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Free Bol loops of exponent two ☆

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ABSTRACT

A Bol loop is a loop that satisfies the identity $x((yz)y) = ((xy)z)y$. In this paper, we give a construction of the free Bol loops of exponent two. We define a canonical form of all their elements and describe their multiplication law based on this form.

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1. Introduction

A *loop* consists of a nonempty set L with a binary operation $*$ such that, for each $a, b \in L$, the equations $a * x = b$ and $y * a = b$ have unique solutions for $x, y \in L$, and there exists an *identity element* $1 \in L$ satisfying $1 * x = x = x * 1$, for any $x \in L$. A (*right*) *Bol loop* is a loop that satisfies the (right) Bol identity

$$x((yz)y) = ((xy)z)y. \quad (1)$$

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One of the most interesting subvarieties of Bol loops is the variety \mathbf{B}_2 of Bol loops of exponent two. Every loop in \mathbf{B}_2 is a Bruck loop, i.e., a Bol loop with the automorphic inverse property $((xy)^{-1} = x^{-1}y^{-1}$, for every x, y in the loop). Many constructions of non-associative loops of \mathbf{B}_2 can be found in the literature (see [7,8] for example), the minimal such loop has order 8. Some of the most important problems involving loops of \mathbf{B}_2 are those related to solvability and existence of simple loops (see [1,3,9,10]). In [10], a class of non-associative simple Bol loops of exponent 2 was constructed. The smallest loop in this class, which is also the smallest non-associative simple loop in \mathbf{B}_2 ([3, Theorem 3]), has order 96.

In this paper, we give a construction of free objects in the variety \mathbf{B}_2 . Let $B(X)$ be the free Bol loop of exponent two with free set of generators X . We construct a subset $R(X)$ of $B(X)$ such that every element $b \in B(X) \setminus \{1\}$ has the canonical form $b = (\dots(b_1b_2)b_3\dots)b_mb_{m-1}\dots)b_2b_1$, where $b_i \in R(X)$ and $b_i \neq b_{i+1}$, for all i , and then we describe the multiplication law of $B(X)$ based on this form. Furthermore, we prove that the nuclei and the center of $B(X)$ are trivial.

2. Preliminaries

Let L be a loop and $x \in L$. The bijections $L_x, R_x : L \rightarrow L$ defined by $(y)L_x = xy$ and $(y)R_x = yx$ are called the *left and right translations* of x in L , respectively. The *right multiplication group* of L is the group $Mlt_r(L) = \langle R_x \mid x \in L \rangle$ and the *right inner mapping group* of L is $Inn_r(L) = \{\phi \in Mlt_r(L) \mid (1)\phi = 1\}$. The subgroup $Inn_r(L)$ of $Mlt_r(L)$ is *core-free*, i.e., the only subgroup of $Inn_r(L)$ that is normal in $Mlt_r(L)$ is the trivial subgroup $\{I_d\}$, where I_d is the identity mapping of L .

The *left, middle and right nuclei* of L , denoted respectively by $N_\lambda(L)$, $N_\mu(L)$ and $N_\rho(L)$, are defined by:

$$N_\lambda(L) = \{a \in L \mid a(xy) = (ax)y \ \forall x, y \in L\},$$

$$N_\mu(L) = \{a \in L \mid x(ay) = (xa)y \ \forall x, y \in L\},$$

$$N_\rho(L) = \{a \in L \mid x(ya) = (xy)a \ \forall x, y \in L\}.$$

The *nucleus* of L is defined by $N(L) = N_\lambda(L) \cap N_\mu(L) \cap N_\rho(L)$ and the *center* of L is the set $\mathcal{Z}(L) = \{a \in N(L) \mid ax = xa \ \forall x \in L\}$. The nuclei of L are subgroups of L and the center of L is an abelian subgroup of L .

Bol loops are loops that satisfy the identity (1). This class of loops contains Moufang loops and groups. Furthermore, Bol loops are power-associative and right alternative, and have the right inverse property. Other basic facts from loop theory and Bol loops can be found in [4,11].

The Baer correspondence ([2]) is an important tool in the study of Bol loops (cf. [1]). From it, we obtain that Bol loops are related to twisted subgroups, as we can see in the next proposition. A subset K of a group G is called a *twisted subgroup* of G if $1 \in K$ and $x^{-1}, xyx \in K$, for all $x, y \in K$.

Proposition 2.1. ([5, Proposition 5.2]) Let (G, H, B) be a Baer triple, i.e., G is a group, H is a subgroup of G and B is a right transversal of H in G . If B is a twisted subgroup of G , then B with the operation $*$ defined by

$$b * b' = c, \text{ where } bb' = hc, \text{ for some } h \in H, \quad (2)$$

is a Bol loop. Conversely, if $(B, *)$ is a Bol loop and H is core-free, then B is a twisted subgroup of G .

If L is a loop, the triple (G, H, B) , where $G = \text{Mlt}_r(L)$, $H = \text{Inn}_r(L)$ and $B = \{R_x \mid x \in L\}$, is a Baer triple. In this condition, L is a Bol loop if and only if B is a twisted subgroup of G [1, 6.1].

Let B be a Bol loop of exponent n and X be a subset of B . We say that X is a *free set of generators* of B if X generates B and every mapping between X and a Bol loop B' of exponent n can be extended to a homomorphism between B and B' . We say that B is a *free Bol loop of exponent n* if it has a free set of generators.

Now consider B as a free Bol loop of exponent two. A subset $T \subset B$ is a *prebasis* of B if for every $b \in B$ there exist $b_1, \dots, b_n \in T$ such that $b = b_1 b_2 \dots b_n b_{n-1} \dots b_2 b_1$. Here and in the following, we will write $v = v_1 v_2 \dots v_n$ if $v = (\dots((v_1 v_2) v_3) \dots) v_n$. A subset $T \subset B$ is an *independent* if for every $a_1, \dots, a_m, b_1, \dots, b_n \in T$, such that $b_i \neq b_j$ and $a_p \neq a_q$, for all i, j, p, q , from $a_1 a_2 \dots a_m a_{m-1} \dots a_2 a_1 = b_1 b_2 \dots b_n b_{n-1} \dots b_2 b_1$, we have that $n = m$ and $a_i = b_i, i = 1, \dots, n$. A subset $T \subset B$ is a *basis* of B if T is an independent prebasis of B . A group G is a *free 2-group* if it is a free product of cyclic groups of order two, i.e., it has the form $G = \prod_{x \in T} \star < x \mid x^2 = 1 >$.

3. Construction of a basis of free Bol loops of exponent two

Let X be a finite ordered set of letters and $P = P(X)$ be the set of all non-associative words on X . We denote the empty word by 1. For $v \in P$, by $\text{Sub}(v)$ we denote the set of all subwords of v . Note that if $v = v_1 v_2$, then $\text{Sub}(v) = \{v\} \cup \text{Sub}(v_1) \cup \text{Sub}(v_2)$.

For $v \in P$, the length of v , denoted by $|v|$, is the number of letters in the word v . Note that $|1| = 0$.

Let $C(X) = \{uu, (uv)v \mid u, v \in P\}$ and $W = W(X) = \{v \in P \mid \text{Sub}(v) \cap C(X) = \emptyset\}$. Define the mapping $\pi : P \rightarrow W$, where, for $v \in P$, $\pi(v)$ is given by induction on $|v|$ using the following rules:

- (i) $\pi(x) = x$, if $x \in X$,
- (ii) If $u, v \in W$, then

$$\pi(uv) = \begin{cases} 1, & \text{if } u = v, \\ a, & \text{if } u = av, \\ uv, & \text{if } uv \in W, \end{cases}$$

- (iii) If $u \notin W$ or $v \notin W$, then $\pi(uv) = \pi(\pi(u)\pi(v))$.

Notice that in the case (iii) we get $|\pi(u)\pi(v)| < |uv|$. Hence this definition is correct.

Lemma 3.1. *Let $u, v, w, v_1, \dots, v_n \in P$ and $a \in W$. Then:*

- (a) $\pi(uv) = \pi(\pi(u)\pi(v))$.
- (b) $\pi(uv.v) = \pi(u.vv) = \pi(u)$.
- (c) $\pi(u) = \pi(v)$ if and only if $\pi(uw) = \pi(vw)$.
- (d) If $\pi(uv_1v_2\dots v_n) = \pi(v)$, then $\pi(u) = \pi(vv_n\dots v_2v_1)$.
- (e) If $\pi(v_1v_2\dots v_n) = a$, then $\pi(av_n\dots v_2v_1) = 1$.
- (f) If $\pi(uv) = \pi(uw)$, then $\pi(v) = \pi(w)$.

