x^2-b^2)}{-\sqrt{x(x^2-a^2)(x^2-b^2)}}\,dx\right\}$$

On the lower part of the dumbbell-shaped curve we have $+\sqrt{}$, on the upper part we must change the sign: $-\sqrt{}$, here we also obviously change the orientation.

Hence

$$\int_{\gamma_1} \phi_1 dz = -\int_0^a \frac{a^2-x^2}{\sqrt{x(a^2-x^2)(b^2-x^2)}}\,dx + B^2\int_0^a \frac{x(b^2-x^2)}{\sqrt{x(a^2-x^2)(b^2-x^2)}}\,dx \equiv -F_1+B^2F_2$$

with positive real values:

$$F_1 = F_1(a,b) > 0 \quad \text{and} \quad F_2 = F_2(a,b) > 0.$$

By similar calculations we get the period table for the first two cycles:

	γ_1	γ_2
$\int \phi_1 dz$	$-F_1+B^2F_2$	$iF_1+iB^2F_2$
$\int \phi_2 dz$	$-iF_1-iB^2F_2$	$-F_1+B^2F_2$

We introduce further the positive real values:

$$F_3 = \int_a^b \frac{x^2-a^2}{\sqrt{x(x^2-a^2)(b^2-x^2)}}dx > 0$$

$$F_4 = \int_a^b \frac{x(b^2-x^2)}{\sqrt{x(x^2-a^2)(b^2-x^2)}}dx > 0$$

And the period table of the other two cycles is:

	γ_3	γ_4
$\int \phi_1 dz$	$-iF_3-iB^2F_4$	$F_3-B^2F_4$
$\int \phi_2 dz$	$F_3-B^2F_4$	$iF_3+iB^2F_4$

d) Last problem. We must choose a,b,B so that

(8) $$F_1 = B^2F_2,$$

(9) $$F_3 = B^2F_4,$$

or, equivalently,

(8) $$F_1 = B^2F_2,$$

(10) $$F_1F_4 = F_2F_3.$$

By choosing a proper real positive B, (8) can always be realized for any a and b. Since a linear transformation does not change the problem, we set a=1 and we want to find a real value b > 1 so that (10) is fulfilled:

$$\int_0^1 \frac{1-x^2}{\sqrt{x(1-x^2)(b^2-x^2)}}dx \int_1^b \frac{x(b^2-x^2)}{\sqrt{x(x^2-1)(b^2-x^2)}}dx \overset{!}{=} \int_0^1 \frac{x(b^2-x^2)}{\sqrt{x(1-x^2)(b^2-x^2)}}dx \int_1^b \frac{x^2-1}{\sqrt{x(x^2-1)(b^2-x^2)}}$$

After parametric transformations we get

$$\int_0^1 \sqrt{\frac{1+x}{(b+x)(b-x)}} \sqrt{\frac{1-x}{x}}dx \int_0^{b-1} \sqrt{\frac{(1+x)(b+1+x)}{2+x}} \sqrt{\frac{b-1-x}{x}}dx$$

$$\overset{!}{\leq} \int_0^1 \sqrt{\frac{(b-1-x)(b+1-x)}{2-x}} \sqrt{\frac{1-x}{x}}dx \int_0^{b-1} \sqrt{\frac{b+1-x}{(b-x)(2b-x)}} \sqrt{\frac{b-1-x}{x}}dx$$

For $b \rightarrow \infty$, the left-hand side becomes smaller than the right-hand side because

$$\text{LHS} \leq \sqrt{\frac{2}{b(b-1)}} \int_0^1 \sqrt{\frac{1-x}{x}}dx \sqrt{2b} \int_0^{b-1} \sqrt{\frac{b-1-x}{x}}dx$$

$$\text{RHS} \geq \sqrt{\frac{(b-1)b}{2}} \int_0^1 \sqrt{\frac{1-x}{x}}dx \sqrt{\frac{1}{2b}} \int_0^{b-1} \sqrt{\frac{b-1-x}{x}}dx$$

For $b = 1+\varepsilon$, $\varepsilon>0$ $\varepsilon \rightarrow 0$, the left-hand side becomes bigger than the right-hand side because for ε sufficiently small

$$\text{LHS} = \int_0^1 \sqrt{\frac{1+x}{(1+\varepsilon+x)(1+\varepsilon-x)}} \sqrt{\frac{1-x}{x}}dx \int_0^{\varepsilon} \sqrt{\frac{(1+x)(2+\varepsilon+x)}{2+x}} \sqrt{\frac{\varepsilon-x}{x}}dx$$

$$\geq \int_0^1 \frac{0.9}{\sqrt{1+\varepsilon-x}} \sqrt{\frac{1-x}{x}}dx \int_0^{\varepsilon} \sqrt{\frac{\varepsilon-x}{x}}dx,$$

$$\text{RHS} = \int_0^1 \sqrt{\frac{(\varepsilon+x)(2+\varepsilon-x)}{2-x}} \sqrt{\frac{1-x}{x}}dx \int_0^{\varepsilon} \sqrt{\frac{2+\varepsilon-x}{(1+\varepsilon-x)(2+2\varepsilon-x)}} \sqrt{\frac{\varepsilon-x}{x}}dx$$

$$\leq \int 1.1 \sqrt{\varepsilon+x} \sqrt{\frac{1-x}{x}}dx \int_0^{\varepsilon} \sqrt{\frac{\varepsilon-x}{x}}dx,$$

and $$0.9 \frac{1}{\sqrt{1+\varepsilon-x}} \geq 1.1 \sqrt{\varepsilon+x} \quad \text{for} \quad 0\leqslant x\leqslant 1.$$

So by the intermediate value theorem, there exists a real value b with $1 < b < \infty$ such that the equality (10) holds.

q.e.d.

§ 4. Some questions

4.1. Are the surfaces in theorems 2 and 3 the only complete minimal surfaces of Enneper's type with one or two handles?

4.2. Is it possible to construct examples of Enneper's type with more than two handles? In the case p=3, it would be natural to start with the Riemann surface $\bar{R}$ produced by

$$W = \sqrt{z(z^2-a^2)(z^2-b^2)(z^2-c^2)} \quad \text{with} \quad 0<a<b<c<\infty$$

And let

$$g(z) = C \frac{W}{z(z^2-b^2)}$$

$$fdz = \frac{z(z^2-b^2)}{W} dz$$

The zeros and poles of g and fdz would be

z	-c	-b	-a	o	a	b	c	∞
g	o	∞	o	∞	o	∞	o	∞
fdz		o,o		o,o		o,o		∞ ∞

But we have a problem with the periods, we have two equations of type (10) and two real parameters. And the method used in §3,d) has, so far, been unable to be extended.

4.3. It would be interesting to construct complete minimal surfaces of Catenoid's type, i.e. N=2, $k_1=k_2=1$. So far, it's unknown to us. W.H. Meeks III has constructed a complete minimal surface with p=1, N=3, $k_1=k_2=k_3=1$ and $C = -12\pi$.

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