Proof. The item (a) follows from the definition of π . The item (c) is a consequence of (a) and (b), and the items (d) and (e) are consequences of (b) and (c). Let us prove (b) and (f).

(b) By (a), we have $\pi(uv.v) = \pi(\pi(\pi(u)\pi(v))\pi(v))$ and $\pi(u.vv) = \pi(u)$. If $\pi(u)\pi(v) \in W$, then $\pi(\pi(\pi(u)\pi(v))\pi(v)) = \pi((\pi(u)\pi(v))\pi(v)) = \pi(u)$. If $\pi(u) = c\pi(v)$, then $\pi(\pi(\pi(u)\pi(v))\pi(v)) = \pi(c\pi(v)) = \pi(u)$.

(f) By (a), we only have to prove the case where $u, v, w \in W$. If either $1 \in \{u, v\}$ or $u = v$, the result is trivial. Suppose that $u, v \in W \setminus \{1\}$ and $u \neq v$. If $u = cv$, for some $c \neq 1$, then $\pi(uw) = c$. Since $|c| < |u|$, we have $uw \notin W$. Thus $u = dw$, for some $d \neq 1$, and we have $cv = u = dw$. Therefore $w = v$.

Now suppose that $uv \in W$. Since $|uv| > |u|$, it follows that $uw \in W$. Hence $uw = uv$ and we have $w = v$. \square

Lemma 3.2. *Let $v = v_1v_2v_3\dots v_m$, where $v_i \in W$ and $v_i \neq v_{i+1}$, for $i = 1, \dots, m-1$. If $|\pi(v)| < |v|$, then there are three possibilities:*

- (a) $v_1v_2 \in W$ and $v_i = v_1v_2v_3\dots v_{i-1}$, for some $i > 2$,
- (b) There exists $v'_1 \in W$ such that $v_1 = v'_1v_jv_{j-1}\dots v_3v_2$, where $1 < j < m$ and $v'_1v_{j+1} \in W$,
- (c) $v_1 = v'_1v_mv_{m-1}\dots v_2$, for some $v'_1 \in W$.

Proof. If $v_1v_2 \in W$, then there exists $i \in \{3, \dots, m\}$ such that $v_1v_2\dots v_{i-1} \in W$ and $v_1v_2\dots v_{i-1}v_i \notin W$. Since $v_i \neq v_{i+1}$, we have $v_i = v_1v_2v_3\dots v_{i-1}$. When $v_1v_2 \notin W$ we have that $v_1 = \alpha v_2$, for some $\alpha \in W$. If $v_1 \neq \beta v_mv_{m-1}\dots v_2$, for every $\beta \in W$, then there exist $v'_1 \in W$ and $j \in \{2, 3, \dots, m-1\}$ such that $v_1 = v'_1v_jv_{j-1}\dots v_3v_2$ and $v'_1 \neq \gamma v_{j+1}$, for every $\gamma \in W$. Hence $v'_1v_{j+1} \in W$. \square

Remark. In the Lemma 3.2 it is possible that $v'_1 = 1$.

The following result is a consequence of Lemma 3.2.

Corollary 3.3. *Let $v = v_1v_2v_3\dots v_m$, where $v_i \in W$ and $v_i \neq v_{i+1}$, for $i = 1, \dots, m-1$. There are four possibilities:*

- (a) $\pi(v) = 1$,

- (b) $\pi(v) = v_l v_{l+1} \dots v_m$, where $\pi(v_1 v_2 \dots v_{l-1}) = 1$ and $1 \leq l \leq m$,
 (c) $\pi(v) = v'_l v_{j+1} v_{j+2} \dots v_m$, where $\pi(v_1 v_2 \dots v_{l-1}) = 1$, $v_l = v'_l v_j v_{j-1} \dots v_{l+1}$, $v'_l \neq 1$ and $1 \leq l < j < m$,
 (d) $\pi(v) = v'_l$, where $\pi(v_1 v_2 \dots v_{l-1}) = 1$, $v_l = v'_l v_m v_{m-1} \dots v_{l+1}$, $v'_l \neq 1$ and $1 \leq l < m$.

Consider $X = \{x_1, x_2, \dots, x_r\}$. We define an order $>$ in W inductively by the following rules:

- (i) $x_i > x_j$, if $i > j$,
 (ii) $u > v$, if $|u| > |v|$,
 (iii) If $|u| = |v|$, $u = u_1 u_2$, $v = v_1 v_2$, then $u > v$ in the following cases:
 (iii.1) $u_2 > v_2$,
 (iii.2) $u_2 = v_2$ and $u_1 > v_1$.

Definition 3.4. For any $y \in P$ there exists unique canonical decomposition $y = y_1 y_2 \dots y_{m-1} y'_m$ such that $|y_1| = 1$. We denote $y^t = y'_m y_{m-1} \dots y_1$. If $y'_m = y_k y_{k-1} \dots y_m$ with $|y_k| = 1$, then $(y^t)^t = y^{tt} = y_1 y_2 \dots y_m \dots y_k$ and $y^{ttt} = y_k y_{k-1} \dots y_m \dots y_1 = y^t$.

Definition 3.5. In notation above, define the following:

- (i) $\|y\| = m$,
 (ii) $y^* = \{x \in P \mid x^{tt} = y^t, \text{ or } x^{tt} = y^{tt}\}$.
 (iii) $y_{(i)} = y_k y_{k-1} \dots y_i (y_1 y_2 \dots y_{i-1})$, $i = 3, \dots, k$,
 (iv) $y^{(i)} = y_1 y_2 \dots y_{i-1} (y_k y_{k-1} \dots y_i)$, $i = 2, \dots, k-1$.

Example 3.6. Let $X = \{a, b, c\}$ and $y = (a(bc))((ca)b)$. Then the canonical decomposition of y is $y = y_1 y_2 y'_3$, where $y_1 = a$, $y_2 = bc$, $y'_3 = (ca)b = y_5 y_4 y_3$, and hence $\|y\| = 3$ and $\|y^t\| = \|y^{tt}\| = 5$. Furthermore, $y^t = (((ca)b)(bc))a$, $y^{tt} = (((a(bc))b)a)c$, $y_{(3)} = y_5 y_4 y_3 (y_1 y_2)$, $y_{(4)} = y_5 y_4 (y_1 y_2 y_3)$, $y_{(5)} = y_5 (y_1 y_2 y_3 y_4)$, $y^{(2)} = y_1 (y_5 y_4 y_3 y_2)$, $y^{(3)} = (y_1 y_2) (y_5 y_4 y_3)$, and $y^{(4)} = (y_1 y_2 y_3) (y_5 y_4)$. Note that $y = y^{(3)}$ and $y^* = \{y^t, y^{tt}, y_{(3)}, y_{(4)}, y_{(5)}, y^{(2)}, y^{(3)}, y^{(4)}\}$.

Define the set of symmetric words of P by $S(X) = \{y_1 y_2 \dots y_m y_{m+1} y_m \dots y_1 \mid y_i \in P, m > 0\}$.

Lemma 3.7. In notation above, we have:

- (a) $y^* = \{y^{tt} = y^{tttt}, y^t, y_{(i+1)}, y^{(i)}, i = 2, \dots, k-1\}$ and $|\{y^{tt}, y^{(i)} \mid i = 2, \dots, k-1\}| = |\{y^t, y_{(i)} \mid i = 3, \dots, k\}| = k-1$.
 (b) If $y^t = y^{tt}$, then $y^* = \{y^{tt}, y^{(i)} = y_{(k-i+2)}, i = 2, \dots, k-1\}$ and $|y^*| = k-1$.
 (c) If $y^t \neq y^{tt}$, then $|y^*| = 2(k-1)$ and $y^* \cap S(X) = \emptyset$.
 (d) If $y^t, y^{tt} \in W$, then $y^* \subset W$.
 (e) $\min\{y^t, y^{tt}\} = \min\{x \mid x \in y^*\}$.

Proof. (a) It is immediate that $\{y^t, y^{tt}, y_{(i+1)}, y^{(i)}, i = 2, \dots, k-1\} \subset y^*$ and $|\{y^{tt}, y^{(i)} \mid i = 2, \dots, k-1\}| = |\{y^t, y_{(i)} \mid i = 3, \dots, k\}| = k-1$.

Let $z \in y^*$. We have that $z^{tt} = z_1 z_2 \dots z_r$, where $z = z_1 z_2 \dots z'_l$, $z'_l = z_r z_{r-1} \dots z_l$ and $|z_1| = |z_r| = 1$. Since $|z_1| = |z_r| = 1$ and $z^{tt} \in \{y^t, y^{tt}\}$, we have $k = r$. If $z^{tt} = y^{tt}$, then $z_i = y_i$, for all i , and so $z \in \{y^{tt}, y^{(i)} | i = 2, \dots, k-1\}$. If $z^{tt} = y^t$, then $z_{k+1-i} = y_i$, for all i , and so $z \in \{y^t, y^{(i)} | i = 3, \dots, k\}$. Therefore $y^* = \{y^t, y^{tt}, y^{(i+1)}, y^{(i)}, i = 2, \dots, k-1\}$.

(b) If $y^t = y^{tt}$, then $y^{(i)} = y_{(k-i+2)}$, for all $i \in \{2, \dots, k-1\}$. Thus the claim follows from (a).

(c) If $y_{(i)} = y^{(j)}$, for some i and j , then a simple calculation shows that $j = k - i + 2$ and $y_l = y_{k+1-l}$, for all l , and so $y^t = y^{tt}$. Hence $|y^*| = 2(k-1)$ by (a). By a similar argument, we can get that $y^{(i)}, y_{(i+1)} \notin S(X)$, for all i .

(d) Let $y^{(i)} = y_1 y_2 \dots y_{i-1} (y_k y_{k-1} \dots y_i) \in y^*$. Since $y^t, y^{tt} \in W$, we have $y_1 y_2 \dots y_{i-1}, y_k y_{k-1} \dots y_i \in W$, $y_{i-1} \neq y_i$ and $y_{i-1} \neq y_k y_{k-1} \dots y_i$. Then $y_1 y_2 \dots y_{i-1} \neq \alpha (y_k y_{k-1} \dots y_i)$, for all $\alpha \in P$. Hence $y^{(i)} \in W$. By similar arguments, we can conclude that $y_{(j)} \in W$, for all j . Therefore, $y^* \subset W$.

(e) It is clear that $y^{tt} = \min\{y^{tt}, y^{(i)} | i = 2, \dots, k-1\}$ and $y^t = \min\{y^t, y_{(i)} | i = 3, \dots, k\}$. Thus the claim follows from (a). \square

Remark. We can define an equivalence relation \sim on $P(X)$ by $x \sim y$ if and only if $x^* \cap y^* \neq \emptyset$. The equivalence classes of this relation can be of three types: O_1 , O_2 and O_3 , where:

- (i) $O_1 \subset W(X) \setminus S(X)$,
- (ii) $O_2 \subset W(X)$ and $y^t = y^{tt} \in S(X)$, for $y \in O_2$,
- (iii) $O_3 \not\subset W(X)$.

Definition 3.8. For $y \in W$, let $y_0 = \min\{y^t, y^{tt}\}$. Define the set $D = D(X) = \{y_0 | y, y_0, y_0^t \in W, y_0^t \neq y_0\}$.

Example 3.9. If $X = \{a, b\}$ with $b > a$ and $W_n = \{y \in W \mid |y| = n\}$, then $D \cap W_5 = \{a, b, ba, ((ba)b)a, (b(ab))a, (b(ba))a, ((ba)(ab))a, ((a(ba))b)a, ((b(ab))a)b, ((b(ba))b)a, (b(a(ab)))a, (b(a(ba)))a, (b(b(ab)))a, (b(b(ba)))a, (b((ab)a)a, (b((ba)b)a)\}$.

Definition 3.10. Define the following sets:

- (i) $R_1 = X = \{x_1, x_2, \dots, x_r\}$,
- (ii) $R_n = R_{n-1} \cup \{y \in D(X) \mid |y| \leq n, y = u_1 u_2 \dots u_m, u_i \in R_{n-1}, i = 1, \dots, m\}$, for $n > 1$,
- (iii) $R(X) = \bigcup_{n \in \mathbb{N}} R_n$.

Notice that $X \subset R(X) \subset W$ and $R(X) \cap S(X) = \emptyset$.

Corollary 3.11. Let $b = b_1 b_2 \dots b_n \in W$, be such that $b_1 \in X$. If $b \in R(X)$, then

$$b < b^t, b^* \subset W, b_n \in X \text{ and } b_i \in R(X), \text{ for } i = 1, \dots, n. \quad (3)$$

Example 3.12. If $X = \{a, b\}$ with $b > a$, then:

$$R_5 = \{a, b, ba, ((ba)b)a, (b(ba))a, ((a(ba))b)a, ((b(ba))b)a\}.$$

Note that $(b((ba)b))a, (b((ab)a))a \in (D \cap W_5) \setminus R_5$, since $(ba)b, (ab)a \in S(X)$.

Definition 3.13. $B(X) = \{1\} \cup \{y \in W(X) | y = y_1 y_2 \dots y_n, y_i \in R(X)\}$.

Remark. Let $y = y_1 y_2 \dots y_n \in P$ be such that $y_i \in R(X)$, for all i . By Lemma 3.2, $y \in W$ if and only if $y_1 y_2 \dots y_{i-1} \neq y_i \neq y_{i+1}$, for $i \in \{1, \dots, n-1\}$.

4. Proof that $R(X)$ is a basis of $B(X)$

For proof that $R(X)$ is a basis of $B(X)$ we need the detailed information about $\pi(b)$ if $b = b_1 \dots b_k \dots b_1$, $b_i \neq b_{i+1}$ and $b_i \in R(X)$. We begin with the following simple fact.

Lemma 4.1. Let $b_1, b_2, \dots, b_k \in P$. Then $\pi(b_1 b_2 \dots b_k b_{k-1} \dots b_1) = 1$ if and only if $\pi(b_k) = 1$.

Proof. We have $\pi(b_1 \dots b_k \dots b_1) = \pi(\pi(b_1) \dots \pi(b_k) \dots \pi(b_1))$. If $\pi(b_k) = 1$, then it is clear that $\pi(b_1 \dots b_k \dots b_1) = 1$.

Now suppose that $\pi(b_1 \dots b_k \dots b_1) = 1$. Omitting all b_j, b_{j+1} such that $\pi(b_j) = \pi(b_{j+1})$, we get that $\pi(\pi(b_1) \dots \pi(b_k) \dots \pi(b_1)) = \pi(a_1 a_2 \dots a_r a_{r-1} \dots a_1)$, where $r \leq k$, $a_r = \pi(b_k)$, $a_i \neq a_{i+1}$ and $a_i \in W \setminus \{1\}$, for all $i < r$.

We will prove that $a_r = 1$ by induction on r . Consider $r > 1$ and define $a_{r+i} = a_{r-i}$, for all i . Let l be the minimal such that $\pi(a_1 a_2 \dots a_l) = 1$. If $l < 2r-1$, then $\pi(a_{l'} a_{l'+1} \dots a_1) = 1$, where $l' = 2r-1-l$, and so $\pi(a_1 a_2 \dots a_{l'}) = 1$ by Lemma 3.1. Thus we only have to consider three cases:

- (i) $l < r$. Then $\pi(a_{l+1} \dots a_r \dots a_{l+1}) = 1$, and hence $a_r = 1$ by the induction hypothesis.
- (ii) $l = r$. Then $\pi(a_1 a_2 \dots a_r) = \pi(a_1 a_2 \dots a_{r-1}) = 1$, and we get $a_r = 1$.
- (iii) $l = 2r-1$. By Lemma 3.2, if $a_r \neq 1$, then either $a_1 = a_1 a_2 \dots a_r a_{r-1} \dots a_2$ or $a_1 = v a_s a_{s-1} \dots a_2$, for some $v \neq 1$ and $s > 0$ such that $2(s-1) = 2r-3$, but both cases are impossible. Hence $a_r = 1$. \square

Lemma 4.2. Let $n > 1$ and $c, w_1, w_2, \dots, w_n \in W \setminus \{1\}$ be such that $cw_1 \in W$, $w_i \in \text{Sub}(c) \cup \text{Sub}(w_1)$ and $w_{i-1} \neq w_i$, for all i . Then $cw_1 w_2 \dots w_n \in W$.

Proof. For $1 \leq m < n$, suppose that $cw_1 w_2 \dots w_m \in W$. Since $w_{m+1} \in \text{Sub}(c) \cup \text{Sub}(w_1)$, we have that $w_{m+1} \neq cw_1 w_2 \dots w_m$. Since $w_m \neq w_{m+1}$, there is no β such that $cw_1 w_2 \dots w_m = \beta w_{m+1}$. Hence $cw_1 w_2 \dots w_m w_{m+1} \in W$. \square

Lemma 4.3. Let $k > 1$ and $w = w_1 w_2 \dots w_k w_{k-1} \dots w_1 \in S$ be such that $w_i \in W \setminus \{1\}$, $w_1 w_2 \in W$ and $w_i \neq w_{i+1}$ for all i . There are two possibilities:

- (a) $\pi(w) = w$ or
- (b) There exists l such that $3 \leq l \leq k$ and $w_l = w_1 w_2 \dots w_{l-1}$.

Proof. If $k = 2$, then $w_1 w_2 w_1 \in W$ since $w_1 w_2 \neq \alpha w_1$, for all $\alpha \in W$. Hence $\pi(w) = w$. Suppose that $k \geq 3$ and $\pi(w) \neq w$, and define $w_{k+i} = w_{k-i}$, for all i . By Lemma 3.2 (a), there exists l such that $2 < l \leq 2k - 1$ and $w_l = w_1 w_2 \dots w_{l-1}$. Since w_l is not a proper subword of itself, we must have $l \leq k$. \square

Proposition 4.4. *Let $b = b_1 b_2 \dots b_k b_{k-1} \dots b_1 \in S$ be such that $b_1 \in W \setminus \{1\}$, $b_i \in R$ and $b_{i-1} \neq b_i$ for all $i > 1$. Then $\pi(b) = \lambda b_1$, where $\lambda = 1$ implies that $k = 1$ or $b_1 \notin R$. Moreover, if $b_1 \in R$, then $\pi(b) \in R$ if and only if $k = 1$.*

Proof. If $k \in \{1, 2\}$ it is easy to see that the claim holds. Suppose that the claim holds for all $k' < k$, where $k \geq 3$. First we will prove the following lemmas.

Lemma 4.5. *Suppose that $b_m = b_1 b_2 \dots b_{m-1}$, where $3 \leq m \leq k$. Then $\pi(b) = e b_{m-1} \dots b_2 b_1 \notin R$.*

Proof. We have three cases:

(i) $m = k$. Thus $b_k = b_1 b_2 \dots b_{k-1}$, and hence $\pi(b) = \pi(b_{k-1} \dots b_2 b_1)$. Since $b_{k-1} \dots b_2 b_1 \in b_k^*$, it follows that $\pi(b) = b_{k-1} \dots b_2 b_1$. Since $b_k \in R$ and $R \cap S = \emptyset$, we have $\pi(b) = b_{k-1} \dots b_2 b_1 \notin R$.

(ii) $m = k - 1$. Thus $b_{k-1} = b_1 b_2 \dots b_{k-2}$ and $\pi(b) = \pi(b_k b_{k-1} \dots b_1)$. Since $|b_{k-1}| > 1$ and $b_k \in R$, we have $b_k b_{k-1} \in W$ by (3). By Lemma 4.2, $b_k b_{k-1} \dots b_1 \in W$, and then $\pi(b) = b_k b_{k-1} \dots b_1$. Since $b_1 b_2 \dots b_{k-1} \notin W$, it follows that $b_1 b_2 \dots b_k \in \pi(b)^* \setminus W$, and hence $\pi(b) \notin R$ by (3).

(iii) $m < k - 1$. Thus $\pi(b) = \pi(b_{m+1} \dots b_k b_{k-1} \dots b_{m+1} \dots b_2 b_1)$. By the induction hypothesis, $\pi(b_{m+1} \dots b_k b_{k-1} \dots b_{m+1}) = \lambda b_{m+1}$, where $\lambda \neq 1$ because $m + 1 < k$ and $b_{m+1} \in R$. Then $\pi(b) = \pi(b_{m+1} \dots b_k b_{k-1} \dots b_{m+1} \dots b_2 b_1) = \pi(\lambda b_{m+1} \dots b_2 b_1)$. If $b_m = \lambda b_{m+1}$, then $\pi(b) = \pi(b_{m-1} \dots b_2 b_1)$. Since $b_{m-1} \dots b_2 b_1 \in b_m^*$, it follows that $\pi(b) = b_{m-1} \dots b_2 b_1$. Furthermore, since $b_m \in R$ and $b_{m-1} \dots b_2 b_1 \neq b_m$, we have $\pi(b) = b_{m-1} \dots b_2 b_1 \notin R$.

When $\lambda b_{m+1} b_m \in W$ we have that $\lambda b_{m+1} b_m \dots b_2 b_1 \in W$ by Lemma 4.2. Then $\pi(b) = \lambda b_{m+1} \dots b_2 b_1$. Since $b_1 b_2 \dots b_m \notin W$, we have $b_1 b_2 \dots b_{m+1} \lambda \in \pi(b)^* \setminus W$, and hence $\pi(b) \notin R$ by (3).

Therefore, we proved Lemma 4.5. \square

Define $b_{k+i} = b_{k-i}$, for all i . Note that $b = b_{2k-1} b_{2k-2} \dots b_1 = b_1 b_2 \dots b_{2k-1}$.

Lemma 4.6. *Suppose that for $n \in \{2, 3, \dots, k\}$, $b_1 \in R$ and we have one of the following situations:*

(a1) $b_n = b'_n b_1 b_2 \dots b_{n-1}$, where $b'_n \neq 1$, or

(a2) $b_n = b_2 \dots b_{n-1}$.

Then $\pi(b_{2k-1} b_{2k-2} \dots b_{n+1}) \neq 1$.

Proof. Suppose by contradiction that $\pi(b_{2k-1} b_{2k-2} \dots b_{n+1}) = 1$. We have two cases:

(i) $n < k$. Let $v = b_1 b_2 \dots b_n$. Then $\pi(v b_{n+1} \dots b_k \dots b_{n+1}) = 1$, and so $\pi(v) = \pi(b_{n+1} \dots b_k \dots b_{n+1})$ by Lemma 3.1. By the induction hypothesis, we get that $\pi(v) = \lambda b_{n+1}$, where $\lambda = 1$ if and only if $n + 1 = k$. Applying Corollary 3.3 to the word $v = b_1 b_2 \dots b_n$, we have two cases $\pi(v) = \alpha b_n$ (in the cases (b) and (c)) or $\pi(v) = b'_l$, $\pi(b_1 \dots b_{l-1}) = 1$, $b_l = b'_l b_n b_{n-1} \dots b_{l+1}$ (case (d)). We note that the case (a) is impossible since $\pi(v) \neq 1$. Let $\pi(v) = \alpha b_n$. If $\alpha = 1$, then $\lambda b_{n+1} \in R$. Since $b_1 \in R$, we get that $n + 1 = k$ by the induction hypothesis, and hence $b_n = \pi(v) = b_{n+1}$, which is a contradiction. Suppose that $\alpha \neq 1$. Since $b_n \neq b_{n+1}$, it follows that $\lambda = 1$, and then $|b_n| = 1$ by (3), which contradicts (a1) and (a2).

Let $\pi(v) = b'_l$ and $l = 1$. Then $b_1 = \lambda b_{n+1} b_n b_{n-1} \dots b_2$. In (a1) this does not occur since $b_1 \in \text{Sub}(b_n)$. Now consider the case (a2). Since $\pi(b_2 b_3 \dots b_n) = 1$ in this case, we get that $b'_1 \in b_1^* \setminus W$, and then $b_1 \notin R$ by (3), a contradiction.

If $\pi(v) = b'_l$ and $1 < l < n$, $\pi(b_1 b_2 \dots b_{l-1}) = 1$ and $b_l = b'_l b_n b_{n-1} \dots b_{l+1}$. Then we have a contradiction since $b_l \in \text{Sub}(b_n)$ in both cases (a1) and (a2).

(ii) $n = k$. By assumption, we have that $\pi(b_1 b_2 \dots b_{k-1}) = 1$, and then $\pi(b_{k-1} \dots b_2 b_1) = 1$ by Lemma 3.1. First, consider the case (a1). Since $b_k = b'_k b_1 b_2 \dots b_{k-1}$ and $\pi(b_{k-1} \dots b_2 b_1) = 1$, it follows that $b'_k \in b_k^* \setminus W$, and then $b_k \notin R$ by (3), a contradiction.

Now consider the case (a2). Since $b_k = b_2 \dots b_{k-1}$, then $|b_{k-1}| = 1$ by (3), and so $b_{k-1} b_{k-2} \in W$. Since $\pi(b_{k-1} \dots b_2 b_1) = 1$, it follows that there exists l such that $b_l = b_{k-1} b_{k-2} \dots b_{l+1}$ by Lemma 3.2. If $l > 1$, then $\pi(b_{k-1} b_{k-2} \dots b_{l+1} b_l) = 1$, and so $b'_k \in b_k^* \setminus W$, which is a contradiction. If $l = 1$, then $b_1 = b'_k$, which is a contradiction since $b_1, b_k \in R$.

Therefore, Lemma 4.6 is proved. \square

Now we can finish the proof of Proposition 4.4. First, let us prove that $\pi(b) = \lambda b_1$, for some $\lambda \in W$, where $\lambda \neq 1$ if $b_1 \in R$. By Lemma 4.1, we have that $\pi(b) \neq 1$, and then there are three possibilities according to Corollary 3.3:

(i) $\pi(b) = b'_n b_m \dots b_1$, where $b_n = b'_n b_{m+1} b_{m+2} \dots b_{n-1}$, $1 \leq m < n \leq 2k - 1$ and $b'_n \neq 1$ if $m = 1$. Thus we have the desired result.

(ii) $\pi(b) = b'_n$, where $b_n = b'_n b_1 b_2 \dots b_{n-1}$, $1 < n < 2k - 1$, $b'_n \neq 1$ and $\pi(b_{2k-1} b_{2k-2} \dots b_{n+1}) = 1$. Since b_n can not be a proper subword of itself, we get $n \leq k$. Furthermore, we get that $b_1 \in R$ by (3). Then we have a contradiction with Lemma 4.6.

(iii) $\pi(b) = b_n b_{n-1} \dots b_1$, where $\pi(b_{2k-1} b_{2k-2} \dots b_{m+1}) = 1$, $b_m = b_{n+1} b_{n+2} \dots b_{m-1}$ and $1 \leq n < m \leq 2k - 1$. If $n > 1$ or $n = 1$ and $b_1 \notin R$, then we have the desired result. Suppose that $\pi(b) = b_1 \in R$. We have two cases:

(iii.1) $|b_2| > 1$. By (3), b_1 can not be of the form $b'_1 b_2$, and then $b_1 b_2 \in W$. By Lemmas 4.3 and 4.5, we get that $\pi(b) \notin R$, which is a contradiction.

(iii.2) $|b_2| = 1$. Note that $b_m = b_2 b_3 \dots b_{m-1}$. Since b_m can not be a proper subword of itself and $b_2 b_3 \dots b_k \dots b_2 \notin R$, it follows that $m \leq k$. Then $\pi(b_{2k-1} b_{2k-2} \dots b_{m+1}) \neq 1$ by Lemma 4.6, which is a contradiction.

Now we only have to prove that $\pi(b) \notin R$ when $b_1 \in R$. Consider that $b_1 \in R$ and $\pi(b) = \lambda b_1$, where $\lambda \neq 1$. If $|b_1| > 1$, then $\pi(b) \notin R$ by (3). If $|b_1| = 1$, then $b_1 b_2 \in W$, and as in (iii.1) we get that $\pi(b) \notin R$. \square

Corollary 4.7. *Let $b = b_1 b_2 \dots b_k b_{k-1} \dots b_1 \in S$ and $g = g_1 g_2 \dots g_n g_{n-1} \dots g_1 \in S$ be such that $b_i, g_j \in R$, $b_{i-1} \neq b_i$ and $g_{j-1} \neq g_j$, for all i and j . If $\pi(b) = \pi(g)$, then $b_1 = g_1$.*

As a consequence of Proposition 4.4 and Lemma 4.5 we have the following result:

Corollary 4.8. *Let $b = v b_1 b_2 \dots b_k b_{k-1} \dots b_1 v \in S$ be such that $v \in W \setminus \{1\}$, $v b_1 \in W$, $b_i \in R$ and $b_{i-1} \neq b_i$ for all $i > 1$. There are two possibilities:*

- (a) $\pi(b) = b$,
- (b) *There exists m such that $2 \leq m \leq k$ and $\pi(b) = \epsilon b_{m-1} \dots b_1 v$, where $\epsilon \in W$.*

Lemma 4.9. *Let $b = v b_1 v \in S$ and $g = v g_1 g_2 \dots g_n g_{n-1} \dots g_1 v \in S$ be such that $v \in W \setminus \{1\}$, $v \neq b_1$, $v \neq g_1$, $b_1, g_j \in R$, and $g_{j-1} \neq g_j$, for all j . If $\pi(b) = \pi(g)$, then $b = g$.*

Proof. If $n = 1$, then $\pi(v b_1) = \pi(v g_1)$. By Lemma 3.1 (f), $b_1 = g_1$, and hence $b = g$. Now suppose that $n > 1$ and the claim holds for every $n' < n$. We will prove this result in two steps:

- (i) First we will prove that there exists $\alpha \in W \setminus \{1\}$ such that $\alpha b_1 \alpha \in W$, $\pi(\alpha b_1 g_1 g_2 \dots g_n g_{n-1} \dots g_1 b_1 \alpha) = \alpha b_1 \alpha$ and either $\alpha b_1 \neq g_1$ or $\alpha = v$. We have two cases:
 - (i.1) $v = \alpha b_1$, with $\alpha \neq 1$. Then $\pi(v b_1 v) = \pi(\alpha v) = \pi(v g_1 g_2 \dots g_n g_{n-1} \dots g_1 v)$, and hence by Lemma 3.1 (c):

$$\pi(\alpha) = \pi(v g_1 g_2 \dots g_n g_{n-1} \dots g_1). \quad (4)$$

Using Lemma 3.1 (c) and $v = \alpha b_1$ in (4), we get $\pi(\alpha b_1 \alpha) = \pi(\alpha b_1 g_1 \dots g_n \dots g_1 b_1 \alpha)$.

- (i.2) $v b_1 v \in W$. By $\pi(v g_1 g_2 \dots g_n g_{n-1} \dots g_1 v) = \pi(v b_1 v) = v b_1 v$ and Lemma 3.1 (e), we get $\pi(v b_1 g_1 g_2 \dots g_n g_{n-1} \dots g_1 v) = 1$. Thus

$$\pi(v b_1 g_1 \dots g_n \dots g_1 b_1 v) = \pi(v b_1 g_1 \dots g_n \dots g_1 v v b_1 v) = \pi(\pi(v b_1 g_1 \dots g_n \dots g_1 v) \pi(v b_1 v)) = v b_1 v$$

and we put $\alpha = v$.

- (ii) Now consider $\alpha \in W \setminus \{1\}$ as in (i). If $b_1 = g_1$, then

$$\pi(\alpha b_1 g_1 g_2 \dots g_n \dots g_1 b_1 \alpha) = \pi(\alpha g_2 \dots g_n \dots g_2 \alpha) = \alpha b_1 \alpha.$$

If $\alpha \neq g_2$, then by induction $b_1 = g_2 \dots g_n \dots g_2$. Since $b_1 \in R$ and $R \cap S = \emptyset$, then $n = 2$ and $b_1 = g_2$, which is a contradiction with $b_1 = g_1 \neq g_2$. In the case $\alpha = g_2$ and $n > 2$, we get $\pi(g_3 \dots g_n \dots g_3) = \alpha b_1 \alpha = g_2 b_1 g_2$. By Corollary 4.7, we have $g_3 = g_2$, a contradiction. Finally, if $\alpha = g_2$ and $n = 2$, we have $\pi(g_2) = g_2 b_1 g_2$. Since $g_2 \in R$, hence $b_1 = g_2$, which is a contradiction with $b_1 = g_1$.

Suppose $b_1 \neq g_1$. By the choice of α , either $\alpha b_1 g_1 \in W$ or $g_1 = \alpha b_1$. We have two cases:
(ii.1) $g_1 = \alpha b_1$. Then $\alpha b_1 \alpha = \pi(\alpha b_1 g_1 \dots g_n g_{n-1} \dots g_1 b_1 \alpha) = \pi(g_2 \dots g_n g_{n-1} \dots g_1 b_1 \alpha)$. Hence by Lemma 3.1 (c) we get $\pi(g_2 \dots g_n g_{n-1} \dots g_1) = \alpha$ and $\alpha g_1 = \alpha b_1 \alpha \in W$. Using the same lemma again, we get $\pi(g_2 \dots g_n g_{n-1} \dots g_2) = \alpha g_1$. If $n = 2$, then $g_2 = \alpha g_1 \in R$, and hence $g_1 \in X$, which is a contradiction with $g_1 = \alpha b_1$. Then $n > 2$. By Proposition 4.4, there exists $\lambda \neq 1$ such that $\alpha g_1 = \lambda g_2$, and then $g_1 = g_2$, a contradiction.

(ii.2) $\alpha b_1 g_1 \in W$. Note that $\alpha b_1 \alpha \neq \alpha b_1 g_1 \dots g_n \dots g_1 b_1 \alpha$. Then there exists m such that $1 < m \leq n$ and $g_m = \alpha b_1 g_1 \dots g_{m-1}$ by Lemma 4.3. We have three more cases:

(ii.2.1) $m = n$. Then $\alpha b_1 \alpha = \pi(g_{n-1} \dots g_1 b_1 \alpha)$. Since $g_{n-1} \dots g_1 b_1 \alpha \in g_n^*$ and $g_n \in R$, it follows that $\alpha b_1 \alpha = g_{n-1} \dots g_1 b_1 \alpha$, which is a contradiction because $g_n^* \cap S(X) \neq \emptyset$.

(ii.2.2) $m = n - 1$. Then $\alpha b_1 \alpha = \pi(\alpha b_1 g_1 \dots g_n \dots g_1 b_1 \alpha) = \pi(g_n g_{n-1} \dots g_1 b_1 \alpha)$. Since $|g_{n-1}| > 1$ and $g_n \in R$, we have $g_n g_{n-1} \in W$. By Lemma 4.2, $g_n g_{n-1} \dots g_1 b_1 \alpha \in W$. Thus $\alpha b_1 \alpha = g_n g_{n-1} \dots g_1 b_1 \alpha$, and hence $\alpha = g_n g_{n-1} \dots g_1$, which is a contradiction because $\alpha \in \text{Sub}(g_{n-1})$.

(ii.2.3) $m < n - 1$. Then $\alpha b_1 \alpha = \pi(g_{m+1} \dots g_n \dots g_1 b_1 \alpha)$. By Proposition 4.4, there exists $\lambda \neq 1$ such that $\alpha b_1 \alpha = \pi(\lambda g_{m+1} g_m \dots g_1 b_1 \alpha)$ and $\lambda g_{m+1} \in W$. If $g_m = \lambda g_{m+1}$, similarly to (ii.2.1) we get a contradiction. If $\lambda g_{m+1} g_m \in W$, similarly to (ii.2.2) we get a contradiction. \square

Lemma 4.10. *Let $b = vb_1 b_2 \dots b_k b_{k-1} \dots b_1 v \in S$ and $g = vg_1 g_2 \dots g_n g_{n-1} \dots g_1 v \in S$ be such that $v \in W \setminus \{1\}$, $v \neq b_1$, $v \neq g_1$, $b_i, g_j \in R$, $b_{i-1} \neq b_i$ and $g_{j-1} \neq g_j$, for all i and j . If $\pi(b) = \pi(g)$, then $b = g$.*

Proof. We will prove this result by induction on k . We can consider $n \geq k > 1$. First we will prove the Affirmation 1:

Affirmation 1. If $b_1 = g_1$, then $b = g$.

Proof of Affirmation 1. We have that $\pi(vb_1 b_2 \dots b_k b_{k-1} \dots b_2) = \pi(vb_1 g_2 \dots g_n g_{n-1} \dots g_2)$. Then $\pi(\pi(vb_1) b_2 \dots b_k b_{k-1} \dots b_2 \pi(vb_1)) = \pi(\pi(vb_1) g_2 \dots g_n g_{n-1} \dots g_2 \pi(vb_1))$. If $b_2 \neq \pi(vb_1)$, then $g_2 \neq \pi(vb_1)$ by Proposition 4.4 and Corollary 4.7, and hence the result follows by the induction hypothesis.

Suppose that $b_2 = \pi(vb_1)$. If $k = 2$, then $\pi(b_2 g_2 \dots g_n g_{n-1} \dots g_2 b_2) = b_2$. By Proposition 4.4, we must have $b_2 = g_2$ and $n = 2$, and then $b = g$. Using a similar argument, we get that $k = 3$ implies $b = g$. Now consider $k > 3$. Then $\pi(b_3 \dots b_k b_{k-1} \dots b_3) = \pi(b_2 g_2 g_3 \dots g_n g_{n-1} \dots g_3 g_2 b_2)$. By $b_2 \neq b_3$ and Corollary 4.7, we must have $b_2 = g_2$ and $b_3 = g_3$, and hence the desired result follows by the induction hypothesis.

Therefore, Affirmation 1 is proved.

By Proposition 4.4, there exist $\alpha, \beta \in W$ such that $\pi(vb_1 \dots b_k \dots b_1 v) = \alpha v$ and $\pi(vg_1 \dots g_n \dots g_1 v) = \beta v$. We have ten cases depending on α and β :

(i) $\alpha = 1$ (or $\beta = 1$).

- (ii) $\alpha b_1, \beta g_1 \in W$ and $\alpha, \beta \neq 1$.
- (iii) $\alpha = \alpha_1 b_1$ and $\beta = \beta_1 g_1$, where $\alpha_1 \neq 1 \neq \beta_1$.
- (iv) $\alpha = b_1$ and $\beta = g_1$.
- (v) $\alpha = b_1, \beta \neq 1$ and $\beta g_1 \in W$.
- (vi) $\alpha = \alpha_1 b_1$ and $\beta = g_1$, where $\alpha_1 \neq 1$.
- (vii) $\alpha \neq 1, \alpha b_1 \in W$ and $\beta = g_1$.
- (viii) $\alpha = b_1$ and $\beta = \beta_1 g_1$, where $\beta_1 \neq 1$.
- (ix) $\alpha = \alpha_1 b_1$ and $\beta g_1 \in W$, where $\alpha_1 \neq 1 \neq \beta$.
- (x) $\alpha b_1 \in W$ and $\beta = \beta_1 g_1$, where $\alpha \neq 1 \neq \beta_1$.

We note that if $vb_1 \in W$, then $\pi(vb_1 \dots b_k \dots b_1 v) = \epsilon b_1 v$ by Corollary 4.8. Analogously, if $vg_1 \in W$, then $\pi(vg_1 \dots g_n \dots g_1 v) = \eta g_1 v$. Hence in the cases (i) and (ii) we have $vb_1, vg_1 \notin W$. Since $v \in W$, then $v = v'b_1 = v''g_1$, for some $v', v'' \in W$, and so $b_1 = g_1$. The same equality $b_1 = g_1$ we have in the cases (iii) and (iv). Then in those cases the claim follows by Affirmation 1.

Case (v): Since $\beta g_1 \in W$, then $vg_1 \notin W$ by Corollary 4.8. Hence $v = \beta' g_1$, for some $\beta' \neq 1$. We have that $\pi(vb_1 \dots b_k \dots b_1 v) = b_1 v$, hence by Lemma 3.1 (c) we get $\pi(vb_1 b_2 \dots b_k \dots b_2) = 1$ and $\pi(b_2 \dots b_k b_{k-1} \dots b_2 b_1 v) = 1$. Using the same lemma again, we get $\pi(b_2 \dots b_k b_{k-1} \dots b_2) = \pi(vb_1)$. If $vb_1 \notin W$, then $v = \gamma b_1 = \beta' g_1$, for some $\gamma \neq 1$, and so $b_1 = g_1$. By Affirmation 1, Lemma 4.10 is proved. Let $vb_1 \in W$. By Proposition 4.4, we get $\pi(b_2 \dots b_k \dots b_2) = \lambda b_2 = vb_1$. If $k > 2$, then $\lambda \neq 1$, and hence $b_1 = b_2$, a contradiction. Then $k = 2$ and $b_2 = vb_1$. From $\pi(vg_1 \dots g_n \dots g_1 v) = b_1 v$, we have $\pi(vg_1 \dots g_n \dots g_1) = b_1$ and $\pi(vg_1 \dots g_n \dots g_1 b_1) = 1$ by Lemma 3.1 (c). Hence $\pi(b_1 g_1 \dots g_n \dots g_1 v) = 1$ and $\pi(b_1 g_1 \dots g_n \dots g_1) = v$. Then $\pi(b_1 g_1 \dots g_n \dots g_1 b_1) = vb_1 = b_2$. If $b_1 \neq g_1$, then $b_2 \notin R$ by Proposition 4.4, which is a contradiction. In the case $b_1 = g_1$ we have Lemma 4.10 by Affirmation 1.

Case (vi): We have that $g_1 = \alpha_1 b_1$ and $\pi(vg_1 g_2 \dots g_n g_{n-1} \dots g_2) = 1$. Then $\pi(g_2 \dots g_n g_{n-1} \dots g_2) = \pi(vg_1)$. We have two cases:

(vi.1) $n = 2$. Then $g_2 = \pi(vg_1)$. Since $g_1 v = \pi(vb_1 \dots b_k \dots b_1 v)$, we have that $\pi(g_1 b_1 \dots b_k \dots b_1 v) = 1$, and then $\pi(g_1 b_1 \dots b_k \dots b_1 g_1) = \pi(vg_1) = g_2$, which is a contradiction with Proposition 4.4.

(vi.2) $n > 2$. By Proposition 4.4, there exists $\lambda \neq 1$ such that $\lambda g_2 = \pi(vg_1)$. Since $g_1 \neq g_2$, it follows that $vg_1 \notin W$. Then there exists $\gamma \neq 1$ such that $v = \gamma g_1$. Since $\pi(vb_1 b_2 \dots b_k b_{k-1} \dots b_1 v) = g_1 v$, we have that $\pi(g_1 b_1 b_2 \dots b_k b_{k-1} \dots b_1 g_1) = \gamma$. By Proposition 4.4, there exists $\lambda' \neq 1$ such that $\lambda' g_1 = \gamma$, which is a contradiction because $v = \gamma g_1 \in W$.

The cases (vii) and (viii) are analogous to the cases (v) and (vi), respectively.

Case (ix): We have that $\pi(vb_1 b_2 \dots b_k b_{k-1} \dots b_2 \alpha_1) = 1$. Then $\pi(\alpha_1 b_2 \dots b_k b_{k-1} \dots b_2 b_1) = v$. Since $\pi(vg_1 g_2 \dots g_n g_{n-1} \dots g_1 v) = \alpha_1 b_1 v$, it follows that $\pi(\alpha_1 b_1 g_1 g_2 \dots g_n g_{n-1} \dots g_1) = v$, and then $\pi(\alpha_1 b_2 \dots b_k b_{k-1} \dots b_2 \alpha_1) = \pi(\alpha_1 b_1 g_1 g_2 \dots g_n g_{n-1} \dots g_1 b_1 \alpha_1)$.

If $b_1 = g_1$, then the claim follows by Affirmation 1. Suppose that $b_1 \neq g_1$. By Proposition 4.4 and Corollary 4.7, we have that $\alpha_1 \neq b_2$, and hence the claim follows by the induction hypothesis.

Case (x): By Corollary 4.8, we have $vb_1 \notin W$. Then there exists $a \neq 1$ such that $v = ab_1$. Since $\pi(vb_1b_2\dots b_kb_{k-1}\dots b_1v) = \pi(vg_1g_2\dots g_ng_{n-1}\dots g_1v)$, it follows that $\pi(ab_2\dots b_kb_{k-1}\dots b_1) = \pi(ab_1g_1g_2\dots g_ng_{n-1}\dots g_1)$. Thus $\pi(ab_2\dots b_kb_{k-1}\dots b_2a) = \pi(ab_1g_1g_2\dots g_ng_{n-1}\dots g_1b_1a)$, and the rest of the proof is analogous to (ix). \square

As a consequence of Proposition 4.4, Corollary 4.7 and Lemma 4.10, we have the following result.

Corollary 4.11. *Let $b = b_1b_2\dots b_kb_{k-1}\dots b_1 \in S$ and $g = g_1g_2\dots g_ng_{n-1}\dots g_1 \in S$ be such that $b_i, g_j \in R$, $b_{i-1} \neq b_i$ and $g_{j-1} \neq g_j$, for all i and j . If $\pi(b) = \pi(g)$, then $b = g$.*

Theorem 4.12. *For every $g \in B(X) \setminus \{1\}$ there exist unique $g_1, \dots, g_m \in R(X)$ such that $g = \pi(g_1g_2\dots g_mg_{m-1}\dots g_1)$ and $g_i \neq g_{i+1}$, for $i = 1, \dots, m-1$.*

Proof. Let $g = a_1a_2\dots a_s \in B$, where $a_s = b_1b_2\dots b_l$, $|a_1| = |b_1| = 1$, and $a_i, b_j \in R$, for all i and j . First we will prove by induction on $|g|$ that there exists $s(g) = g_1g_2\dots g_mg_{m-1}\dots g_1$ such that $\pi(s(g)) = g$ and $g_i \neq g_{i+1}$, for $i = 1, \dots, m-1$. If $|g| = 1$ or $g \in R$, then $s(g) = g$. Now suppose that $|g| > 1$ and $g \notin R$. We have four cases:

(i) $g^t, g^{tt} \in W$ and $g^t \neq g^{tt}$. Then

$$s(g) = \begin{cases} a_s a_{s-1} \dots a_2 a_1 g^t a_1 \dots a_{s-1} a_s, & \text{if } g^t \in R, \\ a_s b_1 b_2 \dots b_l g^{tt} b_l \dots b_1 a_s, & \text{if } g^{tt} \in R. \end{cases}$$

(ii) $g^t \in W \cap S$. Then there exist $c_1, \dots, c_r \in R$ such that $g^t = c_1c_2\dots c_rc_{r-1}\dots c_1$. Thus $g = \pi(a_s a_{s-1} \dots a_2 a_1 c_1 c_2 \dots c_r c_{r-1} \dots c_1 a_1 \dots a_{s-1} a_s)$, and it is clear that we can get $s(g)$ from this equation.

(iii) $g^t \notin W$. Since $\pi(g) \neq 1$, we have that $\pi(g^t) \neq 1$ by Lemma 3.1. Then $1 \leq |\pi(g^t)| < |g|$. It is not difficult to see that $\pi(g^t) \in B$. By the induction hypothesis, there exist $c_1, \dots, c_r \in R$ such that $\pi(g^t) = \pi(c_1c_2\dots c_rc_{r-1}\dots c_1)$, and by using similar arguments as in (ii) we get $s(g)$.

(iv) $g^{tt} \notin W$. This case is analogous to (iii).

Now we need to prove that $s(g)$ is unique. But this is a consequence of Corollary 4.11. \square

5. Main theorem

Definition 5.1. In notation of Theorem 4.12 we put for any $g \in B \setminus \{1\}$:

$$s(g) = g_1g_2\dots g_mg_{m-1}\dots g_1.$$

Now, define a multiplication \circ on the set $B(X)$ by:

(i) $x \circ 1 = 1 \circ x = x$,

(ii) $x \circ y = \pi(xy_1y_2\dots y_my_{m-1}\dots y_1)$, where $s(y) = y_1y_2\dots y_my_{m-1}\dots y_1$.

Notice that the identities $x \circ x = 1$ and $(x \circ y) \circ y = x$ can be easily obtained from the definition above.

Theorem 5.2. *The set $B = B(X)$ with multiplication \circ defined above is a free Bol loop of exponent 2 with free set of generators X .*

Proof. It is clear that X generates B and if B is a Bol loop, then the construction of $R(X)$ and Theorem 4.12 give us a natural way to extend a mapping between X and another Bol loop L of exponent two to a homomorphism from B into L . So we only have to prove that B is a Bol loop. It is possible to prove this directly, but in this case we have to consider many particular cases. We choose the other way based on the connection of Bol loops with twisted subgroups described in the Preliminaries.

Let $G = \prod_{y \in R(X)} \star < R_y | R_y^2 = I_d >$ be a free 2-group. The group G acts on B : $bR_y = b \circ y$ and $bI_d = b$. Then the set $H = \{g \in G | 1^g = 1\}$ is a subgroup of G , where 1 is the empty word of $B(X)$.

Now, let $B' = \{I_d\} \cup \{R_y | y \in R(X)\}^G$. Note that $R_yR_zR_y \in B'$, for all $y, z \in R(X)$, and then B' is a twisted subgroup of G .

Lemma 5.3. $G = HB'$.

Proof of Lemma 5.3. Let $g = \prod_{i=1}^m R_{g_i} \in G$ and $y = g_1g_2\dots g_m$. Then $1^g = \pi(y)$. If $\pi(y) = 1$, then $g \in H$.

Suppose that $\pi(y) \neq 1$ and consider $s(\pi(y)) = y_1y_2\dots y_ky_{k-1}\dots y_1$, where $y_i \in R(X)$. Note that $S(g) = R_{y_1}R_{y_2}\dots R_{y_k}R_{y_{k-1}}\dots R_{y_1} \in B'$. We have that $1^{S(g)} = \pi(s(\pi(y))) = \pi(y)$, and so $\pi(y)^{S(g)} = 1$. Hence $gS(g) \in H$ and $g = (gS(g))S(g) \in HB'$.

Lemma 5.3 is proved.

Lemma 5.4. $H \cap (B'B') = \{I_d\}$.

Proof of Lemma 5.4. Let $b = R_{b_1}R_{b_2}\dots R_{b_m}R_{b_{m-1}}\dots R_{b_1} \in B'$ and $c \in B'$ be such that $bc \in H$. By Lemma 4.1, it follows that $\pi(b_1\dots b_m\dots b_1) \neq 1$, and then $c \neq I_d$. Consider $c = R_{c_1}R_{c_2}\dots R_{c_k}R_{c_{k-1}}\dots R_{c_1}$. Hence

$$(\dots(b_1 \circ b_2)\dots) \circ b_m) \circ b_{m-1})\dots \circ b_1) \circ c_1)\dots) \circ c_k)\dots) \circ c_1 = 1. \quad (5)$$

Since $(x \circ y) \circ y = x$, we get $(\dots(b_1 \circ b_2)\dots) \circ b_m) \circ b_{m-1})\dots \circ b_1) = (\dots(c_1 \circ c_2)\dots) \circ c_k)\dots) \circ c_1$. Then $\pi(b_1b_2\dots b_mb_{m-1}\dots b_1) = \pi(c_1c_2\dots c_k\dots c_1)$. By Corollary 4.11, we get that $m = k$ and $c_i = b_i$, for all i .

Lemma 5.4 is proved.

As a consequence of the Lemmas above and Proposition 2.1, we have the following result.

Proposition 5.5. *(G, H, B') is a Baer triple. Furthermore, B' with the operation $*$ defined by $b * b' = c$, where $bb' = hc$, for some $h \in H$, is a Bol loop of exponent two.*

Now, let us conclude the proof of Theorem 5.2. We just need to prove that $(B, \circ) \cong (B', *)$. Define $\varphi : B' \rightarrow B$ by

$$\varphi(R_{y_1}R_{y_2}\dots R_{y_m}R_{y_{m-1}}\dots R_{y_1}) = \pi(y_1y_2\dots y_my_{m-1}\dots y_1) \text{ and } \varphi(I_d) = 1.$$

By Lemma 4.1 and Theorem 4.12, we get that φ is a bijection.

Let $b = R_{y_1}R_{y_2}\dots R_{y_m}R_{y_{m-1}}\dots R_{y_1} \in B'$ and $c = R_{z_1}R_{z_2}\dots R_{z_n}R_{z_{n-1}}\dots R_{z_1} \in B'$. Consider $y = y_1y_2\dots y_my_{m-1}\dots y_1$, $z = z_1z_2\dots z_nz_{n-1}\dots z_1$ and $u = yz_1\dots z_n\dots z_1$. Note that $\varphi(b) \circ \varphi(c) = \pi(y) \circ \pi(z) = \pi(u)$.

If $\pi(u) = 1$, then $\pi(y) = \pi(z)$. By Corollary 4.11, we have $y = z$. Thus $b = c$ and $\varphi(b * c) = \varphi(I_d) = 1 = \varphi(b) \circ \varphi(c)$.

If $\pi(u) \neq 1$, consider $s(\pi(u)) = u_1u_2\dots u_ru_{r-1}\dots u_1$ and $g = R_{u_1}R_{u_2}\dots R_{u_r}R_{u_{r-1}}\dots R_{u_1}$. By the proof of Lemma 5.3, we have that $b * c = g$. Hence $\varphi(b * c) = \varphi(g) = \pi(u) = \varphi(b) \circ \varphi(c)$. Therefore φ is an isomorphism and we have that (B, \circ) is a Bol loop of exponent two. \square

Proposition 5.6. *H is a core-free subgroup of G .*

Proof. Let $N \leq H$ be such that N is normal in G . Suppose that $N \neq \{I_d\}$. Then there exists $\phi = R_{y_1}R_{y_2}\dots R_{y_n} \in N$, with $n > 1$, $y_i \in R(X)$ and $y_i \neq y_{i+1}$, for all i .

Since $1\phi = 1$, it follows that $\pi(y_1y_2\dots y_n) = 1$. Then

$$\pi(y_n\dots y_2y_1y_2) = y_2. \quad (6)$$

Since $N \triangleleft G$ and $N \leq H$, we have $R_{y_2}\phi R_{y_2} \in H$, and then $1R_{y_2}\phi R_{y_2} = 1$. Thus $1R_{y_2}\phi = 1R_{y_2}$, and hence $y_2\phi = y_2$. By (6), we get $\pi(y_n\dots y_2y_1y_2)\phi = y_2$, and then $y_2 = \pi(y_n\dots y_2y_1y_2y_1y_2\dots y_n)$, which is a contradiction with Proposition 4.4. \square

Now we will determine the nuclei and the center of $B(X)$. Firstly, we need the following lemma.

Lemma 5.7. *Let $x, z \in B(X) \setminus \{1\}$. Then $z = x \circ (x \circ z)$ if and only if $x = z$.*

Proof. Suppose that $x \neq z$ and $z = x \circ (x \circ z)$. Consider $s(z) = z_1z_2\dots z_nz_{n-1}\dots z_1$ and $s(x \circ z) = u_1u_2\dots u_mu_{m-1}\dots u_1$. Then

$$\pi(u_1\dots u_mu_{m-1}\dots u_1) = x \circ z = \pi(xz_1\dots z_n\dots z_1), \quad (7)$$

$$\pi(xu_1\dots u_mu_{m-1}\dots u_1) = x \circ (x \circ z) = z = \pi(z_1\dots z_n\dots z_1) \quad (8)$$

By (7) and (8), we get $\pi(u_1 \dots u_m \dots u_1) = \pi(z_1 \dots z_n \dots z_1 u_1 \dots u_m \dots u_1 z_1 \dots z_n \dots z_1)$. Then $m = n$ and $u_i = z_i$, for all i , by Corollary 4.11. Therefore $x = 1$, a contradiction. \square

As a consequence of Lemma 5.7, we have that $(x \circ (x \circ z)) \circ z \neq 1 = x \circ ((x \circ z) \circ z)$, for every $x, z \in B(X) \setminus \{1\}$ such that $x \neq z$. It follows that $N_\lambda(B)$, $N_\mu(B)$ and $N_\rho(B)$ contain only the identity element 1. Therefore we established the following result.

Corollary 5.8. *The nuclei and the center of $B(X)$ are trivial.*

6. Open problems

We finish this paper with two conjectures.

If $|X| > 1$, it is easy to construct proper subloops of $B(X)$ that are free Bol loops of exponent 2. In the case of free loops (infinite exponent), it is well known that all subloops of these loops are free [4, Corollary 1, pg. 16].

Conjecture 6.1. *Every subloop of a free Bol loop of exponent two is free.*

Let $Y = \{y_1, y_2, \dots, y_n\}$ be a free set of generators of $B(X)$. For $i \in \{1, 2, \dots, n\}$ and $v \in \langle Y \setminus \{y_i\} \rangle$, define $e_{(i,v)}, f_{(i,v)} : B(X) \rightarrow B(X)$ by

$$e_{(i,v)}(y_i) = y_i v, \quad f_{(i,v)}(y_i) = v y_i \quad \text{and} \quad e_{(i,v)}(y_j) = f_{(i,v)}(y_j) = y_j,$$

for every $j \in \{1, 2, \dots, n\} \setminus \{i\}$. The mappings $e_{(i,v)}$ and $f_{(i,v)}$ are automorphisms of $B(X)$ and they are called *elementary automorphisms* of $B(X)$. An automorphism of $B(X)$ is called *tame* if it belongs to the group generated by all elementary automorphisms of $B(X)$. A question concerning free objects in varieties of loops is whether all of their automorphisms are tame. For free Steiner loops the answer to this question is positive [6, Theorem 7].

Conjecture 6.2. *Every automorphism of a free Bol loop of exponent two is tame.*

Data availability

No data was used for the research described in the article.

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References

- [1] M. Aschbacher, On Bol loops of exponent 2, *J. Algebra* 288 (1) (2005) 99–136.
- [2] R. Baer, Nets and groups, *Trans. Am. Math. Soc.* 47 (1939) 110–141.
- [3] B. Baumeister, A. Stein, Self-invariant 1-factorizations of complete graphs and finite Bol loops of exponent 2, *Contrib. Algebr. Geom.* 51 (1) (2010) 117–135.
- [4] R.H. Bruck, *A Survey of Binary Systems*, Springer, 1971.
- [5] T. Foguel, M.K. Kinyon, J.D. Phillips, On twisted subgroups and Bol loops of odd order, *Rocky Mt. J. Math.* 36 (1) (2006) 183–212.
- [6] A. Grishkov, D. Rasskazova, M. Rasskazova, I. Stuhl, Free Steiner triple systems and their automorphism groups, *J. Algebra Appl.* 14 (02) (2015) 1550025.
- [7] H. Kiechle, *Theory of K-Loops*, *Lecture Notes in Mathematics*, vol. 1778, Springer, Berlin, 2002.
- [8] H. Kiechle, G.P. Nagy, On the extension of involutorial Bol loops, *Abh. Math. Semin. Univ. Hamb.* 72 (2002) 235–250.
- [9] G.P. Nagy, Solvability of universal Bol 2-loops, *Commun. Algebra* 26 (2) (1998) 549–555.
- [10] G.P. Nagy, A class of finite simple Bol loops of exponent 2, *Trans. Am. Math. Soc.* 361 (2009) 5331–5343.
- [11] H.O. Pflugfelder, *Quasigroups and Loops: Introduction*, *Sigma Series in Pure Math.*, vol. 7, Heldermann, 1990.